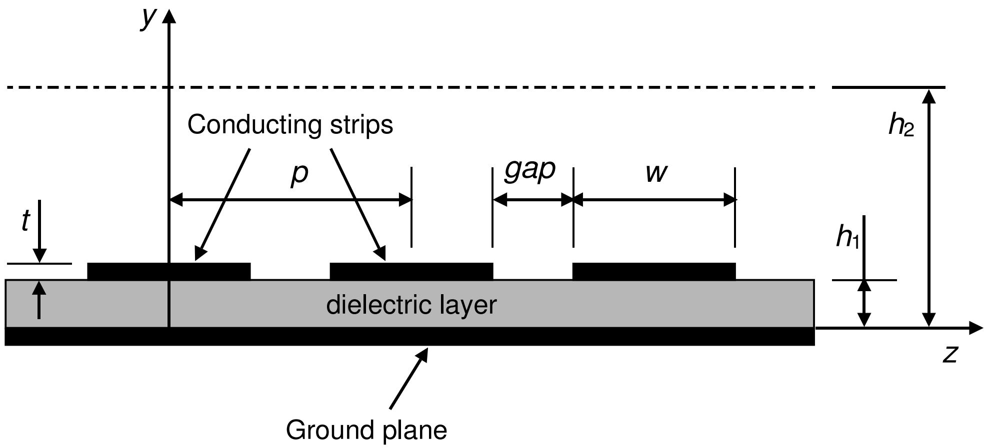
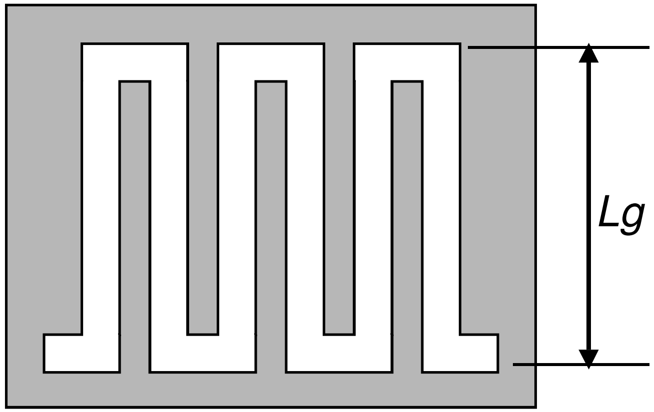
WS 4.2 Meander line

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

The contents of this sheet are provided for educational purposes only and no warranty is expressed or implied that they are suitable for use as professional design tools.

J. A. Weiss, "Dispersion and Field Analysis of a Microstrip Meander-Line Slow-Wave Structure," *Microwave Theory and Techniques, IEEE Transactions on,* vol. 22, pp. 1194-1201, 1974. Note: here ϕ is the phase change between adjacent lines and not the phase change per pitch as used by Weiss. The dimensions in W. E. Courtney, "Printed-circuit RF-keyed crossed-field amplifier," Lincoln Laboratory, Lexington, MA.1975 are apparently the same.





Structure dimensions

$h_1 := 0.062 \cdot in$ $\quad h_2 := 2.6 \cdot h_1$ $\quad Lg := 0.537 \cdot in$ $\quad w := 0.045 \cdot in$ $\quad gap := 0.057 \cdot in$ $\quad p := w + gap$ $\quad \varepsilon_2 := 6.5$

$p = 2.591 \cdot mm$ $\quad h_1 = 1.57 \cdot mm$ $\quad Lg = 13.6 \cdot mm$ $\quad w = 1.14 \cdot mm$ $\quad gap = 1.45 \cdot mm$ $\quad p = 2.59 \cdot mm$ $\quad h_2 = 4.09 \cdot mm$

Choose mesh size for FD calculations $\quad mesh := 0.142 \cdot mm$ $\quad$ Note: In this model t = 0

$$NH1 := \frac{h_1}{mesh} \quad NH2 := \frac{h_2}{mesh} \quad NP := \frac{p}{mesh} \quad NW := \frac{w}{mesh}$$

