WS 13.1 Small Signal Model of a Klystron

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

Define the beam parameters - SLAC 50 MW tube data

Anode voltage	Beam current	Centre frequency	Input power	Magnetic field
$V_a := 315 \cdot kV$	$I_0 := 354 \cdot A$	$f_0 := 2856 \cdot MHz$	$P_{in} := 50 \cdot W$	$B_0 := 0.11 \cdot T$
Tunnel radius	Beam radius	Number of cavities	Field profile parameter (Equation 3.90)	
$a := 15.9 \cdot mm$	$b := 11.0 \cdot mm$	$NCAV := 6$	$kgap := 4$	

kgap = 0 for a uniform field in the gap
kgap = 4 for approximation to a knife edge field

Define the cavity parameters

NB. The first element of each vector is not used. The cavity count starts from 1. A cavity is unloaded if Qe >= 95000.

Cavity frequency	Cavity harmonic	External Q	Unloaded Q	R/Q	Gap length	Gap position
$fc := \begin{pmatrix} 0 \\ 2860 \\ 2870 \\ 2890 \\ 2910 \\ 2970 \\ 2853 \end{pmatrix} \cdot MHz$	$nh := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	$Qe := \begin{pmatrix} 0 \\ 200 \\ 95000 \\ 95000 \\ 95000 \\ 95000 \\ 21 \end{pmatrix}$	$Q0 := \begin{pmatrix} 0 \\ 2000 \\ 2000 \\ 2000 \\ 2000 \\ 2000 \\ 2000 \end{pmatrix}$	$R\_Q := \begin{pmatrix} 0 \\ 80 \\ 75 \\ 87 \\ 96 \\ 96 \\ 85 \end{pmatrix} \cdot \Omega$	$gap := \begin{pmatrix} 0 \\ 0.0068 \\ 0.0072 \\ 0.0082 \\ 0.011 \\ 0.0116 \\ 0.0162 \end{pmatrix} \cdot m$	$zg := \begin{pmatrix} 0 \\ 0 \\ 0.056 \\ 0.111 \\ 0.166 \\ 0.444 \\ 0.555 \end{pmatrix} \cdot m$

The detailed calculations can be hidden to allow the data and results to be viewed on the screen simultaneously

Define the charge/mass ratio of the electron. Note that the primary electric constant and the velocity of light are already defined in Mathcad.

$\eta := 1.759 \cdot 10^{11} \cdot \frac{C}{kg}$ $\quad \varepsilon_0 = 8.854 \times 10^{-12} \cdot \frac{F}{m}$ $\quad c = 2.998 \times 10^{8} \frac{m}{s}$ $\quad \mu Perv := \mu A \cdot V^{-1.5}$ $\quad V_R := \frac{c^2}{\eta}$

$\omega_0 := 2 \cdot \pi \cdot f_0$ $\quad dB := 1$

Calculate tube constants and small-signal parameters

Calculate the beam voltage and velocity allowing for space-charge potential depression and relativity

$$V_0 := \left| \begin{array}{l} V_0 \leftarrow V_a \\ \text{for } n \in 0..3 \\ \quad \left| \begin{array}{l} u_n \leftarrow c\cdot\left[1 - \dfrac{1}{\left(1 + \dfrac{V_n}{V_R}\right)^2}\right]^{0.5} \\ V_{n+1} \leftarrow V_0 - \dfrac{I_0}{2\cdot\pi\cdot\varepsilon_0\cdot u_n}\cdot\left(\dfrac{1}{2} - \ln\left(\dfrac{b}{a}\right)\right) \end{array}\right. \\ \text{return } V_{n+1} \end{array}\right.$$

Equation 1.4

Equation 7.8

$V_0 = 291.1\cdot\text{kV}$

$$u_0 := c\cdot\left[1 - \frac{1}{\left(1 + \frac{V_0}{V_R}\right)^2}\right]^{0.5}$$ Equation 1.4 $u_0 = 2.311 \times 10^8 \frac{m}{s}$

$$\gamma_{rel} := 1 + \frac{V_0}{V_R}$$ Equation 7.12 $\gamma_{rel} = 1.570$

Electronic propagation constant

$$\beta_e(\omega) := \frac{\omega}{u_0} \qquad \lambda_e(\omega) := \frac{2\cdot\pi}{\beta_e(\omega)} \qquad \gamma(\omega) := \frac{\beta_e(\omega)}{\gamma_{rel}}$$

