Worksheet 7.4 Periodic Permanent Magnet Focusing

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

This worksheet calculates the beam edge profile for a p.p.m. focused beam using the method described by Harker.
Harker, K. J. (1955). "Periodic focusing of beams from partially shielded cathodes."
IRE Transactions on Electron Devices **ED-2**(4): 13-19.

The values of δ and m are set and the differential equation governing the beam edge is solved given the initial radius and slope of the beam edge. The axial variation of the magnetic field is as cos θ and the normalised beam radius is R.

The paraxial ray equation referred to the maximum radius is

$$\frac{d^2R}{d\theta^2} = \beta\frac{1}{R} + 2\alpha K\frac{1}{R^3} - 2\alpha\cos^2(\theta)R$$

Equation 7.88

where $\beta_0 z = \theta;\quad \beta = \frac{1}{2}\frac{\beta_p^2}{\beta_0^2};\quad \alpha = \frac{1}{2}\frac{\beta_L^2}{\beta_0^2}$

The parameters α and K are defined, β is set using the slider and the beam profile is displayed. The minimum ripple solution is obtained by finding the maximum value of β for which $R_{max} = 1$

$\alpha := 0.3$

$K := 0.1$

$\beta set :=$

$R_0 := 1$ $\quad R1_0 := 0$ $\quad \beta := 0.187$

Definitions for magnetic field and beam current relative to their uniform values. The effects of different functions can be investigated by selecting them. Note that the steps and ramps start at $\theta = 2.5\pi$ i.e. at a zero of the magnetic field.

B_control :=
Uniform
Step
Ramp

k_control :=
Uniform
Step
Ramp

B_step := 2 $\quad$ k_step := 2

B_slope := 1 $\quad$ k_slope := 1

$$B(\theta) := \left| \begin{array}{l} B \leftarrow \cos(\theta) \\ B \leftarrow \left| \begin{array}{l} B \ \text{ if } \ \theta < 2.5\cdot\pi \\ B \ \text{ if } \ \theta \geq 2.5\cdot\pi \wedge B\_control = 1 \\ B\_step\cdot B \ \text{ if } \ \theta \geq 2.5\cdot\pi \wedge B\_control = 2 \\ \left[1 + \dfrac{B\_slope\cdot(\theta - 2.5\cdot\pi)}{2.\pi}\right]\cdot B \ \text{ if } \ \theta \geq 2.5\cdot\pi \wedge B\_control = 3 \end{array} \right. \\ \text{return } B \end{array} \right.$$

$$k(\theta) := \left| \begin{array}{l} k \leftarrow \left| \begin{array}{l} 1 \ \text{ if } \ \theta < 2.5\cdot\pi \\ 1 \ \text{ if } \ \theta \geq 2.5\cdot\pi \wedge k\_control = 1 \\ k\_step \ \text{ if } \ \theta \geq 2.5\cdot\pi \wedge k\_control = 2 \\ \left[1 + \dfrac{k\_slope\cdot(\theta - 2.5\cdot\pi)}{2.\pi}\right] \ \text{ if } \ \theta \geq 2.5\cdot\pi \wedge k\_control = 3 \end{array} \right. \\ \text{return } k \end{array} \right.$$

The differential equation is solved for the initial conditions R = R_0 and dR/dθ = $R1_0$

$$D(\theta, R1) := \begin{bmatrix} R1_1 \\ \dfrac{\beta\cdot k(\theta)}{R1_0} + \dfrac{2\cdot\alpha\cdot K}{(R1_0)^3} - 2\cdot\alpha B(\theta)^2\cdot R1_0 \end{bmatrix}$$

$$\frac{dR_0}{d\theta} = R_1$$

$$\frac{dR_1}{d\theta} = f(R_0, \alpha, \beta, K)$$

The differential equation is expressed as a pair of simultaneous first order equations.

