## Worksheet 9.1 Pierce Electron Gun Design



This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.



This worksheet calculates the dimensions of a Pierce electron gun. The method is essentially that described by Pierce (Theory and Design of Electron Beams) and Gittins (Power Travelling-Wave Tubes). It also computes approximate shapes for the electrodes and the position of the magnetic pole-piece

The data provided is for the design discussed in Section 9.2.5

**Basic parameters of the design**

| Anode voltage | Beam current | Beam radius | Cathode loading | Ratio of magnetic field to Brillouin field |
|---|---|---|---|---|
| $V_a := 2 \cdot kV$ | $I_0 := 0.17 \cdot A$ | $b := 1.35 \cdot mm$ | $J_c := 0.084 \cdot A \cdot cm^{-2}$ | $B0BB := 2.0$ |

*Parameters of the model*

| Length of the solution space (mm) | Effective anode voltage / anode voltage | Cathode temperature |
|---|---|---|
| $yy_{end} := 40$ | $VF := 0.8$ | $T_k := 1300 \cdot K$ |

The details of the calculation can be concealed to allow the input data and the results to be on the screen simultaneously.

*Define physical constants*

Charge / mass ratio of the electron $\eta := 1.759 \cdot 10^{11} \cdot C \cdot kg^{-1}$

Rest energy of the electron (eV) $V_R := 511 \cdot kV$

Microperveance $\mu P := \mu A \cdot V^{-1.5}$

Boltzmann's constant $k := 1.38 \cdot 10^{-23} \cdot J \cdot K^{-1}$

Elementary charge $q_e := 1.602 \cdot 10^{-19} \cdot C$

*Electrostatic design - Step 1*

Calculate the perveance, beam velocity and $\beta_p$

Perveance $$Perv := \frac{I_0}{V_a^{1.5}}$$ Equation 9.1 $Perv = 1.90 \cdot \mu P$

Beam velocity $$u_0 := c \cdot \left[1 - \frac{1}{\left(1 + \frac{V_a}{V_R}\right)^2}\right]^{0.5}$$ Equation 1.4 $u_0 = 2.645 \times 10^7 \cdot m \cdot s^{-1}$

$$\gamma_R := 1 + \frac{V_a}{V_R}$$ Equation 7.2

Plasma frequency $$\omega_p := \left(\frac{\eta \cdot I_0}{\pi \cdot b^2 \varepsilon_0 \cdot u_0}\right)^{0.5}$$ Equation 7.47

$$\beta_p := \frac{\omega_p}{u_0 \cdot \gamma_R^{\,3}}$$ $\beta_p = 176.5 \cdot m^{-1}$

*Step 2*

Calculate the cathode disc radius and the area convergence

Cathode radius $$r_c := \left(\frac{I_0}{\pi \cdot J_c}\right)^{0.5}$$ $r_c = 0.316 \cdot in$

Area convergence $$Aconv := \left(\frac{r_c}{b}\right)^2$$ $Aconv = 35.3$

*Step 3*

Define α for a spherical space-charge limited diode and $\theta_0$ as functions of $R_c/R$

$$\gamma(Rc\_R) := \ln\left(\frac{1}{Rc\_R}\right)$$ Equation 5.83

$$\alpha(Rc\_R) := \gamma(Rc\_R) - 0.3 \cdot \gamma(Rc\_R)^2 + 0.075 \cdot \gamma(Rc\_R)^3 - 0.14318 \cdot \gamma(Rc\_R)^4 + 0.00216 \cdot \gamma(Rc\_R)^5 - 0.0002679 \cdot \gamma(Rc\_R)^6$$ Equation 5.85

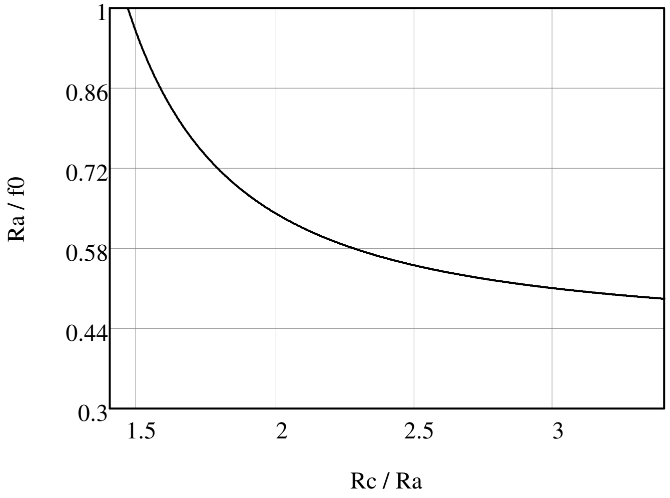
$$\theta 0(Rc\_R) := \operatorname{acos}\left(1 - \frac{9 \cdot \alpha(Rc\_R)^2}{8 \cdot \pi \cdot \varepsilon_0 \cdot \sqrt{2 \cdot \eta}} \cdot Perv\right)$$ Equation 9.4

Note:
Logically *Perv* in this equation should be divided by $VF^{1.5}$ because the potential at $R_a$ is taken to be $V_e$. This change does not make a big difference to the results. Values close to those obtained without the correction can be obtained by making a small adjustment to *VF*.