$\beta_e(\omega_0) = 77.655 \frac{1}{m}$

$\beta_e(\omega_0)\cdot b = 0.854$

$\beta_e(\omega_0)\cdot a = 1.235$

$\gamma(\omega_0) = 49.471 \frac{1}{m}$

$\gamma(\omega_0)\cdot a = 0.787$

Calculate the plasma frequency and the reduced plasma frequency

$$\omega_{p0} := \sqrt{\frac{\eta}{\varepsilon_0}\cdot\frac{I_0}{\pi\cdot b^2\cdot u_0}}$$

Equation 7.47

$$\omega_p := \omega_{p0}\cdot\sqrt{\frac{1}{\gamma_{rel}^{\ 3}}}$$

$$\beta_p := \frac{\omega_p}{u_0}$$

Calculate the Brillouin field and the ratio mB = B_0 / B_B

$$B_B := \sqrt{\frac{2}{\gamma_{rel}}}\cdot\frac{\omega_{p0}}{\eta}$$

Equation 7.55

$$B_B = 0.0574\,T$$

$$mB := \frac{B_0}{B_B}$$

$$mB = 1.916$$

Calculate the plasma frequency reduction factor $p = \omega_q / \omega_p$

$$\tau b(\beta b, m, p) := \beta b\cdot\left[\frac{\frac{1}{p^2} - 1}{\frac{1}{p^2 - 2\cdot\left(m^2 - 1\right)} - 1}\right]^{\frac{1}{2}}$$

Equation 11.63

$$ab := \frac{a}{b}$$

$$fn1(\beta b, A) := \frac{1}{\beta b}\cdot\frac{I1(\beta b)\cdot K0(A\cdot\beta b) + I0(A\cdot\beta b)\cdot K1(\beta b)}{I0(\beta b)\cdot K0(A\cdot\beta b) - I0(A\cdot\beta b)\cdot K0(\beta b)}$$

Equation 11.62 (Right hand side where A = a /b)

$$fn2(\beta b, m, p) := \frac{1 - \frac{1}{p^2}}{\tau b(\beta b, m, p)}\cdot\frac{I1(\tau b(\beta b, m, p))}{I0(\tau b(\beta b, m, p))}$$

Equation 11.62 (Left hand side)

The solution is the value of p for which fn1 = fn2. Since fn2 varies very rapidly with p it is necessary to use the inverse functions to ensure a stable solution. We therefore seek for the value of *p* which makes the function fn zero.

$$\mathrm{fn}(\beta b, A, m, p) := \frac{1}{\mathrm{fn1}(\beta b, A)} - \frac{1}{\mathrm{fn2}(\beta b, m, p)}$$

A guessed value of *p* is used to seed the solution which is then found as a function of βb, A and m using the Mathcad *root* function. For a relativistic beam we assume that βb is replaced by γb though this has not been proved except for confined flow.

$$p := 0.9 \qquad P(\beta b, A, m) := \mathrm{root}(\mathrm{fn}(\beta b, A, m, p), p)$$

$$\omega_q(\omega) := P\left[(\gamma(\omega)\cdot b), \frac{a}{b}, mB\right]\cdot\omega_p \qquad \beta_q(\omega) := \frac{\omega_q(\omega)}{u_0} \qquad \lambda_q(\omega) := \frac{2\cdot\pi}{\beta_q(\omega)} \qquad \frac{\omega_q(\omega_0)}{\omega_0} = 0.073 \qquad \lambda_q(\omega_0) = 1.115\,\mathrm{m}$$

$$\omega_{q0}(\omega) := P\left[(\beta_e(\omega)\cdot b), \frac{a}{b}, mB\right]\cdot\omega_{p0} \qquad SCF := \left[\frac{P\left[(\gamma(\omega_0)\cdot b), \frac{a}{b}, mB\right]}{P\left[(\beta_e(\omega_0)\cdot b), \frac{a}{b}, mB\right]}\right]^2 \qquad SCF = 0.460$$

SCF is used to make a relativistic correction to the space-charge calculation in the large-signal model in WS13.3

