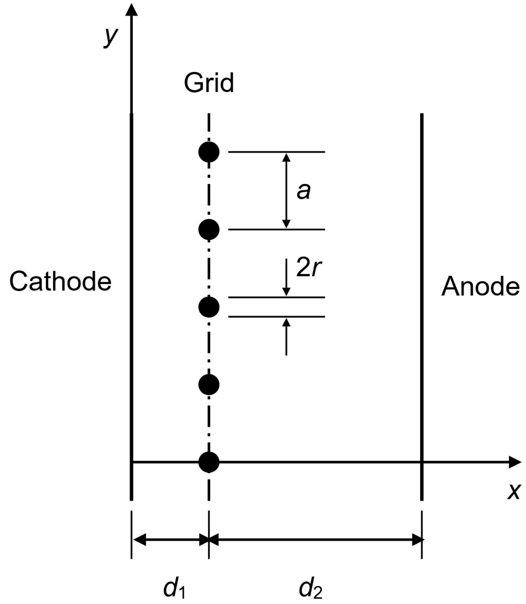
WS 6.1 Electrostatic solution for a triode

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.



Specify the dimensions and the electrode voltages

$a := 1.0 \cdot mm$ $\quad d_1 := 1.0 \cdot mm$ $\quad d_2 := 1.5 \cdot mm$ $\quad r := 0.05 \cdot mm$

$V_a := 100 \cdot V$ $\quad V_g := 10 \cdot V$

The section below can be collapsed to conceal the detailed calculations

Penetration factor in a planar triode with uniform field on the cathode

$$D := -\frac{a}{2\cdot\pi\cdot d_2}\cdot\ln\left(2\cdot\sin\left(\frac{\pi\cdot r}{a}\right)\right)$$

Equation 6.34

$D = 0.123$

$$D1(a,d_2,r) := \frac{\ln\left(\coth\left(\frac{2\cdot\pi\cdot r}{a}\right)\right)}{\left(\frac{2\cdot\pi\cdot d_2}{a}\right) - \ln\left(\cosh\left(\frac{2\cdot\pi\cdot r}{a}\right)\right)}$$

Equation 6.36

$D1(a,d_2,r) = 0.127$

Generate equipotential plot

$$f_g(d_1,x,y) := \frac{a}{4\cdot\pi}\cdot\ln\left[\frac{\cosh\left[\frac{2\cdot\pi\cdot(x+d_1)}{a}\right] - \cos\left(\frac{2\cdot\pi\cdot y}{a}\right)}{\cosh\left[\frac{2\cdot\pi\cdot(x-d_1)}{a}\right] - \cos\left(\frac{2\cdot\pi\cdot y}{a}\right)}\right]$$

Equation 6.24

$$\text{Cathode\_field} := \left| \begin{array}{l} CF \leftarrow \text{"Uniform"} \quad \text{if } \frac{d_1}{a} \ge 1 \\ CF \leftarrow \text{"Non-uniform"} \quad \text{otherwise} \\ \text{return } CF \end{array} \right.$$

$$C_1(d_1) := \frac{a\cdot\varepsilon_0}{(d_1+d_2)\cdot f_g(d_1,d_1,r) - d_1^{\,2}}$$

Equation 6.30

$$\begin{pmatrix} q_g \\ q_a \end{pmatrix} := C_1(d_1)\cdot\begin{pmatrix} d_1+d_2 & -d_1 \\ -d_1 & f_g(d_1,d_1,r) \end{pmatrix}\cdot\begin{pmatrix} V_g \\ V_a \end{pmatrix}$$

Equation 6.29

$$V_t(x,y) := \frac{1}{a\cdot\varepsilon_0}\cdot\left(f_g(d_1,x,y)\cdot q_g + x\cdot q_a\right)$$

Equation 6.23

Matrix of data points for plotting the potential map $n := 50$

$$
\begin{aligned}
Vt\_plot := \;& \text{for } i \in 0..\left(\frac{d_1 + d_2}{a}\right)\cdot n \\
& \quad \text{for } j \in 0..n \\
& \quad\quad x \leftarrow \frac{i}{n}\cdot a \\
& \quad\quad y \leftarrow \frac{j}{n}\cdot a \\
& \quad\quad V_{j,i} \leftarrow \begin{cases} V_g & \text{if } (x = d_1 \wedge y = 0) \vee (x = d_1 \wedge y = a) \\ V_t(x,y) & \text{otherwise} \end{cases} \\
& \text{return } V
\end{aligned}
$$