*Step 4 Define $R_a/f_0$ as a function of $R_c/R$*

$$Ra\_f0(Rc\_R) := \frac{2 \cdot (V_a + V_R)}{3 \cdot (V_a + 2 \cdot V_R)} \cdot \frac{Rc\_R}{\alpha(Rc\_R)} \cdot \frac{d}{dRc\_R}(\alpha(Rc\_R))$$ Equation 9.8

Correction factor for focal length as a function of Rc/Ra and perveance




*Non-relativistic graph agrees with Gittins fig. 5.4 except that he has labelled the y axis wrong*

*Anode lens correction factor*

Effective electron energy $V_e := VF \cdot V_a$

$$\phi(Rc\_R) := \operatorname{atan}\left[\frac{V_e + V_R}{V_e \cdot \left(V_e + 2 \cdot V_R\right)} \cdot \frac{2V_a}{3} \cdot \frac{2 \cdot (1 - \cos(\theta 0(Rc\_R)))}{\sin(\theta 0(Rc\_R))} \cdot \left[\frac{Rc\_R}{\alpha(Rc\_R)} \cdot \frac{d}{dRc\_R}(\alpha(Rc\_R))\right]\right]$$

Equations 9.8 and 9.25

*Step 5*

Calculate the angle of convergence after the beam has passed through the anode lens as a function of Rc/Ra

$$\theta 1(Rc\_R) := \theta 0(Rc\_R) - \phi(Rc\_R)$$

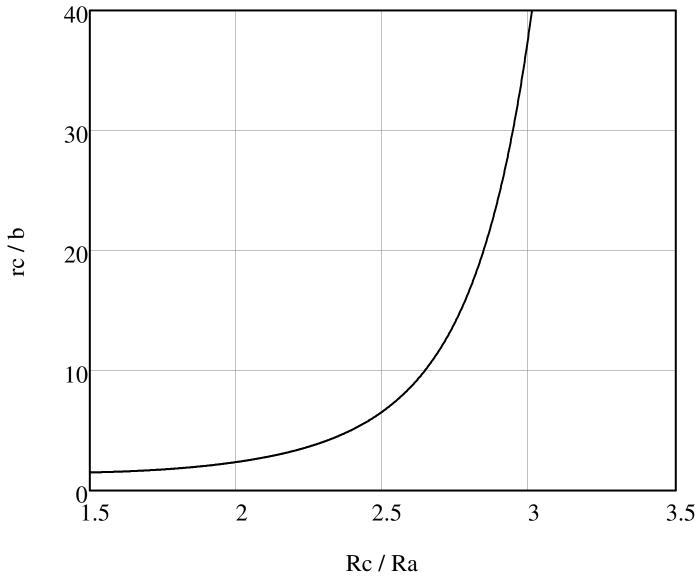
Equation 9.27

Step 6

Define $r_c/b$ as a function of $R_c/R$

$$rc\_b(Rc\_R) := Rc\_R \cdot \exp\left[\left(\frac{\tan(\theta 1(Rc\_R))}{\beta_p \cdot b}\right)^2\right]$$

Equation 9.13



*Step 7  Calculate thermal velocity effects (Section 9.2.4)*

$$AA(Rc\_R) := \int_{1}^{Rc\_R} \frac{1}{(-\alpha(RcR1))^{\frac{2}{3}}}\, dRcR1$$

Equation 9.31

$$x := 1, 1.01 .. 3.8$$

$$BB(ra\_b) := \mathrm{erf}\left(\sqrt{\ln(ra\_b)}\right)$$

Equation 9.32

At the anode

$$\sigma_{b\_norm}(Rc\_R) := \left[\frac{(-\alpha(Rc\_R))^{\frac{2}{3}}}{rc\_b(Rc\_R)} \cdot AA(Rc\_R) + 3 \cdot BB\left(\frac{rc\_b(Rc\_R)}{Rc\_R}\right)\sqrt{\frac{\pi}{2} \cdot (-\alpha(Rc\_R))^{2}}\right]$$