$NH1 = 11.09$ $\quad NH2 = 28.834$ $\quad NP = 18.245$ $\quad NW = 8.049$ $\quad$ Note: these numbers must be close to integers

$$Nh1 := round(NH1) \quad Nh2 := round(NH2) \quad Np := round(NP) \quad Nw := round(NW) \quad Nw2 := \frac{Nw}{2} \quad Np2 := \frac{Np}{2}$$

$Nh1 = 11$ $\quad Nh2 = 29$ $\quad Np = 18$ $\quad Nw = 8$ $\quad$ NB: Np and Nw must be even numbers

This variable is 0 for a conducting top plane and 2 for a symmetry boundary $\quad H2sym := 2$ $\quad$ Set maximum number of iterations $\quad Niter := 1000$

Calculation of capacitances using a finite difference solution of Laplace's equation

$$
\begin{aligned}
CM(\Phi,\varepsilon_2) := \ & \text{for } i \in 0..Np \\
& \quad \text{for } j \in 0..Nh2 \\
& \qquad V_{i,j} \leftarrow 0 \\
& V1 \leftarrow 100 \\
& V_2 \leftarrow \begin{cases} 100 & \text{if } \Phi = 0 \\ 0 & \text{if } \Phi = 0.5\cdot\pi \\ (-100) & \text{if } \Phi = \pi \end{cases} \\
& Rsym \leftarrow \begin{cases} 2 & \text{if } \Phi = 0 \\ 0 & \text{if } \Phi = 0.5\cdot\pi \\ 2 & \text{if } \Phi = \pi \end{cases} \\
& \text{for } n \in 1..Niter \\
& \quad \text{for } i \in 0..Np \\
& \qquad V_{i,0} \leftarrow 0 \\
& \quad \text{for } j \in 1..(Nh1-1) \\
& \qquad V_{0,j} \leftarrow \left(2\cdot V_{1,j} + V_{0,j+1} + V_{0,j-1}\right)\cdot 0.25 \\
& \qquad V_{Np,j} \leftarrow \left(Rsym\cdot V_{Np-1,j} + V_{Np,j+1} + V_{Np,j-1}\right)\cdot 0.25 \\
& \qquad \text{for } i \in 1..Np-1 \\
& \qquad\quad V_{i,j} \leftarrow \left(V_{i+1,j} + V_{i,j+1} + V_{i-1,j} + V_{i,j-1}\right)\cdot 0.25 \\
& \quad \text{for } i \in 0..Nw2 \\
& \qquad V_{i,Nh1} \leftarrow V1 \\
& \quad \text{for } i \in (Nw2+1)..(Np-Nw2-1) \\
& \qquad V_{i,Nh1} \leftarrow \frac{V_{i,Nh1-1}\cdot\varepsilon_2 + 0.5\cdot\left(V_{i-1,Nh1} + V_{i+1,Nh1}\right)\cdot\left(\varepsilon_2+1\right) + V_{i,Nh1+1}}{2\cdot\varepsilon_2 + 2}
\end{aligned}
$$

for $i \in (Np - Nw2)..Np$

$V_{i,Nh1} \leftarrow V_2$

for $j \in (Nh1 + 1)..(Nh2 - 1)$

$V_{0,j} \leftarrow \left(2 \cdot V_{1,j} + V_{0,j+1} + V_{0,j-1}\right) \cdot 0.25$

$V_{Np,j} \leftarrow \left(Rsym \cdot V_{Np-1,j} + V_{Np,j+1} + V_{Np,j-1}\right) \cdot 0.25$

for $i \in 1..Np - 1$

$V_{i,j} \leftarrow \left(V_{i+1,j} + V_{i,j+1} + V_{i-1,j} + V_{i,j-1}\right) \cdot 0.25$

$V_{0,Nh2} \leftarrow \left(2 \cdot V_{1,Nh2} + H2sym \cdot V_{0,Nh2-1}\right) \cdot 0.25$

