WS 14.1 Helix TWT small-signal model

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

This sheet illustrates the basic principles of the continuous travelling-wave tube interaction using small-signal theory (see Section 14.2.1)

Define the beam and helix parameters

Anode voltage	Beam current	Synchronous frequency	Magnetic field/Brillouin field	
$V_a := 6.0 \cdot kV$	$I_0 := 135 \cdot mA$	$f_0 := 11.7 \cdot GHz$	$mB := 2$	
Tunnel radius	Beam radius	Shield radius	Effective permittivity	Cold loss per wavelength
$a := 0.68 \cdot mm$	$b := 0.34 \cdot mm$	$r_s := a \cdot 1.5$	$\varepsilon_2 := 2.5$	$loss := 0.05 \cdot dB$
Section 1 length	Sever length	Section 2 length	Tape helix parameters	
$z_1 := 50 \cdot mm$	$z_s := 5 \cdot mm$	$z_2 := 50 \cdot mm$	$\alpha_C := 1$	$\alpha_L := 1$

The section below can be collapsed to allow the input data and the results to be seen on the screen simultaneously

Physical constants

Charge/mass ratio of the electron $\eta := 1.759 \cdot 10^{11} \frac{C}{kg}$ Perveance $\mu P := \mu A \cdot V^{-1.5}$ $dB \equiv 1$ $ORIGIN \equiv 1$

Calculate the beam velocity and the electronic propagation constant

$$V_0 := \left| \begin{array}{l} V_1 \leftarrow V_a \\ \text{for } n \in 1..5 \\ \quad \left| \begin{array}{l} u_n \leftarrow c \cdot \left[1 - \dfrac{1}{\left(1 + \dfrac{\eta \cdot V_n}{c^2}\right)^2} \right]^{0.5} \\ V_{n+1} \leftarrow V_1 - \dfrac{I_0}{2 \cdot \pi \cdot \varepsilon_0 \cdot u_n} \cdot \left(\dfrac{1}{2} - \ln\left(\dfrac{b}{a}\right)\right) \end{array} \right. \\ \text{return } V_{n+1} \end{array} \right.$$

$$u_0 := c \cdot \left[1 - \frac{1}{\left(1 + \frac{\eta \cdot V_0}{c^2}\right)^2} \right]^{0.5} \qquad Rel := \frac{1}{\sqrt{1 - \frac{u_0^2}{c^2}}}$$

$V_0 = 5.94 \cdot kV$ $u_0 = 4.53 \times 10^7 \, m \cdot s^{-1}$ $Rel = 1.012$

Note the parameters of the model must be dimensionless because Mathcad will not accept mixed dimensions in matrices.

$a := \frac{a}{m}$ $b := \frac{b}{m}$ $r_s := \frac{r_s}{m}$ $z_1 := \frac{z_1}{m}$ $z_s := \frac{z_s}{m}$ $z_2 := \frac{z_2}{m}$

$\omega_0 := 2 \cdot \pi \cdot f_0 \cdot s$ $\beta_e(\omega) := \frac{\omega}{u_0} \cdot m \cdot s^{-1}$ $\gamma_e(\omega) := \frac{\beta_e(\omega)}{Rel}$ $\gamma_e(\omega_0) \cdot b = 0.545$

Calculate the plasma frequency and the reduced plasma frequency

Plasma frequency $\omega_p := \frac{s}{m}\cdot\sqrt{\frac{\eta}{\varepsilon_0}\cdot\frac{I_0}{\pi\cdot b^2\cdot u_0}\cdot\frac{1}{Rel^3}}$

$$\omega_q(\omega) := \left|\begin{array}{l} \gamma a \leftarrow \gamma_e(\omega)\cdot a \\ \gamma b \leftarrow \gamma_e(\omega)\cdot b \\ \tau b(p) \leftarrow \gamma b\cdot\left[\dfrac{\dfrac{1}{p^2}-1}{\dfrac{1}{p^2-2\cdot\left(mB^2-1\right)}-1}\right]^{\frac{1}{2}} \\ fn1 \leftarrow \dfrac{1}{\gamma b}\cdot\dfrac{I1(\gamma b)\cdot K0(\gamma a)+I0(\gamma a)\cdot K1(\gamma b)}{I0(\gamma b)\cdot K0(\gamma a)-I0(\gamma a)\cdot K0(\gamma b)} \\ fn2(p) \leftarrow \dfrac{1-\dfrac{1}{p^2}}{\tau b(p)}\cdot\dfrac{I1(\tau b(p))}{I0(\tau b(p))} \\ fn(p) \leftarrow \dfrac{1}{fn1}-\dfrac{1}{fn2(p)} \\ p0 \leftarrow 0.9 \\ \omega_q \leftarrow root(fn(p0),p0)\cdot\omega_p \end{array}\right.$$

