WS 5.2 Cylindrical space-charge limited diode

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

Physical constants

Charge/mass ratio of the electron $\eta := 1.759 \cdot 10^{11} \cdot C \cdot kg^{-1}$

Rest energy of the electron (eV) $V_R := \frac{c^2}{\eta}$ $V_R = 510.947\,kV$

Normalised variables

$R = \frac{r}{r_c}$ $U = \frac{V}{V_R}$ $k_1 = \frac{-I_L \cdot r_c}{2 \cdot \pi \cdot \varepsilon_0 \cdot c \cdot V_R}$ Equation 5.68

$a^3 = \frac{81}{32} \cdot k_1^2$ Equation 5.74 $a := 1.0$

Langmuir's equations

$$\gamma(R) := \ln(R)$$ Equation 5.69

$$\beta A(R) := \gamma(R) - 0.4\cdot\gamma(R)^2 + 0.0916667\cdot\gamma(R)^3 - 0.014242\cdot\gamma(R)^4 + 0.001679\cdot\gamma(R)^5 - 0.000161\cdot\gamma(R)^6$$ Equation 5.71

$$\beta B(R) := \exp\left(-\frac{\gamma(R)}{2}\right)\cdot\left(\gamma(R) + 0.1\cdot\gamma(R)^2 + 0.016667\cdot\gamma(R)^3 + 0.002424\cdot\gamma(R)^4 + 0.000287\cdot\gamma(R)^5 + 0.000026\cdot\gamma(R)^6\right)$$ Equation 5.72

Normalised voltages using equations (5.71) and (5.72)

$$ULA(a,R) := a\cdot R^{\frac{2}{3}}\cdot\left(\left|\beta A(R)\right|\right)^{\frac{4}{3}} \qquad ULB(a,R) := a\cdot R^{\frac{2}{3}}\cdot\left(\left|\beta B(R)\right|\right)^{\frac{4}{3}}$$ Equation 5.75

Acton's equations

$$UA(a,R) := a\cdot\left(\left|\gamma(R)\right|\right)^{\frac{4}{3}}\cdot\left(1 + 0.1333\cdot\gamma(R) + 0.02444\cdot\gamma(R)^2 + 0.0039236\cdot\gamma(R)^3 + 0.00052966\cdot\gamma(R)^4\right)$$ Equation 5.73

$$UAR(a,R) := a\cdot\left(\left|\gamma(R)\right|\right)^{\frac{4}{3}}\cdot\left(1 + \frac{2}{15}\cdot\gamma(R) + \frac{11}{450}\cdot\gamma(R)^2\right) + a^2\cdot\left(\left|\gamma(R)\right|\right)^{\frac{8}{3}}\cdot\left[\frac{1}{14} + \frac{6}{175}\cdot a\cdot\left(\left|\gamma(R)\right|\right)\right]$$ Equation 5.76

Numerical integration of Equation (5.66)

$$k_1 := \sqrt{\frac{32}{81}\cdot a^3}$$

$$k_1 = 0.629$$

$$\frac{d^2U}{dr^2}+\frac{1}{r}\frac{\partial U}{\partial r}=-\frac{I_L}{2\pi\varepsilon_0 c V_0 r}\left[1-\frac{1}{(1+U)^2}\right]^{-\frac{1}{2}}$$

$$\frac{d^2U}{dR^2}+\frac{1}{R}\frac{\partial U}{\partial R}=\frac{k_1}{R}\left[1-\frac{1}{(1+U)^2}\right]^{-\frac{1}{2}}$$

This can be written as a pair of simultaneous first order equations

$$\frac{d}{dR}U_0 = U_1$$

$$\frac{d}{dR}U_1 = \frac{k_1}{R}\cdot\left[\frac{1+U_0}{\sqrt{2\cdot U_0+(U_0)^2}}\right]-\frac{U_1}{R}$$

$$DR(R,U) := \begin{bmatrix} U_1 \\ \frac{k_1}{R}\cdot\left[\frac{1+U_0}{\sqrt{2\cdot U_0+(U_0)^2}}\right]-\frac{U_1}{R}\end{bmatrix}$$

