# WS 3.4 Fujisawa's model



This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.



This sheet uses the method described in
Fujisawa, K. (1958). "General treatment of klystron resonant cavities." IRE Transactions on Microwave Theory and Techniques **MTT-6**(10): 344-358.

It also models the coupling into a cavity using a loop

### Section 3.5.2 Fujisawa's model of re-entrant cavities

Define physical constants

$\varepsilon_0 = 8.854 \times 10^{-12} \cdot \frac{F}{m}$ $\mu_0 = 1.257 \times 10^{-6} \cdot \frac{H}{m}$ $\eta := 1.759 \cdot 10^{11} \cdot \frac{C}{kg}$ $dB := 1$

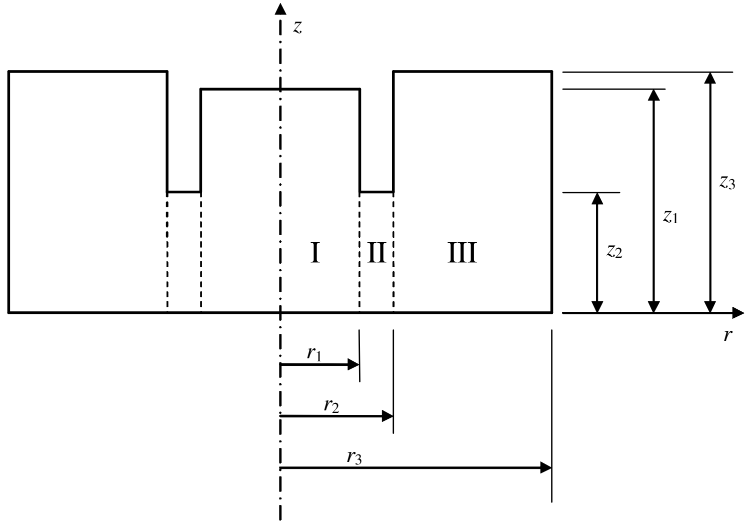
$c = 2.998 \times 10^{8} \frac{m}{s}$ $Z_0 := \sqrt{\frac{\mu_0}{\varepsilon_0}}$ $Y_0 := \frac{1}{Z_0}$

**Define the cavity dimensions**

The cavity is defined as a singly re-entrant cavity divided into three regions numbered I, II and III starting from the axis. The height of region *i* is $z_i$ and its outer radius is $r_i$

$r_1 := 27.5 \cdot mm$ $\quad r_2 := 50 \cdot mm$ $\quad r_3 := 150 \cdot mm$

$z_1 := 200 \cdot mm$ $\quad z_2 := 20.5 \cdot mm$ $\quad z_3 := 99 \cdot mm$




The properties of a doubly re-enetrant cavity with all the height dimensions doubled are calculated from those of the singly re-entrant cavity. Note that the equations in the book assume a doubly re-entrant cavity.

**Calculate the contributions to the capacitance from the three regions**

$$\mathrm{Lim} := \frac{2 \cdot 10^4}{m}$$

$$C_I(r, z) := 2\varepsilon_0 \cdot r \cdot \int_{-\mathrm{Lim}}^{\mathrm{Lim}} \frac{I1(\beta \cdot r)}{\beta \cdot I0(\beta \cdot r)} \cdot \frac{\sin(\beta \cdot z)}{\beta \cdot z} \, d\beta$$ Equation 3.81 $\quad C_I(r_1, z_2) = 0.906 \cdot pF$

$$C_{II} := \frac{\varepsilon_0 \cdot \pi \cdot \left(r_2^{\,2} - r_1^{\,2}\right)}{z_2}$$ Equation 3.82 $\quad C_{II} = 2.366 \cdot pF$

$$C_{III} := 4\cdot\varepsilon_0\cdot r_2\cdot\ln\left[\frac{e\cdot\sqrt{(r_3 - r_2)^2 + z_3^{\,2}}}{2z_2}\right]$$ Equation 3.83 $C_{III} = 3.955\cdot pF$