$V_{Np,Nh2} \leftarrow \left(Rsym \cdot V_{Np-1,Nh2} + H2sym \cdot V_{Np,Nh2-1}\right) \cdot 0.25$

for $i \in 1..Np - 1$

$V_{i,Nh2} \leftarrow \left(V_{i+1,Nh2} + V_{i-1,Nh2} + H2sym \cdot V_{i,Nh2-1}\right) \cdot 0.25$

$$Q1 \leftarrow \left[\sum_{i=0}^{0.5 \cdot Np} V_{i,1} - \left(\frac{V_{0,1} + V_{Np2,1}}{2}\right)\right] \cdot \varepsilon_2$$

$$Q2 \leftarrow \left[\sum_{j=0}^{Nh1} \left(V_{Np2-1,j} - V_{Np2+1,j}\right) - \frac{\left(V_{Np2-1,0} - V_{Np2+1,0}\right)}{2} - \left(\frac{V_{Np2-1,Nh1} - V_{Np2+1,Nh1}}{2}\right)\right] \cdot \frac{\varepsilon_2}{2}$$

$$Q3 \leftarrow \left[\sum_{j=Nh1}^{Nh2} \left(V_{Np2-1,j} - V_{Np2+1,j}\right) - \frac{\left(V_{Np2-1,Nh1} - V_{Np2+1,Nh1}\right)}{2} - \left(\frac{V_{Np2-1,Nh2} - V_{Np2+1,Nh2}}{2}\right)\right] \cdot \frac{1}{2}$$

$$Q4 \leftarrow \left[\sum_{i=0}^{0.5 \cdot Np} V_{i,Nh2-1} - \left(\frac{V_{0,Nh2-1} + V_{Np2,Nh2-1}}{2}\right) - \left[\sum_{i=0}^{0.5 \cdot Np} V_{i,Nh2} - \left(\frac{V_{0,Nh2} + V_{Np2,Nh2}}{2}\right)\right]\right]$$