Triode with island formation

Calculation of the field on the cathode surface for comparison with Bennett and Peterson figs 2 and 3

$$f1_g(d_1, x, y) := \frac{d}{dx} f_g(d_1, x, y)$$ Equation 6.37

$$D_E(d_1, y) := \frac{f_g(d_1, d_1, r) - d_1 \cdot f1_g(d_1, 0, y)}{(d_1 + d_2)\cdot f1_g(d_1, 0, y) - d_1}$$ Equation 6.45

$$D_E(d_1, 0) = 0.12$$

$$E_c(d_1, y) := -\frac{V_g + D_E(d_1, y)\cdot V_a}{d_1 + (d_1 + d_2)\cdot D_E(d_1, y)}$$ Equation 6.46

$$D = 0.123$$

$$E_{c0}(d_1) := -\frac{V_g + D \cdot V_a}{d_1 + (d_1 + d_2) \cdot D}$$

Uniform cathode field for comparison

Calculate cathode current density

$$K_1 := \frac{4 \cdot \varepsilon_0}{9} \cdot \sqrt{2 \cdot 1.759 \cdot 10^{11} \cdot C \cdot kg^{-1}}$$

Triode without island building

$$J_0(d_1, V_g) := K_1 \cdot \sqrt{1 + D} \cdot \frac{(V_g + D \cdot V_a)^{1.5}}{\left[d_1 + D \cdot (d_1 + d_2)\right]^2}$$

Equation 6.56

$$J_0(d_1, V_g) = 0.015 \cdot A \cdot cm^{-2}$$

$$F_1(d_1, V_g, y) := \begin{cases} K_1 \cdot \dfrac{\sqrt{1 + D_E(d_1, y)} \cdot (V_g + D_E(d_1, y) \cdot V_a)^{1.5}}{\left[d_1 + D_E(d_1, y) \cdot (d_1 + d_2)\right]^2} & \text{if } V_g + D_E(d_1, y) \cdot V_a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean current density in a triode with island building

$$J_1(d_1, V_g) := \frac{1}{a} \cdot \int_0^a F_1(d_1, V_g, y)\, dy$$

Equation 6.67

$$J_1(d_1, V_g) = 0.015 \cdot A \cdot cm^{-2}$$

$y1 := 0, 0.01 \cdot a .. 0.5 \cdot a$ $\quad V_1 := -10 \cdot V, -9.5 \cdot V .. 20 \cdot V$

$d_2 = 1.5 \cdot mm$ $a = 1 \cdot mm$ $r = 0.05 \cdot mm$ $V_a = 100\ V$ $V_g = 10\ V$ Cathode_field = "Uniform"