The total capacitance is $C_S := C_I(r_1, z_2) + C_{II} + C_{III}$ $C_S = 7.226\cdot pF$

**Calculate the inductance**

$$L_S := \frac{\mu_0\cdot z_3}{2\pi}\cdot\ln\left(\frac{r_3}{r_2}\right)$$ Equation 3.86 $L_S = 2.175\times 10^{-8}\,H$

**Calculate the resonant frequency and R/Q**

$$\omega_0 := \frac{1}{\sqrt{C_S\cdot L_S}} \qquad f_0 := \frac{\omega_0}{2\cdot\pi}$$ $f_0 = 0.401\cdot GHz$

$$R\_Q_S := \sqrt{\frac{L_S}{C_S}}$$ $R\_Q_S = 54.865\,\Omega$

For a doubly re-entrant cavity $R\_Q_D := 2\cdot R\_Q_S$ $R\_Q_D = 109.73\,\Omega$

**Calculate the surface resistance**

The factor surf can used to adjust the effective conductivity to obtain Q in agreement with experiment

$\sigma := 5.959\cdot 10^7\cdot S\cdot m^{-1}$ $surf := 0.15$ $$R_s := \sqrt{\frac{\omega_0\cdot\mu_0}{2\cdot\sigma\cdot surf}}$$ Equation 3.50 $R_s = 0.013\,\Omega$

**Calculate the stored energy and power dissipation for unit circulating current**

$$W_H := \frac{\mu_0 \cdot z_3}{4 \cdot \pi} \cdot \ln\left(\frac{r_3}{r_2}\right)$$ Equation 3.87

$$P_L := \frac{R_s}{4 \cdot \pi} \cdot \left(\frac{z_3}{r_3} + \frac{z_3 - z_2}{r_2} + \ln\left(\frac{r_3}{r_2}\right)\right)$$ Equation 3.88

Note: the power dissipation is calculated on the curved surfaces and the annular surface so it applies only to a doubly re-entrant cavity