$$C_n \leftarrow \frac{(Q1 + Q2 + Q3 + Q4)}{50}$$

error("Convergence failure") if $n = Niter$

```
    (break) if |C_n - C_n-1| < 0.0001
return C_n
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The capacitances per unit length, divided by ε_0 are

$CM(0,1) = 1.46$ $\quad CM\left(\frac{\pi}{2},1\right) = 3.04$ $\quad CM(\pi,1) = 3.99$ without dielectric

$CM(0,\varepsilon_2) = 9.08$ $\quad CM\left(\frac{\pi}{2},\varepsilon_2\right) = 12.45$ $\quad CM(\pi,\varepsilon_2) = 15.18$ with dielectric

Free space equivalent circuit capacitances calculated using finite differences.
The capacitances are for phase shifts 0, π/2 and π per strip (and double that for the unit cell).

$C_e := CM(0,1)\cdot\varepsilon_0$ $\quad C_{\pi 2} := CM\left(\frac{\pi}{2},1\right)\cdot\varepsilon_0$ $\quad C_o := CM(\pi,1)\cdot\varepsilon_0$ without dielectric

$CK_e := CM(0,\varepsilon_2)\cdot\varepsilon_0$ $\quad CK_{\pi 2} := CM\left(\frac{\pi}{2},\varepsilon_2\right)\cdot\varepsilon_0$ $\quad CK_o := CM(\pi,\varepsilon_2)\cdot\varepsilon_0$ with dielectric

Equivalent circuit parameters without dielectric

$C_0 := C_e \qquad C_1 := 0.25\cdot\left(C_o - C_e\right) \qquad C_2 := 0.25\cdot\left(C_{\pi 2} - C_e - 2\cdot C_1\right)$

$C1(\phi) := C_0 + 2\cdot C_1\cdot(1 - \cos(\phi)) + 2\cdot C_2\cdot(1 - \cos(2\phi))$ Equation 4.39

$L1(\phi) := \dfrac{1}{c^2\cdot C1(\phi)}$ Equation 4.43

Equivalent circuit capacitance with the dielectric substrate

$CK_0 := CK_e \qquad CK_1 := 0.25\cdot\left(CK_o - CK_e\right) \qquad CK_2 := 0.25\cdot\left(CK_{\pi 2} - CK_e - 2\cdot CK_1\right)$