In the non-relativistic approximation

$$DNR(R,U) := \begin{bmatrix} U_1 \\ \frac{k_1}{R}\cdot\left(\frac{1}{\sqrt{2\cdot U_0}}\right)-\frac{U_1}{R}\end{bmatrix}$$

These equations are difficult to integrate numerically because of the singularity in the integrand close to the cathode when U = 0.
However, when R tends to 1, γ tends to zero and U can be approximated from equation (5.73) by

$U = a\cdot\gamma^{\frac{4}{3}}$ so that $R = \exp\left[\left(\frac{U}{a}\right)^{\frac{4}{3}}\right]$ or, approximately $R = 1+\left(\frac{U}{a}\right)^{\frac{4}{3}}$ whence $\frac{dU}{dR} = \frac{4}{3}\cdot a^{\frac{3}{4}}\cdot U^{\frac{1}{4}}$

The solution is started from a point a little way from the cathode using this approximation to determine the initial conditions

Let $U0 := 10^{-6}$ $R_1 := \exp\left[\left(\frac{U0}{a}\right)^{\frac{4}{3}}\right] = 1.000$ $R_{max} := 5$ $R1 := 1, 1.5 .. R_{max}$

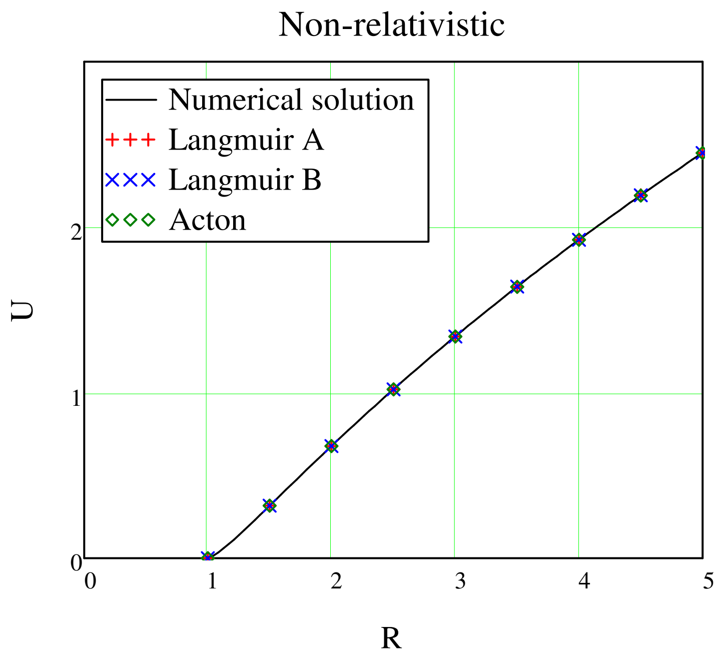
When the cathode lies outside the anode the initial conditions are

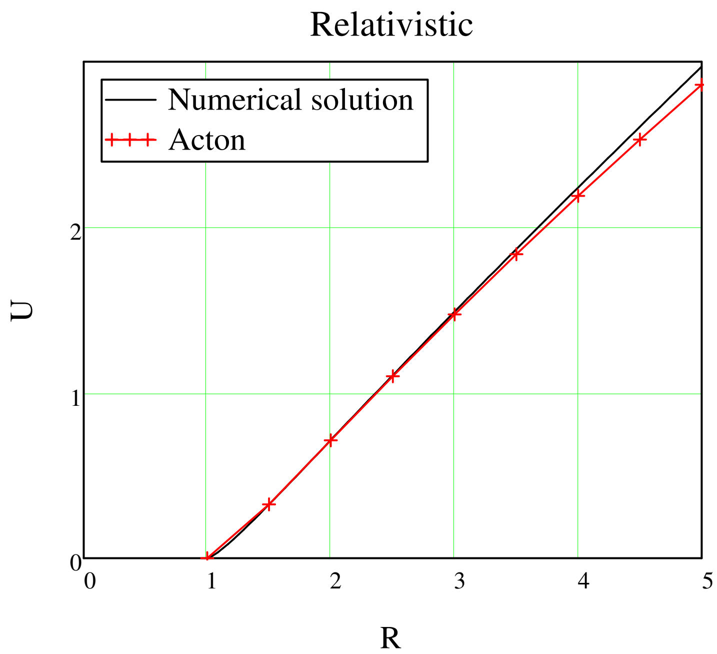
$$U := \begin{pmatrix} U0 \\ \frac{4}{3} \cdot a^{\frac{3}{4}} \cdot U0^{\frac{1}{4}} \end{pmatrix} \qquad U = \begin{pmatrix} 1 \times 10^{-6} \\ 0.042 \end{pmatrix}$$