**Calculate the unloaded Q and the shunt impedance of a doubly re-entrant cavity**

$$Q_D := \frac{\omega_0 \cdot W_H}{P_L}$$ Equation 3.12

$Q_D = 7778$

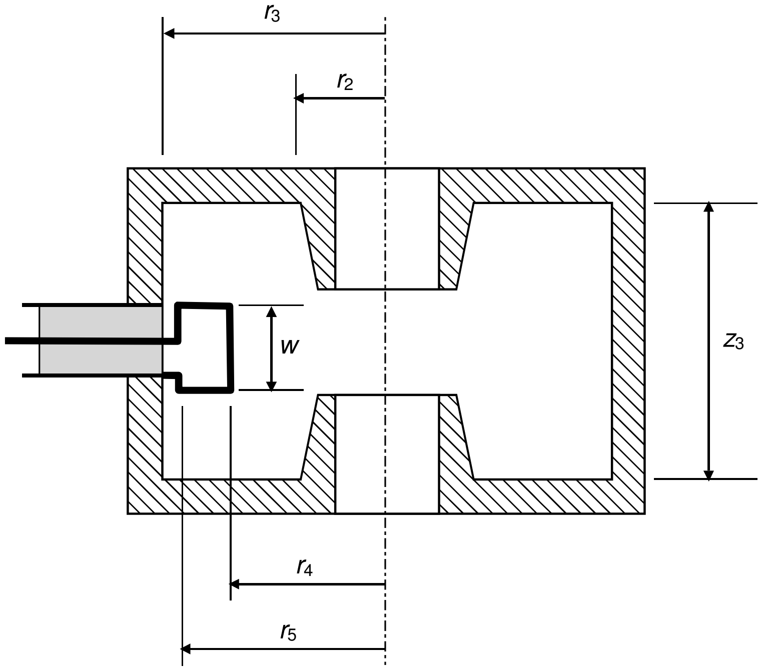
$$R_{CD} := R\_Q_D \cdot Q_D$$

$R_{CD} = 853.5 \cdot k\Omega$

$$r_C := \frac{R_{CD}}{{Q_D}^2}$$

$r_C = 0.014\,\Omega$

**Section 3.6.1 Loop coupling**




Parameters of a doubly re-entrant cavity from Fujisawa's model above

$C_C := 0.5 \cdot C_S$ $\quad L_C := 2 \cdot L_S$ $\quad R_C := R_{CD}$

$C_C = 3.613\,\mathrm{pF}$ $\quad L_C = 0.044\,\mu\mathrm{H}$ $\quad R_C = 853.5\,\mathrm{k\Omega}$

Characteristic impedance of the input transmission line

$$Z_c := 50 \cdot \Omega$$

Dimensions of the loop made of wire with radius $r_0$

$r_4 := 87 \cdot \mathrm{mm}$ $\quad r_5 := 136 \cdot \mathrm{mm}$ $\quad w := 31 \cdot \mathrm{mm}$ $\quad r_0 := 2.5 \cdot \mathrm{mm}$

Calculate the mutual inductance when the plane of the loop is at an angle θ to the axis of the cavity by calculating the flux linked to the loop for unit current circulating in the cavity

$$M_L(\theta) := \frac{\mu_0 \cdot \cos(\theta)}{2 \cdot \pi} \cdot w \cdot \ln\left(\frac{r_5}{r_4}\right) \qquad M_L(0) = 0.0028\,\mu\mathrm{H}$$

The coupling factor is the fraction of the flux in the cavity which is linked to the loop. Using Fujisawa's approximation this is

$$k(\theta) := \frac{M_L(\theta)}{L_C} \qquad k(0) = 0.064$$

**Estimate the self-inductance of the loop in free space**

(a) Loop inductance calculated from the formula for a circular wire loop in Ramo, S., J. R. Whinnery, et al. (1965). *Fields and Waves in Communication Electronics*. New York, Wiley. Section 5.24, equation (2). The RF current flows in a thin surface layer of the wire and the internal inductance is zero.

Loop area $A_L := w \cdot (r_5 - r_4)$

Equivalent circular loop radius $R_L := \sqrt{\frac{A_L}{\pi}}$ $\qquad R_L = 22 \cdot mm$

Note: If the loop is made of wire with a rectangular section its properties can be approximated by using a round wire with the same perimeter.

Loop inductance $La_L := \mu_0 \cdot R_L \cdot \left( \ln\left( \frac{8 \cdot R_L}{r_0} \right) - 2 \right)$ $\qquad La_L = 0.062 \cdot \mu H$

(b) Loop treated as a short-circuited two-wire line using the formula for the characteristic impedance in Ramo, S., J. R. Whinnery, et al. (1965). Fields and Waves in Communication Electronics. New York, Wiley. Table 8.09.

$Z_{CL} := \frac{Z_0}{\pi} \cdot \operatorname{acosh}\left( \frac{w}{2 \cdot r_0} \right)$ $\qquad Z_{CL} = 301\,\Omega$

Electrical length of the line $\phi := \frac{\omega_0}{c} \cdot (r_5 - r_4)$ $\qquad \phi = 23.62 \cdot deg$

$Lb_L := \frac{Z_{CL}}{\omega_0} \cdot \tan(\phi)$ $\qquad Lb_L = 0.052 \cdot \mu H$

(c) Loop inductance from formula in Engala, K. R., A. A. Kishk, et al. (2000). "Simple computation of the coupling coefficient for loop-coupled resonant cavities." *Microwave and Optical Technology Letters* **27**(6): 400-404. Equation (5) corrected.

