WS 13.3 Large Signal Model of a Klystron

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

Disk model

The bunch is represented by a set of rigid disks of equal dimensions whose charges are equal. The motion of ND disks of charge is tracked with time as the independent variable using Runge-Kutta integration. The results of the calculation are transferred into the space domain at nmax equally spaced reference planes. Good results are normally obtained with: ND = 24, nmax = 100. The initial position of the bunch should be such that no electrons have entered the field of the first gap. The final time should be great enough to ensure that all the electrons have moved clear of the output gap.

The space-charge calculation is based on the field of a disc of charge in a conducting tunnel found using the quasi-static approximation. This works well for low beam voltages. For high beam voltages the space-charge is reduced by the factor SCF which is imported from WS 13.1. If SCF = 0 the model is run without space-charge. For simplicity the space-charge force is assumed to be periodic in space and the fields acting on the discs are averaged across them. Electrons are treated as positive, the beam current is positive and the r.f. current is positive at the bunch centre.

The gap voltages for the same input power are imported from WS 13.1 to overcome significance errors at low drive levels. Good results are obtained if these voltages are used except for the last two cavities and any cavities tuned to harmonic frequencies.

Define the parameters of the model - SLAC 50 MW klystron

Anode voltage	Beam current	Frequency	Input power	Input phase	Tunnel radius	Beam radius
$V_a := 315 \cdot kV$	$I_0 := 354 \cdot A$	$f := 2856 \cdot MHz$	$P_{in} := 50 \cdot W$	$\phi_{in} := 0 \cdot deg$	$a := 15.9 \cdot mm$	$b := 11.0 \cdot mm$

Number of cavities

$NCAV := 6$

Small-signal results used up to cavity SCAV

$SCAV := 4$

Space-charge factor imported from WS13.1

$SCF := 0.46$

Number of discs (even)

$ND := 24$

Number of integration steps

$nmax := 500$

Number of reference planes

$NP := 1000$

Gap field parameter

$kgap := 4$

Bunch centre initial position ($\theta = \beta_e z$)

$\theta_0 := -2\pi$

Initial time ($\phi = \omega t$)

$\phi_0 := \theta_0$

Final position

$\theta_f := 18 \cdot \pi$

Final time

$\phi_f := 2 \cdot \theta_f$

Define the cavity parameters

NB. The first element of each vector is not used. The cavity count starts from 1. A cavity is unloaded if Qe >= 95000.

Cavity frequency

$$fc := \begin{pmatrix} 0 \\ 2860 \\ 2870 \\ 2890 \\ 2910 \\ 2970 \\ 2853 \end{pmatrix} \cdot MHz$$

Cavity harmonic

$$nh := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

External Q

$$Qe := \begin{pmatrix} 0 \\ 200 \\ 95000 \\ 95000 \\ 95000 \\ 95000 \\ 21 \end{pmatrix}$$

Unloaded Q

$$Q0 := \begin{pmatrix} 0 \\ 2000 \\ 2000 \\ 2000 \\ 2000 \\ 2000 \\ 2000 \end{pmatrix}$$

R/Q

$$R\_Q := \begin{pmatrix} 0 \\ 80 \\ 75 \\ 87 \\ 96 \\ 96 \\ 85 \end{pmatrix} \cdot \Omega$$

Gap length

$$gap := \begin{pmatrix} 0 \\ 0.0068 \\ 0.0072 \\ 0.0082 \\ 0.011 \\ 0.0116 \\ 0.0162 \end{pmatrix} \cdot m$$

Gap position

$$zg := \begin{pmatrix} 0 \\ 0 \\ 0.056 \\ 0.111 \\ 0.166 \\ 0.444 \\ 0.555 \end{pmatrix} \cdot m$$

The detailed calculations can be hidden to allow the data and results to be viewed on the screen simultaneously

Define the charge/mass ratio of the electron. Note that the primary electric constant and the velocity of light are already defined in Mathcad.

$\eta := 1.759 \cdot 10^{11} \cdot \frac{C}{kg}$ $\quad \varepsilon_0 = 8.854 \times 10^{-12} \cdot \frac{F}{m}$ $\quad c = 2.998 \times 10^8 \frac{m}{s}$ $\quad \mu Perv := \mu A \cdot V^{-1.5}$ $\quad dB := 1$

Calculate tube constants and small-signal parameters

Calculate the beam voltage and velocity allowing for space-charge potential depression and relativity

Equation 7.8

$$V_0 := \left| \begin{array}{l} V_0 \leftarrow V_a \\ \text{for } n \in 0..3 \\ \quad \left| \begin{array}{l} u_n \leftarrow c \cdot \left[1 - \dfrac{1}{\left(1 + \dfrac{\eta \cdot V_n}{c^2}\right)^2} \right]^{0.5} \\ V_{n+1} \leftarrow V_0 - \dfrac{I_0}{2 \cdot \pi \cdot \varepsilon_0 \cdot u_n} \cdot \left(\dfrac{1}{2} + \ln\left(\dfrac{a}{b}\right) \right) \end{array} \right. \\ \text{return } V_{n+1} \end{array} \right.$$