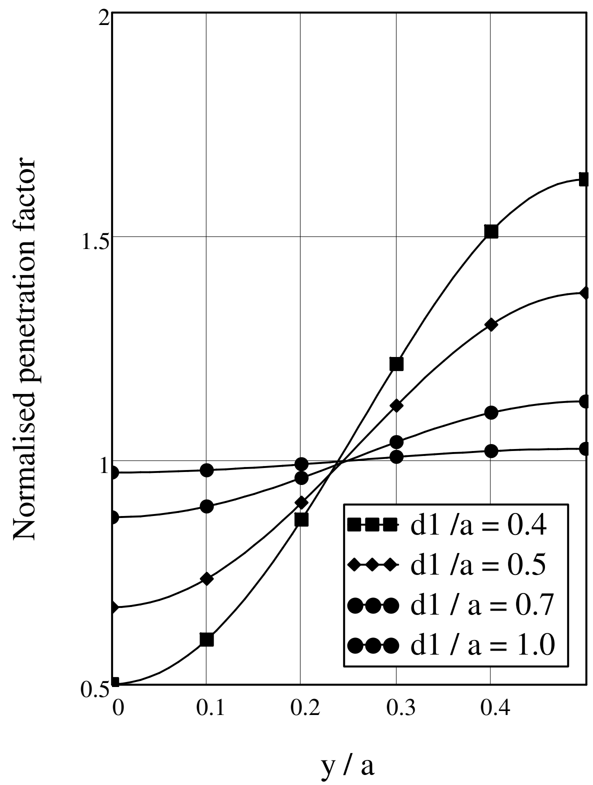
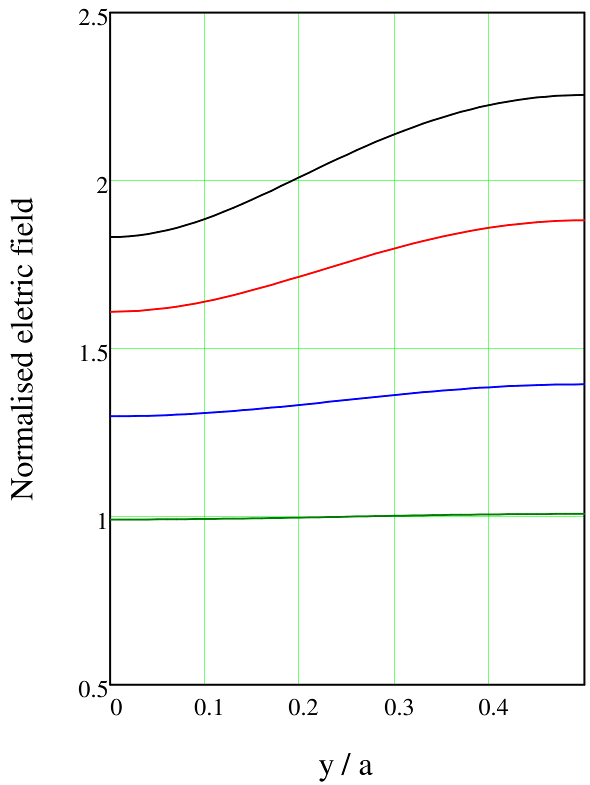
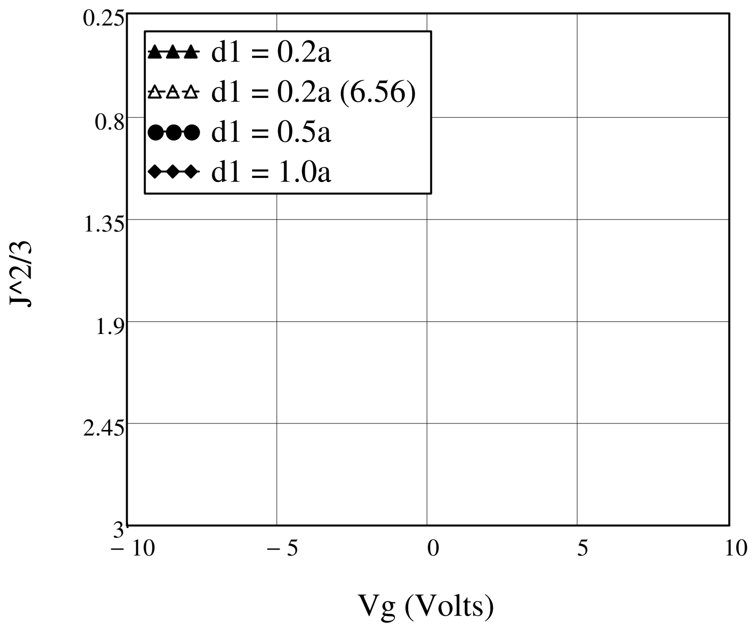