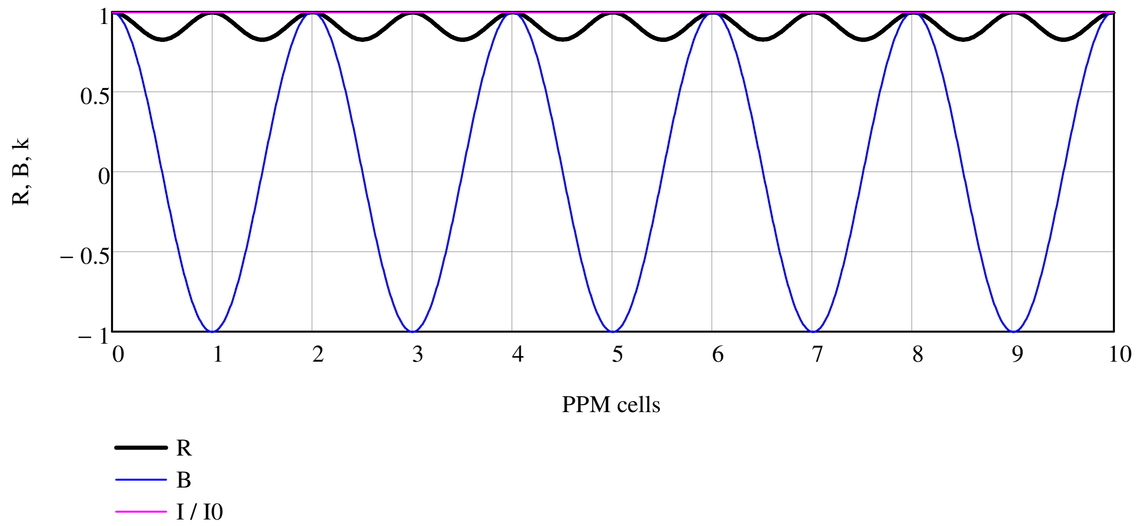
$$R1 := \begin{pmatrix} R_0 \\ R1_0 \end{pmatrix}$$

$$\Theta := \text{AdamsBDF}(R1, 0, 10\cdot\pi, 1000, D)$$

Numerical solution of the differential equation. The matrix Θ contains values of θ in the first column, R in the second column and dR/dθ in the third column.

Find the maximum and minimum radii, the ripple and the mean beam radius

$$R_{min} := \min\left(\Theta^{\langle 1\rangle}\right) \qquad R_{max} := \max\left(\Theta^{\langle 1\rangle}\right) \qquad ripple := \frac{\max\left(\Theta^{\langle 1\rangle}\right) - \min\left(\Theta^{\langle 1\rangle}\right)}{\max\left(\Theta^{\langle 1\rangle}\right) + \min\left(\Theta^{\langle 1\rangle}\right)} \qquad R_{mean} := 0.5\cdot\left(\max\left(\Theta^{\langle 1\rangle}\right) + \min\left(\Theta^{\langle 1\rangle}\right)\right)$$



$\alpha = 0.3$ $\beta = 0.187$ $K = 0.1$ $R_{max} = 1$ $R_{min} = 0.828$ $ripple = 9.4\cdot\%$ $R_{mean} = 0.914$

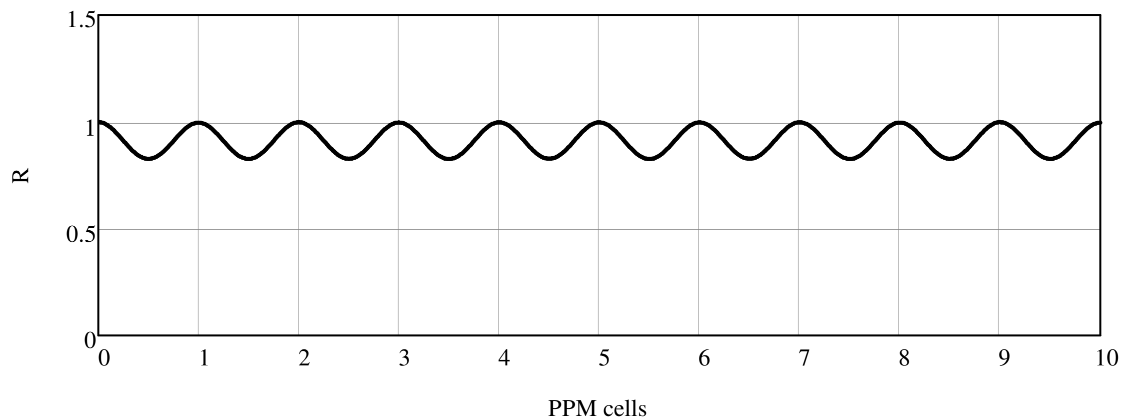


Figure 7.11

Find the minimum ripple solutions

Set the values of α and K, find β for minimum ripple and copy the results into the data tables.
The columns contain α, β and the ripple for each value of K