Equation 1.4

$$u_0 := c \cdot \left[1 - \frac{1}{\left(1 + \frac{\eta \cdot V_0}{c^2}\right)^2} \right]^{0.5} \qquad Rel := \frac{1}{\sqrt{1 - \frac{u_0^2}{c^2}}}$$

$V_0 = 291.1 \cdot kV$ $\quad u_0 = 2.311 \times 10^8 \cdot m \cdot s^{-1}$ $\quad Rel = 1.570$

$\frac{I_0}{V_a^{1.5}} = 2.00 \cdot \mu Perv$ $\quad G_0 := \left| \frac{I_0}{V_a} \right| = 1.124 \cdot \frac{1}{k\Omega}$

Electronic propagation constant $\quad \omega := 2 \cdot \pi \cdot f$ $\quad \beta_e := \frac{\omega}{u_0}$ $\quad \lambda_e := \frac{2 \cdot \pi}{\beta_e}$ $\quad \gamma_e := \sqrt{\beta_e^2 - \frac{\omega^2}{c^2}}$

$\gamma_e \cdot b = 0.5$ $\quad \gamma_e \cdot a = 0.787$

Compute the parameters of the cavities

$\beta eg := \beta_e \cdot gap \qquad \theta g := \beta_e \cdot zg$

Equation 3.5

Shunt conductance

$$Gc := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Gc_n \leftarrow \left(R\_Q_n \cdot Q0_n\right)^{-1} \\ \text{return } Gc \end{array} \right.$$

External conductance

$$Ge := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad \left| \begin{array}{l} Ge_n \leftarrow \left(R\_Q_n \cdot Qe_n\right)^{-1} \text{ if } Qe_n < 95000 \\ Ge_n \leftarrow 0 \text{ otherwise} \end{array} \right. \\ \text{return } Ge \end{array} \right.$$

Cavity admittance at beam harmonic frequency

$$Yc := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Yc_n \leftarrow Gc_n \cdot \left[1 + j \cdot Q0_n \cdot \left(\frac{f \cdot nh_n}{fc_n} - \frac{fc_n}{f \cdot nh_n}\right)\right] \\ \text{return } Yc \end{array} \right.$$

Axial gap coupling factor

Equation 11.36 adapted for non-uniform field

$$\mu d(\beta, gap, k) := \left| \begin{array}{l} \mu d \leftarrow \dfrac{k \cdot \left(\beta \cdot \cosh\left(\frac{gap \cdot k}{2}\right) \cdot \sin\left(\frac{\beta \cdot gap}{2}\right) + k \cdot \sinh\left(\frac{gap \cdot k}{2}\right) \cdot \cos\left(\frac{\beta \cdot gap}{2}\right)\right)}{\beta^2 \cdot \sinh\left(\frac{gap \cdot k}{2}\right) + k^2 \cdot \sinh\left(\frac{gap \cdot k}{2}\right)} \text{ if } k > 0 \\ \mu d \leftarrow \operatorname{sinc}\left(\frac{\beta_e \cdot gap}{2}\right) \text{ otherwise} \end{array} \right.$$

Radial coupling factor averaged across the beam

Equation 11.38

$$\mu_r(\gamma_e) := \frac{2 \cdot I1(\gamma_e \cdot b)}{(\gamma_e \cdot b) \cdot I0(\gamma_e \cdot a)}$$

Small-signal gap coupling factor

Equation 11.35

$$Mg(\beta_e) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Mg_n \leftarrow \mu_r(nh_n \cdot \beta_e) \cdot \mu d\left(\beta_e, gap_n, \frac{kgap}{gap_n}\right) \\ \text{return } Mg \end{array} \right.$$

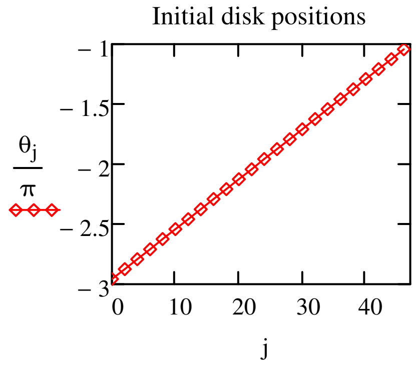
The beam is modelled as ND rigid discs of thickness ΔL. The motions of the electrons at the disc centres are followed.

Normalised disk thickness $\theta_d := \frac{2\cdot\pi}{ND}$ Disk charge $Q := \frac{2\cdot\pi\, I_0}{\omega\cdot ND}$ $Q = 5.165\times 10^{-9}\,C$

Define normalised initial positions and velocities of the discs
The initial positions are symmetrical about the bunch centre.

$$\theta := \left|\begin{array}{l} \text{for } j \in 0,2..2\cdot(ND-1) \\ \quad \left|\begin{array}{l} \theta_j \leftarrow \theta_0 - \pi + \frac{\pi\cdot(j+1)}{ND} \\ \theta_{j+1} \leftarrow 1 \end{array}\right. \\ \theta \end{array}\right.$$

$j := 0,2..2\cdot ND - 1$



Check the D.C. beam current calculated from the disks is correct.

$$\sum_{j=0}^{ND-1} (f\cdot Q) = 354.000\,A \qquad I_0 = 354.000\,A$$

The Space-Charge Field ES(θ) is found from the equations given by J.R. Hechtel, "The effect of potential beam energy on the performance of linear beam devices", *IEEE Transactions on Electron Devices* ED-17, pp.999-1009, Nov. 1970. The potential distribution of a cylinder of charge of radius *b*, length *ΔL* and charge density *ρ* within a conducting cylinder of radius *a* in cylindrical polar coordinates whose origin is at the centre of the cylinder is calculated for a disc charge of 1C

Disc thickness $\Delta L := \frac{\theta_d}{\beta_e}$ $\Delta L = 3.371\cdot mm$ Charge density $\rho_0 := \frac{1}{\pi\cdot b^2\cdot\Delta L}$

Compute the space-charge field at *npts* points

$npts := 100$

The first ten zeros of $J_0(z)$.