Equation 9.30

At the beam waist

$$\sigma_{a\_norm}(Rc\_R) := \left[\frac{(-\alpha(Rc\_R))^{\frac{2}{3}} \cdot AA(Rc\_R)}{Rc\_R}\right]$$

$$J_r(r\sigma, re\sigma) := \exp\left(\frac{-r\sigma^2}{2}\right) \cdot \int_{0}^{re\sigma} x \cdot \exp\left(\frac{-x^2}{2}\right) \cdot I0(x \cdot r\sigma)\, dx$$

Equation 9.29

This integral does not converge if reσ > 25 because of the growth in the Bessel function I0. This could be solved by using a large argument approximation

Plotting data $r\sigma 1 := 0, 0.1 .. 5$ $r\sigma 2 := 0, 0.1 .. 10$ $r\sigma 3 := 5, 5.1 .. 15$ $xb := 1.68, 1.7 .. 3.4$

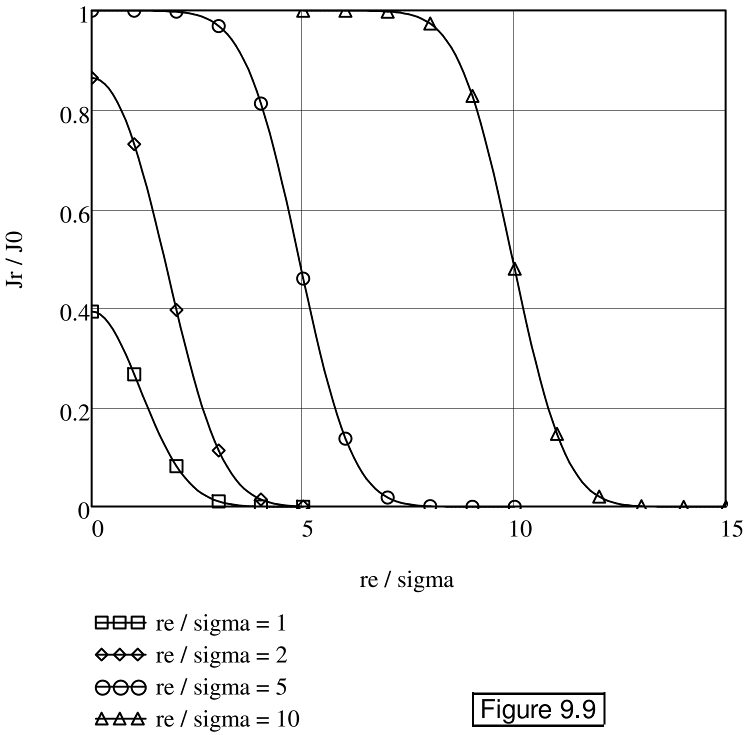



Figure 9.9

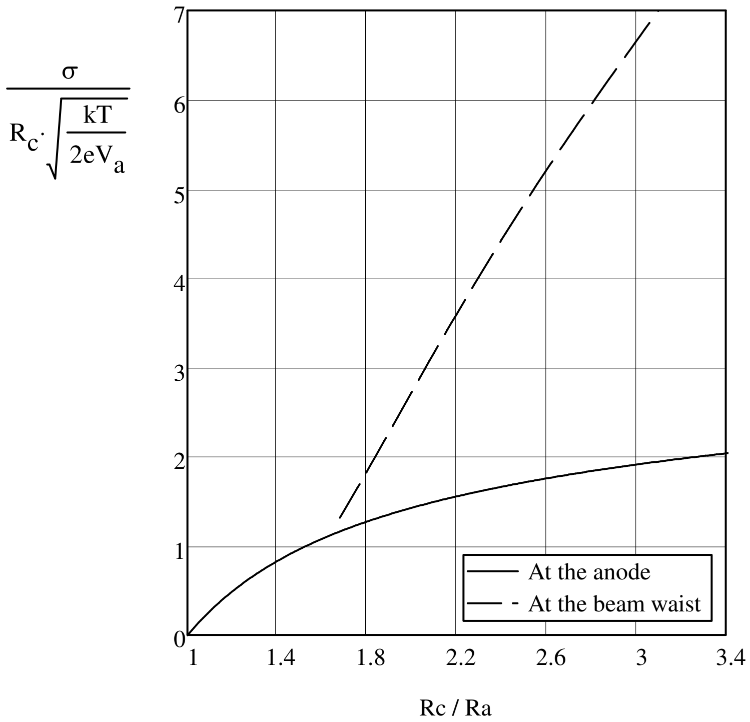



Figure 9.10

Calculate the fraction of current contained within radius r/σ

$$I(rr\sigma, re\sigma) := \frac{2}{re\sigma^2}\left(\int_0^{rr\sigma} r\sigma \cdot J_r(r\sigma, re\sigma)\, dr\sigma\right)$$