The solutions to the differential equations are

$$U1NR := \mathrm{AdamsBDF}\left(U, R_1, R_{max}, 95, DNR\right) \qquad U1R := \mathrm{AdamsBDF}\left(U, R_1, R_{max}, 95, DR\right)$$

The results are in matrices in which the first column is the independent variable, the second column is U and the thrid column its differential

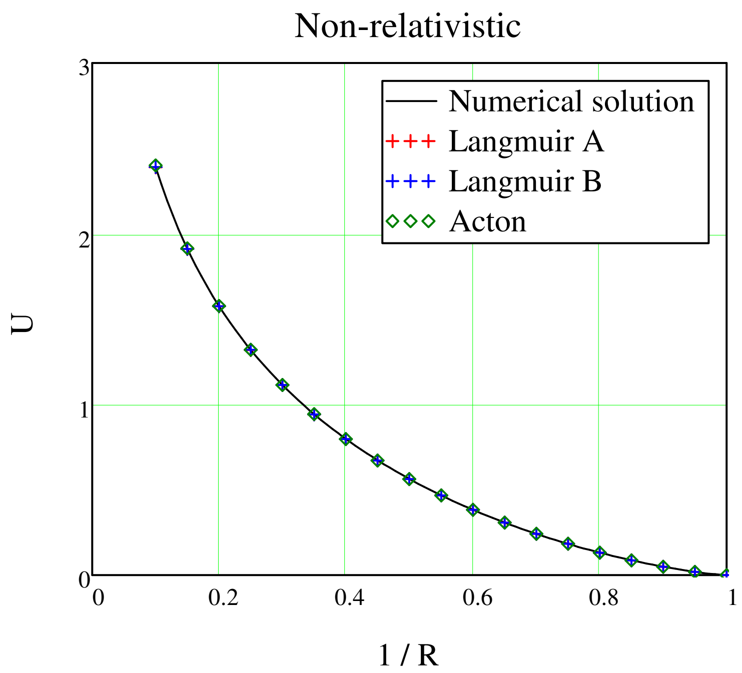


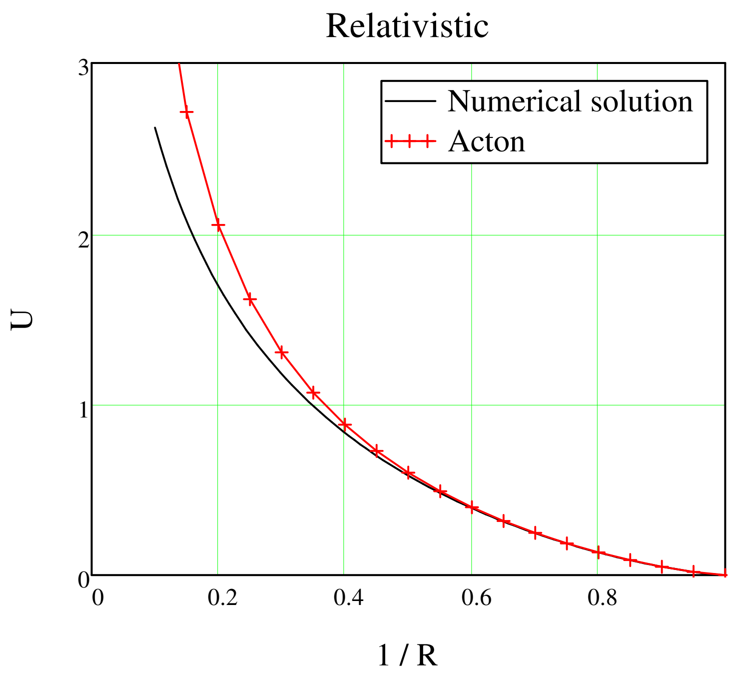


When the anode lies outside the cathode the initial conditions are

$$U := \begin{pmatrix} U0 \\ -\sqrt{2\cdot k_1\cdot\sqrt{2\cdot U0}} \end{pmatrix} \qquad U = \begin{pmatrix} 1\times 10^{-6} \\ -0.042 \end{pmatrix} \qquad R_2 := \frac{1}{R_1} \qquad R2_{min} := 0.1 \qquad R2 := 1, 0.95 .. R2_{min}$$