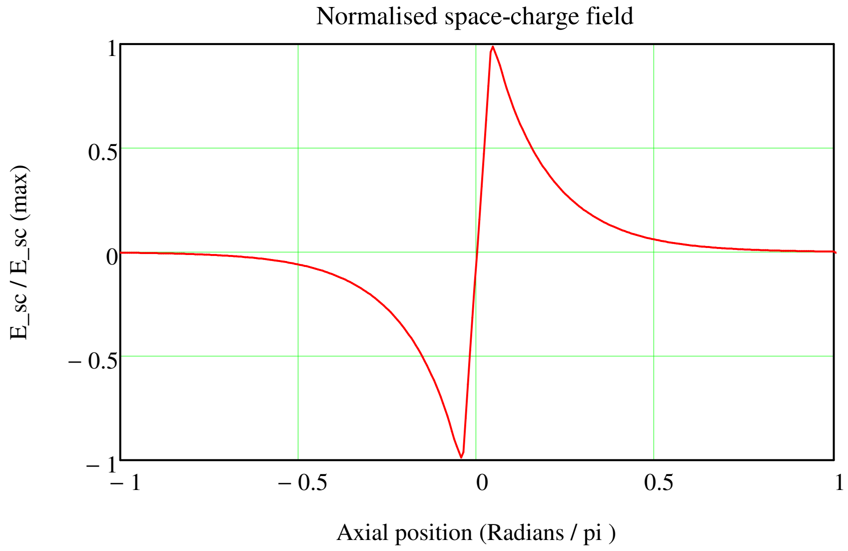
$$\mu B := \frac{1}{a}\cdot(2.405\ \ 5.520\ \ 8.654\ \ 11.791\ \ 14.931\ \ 18.071\ \ 21.212\ \ 24.352\ \ 27.494\ \ 30.635)^T$$

$$
\begin{aligned}
ESn := \ & \text{for } n \in 0..npts \\
& \quad \theta_n \leftarrow \frac{2\cdot n\cdot\pi}{npts} \\
& \quad z_n \leftarrow \frac{\theta_n}{\beta_e} \\
& \quad ES_n \leftarrow \left(\frac{4\cdot\rho_0}{\varepsilon_0}\right)\cdot\sum_{m=0}^{9}\left[\frac{1}{\mu B_m}\cdot\left(\frac{J1(\mu B_m\cdot b)}{\mu B_m\cdot a\cdot J1(\mu B_m\cdot a)}\right)^2\cdot\exp(-\mu B_m\cdot z_n)\cdot\sinh\left(\frac{\mu B_m\cdot\Delta L}{2}\right)\right] \ \text{if } \theta_n \ge 0.5\cdot\theta_d \\
& \quad ES_n \leftarrow \left(\frac{4\cdot\rho_0}{\varepsilon_0}\right)\cdot\sum_{m=0}^{9}\left[\frac{1}{\mu B_m}\cdot\left(\frac{J1(\mu B_m\cdot b)}{\mu B_m\cdot a\cdot J1(\mu B_m\cdot a)}\right)^2\cdot\left(\exp\left(-\mu B_m\cdot\frac{\Delta L}{2}\right)\cdot\sinh(\mu B_m\cdot z_n)\right)\right] \ \text{otherwise} \\
& ES
\end{aligned}
$$

Find the space-charge field as a continuous function by linear interpolation

$$
\begin{aligned}
\theta n := \ & \text{for } n \in 0..npts \\
& \quad \theta_n \leftarrow \frac{2\cdot n\cdot\pi}{npts} \\
& \theta
\end{aligned}
$$

$$
\begin{aligned}
ES(\theta) := \ & Es(\theta) \leftarrow \text{sign}(\theta)\cdot\text{linterp}(\theta n, ESn, |\theta|) \\
& Es(\theta + 2\cdot\pi) \ \text{ if } \theta < -\pi \\
& Es(\theta - 2\cdot\pi) \ \text{ if } \theta > \pi \\
& Es(\theta) \ \text{ otherwise}
\end{aligned}
$$



The space-charge field is set to zero until the electrons reach θ = 0 to avoid non-physical dispersion of the electrons.

$$SCF1(\theta) := \begin{cases} SCF & \text{if } \theta \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The space-charge field of adjacent bunches is included by assuming that the field is periodic in z. This is not correct but tests with an initially unmodulated beam and three wavelengths of electrons give almost identical results for the trajectories and the current harmonics except well beyond the first bunch and at microperveance greater than 2.

