WS 3.1 Resonant Circuits

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

Section 3.2.1 The properties of resonant circuits

The amplitude of the input impedance of a parallel resonant circuit as a function of frequency and Q

$\omega_0 := 1$ $\qquad$ $R_c := 1$

$$Z(\omega, Q) := \frac{R_c}{1 + j \cdot Q \cdot \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}$$ $\boxed{\text{Equation 3.5}}$

$$MZR(\omega, Q) := 20 \cdot \log\left(\frac{|Z(\omega, Q)|}{R_c} \right)$$ (Amplitude of Z) / R in dB

$$P(\omega, Q) := \left(\frac{180}{\pi} \right) \cdot \operatorname{atan}\left(\frac{\operatorname{Im}(Z(\omega, Q))}{\operatorname{Re}(Z(\omega, Q))} \right)$$ Phase of Z in degrees

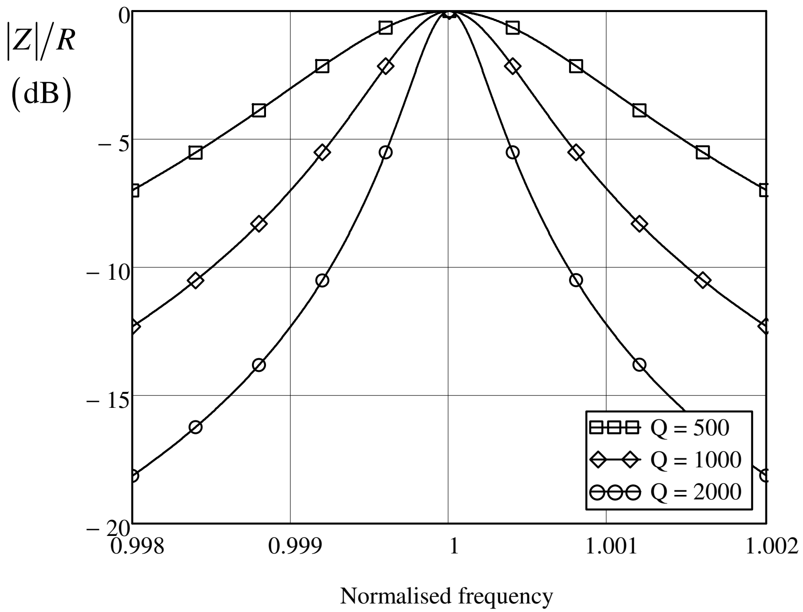


Figure 3.3

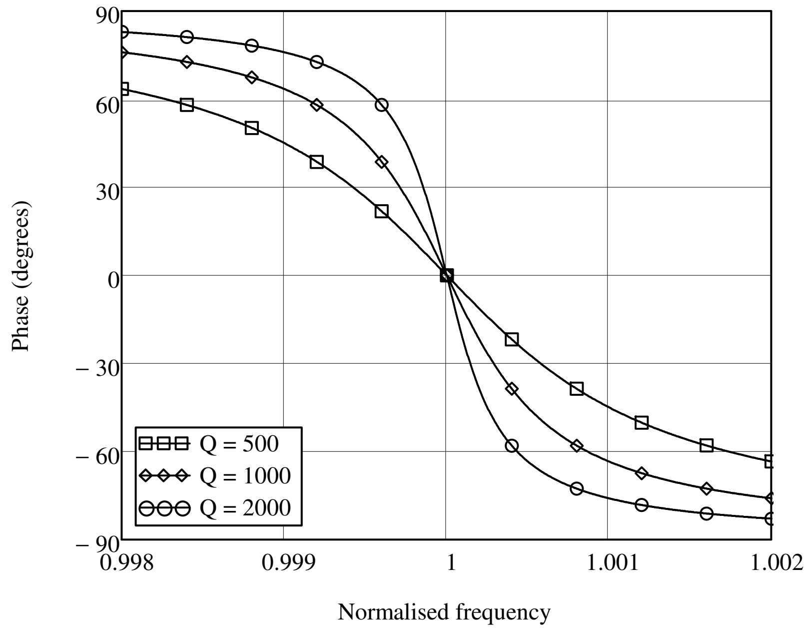


Figure 3.4

Section 3.2.3 Excitation of resonant circuits

$\omega_0 := 2\cdot\pi$ $\omega := 2\cdot\pi$ $RoverQL := 1$ $Q_L := 10$ $Q_U := 1000$ $R' := RoverQL\cdot Q_L$ $R' = 10$