$$U2NR := \mathrm{AdamsBDF}\left(U, R_2, R2_{min}, 100, DNR\right) \qquad U2R := \mathrm{AdamsBDF}\left(U, R_2, R2_{min}, 100, DR\right)$$





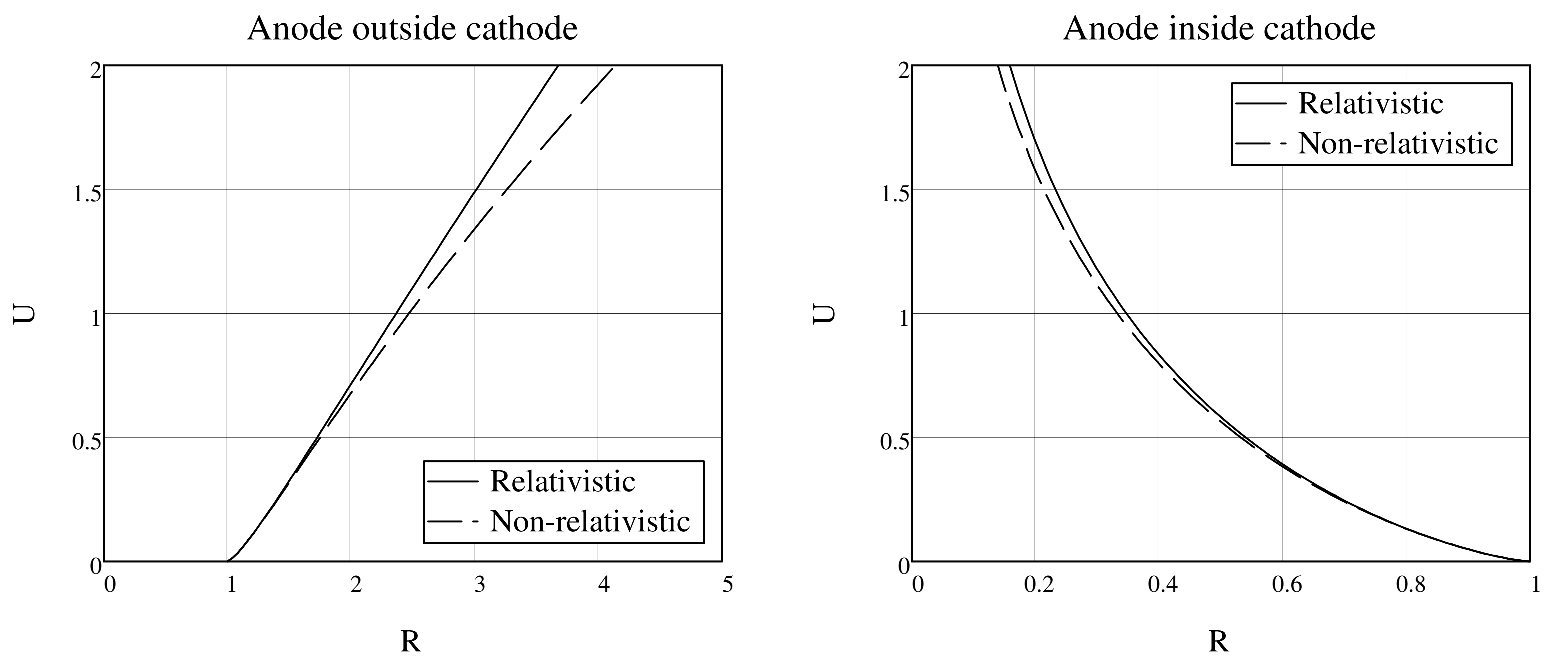


Figure 5.9(a) and (b)