This space-charge calculation is based on a quasi-static analysis. This is adequate for low beam voltages. For higher beam voltages it is important to make a correction based on the use of γ in place of β when calculating the small-signal plasma frequency reduction factor p. Since the square of the plasma frequency is proportional to the charge density in the beam the space-charge field should be multiplied by the correction factor

$$SCF = \left(\frac{p(\gamma)}{p(\beta)}\right)^2$$

This factor is calculated in WS 13.1 and imported. With this correction it is found that the reduced plasma wavelength computed by this sheet is within 2% of that computed by WS 13.1.

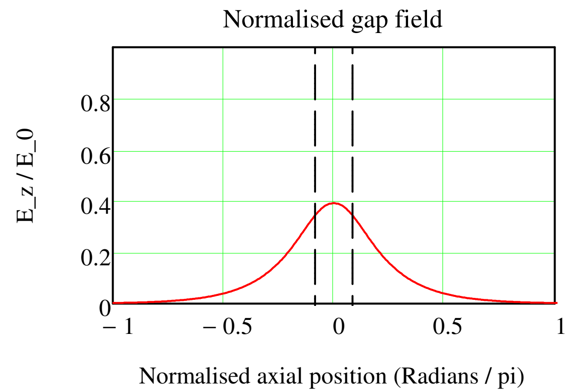
A further problem is that the reduced plasma wavelength is too small for low drive levels. It increases with increasing drive level to the correct value and then decreases again as expected. This is believed to be caused by significance errors. To avoid the problem the voltages for the first few gaps are imported from WS 13.1. The results only change slightly when TOL is changed from the default value to 1E-6.

The Interaction Field is found from the Fourier Transform of the field in the gap. The average of the field over the beam is used. Linear interpolation on the values calculated at regular intervals is used to provide a fast look-up function.

$$
\mathrm{En} := \left|
\begin{array}{l}
\gamma(\beta) \leftarrow \sqrt{\beta^2 - \dfrac{\omega^2}{c^2}} \\
\text{for } ng \in 1..NCAV \\
\quad \text{for } n \in 0..npts \\
\quad\quad \left| \begin{array}{l}
\theta_n \leftarrow \dfrac{2 \cdot n \cdot \pi}{npts} \\
E_{n,ng} \leftarrow \dfrac{V}{\pi} \cdot \displaystyle\int_0^{\frac{20 \cdot \pi \cdot \beta_e}{\beta e g_{ng}}} \frac{2 \cdot I1(\gamma(\beta) \cdot b)}{(\gamma(\beta) \cdot b) \cdot I0(\gamma(\beta) \cdot a)} \cdot \mu d\left(\beta, gap_{ng}, \frac{kgap}{gap_{ng}}\right) \cdot \cos\left(\frac{\beta}{\beta_e} \cdot \theta_n\right) d\beta
\end{array} \right. \\
\text{return } E
\end{array}
\right.
$$

$$
\mathrm{Egap}(\theta) := \left|
\begin{array}{l}
\text{for } ng \in 1..NCAV \\
\quad E_{ng} \leftarrow \mathrm{linterp}\left(\theta n, En^{\langle ng \rangle}, |\theta|\right) \\
\text{return } E
\end{array}
\right.
$$

Plot the normalised field of gap n $\quad ng := 1$



Superimpose the electric fields of the gaps using absolute phase

$$
E_z(\theta, \phi) := \mathrm{Re}\left[\sum_{n=1}^{NCAV} \left(\frac{Vg_n}{V} \cdot \mathrm{Egap}\left(\theta - \theta g_n\right)_n \exp\left(j \cdot nh_n \cdot \phi\right)\right)\right]
$$

CHECK small-signal coupling factors by direct integration of the field.

$$M_g := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad M_n \leftarrow \dfrac{1}{\beta_e \cdot V} \cdot \displaystyle\int_{-2\pi}^{2\pi} Egap(\theta)_n \cdot \cos\left(nh_n \cdot \theta\right) d\theta \\ \text{return } M \end{array} \right.$$

$$M_g = \begin{pmatrix} 0.000 \\ 0.879 \\ 0.877 \\ 0.873 \\ 0.856 \\ 0.852 \\ 0.813 \end{pmatrix} \qquad Mg(\beta_e) = \begin{pmatrix} 0.000 \\ 0.759 \\ 0.757 \\ 0.753 \\ 0.739 \\ 0.735 \\ 0.702 \end{pmatrix}$$

The Coefficients of the Differential Equations for the motions of the electrons are defined.

The rows represent, in order, the position in radians and the normalised velocity of the electrons.

$$D(\phi,\theta) := \left| \begin{array}{l} \text{for } j \in 0,2..2\cdot(ND-1) \\ \quad \left| \begin{array}{l} D_j \leftarrow \theta_{j+1} \\ D_{j+1} \leftarrow \dfrac{\eta}{\omega \cdot u_0} \cdot \left[1 - \left(\dfrac{u_0}{c}\cdot\theta_{j+1}\right)^2\right]^{1.5} \cdot \left[E_z\left(\theta_j,\phi\right) + SCF1\left(\theta_j\right)\cdot Q \cdot \displaystyle\sum_{i=0}^{ND-1} \left(ES\left(\theta_j - \theta_{2\cdot i}\right)\right)\right] \end{array} \right. \\ D \end{array} \right.$$