$CK(\phi) := CK_0 + 2\cdot CK_1\cdot(1 - \cos(\phi)) + 2\cdot CK_2\cdot(1 - \cos(2\phi))$ Equation 4.39

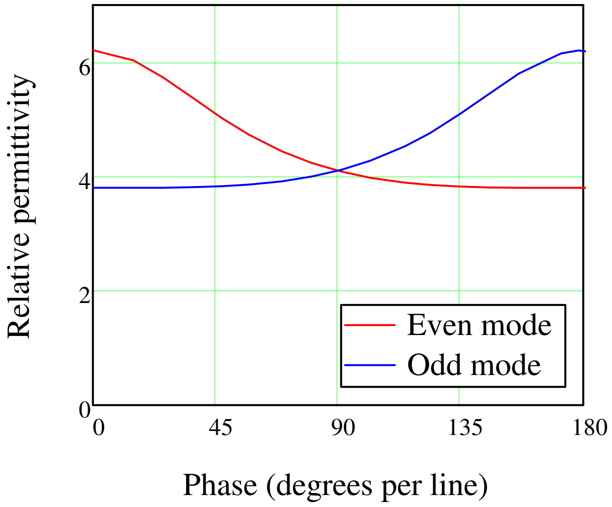
Effective relative permittivity

$\varepsilon_{eff}(\phi) := \dfrac{CK(\phi)}{C1(\phi)}$ Equation 4.47

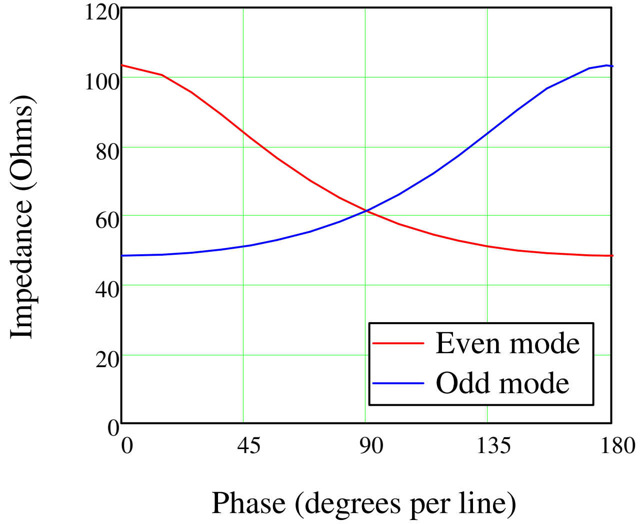
Characteristic impedances of even and odd modes

$Z_e(\phi) := \sqrt{\dfrac{L1(\phi)}{CK(\phi)}} \qquad Z_o(\phi) := Z_e(\pi + \phi)$ Equation 4.50

The best agreement with Weiss is obtained when the capacitances are calculated assuming that the cover plate is a symmetry boundary



Compare Figure 5 in Weiss



Compare Figure 6 in Weiss

Calculate the dispersion diagram

(a) Ignoring coupling between lines and assume that the effective permittivity is that of the substrate

$$\omega 0(\phi) := \frac{c\cdot\phi}{\sqrt{(1+\varepsilon_2)\cdot 0.5}\cdot(p+Lg)} \qquad f0(\phi) := \frac{\omega 0(\phi)}{2\cdot\pi\cdot GHz} \qquad f0(\pi) = 4.769$$

Path length is L + p

(b) Including coupling between lines the phase velocities for the even and odd modes are

$$v_{pe}(\phi) := \frac{c}{\sqrt{\varepsilon_{eff}(\phi)}} \qquad v_{po}(\phi) := \frac{c}{\sqrt{\varepsilon_{eff}(\pi+\phi)}}$$

Equation 4.48

Define the functions in Equation 4.57 for the lower and upper branches of the dispersion diagram

$$F1(\omega,\phi) := \tan\left(\frac{\phi}{2}\right)^2 - \frac{Z_o(\phi)}{Z_e(\phi)}\cdot\tan\left(\frac{\omega\cdot Lg}{2\cdot v_{pe}(\phi)}\right)\cdot\tan\left(\frac{\omega\cdot Lg}{2\cdot v_{po}(\phi)}\right)$$