$\alpha := 0.45$ $\qquad K := 0.2$

$$\mathrm{Ripple}(\alpha,\beta,K) := \left|\begin{array}{l} R1 \leftarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ D(\theta, R1) \leftarrow \begin{bmatrix} R1_1 \\ \dfrac{\beta}{R1_0} + \dfrac{2\cdot\alpha\cdot K}{\left(R1_0\right)^3} - 2\cdot\alpha\cos(\theta)^2\cdot R1_0 \end{bmatrix} \\ \Theta \leftarrow \mathrm{rkfixed}(R1, 0, 10\cdot\pi, 1000, D) \\ \mathrm{Ripple} \leftarrow \dfrac{\max\left(\Theta^{\langle 1\rangle}\right) - \min\left(\Theta^{\langle 1\rangle}\right)}{\max\left(\Theta^{\langle 1\rangle}\right) + \min\left(\Theta^{\langle 1\rangle}\right)} \\ \mathrm{return\ \ Ripple} \end{array}\right.$$

$\mathrm{Rip}(\beta) := \left|\mathrm{Ripple}(\alpha,\beta,K)\right|$ $\qquad \beta_{min} := \mathrm{Minimize}(\mathrm{Rip}, \beta)$ $\qquad \beta_{min} = 0.048$

$\mathrm{Ripple}\left(\alpha, \beta_{min}, K\right) = 0.202$

K = 0 $X0 :=$

	0	1	2
0	0	0	0
1	0.1	0.096	0.026
2	0.2	0.184	0.056
3	0.3	0.263	0.089
4	0.4	0.332	0.126
5	0.5	0.389	0.169
6	0.6	0.434	0.22
7	0.7	0.463	0.281
8	0.8	0.472	0.358
9	0.9	0.452	0.467

K = 0.2 $X2 :=$

	0	1	2
0	0	0	0
1	0.05	0.029	0.013
2	0.1	0.054	0.027
3	0.15	0.075	0.042
4	0.2	0.093	0.059
5	0.25	0.105	0.078
6	0.3	0.11	0.099
7	0.35	0.107	0.124
8	0.4	0.091	0.156
9	0.45	0.048	0.202

K = 0.1 $X1 :=$

	0	1	2
0	0	0	0
1	0.1	0.075	0.027
2	0.2	0.139	0.057
3	0.3	0.188	0.093
4	0.4	0.217	0.138
5	0.5	0.213	0.201
6	0.55	0.182	0.25

K = 0.3 $X3 :=$

	0	1	2
0	0	0	0
1	0.05	0.018	0.013
2	0.1	0.033	0.027
3	0.15	0.042	0.043
4	0.2	0.046	0.061
5	0.25	0.043	0.081
6	0.3	0.03	0.106
7	0.35	0	0.139

K = 0.4 $X4 :=$

	0	1	2
0	0	0	0
1	0.05	$8\cdot10^{-3}$	0.013
2	0.1	0.011	0.028
3	0.15	$9\cdot10^{-3}$	0.044
4	0.19	$2\cdot10^{-3}$	0.059