$\beta_q(\omega) := \frac{\omega_q(\omega)}{u_0}\cdot m\cdot s^{-1}$

$\frac{\omega_q(\omega_0)}{\omega_0} = 0.058$

Propagation constants of the fast and slow space-charge waves

$$\beta_f(\omega) := \beta_e(\omega) - \beta_q(\omega) \qquad \beta_s(\omega) := \beta_e(\omega) + \beta_q(\omega) \qquad \gamma_f(\omega) := \frac{\beta_f(\omega)}{Rel} \qquad \gamma_s(\omega) := \frac{\beta_s(\omega)}{Rel}$$

Electronic admittance

$$Y_e(\omega) := \frac{I_0 \cdot \Omega}{Rel \cdot (Rel + 1) \cdot V_0} \cdot \frac{\omega}{\omega_q(\omega)}$$

Equation 11.80

$$Z_e(\omega) := \frac{1}{Y_e(\omega)}$$

Phase velocity of the slow space-charge wave

$$vp_s(\omega) := \frac{\omega}{\beta_s(\omega)}$$

Sheath helix with a shield and dielectric supports

$$L_1(\gamma, \psi) := \left| \begin{array}{l} \gamma a \leftarrow \gamma \cdot a \\ \gamma s \leftarrow \gamma \cdot r_s \\ L_0 \leftarrow \dfrac{\mu_0 \cdot m}{2 \cdot \pi \cdot H} \cdot I1(\gamma a) \cdot K1(\gamma a)\cot(\psi)^2 \\ L_1 \leftarrow L_0 \cdot \left(1 - \dfrac{I1(\gamma a) \cdot K1(\gamma s)}{I1(\gamma s) \cdot K1(\gamma a)}\right) \end{array} \right.$$

$$C_1(\gamma) := \left| \begin{array}{l} \gamma a \leftarrow \gamma \cdot a \\ \gamma s \leftarrow \gamma \cdot r_s \\ C_0 \leftarrow \dfrac{2 \cdot \pi \cdot \varepsilon_0 \cdot m}{I0(\gamma a) \cdot K0(\gamma a) \cdot F} \\ D \leftarrow \gamma a \cdot I0(\gamma a) \cdot K1(\gamma a) \cdot \left(1 + \dfrac{I1(\gamma a) \cdot K0(\gamma s)}{K1(\gamma a) \cdot I0(\gamma s)}\right) \\ \varepsilon_{eff} \leftarrow 1 + \left(\varepsilon_2 - 1\right) \cdot D \\ C_1 \leftarrow C_0 \cdot \varepsilon_{eff} \cdot \left(1 - \dfrac{I0(\gamma a) \cdot K0(\gamma s)}{I0(\gamma s) \cdot K0(\gamma a)}\right)^{-1} \end{array} \right.$$

Phase velocity

$$vp(\gamma, \psi) := \frac{1}{\sqrt{\alpha_L \cdot \alpha_C \cdot L_1(\gamma, \psi) \cdot C_1(\gamma)}}$$

Find the pitch angle which gives the required phase velocity at synchronism

$$v_s := \frac{\omega_0}{\beta_s(\omega_0)} \qquad \psi_a := 10\cdot\text{deg} \qquad \psi := \text{root}\left[\left(vp\left(\gamma_s(\omega_0),\psi_a\right) - v_s\right),\psi_a\right]$$

Check synchronism

$$vp\left(\gamma_s(\omega_0),\psi\right) - v_s = 0$$

$$\psi = 10.7\cdot\text{deg}$$

Helix pitch

$$p_h := 2\cdot\pi\cdot a\cdot\tan(\psi)\cdot m$$

$$p_h = 0.81\cdot\text{mm}$$

The normalised phase shift per turn at the synchronous point is

$$\frac{\beta_s(\omega_0)\cdot p_h}{\pi\cdot m} = 0.443$$

Propagation constant

$$\beta_0(\omega) := \left|\begin{array}{l} \gamma(\beta) \leftarrow \sqrt{\beta^2 - \frac{\omega^2}{c^2}\cdot\frac{m^2}{s^2}} \\ \beta \leftarrow \beta_s(\omega_0) \\ \beta_0 \leftarrow \text{root}\left(\frac{\beta}{\sqrt{L_1(\gamma(\beta),\psi)\cdot C_1(\gamma(\beta))}} - \omega, \beta\right) \end{array}\right.$$