Define the driving current

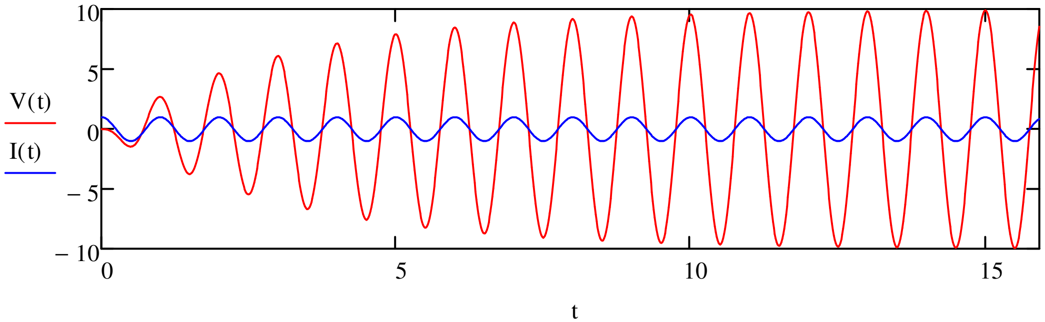
$I_0 := 1$ $I(t) := I_0\cdot\cos(\omega\cdot t)$

The differential equation governing the voltage is (3.23) with the initial conditions

Given

$$\frac{d^2}{dt^2}V(t) = \frac{-\omega_0}{Q_L}\cdot\left(\frac{d}{dt}V(t)\right) - \omega_0{}^2\cdot V(t) + \omega_0\cdot RoverQL\cdot\frac{d}{dt}I(t) \qquad V(0) = 0 \qquad V'(0) = 0$$

Solve the differential equation from t = 0 to $t_{end} := 10\cdot\frac{Q_L}{\omega_0}$ $V := \mathrm{Odesolve}(t, t_{end})$



Build-up of stored energy in a resonator

Time constant $\tau(K) := \dfrac{2 \cdot Q_U}{\omega_0 \cdot (1 + K)}$ Equation 3.25

Stored energy when critically coupled $W_0 := 1$

Stored energy $W(t,K) := W_0 \cdot \dfrac{4 \cdot K}{(1 + K)^2} \cdot \left(1 - \exp\left(\dfrac{-t}{\tau(K)}\right)\right)^2$ Equation 3.27

Input match $S_{11}(t,K) := \left| \left(\dfrac{2 \cdot K}{1 + K}\right) \cdot \left(1 - \exp\left(\dfrac{-t}{\tau(K)}\right)\right) - 1 \right|$ Equation 3.34