Fast and slow wave propagation constants

$$\beta_f(\omega) := \beta_e(\omega) - \beta_q(\omega) \qquad \beta_s(\omega) := \beta_e(\omega) + \beta_q(\omega) \qquad \beta_f(\omega_0) = 72.021\,\frac{1}{\mathrm{m}} \qquad \beta_s(\omega_0) = 83.289\,\frac{1}{\mathrm{m}}$$

$$\gamma_f(\omega) := \sqrt{\beta_f(\omega)^2 - \frac{\omega^2}{c^2}} \qquad \gamma_s(\omega) := \sqrt{\beta_s(\omega)^2 - \frac{\omega^2}{c^2}}$$

Electronic admittance

$$Y_e(\omega) := \frac{I_0}{\gamma_{rel}\cdot(\gamma_{rel}+1)\cdot V_a}\cdot\frac{\omega}{\omega_q(\omega)} \qquad Y_e(\omega_0) = 3.840\times 10^{-3}\,\frac{1}{\Omega}$$

Calculate the beam loading admitance <u>assuming a uniform field in the gap</u>

$$\mathrm{ReI}(\beta,\gamma,a,b,gap) := \frac{2\cdot\sin(0.5\cdot\beta\cdot gap)^2\left(\mathrm{I0}(\gamma\cdot b)^2 - \mathrm{I1}(\gamma\cdot b)^2\right)}{(\beta\cdot gap)^2\cdot\mathrm{I0}(\gamma\cdot a)^2}$$

Equation 11.101

Beam loading admittance

$$\mathrm{G2}(gap,\omega) := \frac{Y_e(\omega)}{2}\cdot\left(\mathrm{ReI}\left(\beta_f(\omega),\gamma_f(\omega),a,b,gap\right) - \mathrm{ReI}\left(\beta_s(\omega),\gamma_s(\omega),a,b,gap\right)\right)$$

Equation 11.100

Choose the number of terms in the summations for the gap susceptance $nmax := 2$

$$\lambda := \begin{array}{|l} \text{for } n\in 1..nmax \\ \quad \begin{array}{|l} x \leftarrow n\cdot\pi \\ \lambda_n \leftarrow \mathrm{root}(\mathrm{J0}(x),x) \end{array} \\ \text{return } \lambda \end{array}$$

Zeroes of J0

$$r(\omega,gap) := \begin{array}{|l} \text{for } n\in 1..nmax \\ \quad r_n \leftarrow \frac{\lambda_n\cdot gap}{a}\cdot\sqrt{1-\left(\frac{a\cdot\beta_e(\omega)\cdot b}{\lambda_n\cdot gap}\right)^2\cdot\left(\frac{u_0}{c}\right)^2} \\ \text{return } r \end{array}$$

Equation 11.103

$$\mathrm{N1}(\omega,gap) := \begin{array}{|l} \text{for } n\in 1..nmax \\ \quad N_n \leftarrow \left(\frac{\lambda_n}{r(\omega,gap)_n\cdot\frac{a}{gap}}\right)^2 \\ \text{return } N \end{array}$$

$$x(gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad x_n \leftarrow \left(\dfrac{gap \cdot J0\left(\dfrac{b \cdot \lambda_n}{a}\right)}{a \cdot J1(\lambda_n)} \right)^2 \\ \text{return } x \end{array} \right.$$

Equation 11.106

$$y(gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad x_n \leftarrow \left(\dfrac{gap \cdot J1\left(\dfrac{b \cdot \lambda_n}{a}\right)}{a \cdot J1(\lambda_n)} \right)^2 \\ \text{return } x \end{array} \right.$$

Equation 11.107

$$A1(\omega, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad A_n \leftarrow 2 \cdot N1(\omega, gap)_n \cdot \left(x(gap)_n + y(gap)_n\right) \cdot \dfrac{\left(r(\omega, gap)_n - 1 + \exp\left(-r(\omega, gap)_n\right)\right)}{r(\omega, gap)_n} \\ \text{return } A \end{array} \right.$$

Equation 11.104

$$B1(\omega, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad B_n \leftarrow N1(\omega, gap)_n \cdot \left(x(gap)_n + y(gap)_n\right) \cdot \left[\dfrac{3 - 2 \cdot r(\omega, gap)_n - \left(3 + r(\omega, gap)_n\right) \cdot \exp\left(-r(\omega, gap)_n\right)}{\left(r(\omega, gap)_n\right)^3} \right] \\ \text{return } B \end{array} \right.$$