Find r/σ enclosing *trans*% of the beam current as a function of re/σ

$rr := 1.$ $\quad r\sigma(re\sigma, trans) := \mathrm{root}(I(rr, re\sigma) - trans, rr, 2, 25)$ $\quad xr := 0.1, 0.2 .. 10$

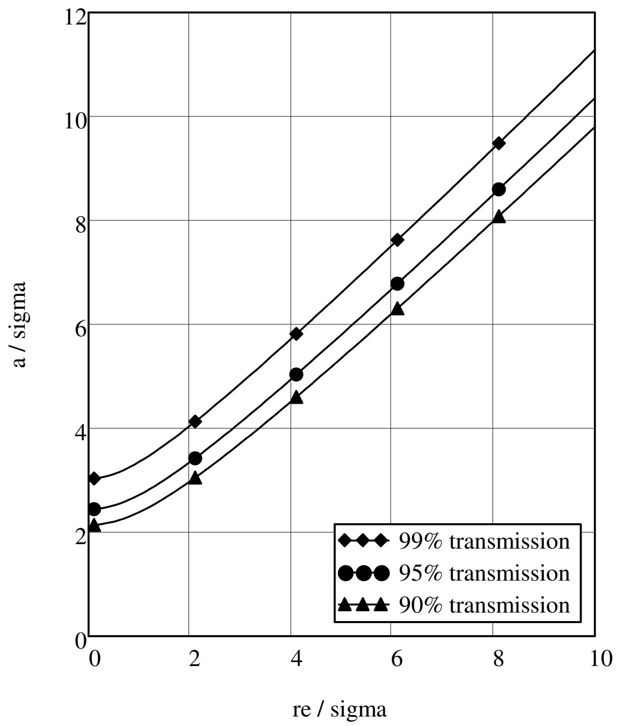



Figure 9.11

*Step 8*

Find the ratio $R_c/R_a$ for the given values of $r_c$ and b starting with a guessed value of $R_c/R$

$$Rc\_R := 2.2 \qquad Rc\_Ra := \mathrm{root}\left[\left(rc\_b(Rc\_R) - \frac{r_c}{b}\right), Rc\_R, 1.5, 3.5\right] \qquad Rc\_Ra = 2.458$$

*Check* $\frac{r_c}{b} = 5.945$ $\quad rc\_b(Rc\_Ra) = 5.945$

*Step 9 Compute the gun parameters from the value of $R_c$ / $R_a$*

Angle of convergence $\theta_0 := \theta0(Rc\_Ra)$ $\quad \theta_0 = 38.40 \cdot deg$

Spherical radius of the cathode $R_c := \frac{r_c}{\sin(\theta_0)}$ $\quad R_c = 12.92 \cdot mm$

Spherical radius of the anode $R_a := \frac{R_c}{Rc\_Ra}$ $\quad R_a = 5.26 \cdot mm$

Beam radius at the anode $r_a := R_a \cdot \sin(\theta_0)$ $\quad r_a = 3.27 \cdot mm$

Radius of the anode aperture $r_{anode} := 1.2 \cdot r_a$ $\quad r_{anode} = 0.154 \cdot in$

Cathode-anode separation $z_{ac} := R_c - R_a \cdot \cos(\theta_0)$ $\quad z_{ac} = 8.80 \cdot mm$

Beam angle at the exit from the anode $\theta_1 := \theta1(Rc\_Ra)$ $\quad \theta_1 = 12.62 \cdot deg$

*Step 10 Find the distance from the anode to the beam waist.*

The universal beam spreading curve is defined by

$$\beta pz(r\_b) := \int_0^{\sqrt{\ln(r\_b)}} 2\cdot\exp\left(x^2\right) dx$$

using the substitution

$$x = \sqrt{\ln\left(\frac{r}{b}\right)}$$

Equation 7.74

$$z_m := \frac{\beta pz\left(\frac{r_a}{b}\right)}{\beta_p}$$

$$\beta pz\left(\frac{r_a}{b}\right) = 2.616$$

$$z_m = 14.82\cdot mm$$

Distance from cathode to the beam waist

$$z_{cm} := R_c - R_a\cdot\cos\left(\theta_0\right) + z_m$$

$$z_{cm} = 23.63\cdot mm$$

*Plot the electrodes and the electrostatic beam edge*

Define a parameter to scan each section of line $x := 0, 0.01 .. 1$

Plot the cathode

$$r1(x) := \frac{R_c}{mm}\cdot\sin\left(\theta_0\cdot x\right) \qquad z1(x) := \frac{R_c}{mm}\cdot\left(1-\cos\left(\theta_0\cdot x\right)\right)$$

Plot the anode

$$r2(x) := \frac{R_a}{mm}\cdot\sin\left(\theta_0\cdot x\right) \qquad z2(x) := \frac{R_c}{mm} - \frac{R_a}{mm}\cdot\cos\left(\theta_0\cdot x\right)$$