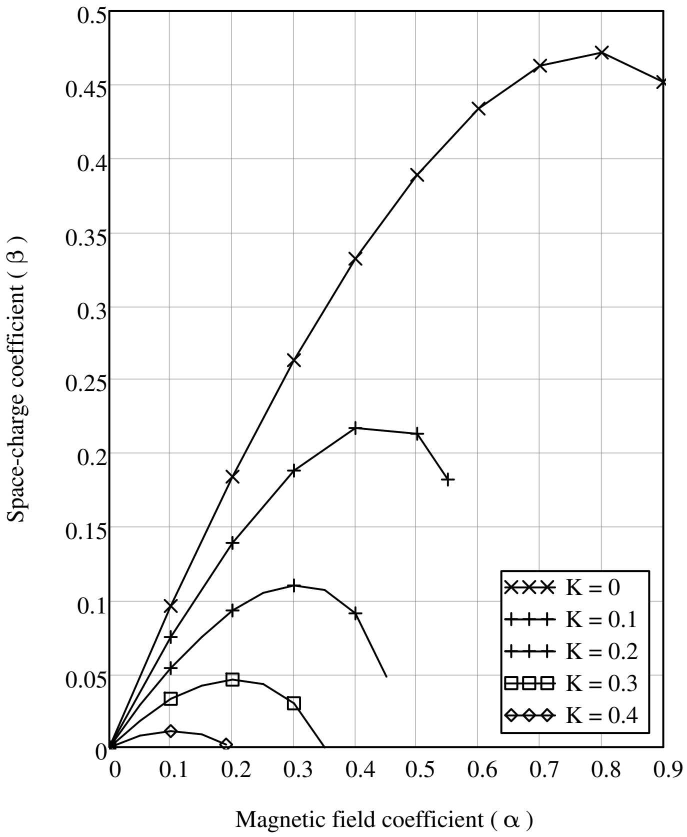


Figure 7.12

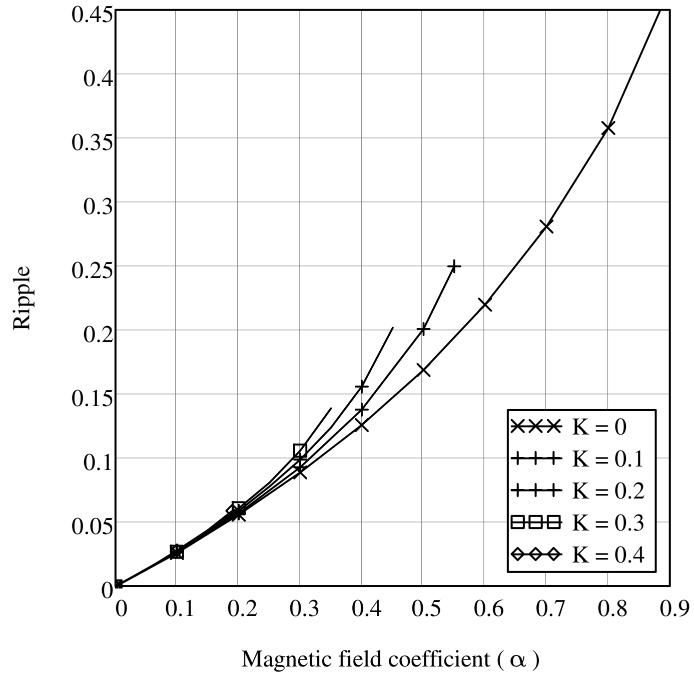


Figure 7.13

Find and plot the maximum and minimum radii for minimum ripple solutions when the current is increased by a factor k for a typical value of α. Details are in the collapsed region.

$\alpha = 0.1$

$K = 0$

$Y10 :=$

	0 (k)	1 (R_{max})	2 (R_{min})
0	0.25	1	0.14
1	0.5	1	0.43
2	0.75	1	0.703
3	1	1	0.947
4	1.25	1.234	0.95
5	1.5	1.454	0.951
6	1.75	1.662	0.953
7	2	1.859	0.954

$\alpha = 0.1$

$K = 0.2$

$Y12 :=$

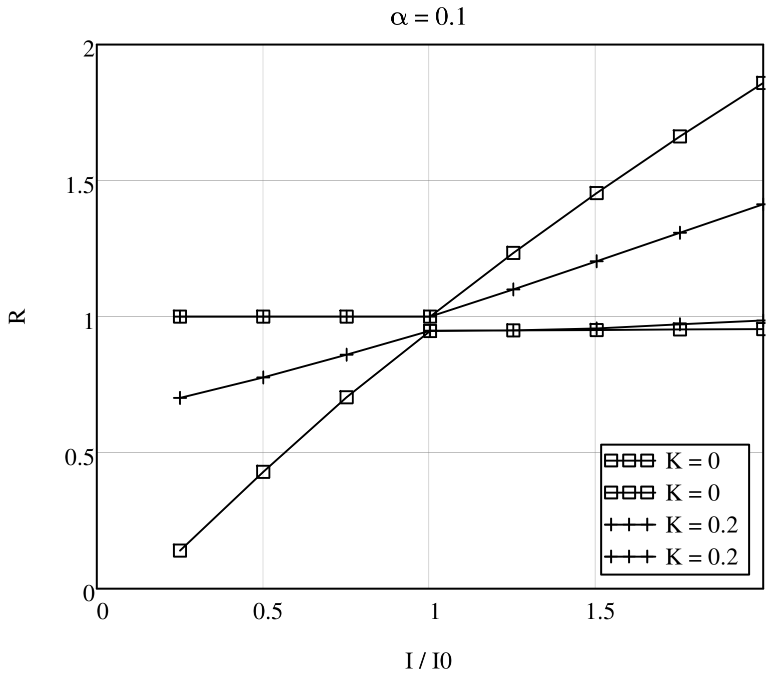
	0 (k)	1 (R_{max})	2 (R_{min})
0	0.25	1	0.701
1	0.5	1	0.776
2	0.75	1	0.86
3	1	1	0.948
4	1.25	1.1	0.949
5	1.5	1.203	0.956
6	1.75	1.308	0.972
7	2	1.413	0.986