Phase velocity and group velocity

$$v_p(\omega) := \frac{\omega}{\beta_0(\omega)} \qquad v_g(\omega) := \left(\frac{d}{d\omega}\beta_0(\omega)\right)^{-1}$$

Check synchronism

$$\beta_0(\omega_0) - \beta_s(\omega_0) = 0$$

Frequency at the pi mode

$$f_\pi := \frac{s^{-1}}{2\cdot\pi}\cdot\text{root}\left(\frac{\beta_s(\omega_0)\cdot p_h}{\pi\cdot m} - 1, \omega_0\right)$$

$$f_\pi = 26.73\cdot\text{GHz}$$

Transverse impedance, characteristic impedance and Pierce impedance

$$Z_t(\gamma) := \sqrt{\frac{\alpha_L \cdot L_1(\gamma,\psi)}{\alpha_C \cdot C_1(\gamma)}}$$

$$Z_c(\omega) := \left| \begin{array}{l} \gamma \leftarrow \sqrt{\beta_0(\omega)^2 - \dfrac{\omega^2}{c^2} \cdot \dfrac{m^2}{s^2}} \\ \dfrac{v_p(\omega)}{v_g(\omega)} \cdot Z_t(\gamma) \end{array} \right.$$

$$Y_c(\omega) := \frac{1}{Z_c(\omega)}$$

$$\gamma_0(\omega) := \sqrt{\beta_0(\omega)^2 - \frac{\omega^2}{c^2} \cdot m^2 \cdot s^{-2}}$$

$$Z_p(\omega) := \frac{Z_c(\omega)}{I0\left(\gamma_0(\omega) \cdot a\right)^2}$$

Compute the coupling factor corresponding to the propagation constant of the growing wave

$$\gamma_1(\omega) := \frac{1}{2} \cdot \left(\gamma_0(\omega) + \gamma_s(\omega)\right)$$

$$\mu_c(\omega) := \frac{2}{\gamma_1(\omega) \cdot b} \cdot \frac{I1\left(\gamma_1(\omega) \cdot b\right)}{I0\left(\gamma_1(\omega) \cdot a\right)}$$

Adjust the propagation constant on the helix to include loss

$$\beta_0(\omega) := \beta_0(\omega) \cdot \left(1 - j \cdot \frac{loss}{40 \cdot \pi \cdot \log(e)}\right)$$

$\beta_0(\omega_0) = 1717 - 1.6i$ $\beta_e(\omega_0) = 1623$ $\beta_q(\omega_0) = 94.314$ $Z_e(\omega_0) = 5201$ $Z_c(\omega_0) = 58.7$ $\mu_c(\omega_0) = 0.765$

Coupled-mode matrix

$$CM(\beta_0, \beta_q, \beta_e, \mu_c, Z_c, Y_e) := \begin{bmatrix} \beta_0 & 0 & \frac{1}{2}\cdot\mu_c\cdot Z_c\cdot Y_e\cdot(\beta_e - \beta_q) & -\frac{1}{2}\cdot\mu_c\cdot Z_c\cdot Y_e\cdot(\beta_e + \beta_q) \\ 0 & -\beta_0 & -\frac{1}{2}\cdot\mu_c\cdot Z_c\cdot Y_e\cdot(\beta_e - \beta_q) & \frac{1}{2}\cdot\mu_c\cdot Z_c\cdot Y_e\cdot(\beta_e + \beta_q) \\ \frac{1}{2}\cdot\mu_c\cdot\beta_0 & -\frac{1}{2}\cdot\mu_c\cdot\beta_0 & (\beta_e - \beta_q) & 0 \\ \frac{1}{2}\cdot\mu_c\cdot\beta_0 & -\frac{1}{2}\cdot\mu_c\cdot\beta_0 & 0 & (\beta_e + \beta_q) \end{bmatrix}$$