Equation 11.105

$$ImI(\omega, \beta, \gamma, gap) := -\left[\begin{array}{l} \dfrac{(\beta \cdot gap - \sin(\beta \cdot gap))\left(I0(\gamma \cdot b)^2 - I1(\gamma \cdot b)^2\right)}{(\beta \cdot gap)^2 \cdot I0(\gamma \cdot a)^2} \ldots \\ + 2 \cdot \beta \cdot gap \cdot \left[\displaystyle\sum_{n=1}^{nmax} \left[\dfrac{A1(\omega, gap)_n}{\left[(\beta \cdot gap)^2 + \left(r(\omega, gap)_n\right)^2\right]^2} - \dfrac{B1(\omega, gap)_n}{(\beta \cdot gap)^2 + \left(r(\omega, gap)_n\right)^2} \right] \right] \end{array} \right]$$

Equation 11.102

Beam loading susceptance

$$B2(gap,\omega) := \frac{Y_e(\omega)}{2}\cdot\left(ImI\left(\omega,\beta_f(\omega),\gamma_f(\omega),gap\right) - ImI\left(\omega,\beta_s(\omega),\gamma_s(\omega),gap\right)\right)$$

Beam loading admittance

$$Y_b(gap,\omega) := G2(gap,\omega) + j\cdot B2(gap,\omega)$$

$$Y_b\left(10\cdot mm,\omega_0\right) = \left(1.220\times 10^{-4} + 2.937i\times 10^{-5}\right)\frac{1}{\Omega}$$

Compute the properties of the cavities

Shunt conductance

$$Gc := \left|\begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Gc_n \leftarrow \left(R\_Q_n\cdot Q0_n\right)^{-1} \\ \text{return } Gc \end{array}\right.$$

External conductance

$$Ge := \left|\begin{array}{l} \text{for } n \in 1..NCAV \\ \quad \left|\begin{array}{l} Ge_n \leftarrow \left(R\_Q_n\cdot Qe_n\right)^{-1} \text{ if } Qe_n < 95000 \\ Ge_n \leftarrow 0 \text{ otherwise} \end{array}\right. \\ \text{return } Ge \end{array}\right.$$

Cavity admittance

$$Yc(\omega) := \left|\begin{array}{l} f \leftarrow \frac{\omega}{2\cdot\pi} \\ \text{for } n \in 1..NCAV \\ \quad Yc_n \leftarrow Gc_n\cdot\left[1 + j\cdot Q0_n\cdot\left(\frac{f}{fc_n} - \frac{fc_n}{f}\right)\right] \\ \text{return } Yc \end{array}\right.$$

Beam loading conductance

$$Yb(\omega) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Yb_n \leftarrow Y_b\left(gap_n, \omega\right) \\ \text{return } Yb \end{array} \right.$$

Total conductance

$$Y_T(\omega) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Y_n \leftarrow \left(Yb(\omega)_n\right) + Yc(\omega)_n + Ge_n \\ \text{return } Y \end{array} \right.$$

Axial gap coupling factors

$$\mu d(\beta, gap, k) := \left| \begin{array}{ll} \text{sinc}\left(\dfrac{\beta \cdot gap}{2}\right) & \text{if } k = 0 \\ \dfrac{k \cdot \left(\beta \cdot \cosh\left(\dfrac{gap \cdot k}{2}\right) \cdot \sin\left(\dfrac{\beta \cdot gap}{2}\right) + k \cdot \sinh\left(\dfrac{gap \cdot k}{2}\right) \cdot \cos\left(\dfrac{\beta \cdot gap}{2}\right)\right)}{\beta^2 \cdot \sinh\left(\dfrac{gap \cdot k}{2}\right) + k^2 \cdot \sinh\left(\dfrac{gap \cdot k}{2}\right)} & \text{otherwise} \end{array} \right.$$

Axial gap coupling factor for field shape parameter *k*.
See equations 3.90 and 11.36