Definitions of normalised variables

$$\phi = \omega \cdot t \qquad \theta = \beta_e \cdot z \qquad \theta' = \frac{v}{u_0}$$

$$\frac{d}{dt}z = v \qquad \frac{d}{d\phi}\theta = \frac{v}{u_0}$$

$$\frac{d}{dt}v = -\eta \cdot E \qquad \frac{d}{d\phi}\frac{v}{u_0} = -\frac{\eta \cdot E}{\omega \cdot u_0}$$

The Equations are Solved using with *nmax* time steps starting from ϕ_0 which is defined in such a way that the centre electron would cross the gap centre at t = 0 if it travelled with a constant velocity u_0. The final time is ϕ_f.

$$Z := rkfixed\left(\theta, \phi_0, \phi_f, nmax, D\right)$$

The results are in a single table (Z) in which the first column (0) is the time and the other columns (1-12) are the positions and velocities of the electrons in the same order as before at each value of n.

Extract the vector of phases, the matrices containing the normalised positions and velocities of the disks and the vector of the final velocities of the electrons

```
φn :=  | for n ∈ 0..nmax
       |   φ_n ← Z_{n,0}
       | φ

θn :=  | for j ∈ 0..(ND − 1)
       |   for n ∈ 0..nmax
       |     θ_{n,j} ← Z_{n,2·j+1}
       | θ

un :=  | for j ∈ 0..(ND − 1)
       |   for n ∈ 0..nmax
       |     u_{n,j} ← Z_{n,2·j+2}
       | u

umax := | for j ∈ 0..(ND − 1)
        |   u_j ← Z_{nmax,2·j+2}
        | u
```

Check for reflected electrons

```
MESSAGE := | M ← "No reflected electrons "
           | for j ∈ 0..(ND − 1)
           |   if un_{nmax,j} ≤ 0
           |     | M ← "REFLECTED ELECTRONS"
           |     | break
           | return M
```

The Kinetic Energy of the bunch at each time step is calculated using the relativistically correct formulae by summing the energies of the disks. The figure becomes unstable when siginificant cross-overs occur. It is essential to ensure that the final time is great enough for all electrons to have left the interaction region.

$$KE := \left| \begin{array}{l} \text{for } n \in 0..nmax \\ \quad KE_n \leftarrow \frac{Q \cdot c^2}{\eta} \cdot \sum_{j=0}^{ND-1} \left[\frac{1}{\sqrt{1 - \frac{(un_{n,j} \cdot u_0)^2}{c^2}}} - 1 \right] \\ KE \end{array} \right.$$

$P_{DC} := V_a \cdot I_0$ $\qquad P_{DC} = 111.5 \cdot MW$

Check that the frequency times the intial KE is equal to the DC beam power allowing for space-charge potential depression.

$V_0 \cdot I_0 = 103.0 \cdot MW$ $\qquad KE_0 \cdot f = 103.0 \cdot MW$

Define a set of equally-spaced reference planes in θ and compute the phases and velocities at which the electrons cross them using linear interpolation. Also find the times relative to an electron travelling at constant velocity u_0.

Plane positions

$$\theta p := \left| \begin{array}{l} \Delta\theta \leftarrow \dfrac{\theta_f - \theta_0}{NP} \\ \text{for } p \in 0..NP \\ \quad \theta_p \leftarrow p\cdot\Delta\theta + \theta_0 \\ \text{return } \theta \end{array} \right.$$

$$\Delta\theta := \frac{\theta_f - \theta_0}{NP}$$

Absolute phase

$$\phi p := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } p \in 0..NP \\ \quad\quad \text{for } n \in 1..nmax \\ \quad\quad\quad \left| \begin{array}{l} flag \leftarrow 0 \\ flag \leftarrow 1 \ \text{ if } \theta n_{n,j} > \theta p_p \\ \phi p_{p,j} \leftarrow \phi n_{n-1} + \dfrac{\theta p_p - \theta n_{n-1,j}}{\theta n_{n,j} - \theta n_{n-1,j}}\cdot\left(\phi n_n - \phi n_{n-1}\right) \ \text{ if } flag = 1 \\ (\text{break}) \ \text{ if } flag = 1 \end{array} \right. \\ \text{return } \phi p \end{array} \right.$$

Relative phase

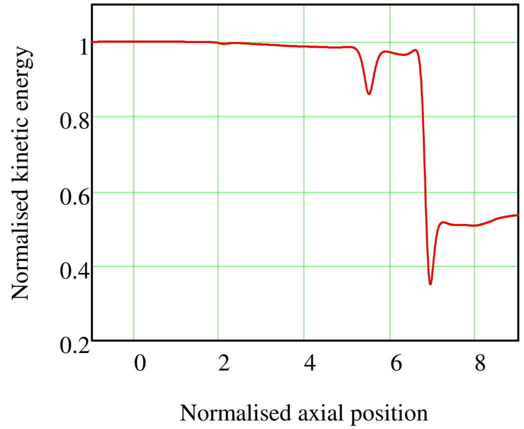
$$\phi r := \left| \begin{array}{l} \text{for } j \in 0..ND-1 \\ \quad \text{for } p \in 0..NP \\ \quad\quad \phi r_{p,j} \leftarrow \phi p_{p,j} - \theta p_p \\ \text{return } \phi r \end{array} \right.$$