Equation 11.132

This equation is incorrect in the book

The eigenvalues of the matrix are

$$\beta n2(\omega) := \mathrm{eigenvals}(CM(\beta_0(\omega), \beta_q(\omega), \beta_e(\omega), \mu_c(\omega), Z_c(\omega), Y_e(\omega)))$$

$$\beta(\omega) := \left| \begin{array}{l} \beta \leftarrow \mathrm{sort}(\beta n2(\omega)) \\ \beta \leftarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_4 \\ \beta_3 \end{pmatrix} \text{ if } \mathrm{Im}(\beta_4) < 0 \\ \beta \end{array} \right.$$

Note: The roots have been sorted in the order:

(1) Backward wave
(2) Fast wave
(3) Decaying wave
(4) Growing wave

$$\beta(\omega_0) = \begin{pmatrix} -1717 + 2i \\ 1511 - 0i \\ 1726 - 67i \\ 1726 + 65i \end{pmatrix}$$

Two wave approximation

$$\beta 2(\omega) := \frac{\beta_0(\omega) + \beta_s(\omega)}{2} + \frac{j}{2}\cdot\sqrt{\mu_c(\omega)^2\cdot Z_c(\omega)\cdot Y_e(\omega)\cdot\beta_0(\omega)\cdot\beta_s(\omega) - (\beta_0(\omega) - \beta_s(\omega))^2}$$

Equation 11.145

$$\beta 2(\omega_0) = 1717 + 69i$$

The connection matrix which relates the amplitudes of the coupled modes to those of the uncoupled modes is

$$
CC(\omega) := \left|
\begin{array}{l}
Yc \leftarrow Y_c(\omega) \\
Ye \leftarrow Y_e(\omega) \\
\mu c \leftarrow \mu_c(\omega) \\
C1 \leftarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ Yc & -Yc & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & Ye & -Ye \end{pmatrix} \\
\beta\beta \leftarrow \beta(\omega) \\
\beta 0 \leftarrow \beta_0(\omega) \\
\beta e \leftarrow \beta_e(\omega) \\
\beta q \leftarrow \beta_q(\omega) \\
C2 \leftarrow \left|
\begin{array}{l}
\text{for } j \in 1..4 \\
\quad \left|
\begin{array}{l}
C2_{1,j} \leftarrow 1 \\
C2_{2,j} \leftarrow \dfrac{\beta\beta_j \cdot Yc}{\beta 0} \\
C2_{4,j} \leftarrow \left[\dfrac{\left(\beta\beta_j\right)^2 - \beta 0^2}{\beta\beta_j \cdot \beta 0}\right] \cdot \dfrac{Yc}{\mu c} \\
C2_{3,j} \leftarrow C2_{4,j} \cdot \dfrac{\beta e - \beta\beta_j}{\beta q \cdot Ye}
\end{array}
\right. \\
C2
\end{array}
\right. \\
CC \leftarrow C2^{-1} \cdot C1
\end{array}
\right.
$$

Equation 14.14

Equations 14.26 to 14.29

Equation 14.31

The transfer matrix which relates the uncoupled modes at the start of a section of length z to those at the end of the section is

$$TT(\omega, z) := \left| \begin{array}{l} \beta z \leftarrow \beta(\omega)\cdot z \\ SS \leftarrow \begin{pmatrix} \exp(-j\cdot\beta z_1) & 0 & 0 & 0 \\ 0 & \exp(-j\cdot\beta z_2) & 0 & 0 \\ 0 & 0 & \exp(-j\cdot\beta z_3) & 0 \\ 0 & 0 & 0 & \exp(-j\cdot\beta z_4) \end{pmatrix} \\ T \leftarrow CC(\omega)^{-1}\cdot SS\cdot CC(\omega) \end{array} \right.$$

Equation 14.32

Equation 14.33

Transfer matrices for the sections of the tube

$T1(\omega) := TT(\omega, z_1)$ $\quad$ $T2(\omega) := TT(\omega, z_2)$

The transfer matrix which relates the uncoupled modes at the start of a section of length z_1 to the wave amplitudes at the end of the section is

$$RR(\omega) := \left| \begin{array}{l} \beta z \leftarrow \beta(\omega)\cdot z_1 \\ SS \leftarrow \begin{pmatrix} \exp(-j\cdot\beta z_1) & 0 & 0 & 0 \\ 0 & \exp(-j\cdot\beta z_2) & 0 & 0 \\ 0 & 0 & \exp(-j\cdot\beta z_3) & 0 \\ 0 & 0 & 0 & \exp(-j\cdot\beta z_4) \end{pmatrix} \\ RR \leftarrow SS\cdot CC(\omega) \end{array} \right.$$