Figure 6.5 was obtained with
$a = 1,\ d_2 = 5,\ r = 0.05,\ V_g = -2V$



Variation of current density over the cathode surface when $d1 := 0.2 \cdot a$



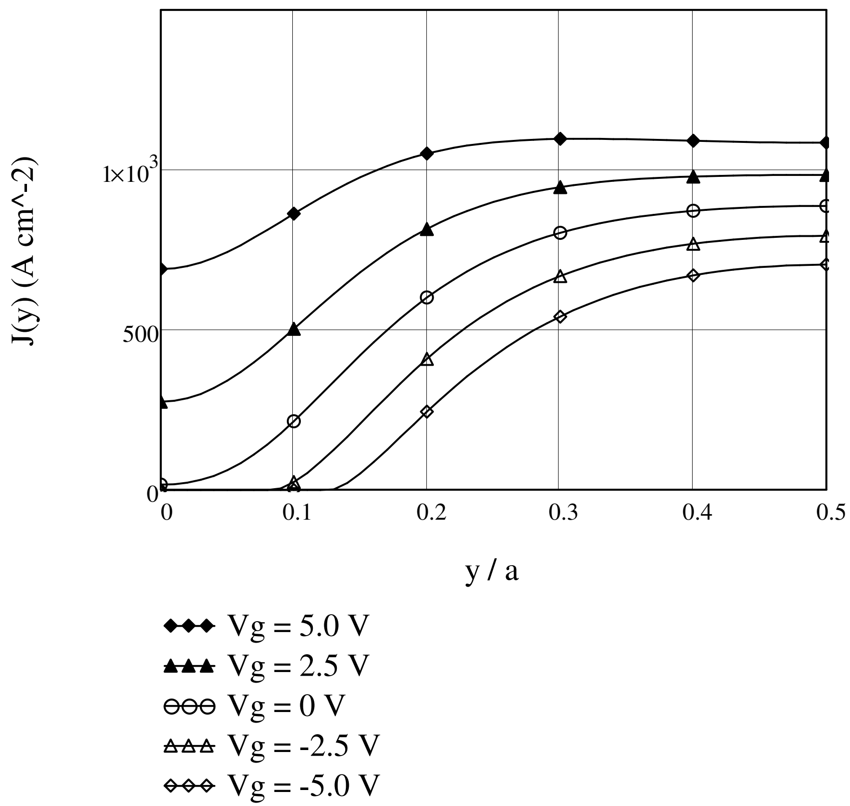


Figure 6.10 was obtained with
a = 1 mm, d_2 = 5 mm, and r = 0.05 mm

The section below can be collapsed to conceal the detailed calculations

Theoretical expressions for the grid current fraction

$$\delta 1 := \frac{a}{\frac{a}{\pi\cdot\left(1 + D1\left(a, d_2, r\right)\right)}\cdot\frac{r}{2\cdot d_1}\cdot\left(1 + \ln\left(\frac{4d_1}{r}\right)\right) + 2\cdot r} - 1$$

Equation 6.62

$$\mu := \frac{1}{D1\left(a, d_2, r\right)}$$

Let $V_{eg\_g} = \frac{V_{eg}}{V_g}$ $\quad V_{ag} = \frac{V_a}{V_g}$ $\quad V1_g\left(V_{ag}\right) := \frac{V_a}{V_{ag}}$

$$V_{eg\_g}\left(V_{ag}\right) := \frac{V_{ag} + \mu}{1 + \mu + \frac{d_2}{d_1}}$$

Equation 6.63

$$I1_{ag}\left(V_{ag}\right) := \frac{a}{2\cdot r}\cdot\sqrt{V_{eg\_g}\left(V_{ag}\right)} - 1$$

Equation 6.65

$$I_{ag}\left(V_{ag}\right) := \frac{a}{2\cdot r}\cdot\sqrt{V_{eg\_g}\left(V_{ag}\right)}\cdot\left(\frac{2\cdot\ln\left(\frac{a}{2\cdot\pi\cdot r}\right)}{2\cdot\ln\left(\frac{a}{2\cdot\pi\cdot r}\right) + \frac{1}{V_{eg\_g}\left(V_{ag}\right)} - 1}\right) - 1$$

Equation 6.66

Calculation of the electron trajectories

Define dimensionless parameters because Mathcad does not allow mixed dimensions in matrices

$$a := \frac{a}{mm} \qquad d_1 := \frac{d_1}{mm} \qquad d_2 := \frac{d_2}{mm} \qquad r := \frac{r}{mm}$$

The working equations are

$$D1(a, d_2, r) := \frac{\ln\left(\coth\left(\frac{2\cdot\pi\cdot r}{a}\right)\right)}{\left(\frac{2\cdot\pi\cdot d_2}{a}\right) - \ln\left(\cosh\left(\frac{2\cdot\pi\cdot r}{a}\right)\right)}$$

Equation 6.36

$$\begin{pmatrix} q_g \\ q_a \end{pmatrix} := \frac{1}{d_2}\cdot\frac{a\cdot\varepsilon_0}{d_1 + D1(a, d_2, r)\cdot(d_1 + d_2)}\cdot\begin{pmatrix} d_1 + d_2 & -d_1 \\ -d_1 & d_1 + D1(a, d_2, r)\cdot d_2 \end{pmatrix}\cdot\begin{pmatrix} V_g \\ V_a \end{pmatrix}$$

$$f(x, y) := \frac{a}{4\cdot\pi}\cdot\ln\left[\frac{\cosh\left[\frac{2\cdot\pi\cdot(x + d_1)}{a}\right] - \cos\left(\frac{2\cdot\pi\cdot y}{a}\right)}{\cosh\left[\frac{2\cdot\pi\cdot(x - d_1)}{a}\right] - \cos\left(\frac{2\cdot\pi\cdot y}{a}\right)}\right]$$