Plot the initial beam cone

$$r12(x) := \frac{x\cdot R_a}{mm}\cdot\sin\left(\theta_0\right) \qquad z12(x) := \frac{R_c}{mm} - \frac{R_a}{mm}\cdot x\cdot\cos\left(\theta_0\right)$$

Plot the beam edge in two sections (cathode - anode and anode - waist)

$$r3(x) := \frac{r_c}{mm} - \frac{x \cdot (r_c - r_a)}{mm} \qquad z3(x) := \frac{(R_c - R_a)}{mm} \cdot x \cdot \cos(\theta_0) + \frac{R_c}{mm} \cdot (1 - \cos(\theta_0))$$

$$r4(x) := \frac{r_a}{mm} - \frac{x}{mm} \cdot (r_a - b) \qquad z4(x) := z3(1) + \left( \frac{\beta pz\left(\frac{r_a}{b}\right)}{\beta_p \cdot mm} \right) - \frac{\beta pz\left(\frac{r4(x) \cdot mm}{b}\right)}{\beta_p \cdot mm}$$

*Step 11 Calculate thermal velocity effects*

Volt equivalent of cathode temperature $\quad V_T := \frac{k \cdot T_k}{q_e} \quad V_T = 0.112\,V$

At the anode $\quad \sigma_a := \sigma_{a\_norm}(Rc\_Ra) \cdot R_c \cdot \sqrt{\frac{V_T}{2 \cdot V_a}} \quad \sigma_a = 0.116 \cdot mm \quad \frac{r_a}{\sigma_a} = 28.202$

At the beam waist $\quad \sigma_b := \sigma_{b\_norm}(Rc\_Ra) \cdot R_c \cdot \sqrt{\frac{V_T}{2 \cdot V_a}} \quad \sigma_b = 0.319 \cdot mm \quad \frac{b}{\sigma_b} = 4.233$

Radius for 99% transmission $\quad rb99 := r\sigma\left(\frac{b}{\sigma_b}, 0.99\right) \cdot \sigma_b \quad b = 1.35 \cdot mm \quad rb99 = 1.89 \cdot mm$

Beam filling factor $\quad \frac{b}{rb99} = 0.71$

*Step 12* *Calculate the theoretical shapes of the focus electrode and the anode*

Formulae given by Sar-El, Nucl. Instrum. Meth. Vol.203, pp.21-33 (1982). Note that the parameter R is the inverse of that used by Sar-El.

Define the constants of Langmuir's series for $\alpha$, the revised parameter $\gamma$ and the revised function $\alpha 1$.

$C1 :=$

| | 0 |
|---|---|
| 0 | ... |

$$\gamma 1(R, \theta) := \ln(R) + j \cdot \left(\theta - \theta_0\right)$$ Equation 9.14

$$\alpha 1(R, \theta) := \sum_{n=0}^{10} \left(C1_n \cdot \gamma 1(R, \theta)^n\right)$$

Calculate the electrostatic potential Ve as a function of R and, hence, find the equipotential surface at the potential $V_s$.

$$Ve(R, \theta) := Re\left[\left(\alpha 1(R, \theta)^2\right)^{\frac{2}{3}}\right]$$ Equation 9.15

$$RC\left(V_s, \theta\right) := root\left[\left(Ve(R, \theta) - V_s\right), R, 0.001, 1\right]$$

*Plot the shapes of the focus electrode and the anode*

Plot the focus electrode ($V_s$ = 0) $r5(\theta) := \frac{R_c}{mm} \cdot RC(0, \theta) \cdot \sin(\theta)$ $z5(\theta) := \frac{R_c}{mm} \cdot (1 - RC(0, \theta) \cdot \cos(\theta))$

Plot the anode nose ($V_s = V_e$)

$$V_s := Ve\left(\frac{1}{Rc\_Ra}, \theta_0\right)$$

$$r6(\theta) := \frac{R_c}{mm} \cdot RC\left(V_s, \theta\right) \cdot \sin(\theta) \qquad z6(\theta) := \frac{R_c}{mm} \cdot \left(1 - RC\left(V_s, \theta\right) \cdot \cos(\theta)\right)$$

Scan a range of angles to plot the curves $\theta := \theta_0, \theta_0 + 0.01 .. 0.35 \cdot \pi$

*Step 13 Magnetic design*

Find the final radius of the flux line which passes through the cathode edge for the chosen ratio of $B_0$ to $B_B$ and find the value of the coordinate ζ at the cathode edge.

$$\rho_0 := b \cdot \left(1 - \frac{1}{B0BB^2}\right)^{\frac{1}{4}}$$

Equation 9.44

$$f1(\zeta) := 1 - \frac{1}{\pi} \cdot \left(\operatorname{acot}(\zeta) - \frac{\zeta}{1 + \zeta^2}\right) \qquad \rho(\zeta) := \rho_0 \cdot f1(\zeta)^{-0.5}$$

Equation 9.40

$$\zeta_c := \operatorname{root}\left[\left(\rho(\zeta) - r_c\right), \zeta, -100, 100\right] \qquad \zeta_c = -1.852$$

Set the slope of the flux line at the cathode rim equal to the angle of convergence of the beam and find the size of the aperture in the pole-piece.