$a_1 := 31 \cdot mm$ $\qquad b_1 := r_5 - r_4$ $\qquad d_1 := \sqrt{{a_1}^2 + {b_1}^2}$

$$Lc_L := \frac{\mu_0}{\pi} \cdot \left[ a_1 \cdot \ln\left[ \frac{2 \cdot a_1 \cdot b_1}{r_0 \cdot (a_1 + d_1)} \right] + b_1 \cdot \ln\left[ \frac{2 \cdot a_1 \cdot b_1}{r_0 \cdot (b_1 + d_1)} \right] - 2 \cdot (a_1 + b_1 - d_1) + \frac{(a_1 + b_1)}{4} \right]$$

$Lc_L = 0.070 \cdot \mu H$

Note: The loop inductances calculated by these three methods agree with one another to around +/- 20%.

When the loop is placed in the cavity the flux generated by it is contained by the walls of the cavity. This increases the reluctance of the flux path and decreases the flux generated by unit current and, hence, the self-inductance of the loop. The loop inductance can be measured experimentally by using a frequency distant from the resonant frequency of the cavity. The measured value of the loop inductance can then be used to model its effects.

In the absence of an experimental result we can get a very rough idea of its value by noting the the flux density at the centre of a round wire loop of radius a in free space is

$$B = \frac{\mu_0 \cdot I}{2 \cdot a}$$

If the flux line through the centre of the loop is constrained by the cavity to be a circle with the same diameter as the loop and the magnetic field is assumed to be uniform along this line, then at the centre of the loop

$$B = \frac{\mu_0 \cdot I}{2 \cdot \pi a}$$

suggesting that the inductance calculated in free space should be divided by π.

Taking the average of the values of the loop inductance in free space calculated above

$$L_L := \frac{La_L + Lb_L + Lc_L}{3}$$

The input impedance of the cavity is

$$Z_{in}(\omega, \theta, LF) := j \cdot \omega \cdot LF \cdot L_L + \frac{k(\theta)^2 \cdot R_C}{1 + j \cdot Q_D \cdot \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}$$

Equation 3.91

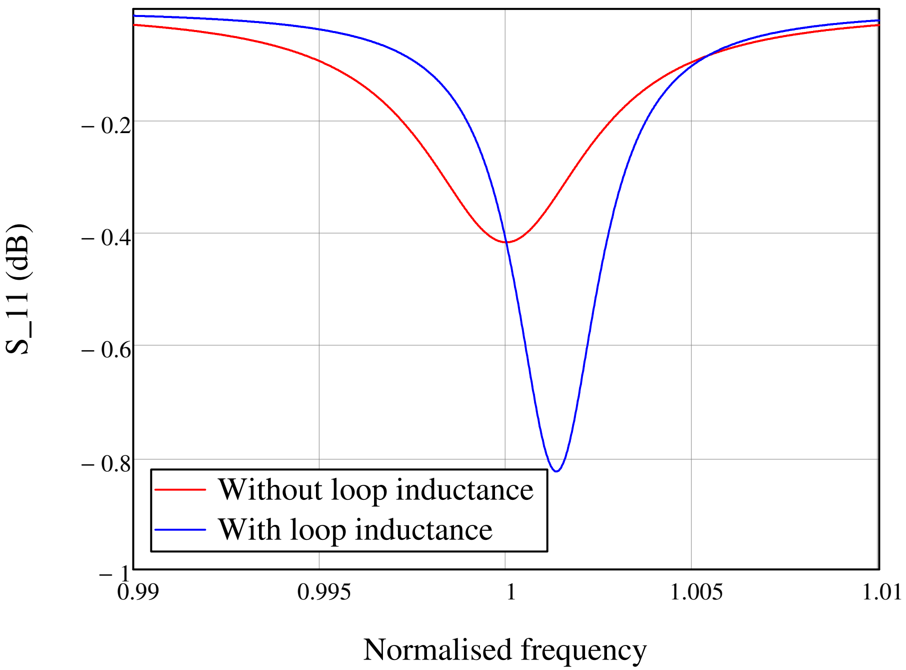
where LF is the correction factor for the inductance. The voltage reflection coefficient and the scattering parameter at the input to the cavity are

$$\rho(\omega, \theta, LF) := \frac{Z_{in}(\omega, \theta, LF) - Z_c}{Z_{in}(\omega, \theta, LF) + Z_c} \qquad S_{11}(\omega, \theta, LF) := 20 \cdot \log(|\rho(\omega, \theta, LF)|)$$