$$F2(\omega,\phi) := \tan\left(\frac{\phi}{2}\right)^2 - \frac{Z_o(\phi)}{Z_e(\phi)}\cdot\cot\left(\frac{\omega\cdot Lg}{2\cdot v_{pe}(\phi)}\right)\cdot\cot\left(\frac{\omega\cdot Lg}{2\cdot v_{po}(\phi)}\right)$$

$$F3(\omega,\phi) := \cot\left(\frac{\phi}{2}\right)^2 - \frac{Z_e(\phi)}{Z_o(\phi)}\cdot\cot\left(\frac{\omega\cdot Lg}{2\cdot v_{pe}(\phi)}\right)\cdot\cot\left(\frac{\omega\cdot Lg}{2\cdot v_{po}(\phi)}\right)$$

$$F4(\omega,\phi) := \cot\left(\frac{\phi}{2}\right)^2 - \frac{Z_e(\phi)}{Z_o(\phi)}\cdot\tan\left(\frac{\omega\cdot Lg}{2\cdot v_{pe}(\phi)}\right)\cdot\tan\left(\frac{\omega\cdot Lg}{2\cdot v_{po}(\phi)}\right)$$

Solve for the lower and upper branches

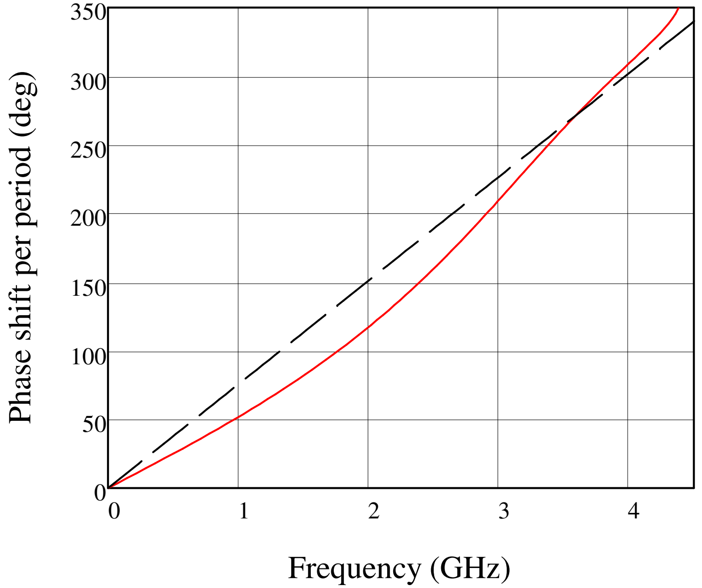
$$\omega_1 := 0.5\cdot\omega 0(\pi) \qquad \omega_2 := 1.4\cdot\omega 0(\pi)$$

$$\omega 1(\phi) := \begin{cases} \mathrm{root}\left(F1(\omega_1,\phi),\omega_1\right) & \text{if } \phi \le 0.5\cdot\pi \\ \mathrm{root}\left(F3(\omega_1,\phi),\omega_1\right) & \text{otherwise} \end{cases}$$

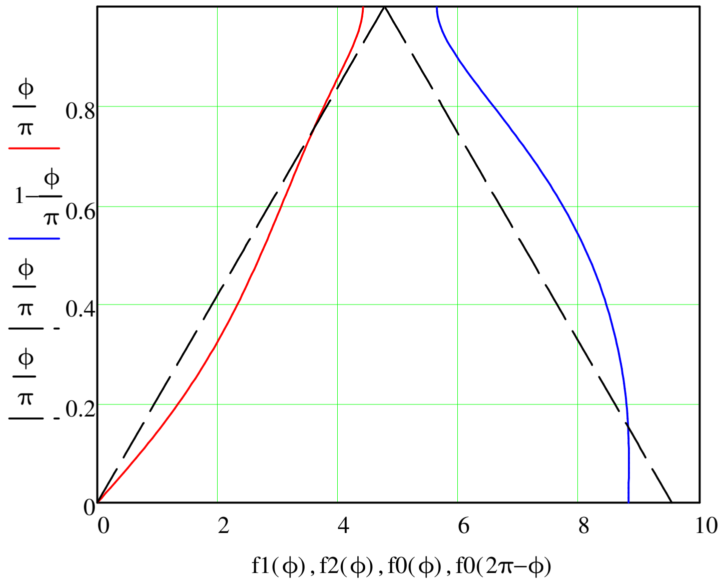
$$\omega 2(\phi) := \begin{cases} \mathrm{root}\left(F2(\omega_2,\phi),\omega_2\right) & \text{if } \phi \le 0.5\cdot\pi \\ \mathrm{root}\left(F4(\omega_2,\phi),\omega_2\right) & \text{otherwise} \end{cases}$$

$$f1(\phi) := \frac{\omega 1(\phi)}{2\cdot\pi\cdot GHz} \qquad f2(\phi) := \frac{\omega 2(\phi)}{2\cdot\pi\cdot GHz}$$

$\phi := 0, 0.01\cdot\pi .. \pi$



Compare Figure 6 in Courtney



Compare Figure 7 in Weiss

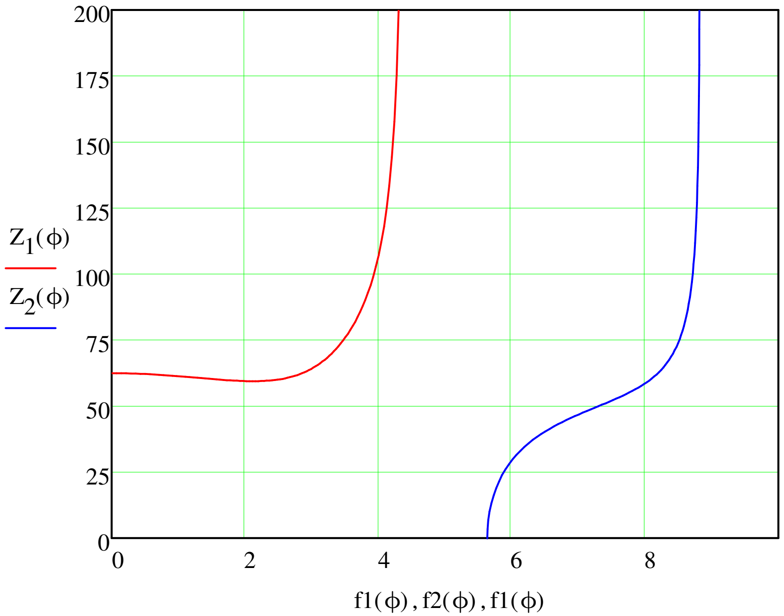
The best agreement with Weiss is obtained when it is assumed that the effective path length is L + p whereas he apparently chose a path length of L.