$d\rho\_d\zeta(\zeta) := \left(\frac{d}{d\zeta}\rho(\zeta)\right)$  $d\rho\_d\zeta(\zeta_c) = -5.308 \cdot mm$  $\frac{d\rho\_d\zeta(\zeta_c)}{dz\_d\zeta(\zeta_c, a0)} = \frac{d\rho}{dz}$

$z(\zeta, a) := \zeta \cdot a$  $dz\_d\zeta(\zeta, a) := \frac{d}{d\zeta} z(\zeta, a)$

$a0 := r_c$  $ap := \mathrm{root}\left(\frac{d\rho\_d\zeta(\zeta_c)}{dz\_d\zeta(\zeta_c, a0)} + \tan(\theta_0), a0\right)$  $ap = 6.698 \cdot mm$

Position of cathode relative to the pole-piece

$y_c := z(\zeta_c, ap)$  $y_c = -12.41 \cdot mm$

Plot the flux line through the cathode edge in coordinates referred to the centre of the cathode

$r7(\zeta) := \frac{\rho(\zeta)}{mm}$  $z7(\zeta) := \frac{-y_c + z(\zeta, ap)}{mm} + z1(1)$

Pole-piece position relative to the cathode centre  $z_{cp} := z7(0) \cdot mm$  $z_{cp} = 15.20 \cdot mm$

*Step 14*

Solve the differential equation for the beam edge starting from the anode in the presence of the magnetic focusing field. Note that the position of the pole-piece is now defined using $y_c$ = ap*$\zeta_c$ to ensure that the cathode flux is consistent with the rest of the calculation.

Brillouin field $$B_B := \sqrt{\frac{2}{\gamma_R}\cdot\frac{\omega_p}{\eta}}$$ Equation 7.55 $$B_B = 0.038\,T$$

Focusing field $$B_0 := B0BB\cdot B_B$$ $$B_0 = 0.076\,T$$

$$\omega_L := \frac{B0BB}{\sqrt{2\cdot\gamma_R}}\cdot\omega_p$$ Equation 7.56 $$\beta_L := \frac{\omega_L}{u_0}$$

$$K_c := 1 - \frac{1}{B0BB^2}$$ Equation 7.57 $$K_c = 0.75$$

Cathode flux $$\Phi_c := \sqrt{K_c}\cdot\pi\cdot b^2\cdot B_0$$ Equation 7.51 $$\Phi_c = 3.76\times10^{-7}\,Wb$$

Define the values of βpb, βLb and of b in millimetres because the Mathcad ODE solver does not accept dimensioned variables. Set up the differential equation and solve it subject to the initial conditions $r = r_a$ and dr/dz = - tan( θ1 ) at the edge of the anode for axial positions relative to the anode edge (denoted by yy) in the range 0 < yy < $yy_{end}$.

$$\beta pb := \beta_p\cdot b \qquad \beta Lb := \beta_L\cdot b \qquad bmm := \frac{b}{mm} \qquad apmm := \frac{ap}{mm}$$

$$f2(yy,r) := \left[\frac{\beta pb^2}{2}\cdot\frac{1}{r} + \beta Lb^2\cdot K_c\cdot\frac{bmm^2}{r^3} - \frac{\beta Lb^2}{bmm^2}\cdot f1\left[\frac{\left(yy + \zeta_c\cdot apmm - z1(1) + z2(1)\right)}{apmm}\right]^2\cdot r\right]$$

Given $\quad R2''(yy) = f2(yy, R2(yy)) \qquad R2(0) = \frac{r_a}{mm} \qquad R2'(0) = -\tan(\theta_1)$

$R2 := \text{Odesolve}(yy, yy_{end})$

$yy := 0, 0.1 .. yy_{end}$

Calculate and plot the magnetic field relative to the cathode centre

$$f3(z) := f1\left[\frac{(z + \zeta_c \cdot apmm - z1(1))}{apmm}\right]$$

Field and flux at the beam waist $\quad BwB0 := f3\left(\frac{z_m}{mm}\right) \qquad BwB0 = 0.464$

$$\Phi_w := \pi \cdot b^2 \cdot BwB0 \cdot B_0$$

The details of the calculation can be concealed to allow the input data and the results to be seen more clearly