Normalised velocities (u / u_0) of the electrons at the reference planes

$$up := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } p \in 0..NP \\ \quad\quad \text{for } n \in 1..nmax \\ \quad\quad\quad \left| \begin{array}{l} flag \leftarrow 0 \\ flag \leftarrow 1 \ \text{ if } \theta n_{n,j} > \theta p_p \\ up_{p,j} \leftarrow un_{n-1,j} + \dfrac{\theta p_p - \theta n_{n-1,j}}{\theta n_{n,j} - \theta n_{n-1,j}}\cdot\left(un_{n,j} - un_{n-1,j}\right) \ \text{ if } flag = 1 \\ (\text{break}) \ \text{ if } flag = 1 \end{array} \right. \\ \text{return } up \end{array} \right.$$

Sum of the kinetic energies of the electrons at each plane.

$$KEp := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad KE_p \leftarrow \dfrac{Q\cdot c^2}{\eta}\cdot\displaystyle\sum_{j=0}^{ND-1}\left[\frac{1}{\sqrt{1 - \dfrac{\left(up_{p,j}\cdot u_0\right)^2}{c^2}}} - 1\right] \\ KE \end{array} \right.$$



Calculate the complex current harmonics at each plane by superimposing the Fourier components of the discs.

$$
\begin{aligned}
Ip := \;& \text{for } p \in 0..NP \\
& \quad Ip_{p,0} \leftarrow \frac{Q}{\Delta L}\cdot\frac{1}{2\cdot\pi}\cdot\sum_{j=0}^{ND-1}\theta_d \\
& \quad \text{for } j \in 0..ND-1 \\
& \quad\quad up_{p,j} \leftarrow 10^{-6} \text{ if } up_{p,j} = 0 \\
& \quad \text{for } h \in 1..0.5\cdot ND \\
& \quad\quad Ip_{p,h} \leftarrow \frac{Q}{\Delta L}\cdot\frac{2}{\pi\cdot h}\cdot\sum_{j=0}^{ND-1}\left(up_{p,j}\cdot\sin\left(\frac{h\cdot\theta_d}{2\cdot up_{p,j}}\right)\cdot\exp\left(-j\cdot h\cdot\phi p_{p,j}\right)\right) \\
& \text{return } Ip\cdot u_0
\end{aligned}
$$

Instantaneous current $= \dfrac{Q\cdot up_{p,j}}{\Delta L}$

Pulse duration $= \dfrac{\theta_d}{up_{p,j}}$

Find the variation of current with time at plane *p* by Fourier synthesis

$$IP(p,\phi) := \mathrm{Re}\left[\sum_{n=0}^{0.5\cdot ND}\left(Ip_{p,n}\cdot\exp(j\cdot n\cdot\phi)\right)\right]$$

Fundamental RF current and its relative phase at the reference planes

$$Ip1 := Ip^{\langle 1 \rangle}$$

$$argrI1 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad argrI1_p \leftarrow arg\left(Ip1_p \cdot exp\left(j \cdot \theta p_p\right)\right) \\ \text{return } argrI1 \end{array} \right.$$

Function for plotting the positions of the cavities

$$\phi g(\theta) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad \phi g \leftarrow 2 \text{ if } \left|\theta - \theta g_n\right| \le \frac{\beta eg_n}{2} \\ \text{return } 2\phi g - 2 \end{array} \right.$$

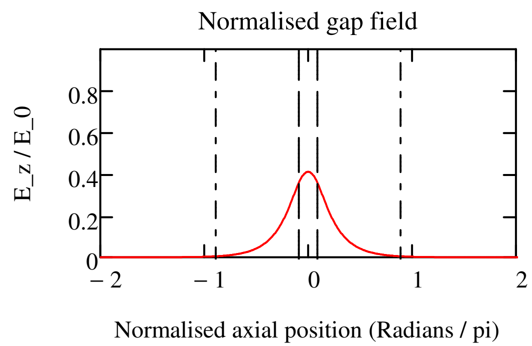
Calculation of the induced curent in each cavity

Find the serial number of the plane at the centre of each cavity and plane numbers at the edges of the gap field. The gap field is assumed to be zero at 10g from the gap centre. Note: θc2 for the last cavity must be less than θf.

$$pc := round\left(\frac{\theta g - \theta_0}{\Delta\theta}\right) \qquad pc1 := round\left(\frac{\theta g - \theta_0 - 10 \cdot \beta eg}{\Delta\theta}\right) \qquad pc2 := round\left(\frac{\theta g - \theta_0 + 10 \cdot \beta eg}{\Delta\theta}\right)$$

Check that field of a cavity is zero at planes pc1 and pc2

Cavity number $cn := 2$ $\quad pc1_{cn} = 80 \quad pc_{cn} = 169 \quad pc2_{cn} = 258.00 \quad \theta c1 := \left(pc1_{cn} - pc_{cn}\right) \cdot \Delta\theta \quad \theta c2 := \left(pc2_{cn} - pc_{cn}\right) \cdot \Delta\theta$