Equation 6.24

$$V_t(x, y) := \frac{1}{a\cdot\varepsilon_0}\cdot\left(f(x, y)\cdot q_g + x\cdot q_a\right)$$

Dimensionless electric field components and charge/mass ratio of the electron

$$E_x(x,y) := -\frac{d}{dx}V_t(x,y)\cdot\frac{1}{V} \qquad E_y(x,y) := -\frac{d}{dy}V_t(x,y)\cdot\frac{1}{V} \qquad \eta := -1.759\cdot 10^{11}$$

Calculate electron trajectory as a function of time. In this calculation all the variables must be dimensionless because Mathcad does not accept matrices in which the dimensions are not all the same.

$$s = \frac{1}{2}\cdot a\cdot t^2 \qquad t = \sqrt{\frac{2\cdot s}{a}} \qquad t_{max} := 1.5\cdot\sqrt{\frac{2\cdot V}{-\eta\cdot V_a}}\cdot\left(d_1 + d_2\right) \qquad t_{max} = 1.264\times 10^{-6}$$

Define the simultaneous first order differential equations of motion for the electrons

$$Dt(t,z) := \begin{pmatrix} \eta\cdot E_x\left(z_1,z_3\right) \\ z_0 \\ \eta\cdot E_y\left(z_1,z_3\right) \\ z_2 \end{pmatrix}$$

$$\frac{d}{dt}v_x = \eta\cdot E_x$$

$$\frac{d}{dt}x = v_x$$

$$\frac{d}{dt}v_y = \eta\cdot E_x$$

$$\frac{d}{dt}y = v_y$$

$nsteps := 1000$ $\qquad$ $ntraj := 20$

Numerical integration of the equations of motion subject to the initial conditions specified. The results are in the matrix Z of which the first column is time in *nsteps* and the remaining columns in groups of four are v_x, x, v_y, and y for each of the *ntraj* electron trajectories in one half of the triode between a pair of grid wires.

$$Z := \left| \begin{array}{l} \text{for } n \in 0..ntraj \\ \quad \left| \begin{array}{l} init \leftarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{a \cdot n}{2 \cdot ntraj} \end{pmatrix} \\ Z_n \leftarrow \text{AdamsBDF}(init, 0, t_{max}, nsteps, Dt) \end{array} \right. \\ \text{return } Z \end{array} \right.$$

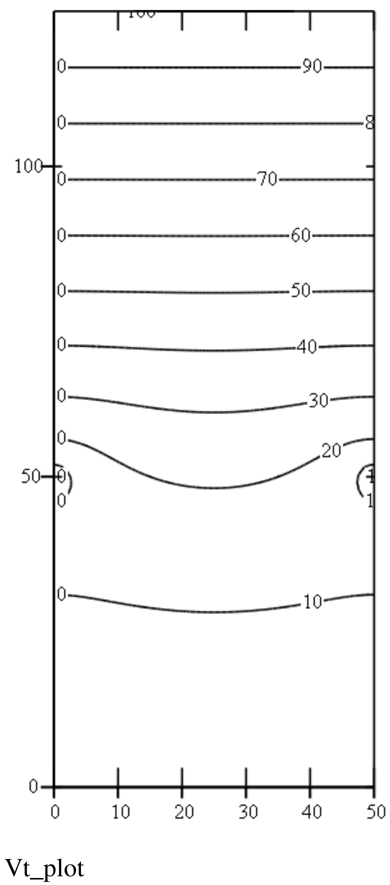
Unpack the trajectory matrix Z and add the symmetrical trajectories