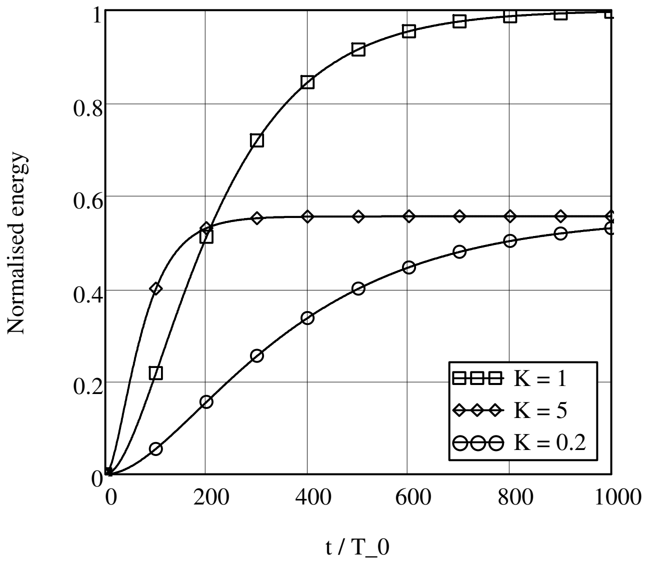


Figure 3.7

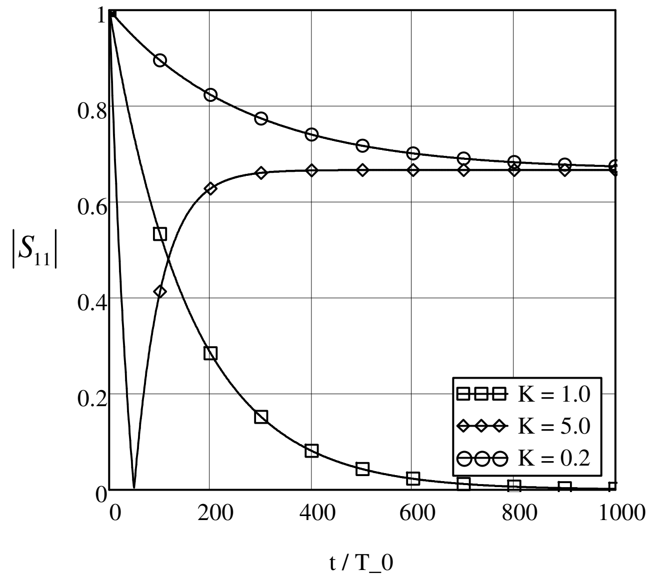
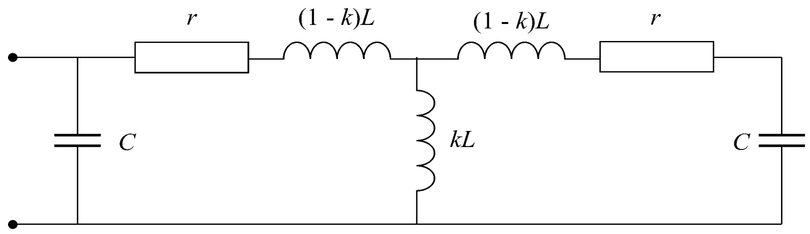


Figure 3.8

Section 3.2.4 Coupled Resonant Circuits

This sheet models two identical parallel resonant circuits coupled by a mutual inductor kL which is connected as shown. This is equivalent to mutual inductive coupling with M = kL



Define the microwave parameters of the uncoupled resonators

$\omega_0 := 1$ $\qquad$ $RoverQ := 100$ $\qquad$ $Q := 1000$

Compute the equivalent circuit parameters

$C := \dfrac{1}{\omega_0 \cdot RoverQ}$ $\qquad$ $L := \dfrac{RoverQ}{\omega_0}$ $\qquad$ $R := RoverQ \cdot Q$ $\qquad$ $r := \dfrac{R}{Q^2}$

$C = 0.01$ $\qquad$ $L = 100$ $\qquad$ $R = 1 \times 10^5$

Compute the impedance matrix of the circuit

$$Z_1(\omega) := j \cdot \omega \cdot L + r + \frac{1}{j \cdot \omega \cdot C}$$

$$ZM(\omega, k) := \begin{pmatrix} \dfrac{1}{j \cdot \omega \cdot C} & \dfrac{-1}{j \cdot \omega \cdot C} & 0 \\ \dfrac{-1}{j \cdot \omega \cdot C} & Z_1(\omega) & -j \cdot k \cdot \omega \cdot L \\ 0 & -j \cdot k \cdot \omega \cdot L & Z_1(\omega) \end{pmatrix}$$