Induced current in each cavity

$$\mathrm{Iind} := \left| \begin{array}{l} \text{for } n \in 2..\mathrm{NCAV} \\ \quad \mathrm{Iind}_n \leftarrow \dfrac{-\Delta\theta}{\beta_e \cdot V} \cdot \left[\displaystyle\sum_{p = pc1_n}^{pc2_n} \left(\mathrm{Ip}_{p,nh_n} \cdot \mathrm{Egap}\left(\theta p_p - \theta g_n\right) n \right) \right] \\ \text{return } \mathrm{Iind} \end{array} \right.$$

Equation 11.173
The integral is approximated by a sum

Revised gap voltage in each cavity

$$\mathrm{Vgc} := \left| \begin{array}{l} \text{for } n \in 1..\mathrm{SCAV} \\ \quad \mathrm{Vgc}_n \leftarrow 0.5\left(\mathrm{Vgss}_n + \mathrm{Vg}_n\right) \\ \text{for } n \in \mathrm{SCAV}+1..\mathrm{NCAV} \\ \quad \mathrm{Vgc}_n \leftarrow 0.5 \cdot \left[\dfrac{\mathrm{Iind}_n}{\left(\mathrm{Yc}_n + \mathrm{Ge}_n\right)} + \mathrm{Vg}_n \right] \\ \text{return } \mathrm{Vgc} \end{array} \right.$$

$$\mathrm{ModVg} := \left| \begin{array}{l} \text{for } n \in 1..\mathrm{NCAV} \\ \quad \mathrm{MVg}_n \leftarrow \left|\mathrm{Vgc}_n\right| \\ \text{return } \mathrm{MVg} \end{array} \right.$$

$$\mathrm{ArgrVgc} := \left| \begin{array}{l} \text{for } n \in 1..\mathrm{NCAV} \\ \quad \mathrm{AVg}_n \leftarrow \arg\left(\mathrm{Vgc}_n \cdot \exp\left(j \cdot \theta g_n\right)\right) + \pi \\ \text{return } \mathrm{AVg} \end{array} \right.$$

Weighted gap voltages to be used for the next iteration

$$\mathrm{Vg1} := (1 - \mathrm{weight}) \cdot \mathrm{Vg} + \mathrm{weight} \cdot \mathrm{Vgc}$$

RF output power from the change in KE

$$P_{KE} := \left(\mathrm{KE}_0 - \mathrm{KE}_{nmax}\right) \cdot f$$

RF output power from the output gap voltage

$$P_{out} := \frac{1}{2} \cdot \left(\mathrm{ModVg}_{NCAV}\right)^2 \cdot \mathrm{Ge}_{NCAV}$$

Gain

$$\mathrm{Gain} := 10 \cdot \log\left(\frac{P_{out}}{P_{in}}\right)$$

Efficiency

$$\mathrm{Efficiency} := \frac{P_{out}}{P_{DC}}$$

Relative change in cavity voltages

$$\mathrm{Err} := \left| \begin{array}{l} \text{for } n \in 2..\mathrm{NCAV} \\ \quad \mathrm{Err}_n \leftarrow \dfrac{\left|\mathrm{Vg}_n - \mathrm{Vgc}_n\right|}{\left|\mathrm{Vgc}_n\right|} \\ \text{return } \mathrm{Err} \end{array} \right.$$

Power dissipated in each cavity

$$\mathrm{Pc} := \left| \begin{array}{l} \text{for } n \in 1..\mathrm{NCAV} \\ \quad \mathrm{Pc}_n \leftarrow \dfrac{1}{2} \cdot \left(\left|\mathrm{Vg}_n\right|\right)^2 \cdot \mathrm{Gc}_n \\ \text{return } \mathrm{Pc} \end{array} \right.$$

Energy balance

$$\mathrm{Energy\_balance} := \frac{P_{out} + \mathrm{KE}_{nmax} \cdot f + \displaystyle\sum_{n=1}^{NCAV} \mathrm{Pc}_n}{I_0 \cdot V_0 + P_{in}} - 1$$

Results

The results of the calculation are in the green cells below. On the first pass the voltages in the yellow matrix are set to zero. After each pass the figures from the green matrix are copied into the yellow matrix. The process in repeated for successive passes until the difference in the vector voltages measured by the 'Error' is considered to be small enough. The process is repeated until it converges. Note: the first element of each matrix is not used. The gap voltages Vg1 are the weighted mean of initial and the revised gap voltages. Normally weight = 1 but other values (e.g. 0.5) sometimes lead to more rapid convergence of the results.

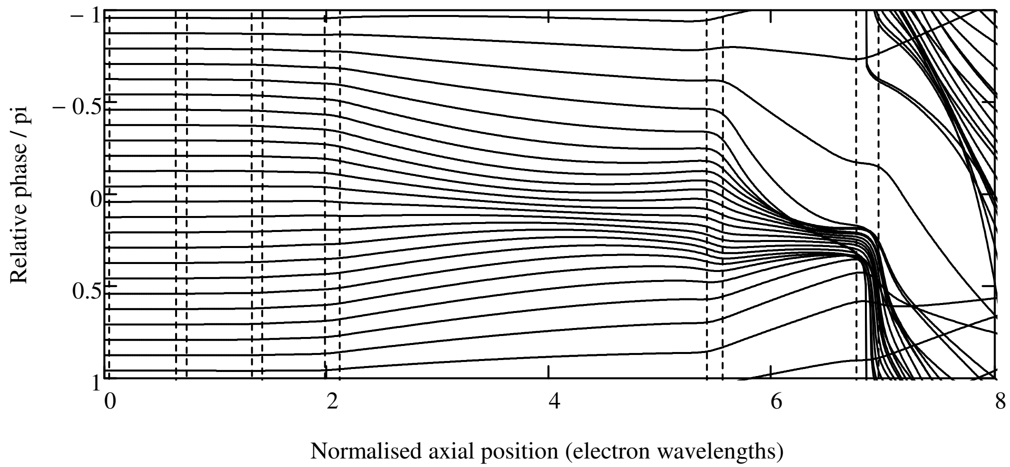
MESSAGE = "No reflected electrons "