Matrix representation of the input section

The amplitude of the input signal is unity, the beam is initially unmodulated and it is assumed that the amplitude of the backward wave amplitude at the output is zero. Thus three boundary conditions are defined at the input and one at the output.

$$U1(\omega) := \begin{pmatrix} 1 \\ \dfrac{-T1(\omega)_{2,1}}{T1(\omega)_{2,2}} \\ 0 \\ 0 \end{pmatrix}$$

Equation 14.35

This equation assumes that the backward wave amplitude in the output transmission line is zero and the the line is matched to the circuit in the absence of the beam. When the beam is present there is a change in the matches at the ends of the circuit resulting in gain ripples. This can be observed by enabling this equation and disabling the one below.

$$U1(\omega) := \begin{pmatrix} 1 \\ \dfrac{-RR(\omega)_{1,1}}{RR(\omega)_{1,2}} \\ 0 \\ 0 \end{pmatrix}$$

In a helix TWT the end of the first section is normally terminated by an internal attenuator so that the amplitude of the backward circuit wave is zero at the end of the section. This is represented by the equation to the left which is not given in the book.

Then at the end of the section

$$W2(\omega) := RR(\omega)\cdot U1(\omega)$$

$$W2(\omega_0) = \begin{pmatrix} 0.0 \\ 0.1 - 0.0i \\ -0.0 + 0.0i \\ 0.5 + 12.0i \end{pmatrix}$$

Note: Backward wave = 0 as required

$$U2(\omega) := T1(\omega)\cdot U1(\omega)$$

Equation 14.33

The gain of the section is $\mathrm{Gain1}(\omega) := 20\cdot\log\left(\left|\mathrm{U2}(\omega)_1\right|\right)$ $\mathrm{Gain1}(1.0\cdot\omega_0) = 21.7\cdot\mathrm{dB}$

The returned signal at the input is $20\cdot\log\left(\left|\mathrm{U1}(\omega_0)_2\right|\right) = -92.4\cdot\mathrm{dB}$

The coupled modes at the start of the section are $\mathrm{W1}(\omega) := \mathrm{CC}(\omega)\cdot\mathrm{U1}(\omega)$

The launching loss is $20\cdot\log\left(\left|\mathrm{W1}(\omega_0)_4\right|\right) = -6.8\cdot\mathrm{dB}$

The cold loss through the section is $20\cdot\log\left(\exp\left(\mathrm{Im}\left(\beta_0(\omega_0)\right)\cdot z_1\right)\right) = -0.7\cdot\mathrm{dB}$

Matrix representation of a sever

The propagation matrix for the uncoupled modes through the sever is

$$
\mathrm{Ts}(\omega) := \left|\begin{array}{l}
\beta f \leftarrow \beta_f(\omega) \\
\beta s \leftarrow \beta_s(\omega) \\
\mathrm{Ts} \leftarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \exp(-j\cdot\beta f\cdot z_s) & 0 \\ 0 & 0 & 0 & \exp(-j\cdot\beta s\cdot z_s) \end{pmatrix}
\end{array}\right.
$$

Matrix representation of the second section

The amplitude of the input signal on the structure is zero as is the backward wave in the output transmission line.
The section is driven by the fast and slow waves on the beam carried forward from the previous section.

$$U3(\omega) := \left| \begin{array}{l} U3 \leftarrow Ts(\omega)\cdot U2(\omega) \\ U3_2 \leftarrow \dfrac{-\left(T2(\omega)_{2,3}\cdot U3_3 + T2(\omega)_{2,4}\cdot U3_4\right)}{T2(\omega)_{2,2}} \\ U3 \end{array} \right.$$

Equation 14.41

This equation assumes that the output of the section is matched in the absence of the beam. When the beam is present the change in the matches produce gain ripples. These can be demonstrated by enabling the equation to the left and disabling the equation below.

$$U3(\omega) := Ts(\omega)\cdot U2(\omega)$$

$$U3(\omega_0) = \begin{pmatrix} 0.0 \\ 0.0 \\ 37.9 - 1.6i \\ -68.2 - 99.2i \end{pmatrix}$$

The gain ripples disappear if it is assumed that the forward and backward uncoupled waves at the start of the circuit are both zero.