**Summary of the results**

$\mathrm{Perv} = 1.90 \cdot \mu\mathrm{P}$

$\mathrm{Aconv} = 35.3$

$R_c = 12.92 \cdot \mathrm{mm}$

$R_a = 5.26 \cdot \mathrm{mm}$

$r_c = 8.026 \cdot \mathrm{mm}$

$r_a = 3.265 \cdot \mathrm{mm}$

$\theta_0 = 38.40 \cdot \mathrm{deg}$

$\theta_1 = 12.62 \cdot \mathrm{deg}$

$z_{ac} = 8.8 \cdot \mathrm{mm}$

$z_m = 14.82 \cdot \mathrm{mm}$

$r_{anode} = 3.92 \cdot \mathrm{mm}$

$z_{cm} = 23.63 \cdot \mathrm{mm}$

$B_0 = 0.076\,\mathrm{T}$

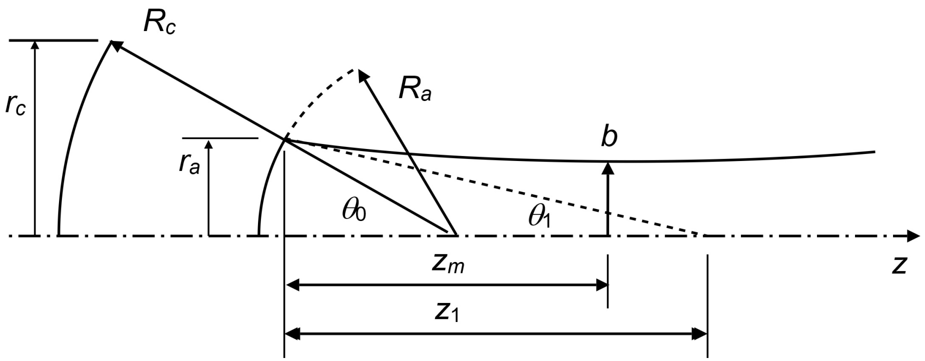
$\Phi_c = 3.76 \times 10^{-7}\,\mathrm{Wb}$