Gap coupling factors for fast and slow space-charge waves

$$Mf(\omega) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Mf_n \leftarrow \dfrac{2 \cdot I1\left(\gamma_f(\omega) \cdot b\right)}{\left(\gamma_f(\omega) \cdot b\right) \cdot I0\left(\gamma_f(\omega) \cdot a\right)} \cdot \mu d\left(\gamma_f(\omega), gap_n, \dfrac{kgap}{gap_n}\right) \\ \text{return } Mf \end{array} \right.$$

$$Ms(\omega) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Ms_n \leftarrow \dfrac{2 \cdot I1\left(\gamma_s(\omega) \cdot b\right)}{\left(\gamma_s(\omega) \cdot b\right) \cdot I0\left(\gamma_s(\omega) \cdot a\right)} \cdot \mu d\left(\gamma_s(\omega), gap_n, \dfrac{kgap}{gap_n}\right) \\ \text{return } Ms \end{array} \right.$$

Compute the gap voltages

$$
\mathrm{Vg}(\omega) := \left|
\begin{array}{l}
\mathrm{Vg}_1 \leftarrow \sqrt{\dfrac{8\cdot \mathrm{Ge}_1}{\left(\left|\mathrm{Ge}_1 + \mathrm{Yc}(\omega)_1 + \mathrm{Yb}(\omega)_1\right|\right)^2}\cdot P_{in}} \\
\mathrm{WVf}_1 \leftarrow 0 \\
\mathrm{WVs}_1 \leftarrow 0 \\
\text{for } n \in 2..\mathrm{NCAV} \\
\quad \left|
\begin{array}{l}
\mathrm{WVf}_n \leftarrow \left(\mathrm{WVf}_{n-1} + \mathrm{Mf}(\omega)_{n-1}\cdot\dfrac{\mathrm{Vg}_{n-1}}{2}\right)\cdot \exp\left[-j\cdot\beta_f(\omega)\cdot\left(zg_n - zg_{n-1}\right)\right] \\
\mathrm{WVs}_n \leftarrow \left(\mathrm{WVs}_{n-1} + \mathrm{Ms}(\omega)_{n-1}\cdot\dfrac{\mathrm{Vg}_{n-1}}{2}\right)\cdot \exp\left[-j\cdot\beta_s(\omega)\cdot\left(zg_n - zg_{n-1}\right)\right] \\
\mathrm{Vg}_n \leftarrow -\left(\dfrac{Y_e(\omega)}{Y_T(\omega)}\right)_n\cdot\left(\mathrm{Mf}(\omega)_n\cdot \mathrm{WVf}_n - \mathrm{Ms}(\omega)_n\cdot \mathrm{WVs}_n\right)
\end{array}\right. \\
\text{return } \mathrm{Vg}
\end{array}\right.
$$

$$
\mathrm{Mf}(\omega_0) = \begin{pmatrix} 0.000 \\ 0.924 \\ 0.924 \\ 0.922 \\ 0.918 \\ 0.916 \\ 0.906 \end{pmatrix}
\qquad
\mathrm{Ms}(\omega_0) = \begin{pmatrix} 0.000 \\ 0.852 \\ 0.851 \\ 0.848 \\ 0.839 \\ 0.837 \\ 0.816 \end{pmatrix}
$$

If the phase of Vg1 is -90 deg then the relative phase of the bunch centre is close to zero.

$$P_{out}(\omega) := \frac{1}{2} \cdot \left| \left(Vg(\omega)_{NCAV} \right)^2 \right| \cdot Ge_{NCAV}$$

$$Gain(\omega) := 10 \cdot \log\left(\frac{P_{out}(\omega)}{P_{in}} \right)$$

$$P_{out}(\omega_0) = 325.39 \cdot MW$$

$$Gain(\omega_0) = 68.1 \cdot dB$$

Space-charge factor for transfer to the large-signal model in WS13.3

$$SCF = 0.460$$

Gap voltages for transfer to the large-signal model in WS13.3

$$Vg(\omega_0) = \begin{pmatrix} 0.000 \\ 0.837 \\ 3.376 + 3.583i \\ -13.925 + 13.735i \\ -42.167 - 53.087i \\ 101.505 + 311.629i \\ -999.741 - 402.708i \end{pmatrix} \cdot kV$$

$$f := 2.83 \cdot GHz, 2.84 \cdot GHz .. 2.91 \cdot GHz$$