The vector of uncoupled waves at the output of the section is

$$U4(\omega) := T2(\omega)\cdot U3(\omega)$$

The overall gain of the tube is $Gain2(\omega) := 20\cdot\log\left(\left|U4(\omega)_1\right|\right)$ Equation 14.42 $Gain2(\omega_0) = 45.0\cdot dB$

The Sever Loss can be computed from the vector of coupled waves at the start of the second section

$$W3(\omega) := CC(\omega)\cdot U3(\omega)$$

$$20\cdot\log\left(\left|W3(\omega_0)_4\right|\right) - 20\cdot\log\left(\left|U2(\omega_0)_1\right|\right) = -5.0\cdot dB$$

Computation of the signal growth along the length of the tube

The amplitudes of the forward waves in dB as a function of position along the structure are computed from the forward wave amplitudes at the start of each section

$WA := W1(\omega_0)$ $\qquad \beta\beta := \beta(\omega_0)$ $\qquad VA(z) := 20\cdot\log\left[\left|\sum_{j=2}^{4}\left(WA_j\cdot\exp(-j\cdot\beta\beta_j\cdot z)\right)\right|\right]$ $\qquad z1 := 0, 0.001 .. z_1$

$WB := W3(\omega_0)$ $\qquad VB(z) := 20\cdot\log\left[\left|\sum_{j=2}^{4}\left[WB_j\cdot\exp\left[-j\cdot\beta\beta_j\cdot(z - z_1 - z_s)\right]\right]\right|\right]$ $\qquad z2 := (z_1 + z_s), (z_1 + z_s + 0.001) .. (z_1 + z_s + z_2)$

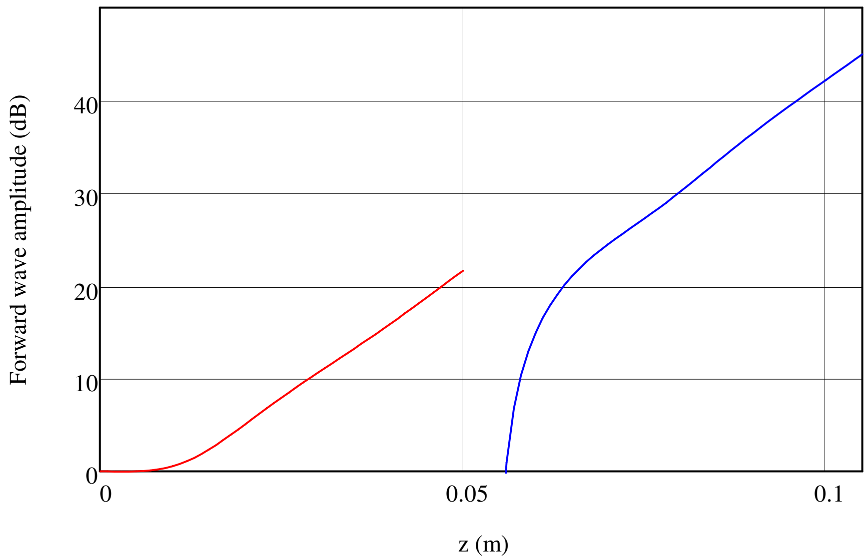
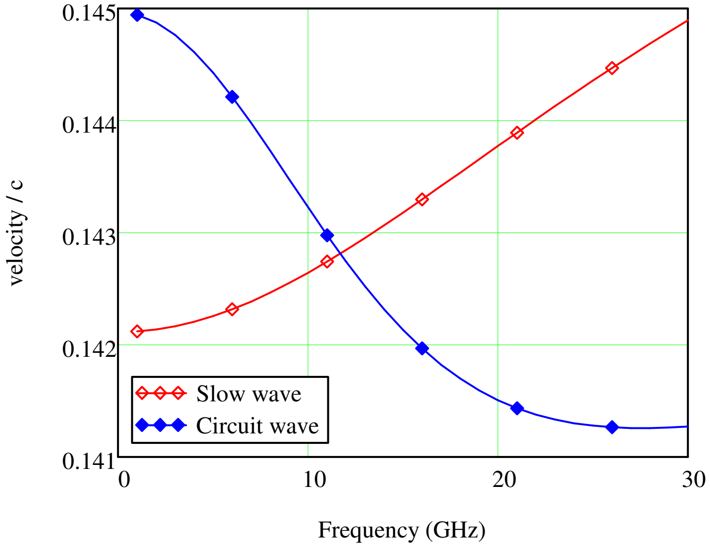
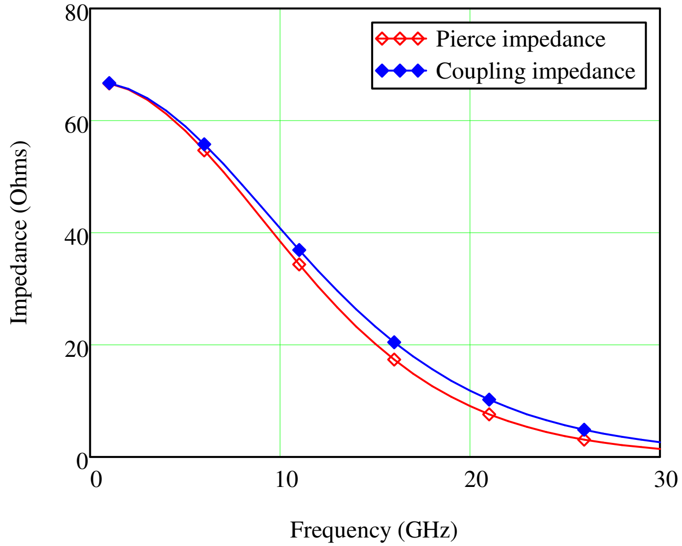