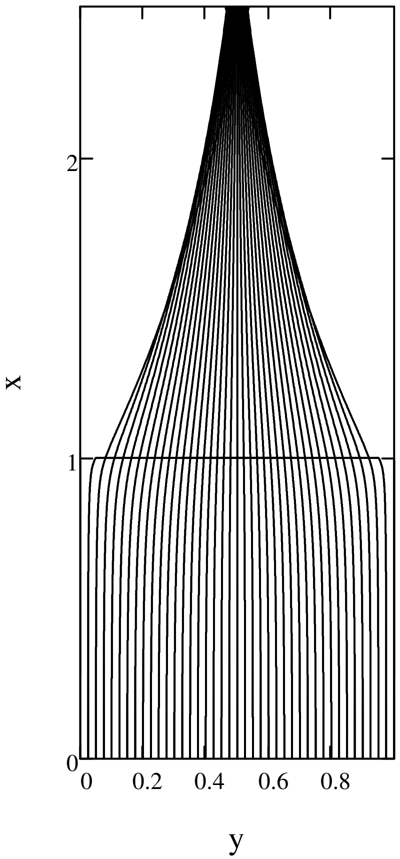
$$xn := \left| \begin{array}{l} nmax \leftarrow 2 \cdot ntraj \\ \text{for } n \in 0..nmax \\ \quad \text{for } i \in 0..nsteps \\ \qquad xn_{i,n} \leftarrow \left| \begin{array}{ll} \left[\left(Z_n\right)^{\langle 2 \rangle} \right]_i & \text{if } n \le ntraj \\ \left[\left(Z_{nmax-n}\right)^{\langle 2 \rangle} \right]_i & \text{otherwise} \end{array} \right. \\ \text{return } xn \end{array} \right.$$

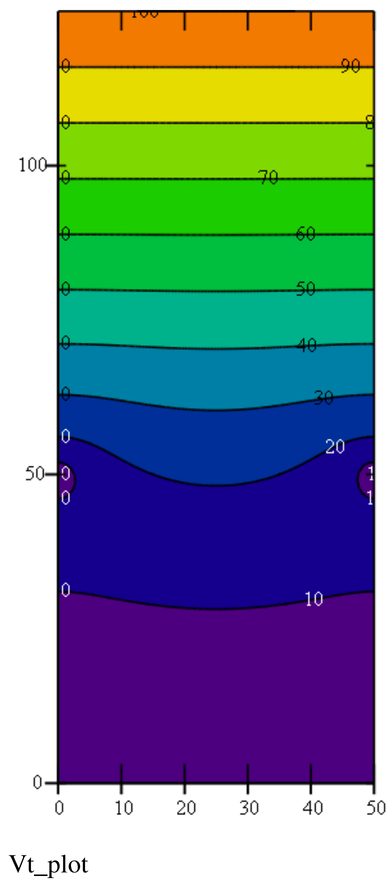
$$yn := \left| \begin{array}{l} nmax \leftarrow 2 \cdot ntraj \\ \text{for } n \in 0..nmax \\ \quad \text{for } i \in 0..nsteps \\ \qquad yn_{i,n} \leftarrow \left| \begin{array}{ll} \left[\left(Z_n\right)^{\langle 4 \rangle} \right]_i & \text{if } n \le ntraj \\ a - \left[\left(Z_{nmax-n}\right)^{\langle 4 \rangle} \right]_i & \text{otherwise} \end{array} \right. \\ \text{return } yn \end{array} \right.$$

Stop a trajectory if it is intercepted by the grid

$$
\begin{aligned}
xm :=\ & \left|\begin{array}{l}
xm \leftarrow xn \\
\text{for } n \in 0..2 \cdot ntraj \\
\quad \left|\begin{array}{l}
gridflag \leftarrow 0 \\
intflag \leftarrow 0 \\
\text{for } i \in 0..nsteps \\
\quad \left|\begin{array}{l}
gridflag \leftarrow \left|\begin{array}{ll} 1 & \text{if } xn_{i,n} \ge d_1 \wedge xn_{i-1,n} < d_1 \\ 0 & \text{otherwise} \end{array}\right. \\
intflag \leftarrow 1 \quad \text{if } gridflag = 1 \wedge \left(yn_{i,n} \le r \vee yn_{i,n} > a - r\right) \\
xm_{i,n} \leftarrow d_1 \quad \text{if } intflag = 1
\end{array}\right.
\end{array}\right. \\
\text{return } xm
\end{array}\right.
\end{aligned}
$$







$V_a = 100\ V$

$V_g = 10\ V$

Figure 6.4 was obtained with
a = 1, d1 = 1, d2 = 1.5, r = 0.05

Figure 6.8 was obtained with
a = 1, d1 = 1, d2 = 1.5, r = 0.05

Note: the aspect ratios of these plots are fixed and do not vary with the parameters of the triode.