Loop angle $\theta := 39 \cdot deg$

Loop correction factor $LF := \frac{1}{\pi}$

Investigate the effect of changing the loop angle



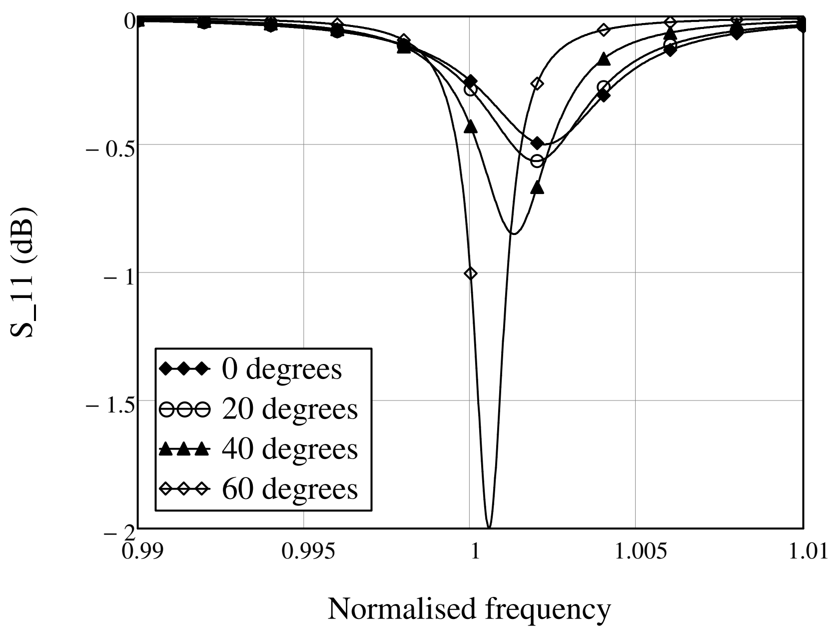




Figure 3.20

Note that this figure differs slightly from the one in the book because of a difference in the assumption about the self-inductance of the loop in free space.

Calculate the loop reactance, the external resistance, and the external Q of the loaded cavity assuming that the source impedance is equal to the characteristic impedance of the input line.

$$X_L := \omega_0 \cdot L_L \cdot LF \qquad R_E := \frac{Z_c^{\,2} + X_L^{\,2}}{k(\theta)^2 \cdot Z_c} \qquad \boxed{\text{Equation 3.93}} \qquad Q_E := \frac{R_E}{\omega_0 \cdot L_C} \qquad Q_E = 368$$

The resonant frequency ( $\omega_1$ ) of the loaded cavity is the solution of

$$\omega C_C - \frac{1}{\omega L_C} - \frac{k^2 X}{\left(R_S^2 + X^2\right)} = 0 \qquad \boxed{\text{Equation 3.95}}$$

$$Fn(\omega) := \omega \cdot C_C - \frac{1}{\omega \cdot L_C} - \frac{X_L \cdot k(\theta)^2}{\left(Z_c^{\,2} + X_L^{\,2}\right)} \qquad \omega := \omega_0 \qquad \omega_1 := \mathrm{root}(Fn(\omega), \omega) \qquad \frac{\omega_1}{2 \cdot \pi} = 0.402\,\mathrm{GHz}$$

$$\frac{\omega_1}{\omega_0} = 1.00134$$

At resonance $\qquad S_{11}(\omega_1, \theta, LF) = -0.825 \cdot dB$