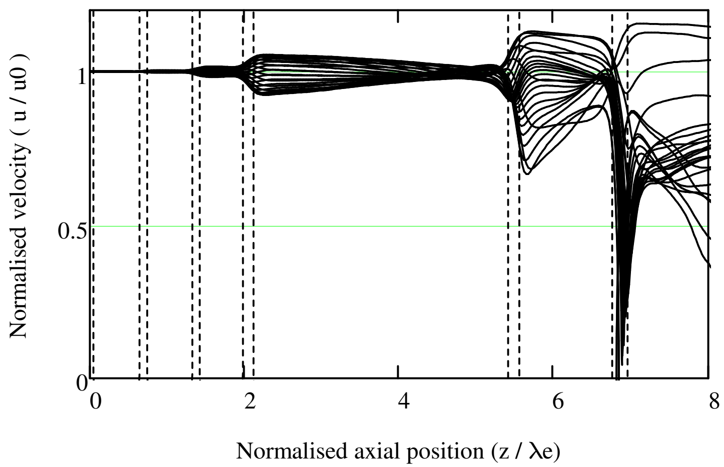
$weight \equiv 1$

Gap voltages imported from WS13.1

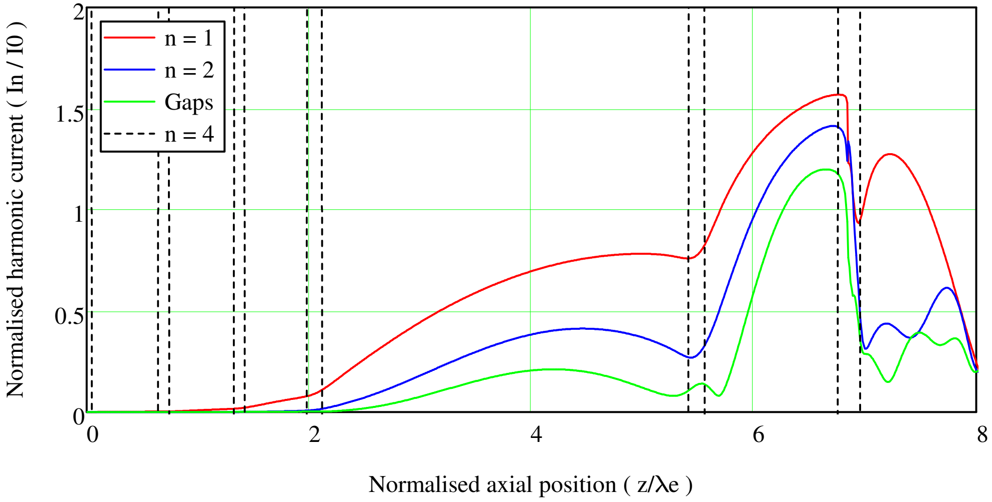
$$Vg \equiv \begin{pmatrix} 0.000 \\ 0.837 \\ 3.376 + 3.583i \\ -13.925 + 13.735i \\ -42.167 - 53.087i \\ 129.825 + 237.557i \\ -410.708 - 58.236i \end{pmatrix} kV \quad Vg1 = \begin{pmatrix} 0.000 \\ 0.837 \\ 3.376 + 3.583i \\ -13.925 + 13.735i \\ -42.167 - 53.087i \\ 129.830 + 237.513i \\ -410.636 - 58.074i \end{pmatrix} \cdot kV \quad Err = \begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{pmatrix} \cdot \% \quad Pc = \begin{pmatrix} 0.000 \\ 0.002 \\ 0.081 \\ 1.099 \\ 11.969 \\ 190.854 \\ 506.096 \end{pmatrix} \cdot kW \quad Vgss \equiv \left[\begin{pmatrix} 0.000 \\ 0.837 \\ 3.376 + 3.583i \\ -13.925 + 13.735i \\ -42.167 - 53.087i \\ 101.505 + 311.629i \\ -999.741 - 402.708i \end{pmatrix}\right] \cdot kV$$

$P_{out} = 48.18 \cdot MW$ $\quad P_{KE} = 47.19 \cdot MW$ $\quad Gain = 59.8 \cdot dB$ $\quad Efficiency = 43.2 \cdot \%$ $\quad Energy\_balance = 1.6 \cdot \%$





p1 := 0 .. NP



Normalised harmonic current (In / I0)
n = 1
n = 2
Gaps
n = 4
0
0.5
1
1.5
2
0
2
4
6
8
Normalised axial position (z/λe)