$\alpha = 0.1$

$K = 0.1$

$Y11 :=$

	0 (k)	1 (R_{max})	2 (R_{min})
0	0.25	1	0.542
1	0.5	1	0.661
2	0.75	1	0.803
3	1	1	0.948
4	1.25	1.161	0.954
5	1.5	1.32	0.952
6	1.75	1.477	0.951
7	2	1.63	0.951



Compare Figure 7.5

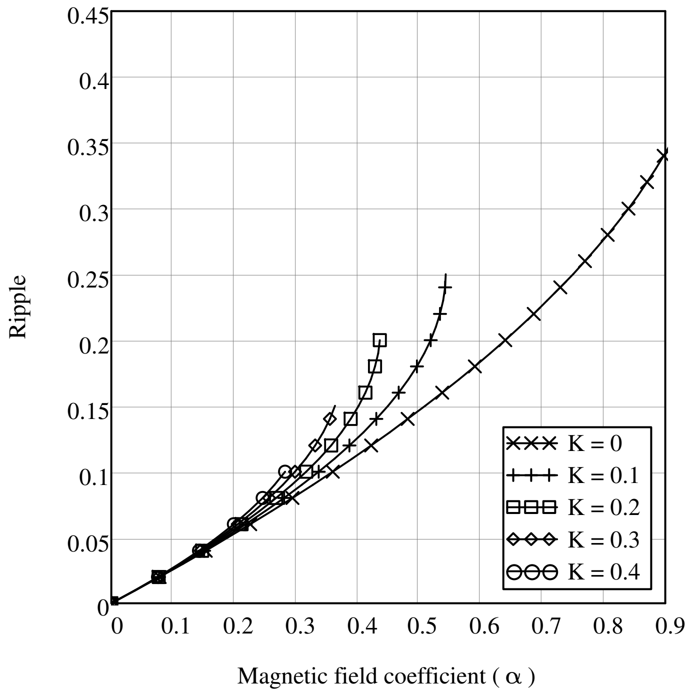
Approximate solution

Define α as a function of δ and K using (7.98), (7.99) and (7.100).

$$\alpha\alpha\left(\delta, K_c\right) := 4\cdot\delta\cdot(1-\delta)\cdot\left[1-2\cdot K_c\cdot(1+\delta)^4\cdot\left[1-\frac{1}{(1-\delta)^2}\right]\right]^{-1}$$

Plot δ against α for a range of values of K for comparison with Figure 7.13. The agreement is good up to α = 0.4

$\delta 0 := 0, 0.01 .. 0.35$ $\delta 1 := 0, 0.01 .. 0.25$ $\delta 2 := 0, 0.01 .. 0.2$ $\delta 3 := 0, 0.01 .. 0.15$ $\delta 4 := 0, 0.01 .. 0.1$



Find δ as a function of α and K and plot β as a function of α for a range of values of K for comparison with Figure 7.12. The agreement is excellent up to α = 0.4

$\delta 2 := 0 \qquad \delta\delta(\alpha, K) := \mathrm{root}(\alpha\alpha(\delta 2, K) - \alpha, \delta 2)$

$$\beta\beta(\alpha, K) := \alpha \cdot \frac{\left[1 - 2 \cdot K \cdot (1 + \delta\delta(\alpha, K))^4\right]}{(1 + \delta\delta(\alpha, K))^2}$$

$\alpha 0 := 0, 0.02 .. 0.9 \qquad \alpha 1 := 0, 0.02 .. 0.7 \qquad \alpha 2 := 0, 0.02 .. 0.45 \qquad \alpha 3 := 0, 0.02 .. 0.4 \qquad \alpha 4 := 0, 0.02 .. 0.2$