Equation 3.35

Inductively coupled circuits with *M = kL*

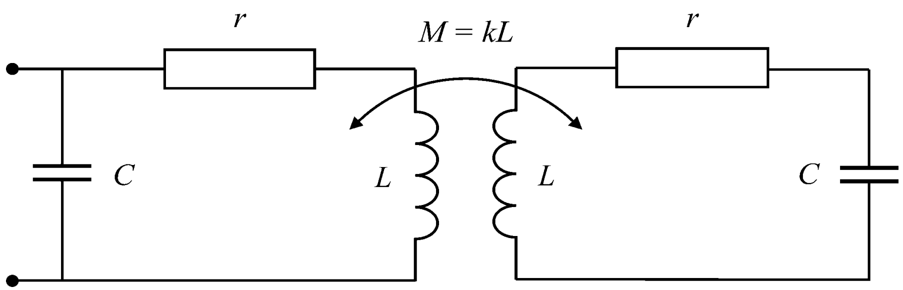


Figure 3.9

Compute the impedance matrix of the circuit

$$Z1(\omega) := \frac{1}{j\cdot\omega\cdot C} + r + j\cdot\omega\cdot L$$

$$ZM(\omega,k) := \begin{pmatrix} \frac{1}{j\cdot\omega\cdot C} & \frac{-1}{j\cdot\omega\cdot C} & 0 \\ \frac{-1}{j\cdot\omega\cdot C} & Z1(\omega) & -j\cdot k\cdot\omega\cdot L \\ 0 & -j\cdot k\cdot\omega\cdot L & Z1(\omega) \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ 0 \\ 0 \end{pmatrix} = ZM(\omega,k)\cdot\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}$$

Note: The impedance matrix is identical to that for the circuits coupled by a mutual inductance

Define the admittance matrix

$$YM(\omega,k) := ZM(\omega,k)^{-1}$$

$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = YM(\omega,k)\cdot\begin{pmatrix} V_1 \\ 0 \\ 0 \end{pmatrix}$$

Calculate the input admittance and impedance

$$Y_{in}(\omega,k) := YM(\omega,k)_{0,0}$$

$$Z_{in}(\omega,k) := Y_{in}(\omega,k)^{-1}$$

Calculate the amplitude of the input impedance, normalised to R, in decibels and the input phase in degrees as functions of ω and k

$$\rho(\omega,k) := \frac{R - Z_{in}(\omega,k)}{R + Z_{in}(\omega,k)}$$

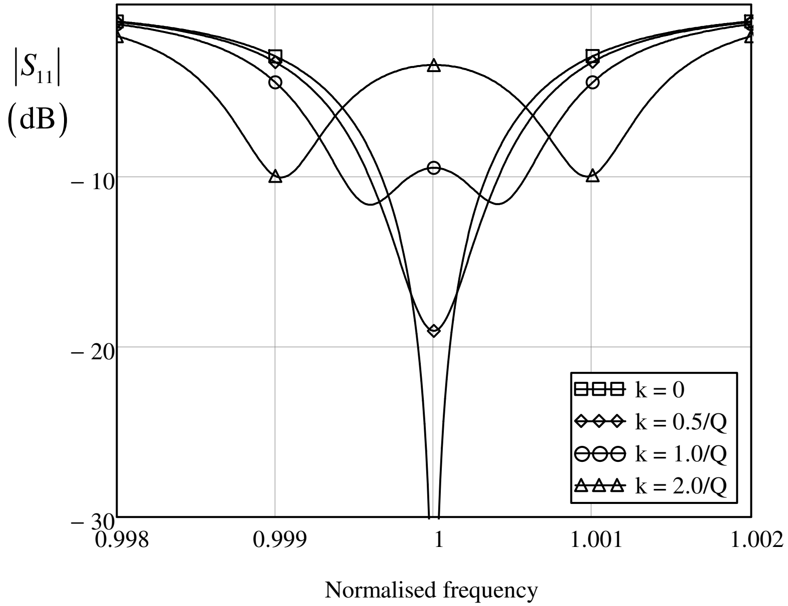
$$S_{11}(\omega,k) := 20 \cdot \log(|\rho(\omega,k)|)$$

$$\mathrm{ModZ}(\omega,k) := 20 \cdot \log\left(\left(\frac{|Z_{in}(\omega,k)|}{R}\right)\right)$$

$$Ph_{in}(\omega,k) := \left(\frac{180}{\pi}\right) \cdot \mathrm{atan}\left(\frac{\mathrm{Im}(Z_{in}(\omega,k))}{\mathrm{Re}(Z_{in}(\omega,k))}\right)$$