Note: identical results are obtained using Equation (22) in Weiss.

Calculate the image (i.e. characteristic or iterative) impedances of the lower and upper modes. (Equation 26 in Weiss).

$$k_e(\phi) := \frac{\omega 1(\phi)}{v_{pe}(\phi)} \qquad k_o(\phi) := \frac{\omega 1(\phi)}{v_{po}(\phi)}$$

For the lower band

$$Z_1(\phi) := \sqrt{Z_e(\phi)\cdot Z_o(\phi)\cdot\frac{\tan\left(\frac{k_o(\phi)\cdot Lg}{2}\right)}{\tan\left(\frac{k_e(\phi)\cdot Lg}{2}\right)}} \qquad Z_2(\phi) := \sqrt{Z_e(\phi)\cdot Z_o(\phi)\cdot\frac{\tan\left(\frac{\omega 2(\phi)\cdot Lg}{2\cdot v_{pe}(\phi)}\right)}{\tan\left(\frac{\omega 2(\phi)\cdot Lg}{2\cdot v_{po}(\phi)}\right)}}$$



Compare Figure 8 in Weiss

Calculate the Pierce impedance at height d assuming that the field is constant in the gap.

Define the amplitude of the even mode voltage relative to ground (a in Weiss) $V_a := 1 \cdot V$

$$b\_a1(\phi) := -\sqrt{\frac{Z_o(\phi)\cdot \sin(k_e(\phi)\cdot Lg)}{Z_e(\phi)\cdot \sin(k_o(\phi)\cdot Lg)}}$$ Weiss: equation (25)

Phase velocity and group velocity $v_p(\phi) := \dfrac{\omega 1(\phi)\cdot p}{\phi}$ $v_g(\phi) := p\cdot\left(\dfrac{d}{d\phi}\omega 1(\phi)\right)$ $\beta_0(\phi) := \dfrac{\phi}{p}$

$$P(\phi) := \frac{2Lg}{p}\cdot v_g(\phi)\cdot {V_a}^2\cdot\left[\frac{1}{v_{pe}(\phi)\cdot Z_e(\phi)}\cdot\left(1 + \mathrm{sinc}(k_e(\phi)\cdot Lg)\right) + \frac{1}{v_{po}(\phi)\cdot Z_e(\phi)}\cdot\left(\frac{\sin(k_e(\phi)\cdot Lg)}{\sin(k_o(\phi)\cdot Lg)}\right)\cdot\left(1 - \mathrm{sinc}(k_o(\phi)\cdot Lg)\right)\right]$$ Equation 4.60