The section below can be collapsed to conceal the detailed calculations

Find the trajectory which just touches the grid using the bisection method

```
Z := | y0 ← 0.5·a
     | y1 ← 0
     | y_0 ← y0
     | for n ∈ 0..ntraj
     |   | intercept ← 0
     |   | Z0 ← (0, 0, 0, y_n)^T
     |   | Z_n ← rkfixed(Z0, 0, t_max, nsteps, Dt)
     |   | for i ∈ 0..1000
     |   |   | xn_i ← [(Z_n)^<2>]_i
     |   |   | yn_i ← [(Z_n)^<4>]_i
     |   |   | intercept ← 1  if [(d_1 − xn_i)^2 + (yn_i)^2] ≤ r^2
     |   |   | (break)  if intercept = 1 ∨ xn_i > d_1
     |   | y1 ← y_n  if intercept = 1
     |   | y0 ← y_n  if intercept ≠ 1
     |   | y_(n+1) ← 0.5·(y1 + y0)
     | return Z
```

Calculate the fraction of current intercepted by the grid assuming uniform cathode emission and calculate the anode current

$y_g := \left[\left(Z_{ntraj}\right)^{\langle 4\rangle}\right]_0$

$y_g = 0.028$

$I_{g0} := \dfrac{2\cdot y_g}{a}$

$I_{a0} := 1 - I_{g0}$

$I_{a0} = 0.943$

$I_{g0} = 0.057$

Check convergence of results

$n := 0..ntraj$

$\left[\left(Z_n\right)^{\langle 4\rangle}\right]_0 =$

0.5000
0.2500
0.1250
0.0625
0.0313
0.0156
0.0234
0.0273
0.0293
0.0283
...

Unpack the trajectory matrix Z and add the symmetrical trajectories

$$xn := \left| \begin{array}{l} nmax \leftarrow 2\cdot ntraj \\ \text{for } n \in 0..2\cdot ntraj \\ \quad \text{for } i \in 0..nsteps \\ \quad\quad xn_{i,n} \leftarrow \left| \begin{array}{ll} \left[\left(Z_n\right)^{\langle 2\rangle}\right]_i & \text{if } n \le ntraj \\ \left[\left(Z_{nmax-n}\right)^{\langle 2\rangle}\right]_i & \text{otherwise} \end{array}\right. \\ \text{return } xn \end{array}\right.$$

$$yn := \left| \begin{array}{l} nmax \leftarrow 2\cdot ntraj \\ \text{for } n \in 0..2\cdot ntraj \\ \quad \text{for } i \in 0..nsteps \\ \quad\quad yn_{i,n} \leftarrow \left| \begin{array}{ll} \left[\left(Z_n\right)^{\langle 4\rangle}\right]_i & \text{if } n \le ntraj \\ a - \left[\left(Z_{nmax-n}\right)^{\langle 4\rangle}\right]_i & \text{otherwise} \end{array}\right. \\ \text{return } yn \end{array}\right.$$

Stop trajectories at the grid if they are intercepted

$$xm := \left| \begin{array}{l} xm \leftarrow xn \\ \text{for } n \in 0..2\cdot ntraj \\ \quad \left| \begin{array}{l} gridflag \leftarrow 0 \\ intflag \leftarrow 0 \\ \text{for } i \in 0..nsteps \\ \quad \left| \begin{array}{l} gridflag \leftarrow \left| \begin{array}{ll} 1 & \text{if } xn_{i,n} \ge d_1 \wedge xn_{i-1,n} < d_1 \\ 0 & \text{otherwise} \end{array}\right. \\ intflag \leftarrow 1 \ \text{ if } gridflag = 1 \wedge \left(yn_{i,n} \le r \vee yn_{i,n} > a - r\right) \\ xm_{i,n} \leftarrow d_1 \ \text{ if } intflag = 1 \end{array}\right. \end{array}\right. \\ \text{return } xm \end{array}\right.$$

Find grid and anode current densities when there is island formation

$$J_c := \frac{2}{a\cdot mm}\cdot\int_0^{0.5\cdot a\cdot mm} F_1\left(d_1\cdot mm, V_g, y\right) dy$$

$$J_c = 152.51\,\frac{A}{m^2}$$

$$J_g := \frac{2}{a\cdot mm}\cdot\int_0^{y_g\cdot mm} F_1\left(d_1\cdot mm, V_g, y\right) dy$$