Plotting range

$$\omega := 0.998, 0.99801 .. 1.002$$



Input impedance of coupled resonators Q = 1000

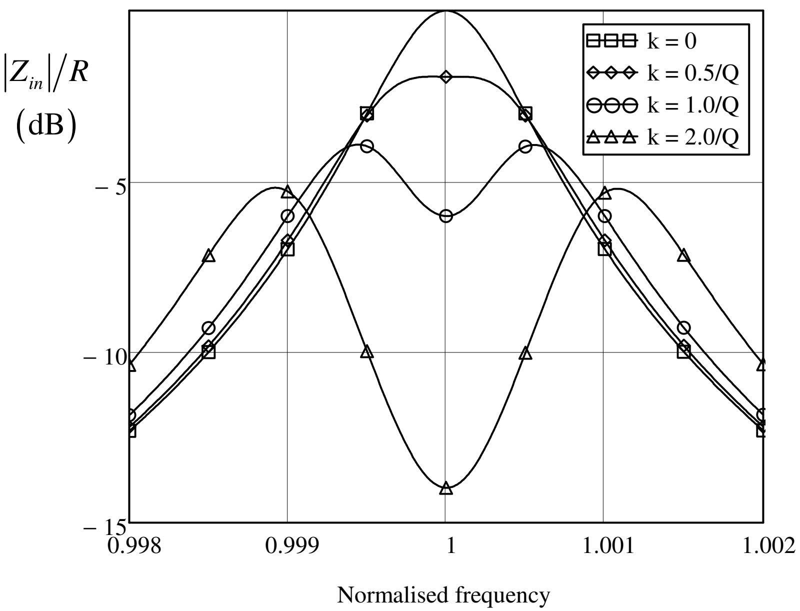


Figure 3.10

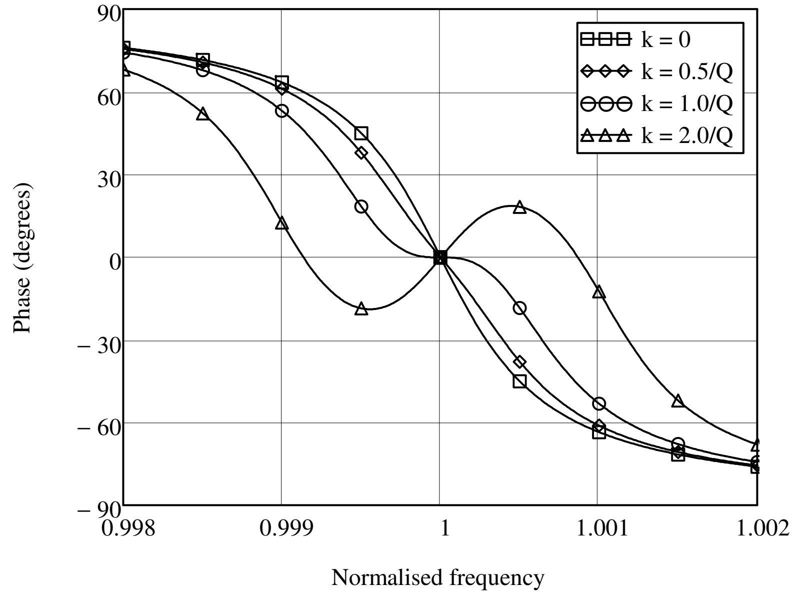


Figure 3.11