Electric field in the gap $E_g(\phi, x) := \dfrac{4\cdot V_a}{gap}\cdot\left(\cos(k_e(\phi)\cdot x)\cdot\sin\left(\dfrac{\phi}{2}\right) + b\_a1(\phi)\cdot\sin(k_o(\phi)\cdot x)\cdot\cos\left(\dfrac{\phi}{2}\right)\right)$ Equation 4.61

On the axis the amplitude of the fundamental space harmonic is $E_0(\phi) := E_g(\phi, 0)\cdot \mathrm{sinc}\left(\beta_0(\phi)\cdot\dfrac{gap}{2}\right)\cdot\dfrac{gap}{p}$ Equation 4.62

Pierce impedance on the centre line of the structure at height d

$$Z_{P0}(\phi, d) := \frac{E_0(\phi)^2}{2\cdot\beta_0(\phi)^2\cdot P(\phi)}\cdot\left[\frac{\cosh\left[\beta_0(\phi)\cdot(h_2 - d)\right]}{\cosh\left[\beta_0(\phi)\cdot(h_2 - h_1)\right]}\right]^2$$ Equation 4.63

Plotting ranges $\phi 1 := 0.01\cdot\pi, 0.02\cdot\pi .. \pi$ $x1 := \dfrac{-Lg}{2}, -0.8\cdot\dfrac{Lg}{2} .. \dfrac{Lg}{2}$

Data from Courtney

Dispersion curve
(Figure 6)

$$\phi_c := \frac{350}{2\cdot 134}\cdot\begin{pmatrix}20\\34\\48\\62\\75\\89\\103\\117\end{pmatrix}\cdot deg \qquad f_c := \frac{4}{152}\cdot\begin{pmatrix}35\\56\\75\\91\\107\\121\\134\\144\end{pmatrix}$$

Impedance curve
(Figure 7)

$$Z_{PC} := \begin{pmatrix}80\\60\\40\\30\\20\\10\\8\\6\\4\\3\end{pmatrix} \qquad f_C := \frac{6}{107}\cdot\begin{pmatrix}21\\24\\29\\35\\42\\54\\57\\61\\65\\67\end{pmatrix}$$

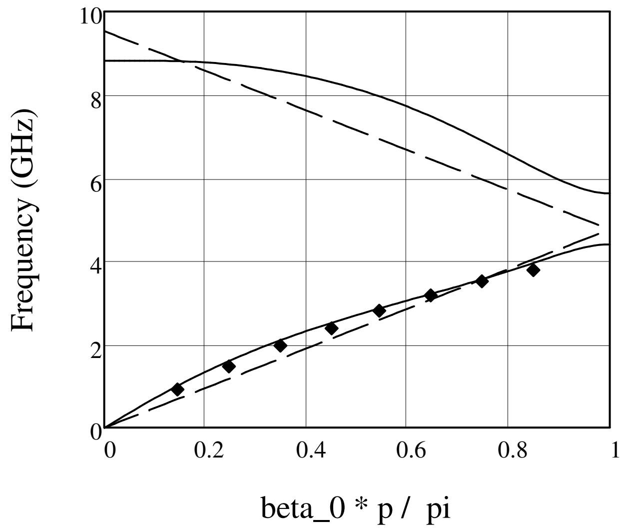


Figure 4.11

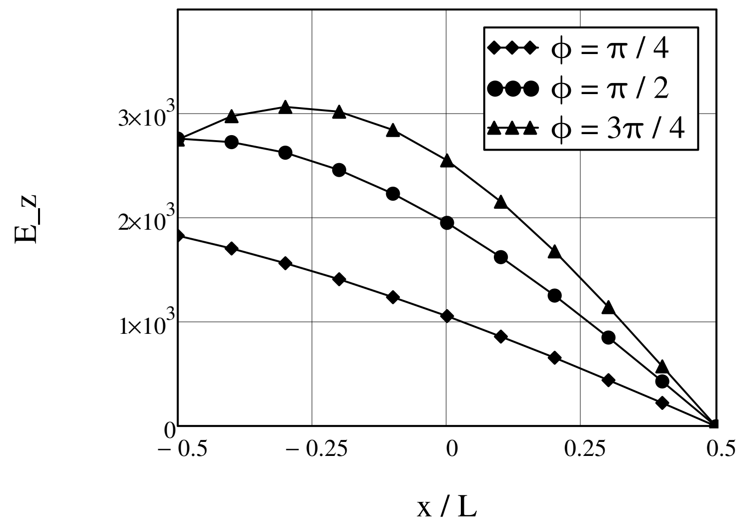
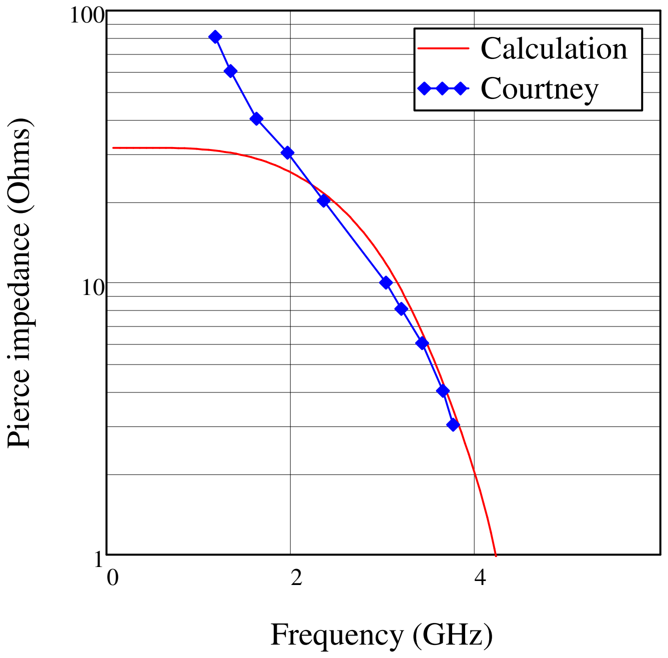


Figure 4.12

Pierce impedance calculated at $d := h_1 + 0.5 \cdot gap$



Note: It is not quite clear how the impedance in Courtney is defined. Here d has been adjusted to fit the calculated curve to the experimental data.