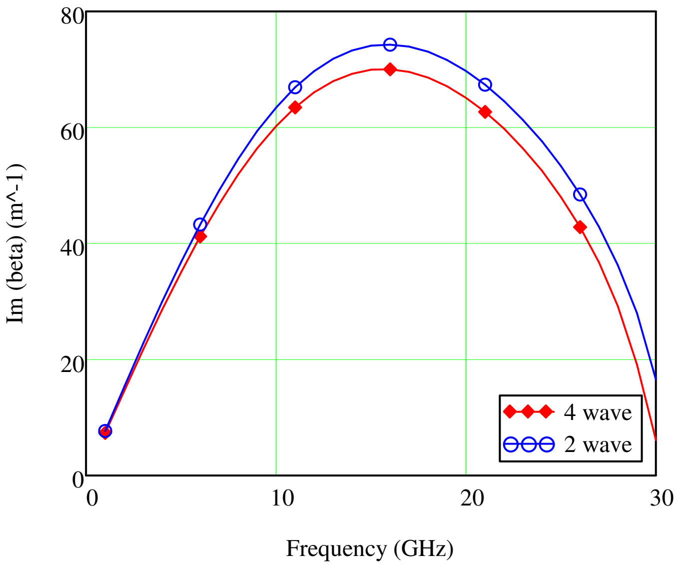
Figure 14.9

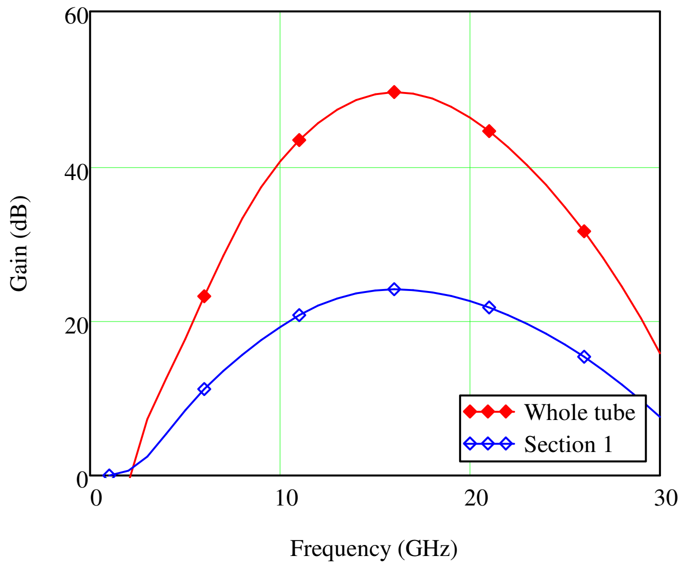
Plotting frequencies in GHz $f1 := 1, 2 .. 30$

$$\omega 1 := \left| \begin{array}{l} \text{for } f \in 1, 2 .. 30 \\ \quad \omega 1_f \leftarrow 2 \cdot \pi \cdot f \cdot 10^9 \\ \omega 1 \end{array} \right.$$









Note: The computation of this graph takes some time