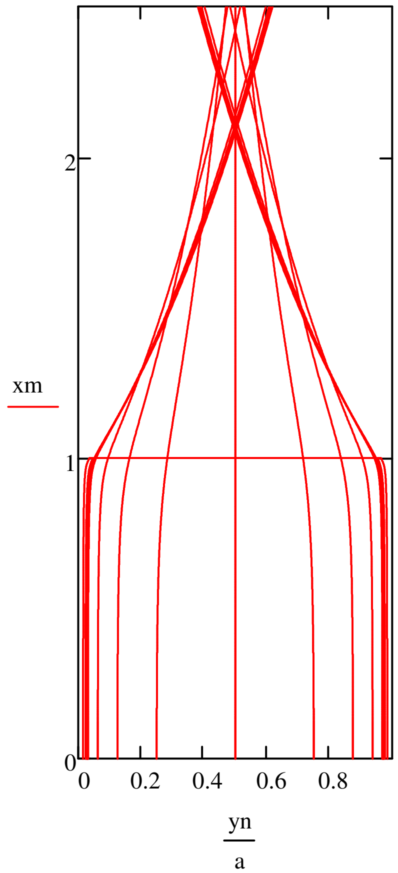
$$J_g = 8.568\,\frac{A}{m^2}$$

$$J_a := J_c - J_g$$

Plot the successive trajectories used to find the one which just touches the grid wires

$a = 1$ $d_1 = 1$ $d_2 = 1.5$ $r = 0.05$

$V_a = 100\ V$ $V_g = 10\ V$



$$RootV_{ga} := \sqrt{\frac{V_g}{V_a}}$$

Enter the values of RootVga (Ig / Ia) in the data matrices below for plotting.

$RootV_{ga} = 0.316$

$$RootVga := \begin{pmatrix} 1 \\ 0.9 \\ 0.8 \\ 0.7 \\ 0.6 \\ 0.5 \\ 0.4 \\ 0.3 \\ 0.2 \\ 0.1 \end{pmatrix} \qquad I_{ga} := \begin{pmatrix} 0.136 \\ 0.131 \\ 0.125 \\ 0.118 \\ 0.108 \\ 0.094 \\ 0.077 \\ 0.056 \\ 0.030 \\ 0 \end{pmatrix}$$

Grid current fraction without island formation

$$\frac{I_{g0}}{I_{a0}} = 0.060$$

Grid current fraction with island formation

$$\frac{J_g}{J_a} = 0.060$$

Compare estimates of the grid current fraction assuming uniform current density on the cathode.

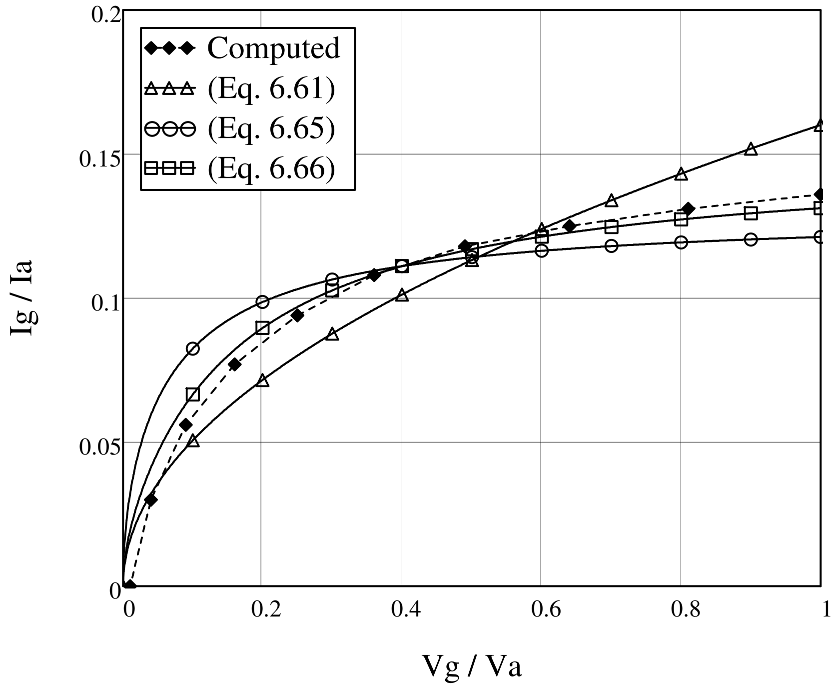


Figure 6.9 was obtained with
a = 1, d1 = 1, d2 = 1.5, r = 0.05, Va = 100V