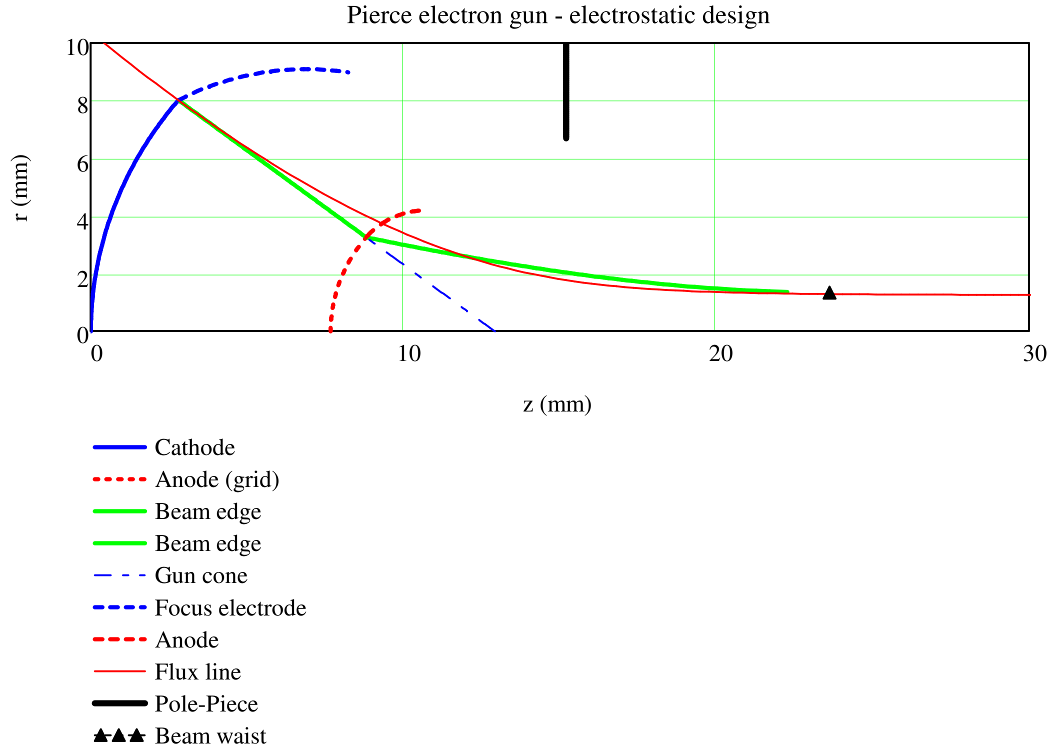
$\mathrm{ap} = 6.70 \cdot \mathrm{mm}$

$z_{cp} = 15.20 \cdot \mathrm{mm}$

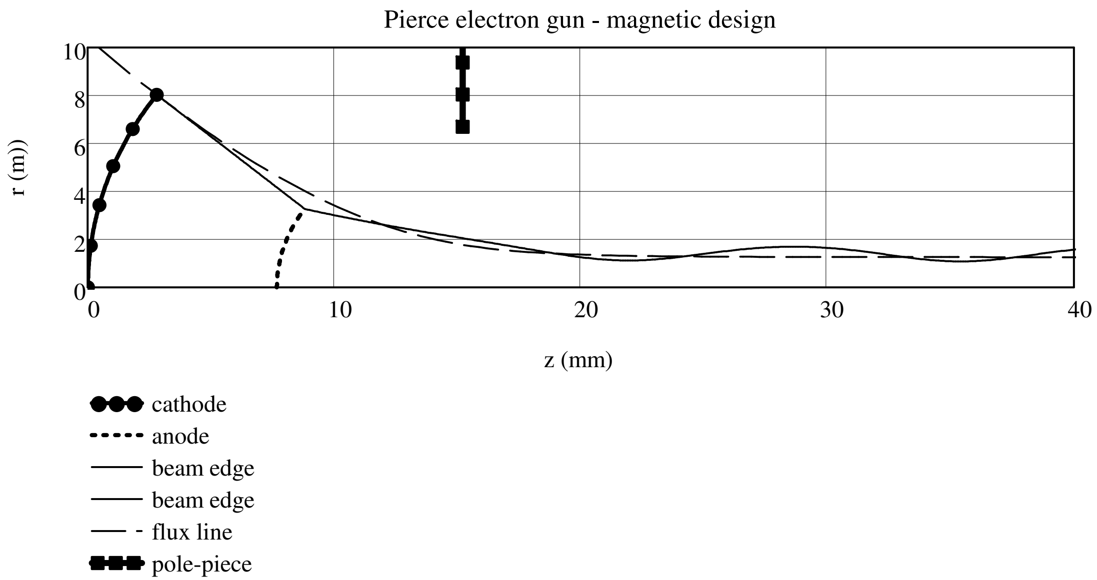
$\mathrm{rb99} = 1.89 \cdot \mathrm{mm}$

$\dfrac{\mathrm{rb99}}{b} = 1.403$

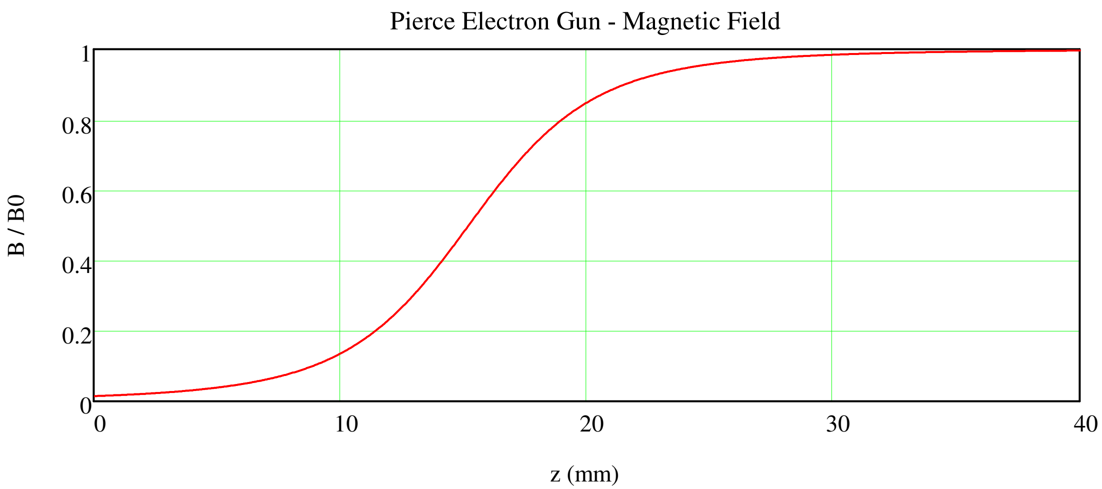



Note: These results differ slightly from those in Table 9.1 because some small adjustments have been made to the model since those results were calculated.



Pierce electron gun - magnetic design
r (m))
z (mm)
0
10
20
30
40
2
4
6
8
cathode
anode
beam edge
beam edge
flux line
pole-piece

Pierce Electron Gun - Magnetic Field
B / B0
z (mm)
0
0.2
0.4
0.6
0.8
1
10
20
30
40