

## WS 5.2 Cylindrical space-charge limited diode

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This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

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### Physical constants

Charge/mass ratio of the electron

$$\eta := 1.759 \cdot 10^{11} \cdot \text{C} \cdot \text{kg}^{-1}$$

Rest energy of the electron (eV)

$$V_R := \frac{c^2}{\eta} \quad V_R = 510.947 \text{ kV}$$

### Normalised variables

$$R = \frac{r}{r_c}$$

$$U = \frac{V}{V_R}$$

$$k_1 = \frac{-I_L \cdot r_c}{2 \cdot \pi \cdot \epsilon_0 \cdot c \cdot V_R}$$

Equation 5.68

$$a^3 = \frac{81}{32} \cdot k_1^2$$

Equation 5.74

$$a := 1.0$$

**Langmuir's equations**

$$\gamma(R) := \ln(R)$$

Equation 5.69

$$\beta A(R) := \gamma(R) - 0.4 \cdot \gamma(R)^2 + 0.0916667 \cdot \gamma(R)^3 - 0.014242 \cdot \gamma(R)^4 + 0.001679 \cdot \gamma(R)^5 - 0.000161 \cdot \gamma(R)^6$$

Equation 5.71

$$\beta B(R) := \exp\left(-\frac{\gamma(R)}{2}\right) \cdot \left(\gamma(R) + 0.1 \cdot \gamma(R)^2 + 0.016667 \cdot \gamma(R)^3 + 0.002424 \cdot \gamma(R)^4 + 0.000287 \cdot \gamma(R)^5 + 0.000026 \cdot \gamma(R)^6\right)$$

Equation 5.72

Normalised voltages using equations (5.71) and (5.72)

$$ULA(a, R) := a \cdot R^{\frac{2}{3}} \cdot \left(|\beta A(R)|\right)^{\frac{4}{3}}$$

$$ULB(a, R) := a \cdot R^{\frac{2}{3}} \cdot \left(|\beta B(R)|\right)^{\frac{4}{3}}$$

Equation 5.75

**Acton's equations**

$$UA(a, R) := a \cdot \left(|\gamma(R)|\right)^{\frac{4}{3}} \cdot \left(1 + 0.1333 \cdot \gamma(R) + 0.02444 \cdot \gamma(R)^2 + 0.0039236 \cdot \gamma(R)^3 + 0.00052966 \cdot \gamma(R)^4\right)$$

Equation 5.73

$$UAR(a, R) := a \cdot \left(|\gamma(R)|\right)^{\frac{4}{3}} \cdot \left(1 + \frac{2}{15} \cdot \gamma(R) + \frac{11}{450} \cdot \gamma(R)^2\right) + a^2 \cdot \left(|\gamma(R)|\right)^{\frac{8}{3}} \cdot \left[\frac{1}{14} + \frac{6}{175} \cdot a \cdot \left(|\gamma(R)|\right)\right]$$

Equation 5.76

**Numerical integration of Equation (5.66)**

$$k_1 := \sqrt{\frac{32}{81}} \cdot a^3$$

$$k_1 = 0.629$$

$$\frac{d^2U}{dr^2} + \frac{1}{r} \frac{\partial U}{\partial r} = -\frac{I_L}{2\pi\epsilon_0 c V_0 r} \left[ 1 - \frac{1}{(1+U)^2} \right]^{\frac{1}{2}}$$

This can be written as a pair of simultaneous first order equations

$$\frac{d}{dR} U_0 = U_1$$

$$\frac{d^2U}{dR^2} + \frac{1}{R} \frac{\partial U}{\partial R} = \frac{k_1}{R} \left[ 1 - \frac{1}{(1+U)^2} \right]^{\frac{1}{2}}$$

$$\frac{d}{dR} U_1 = \frac{k_1}{R} \cdot \left[ \frac{1 + U_0}{\sqrt{2 \cdot U_0 + (U_0)^2}} \right] - \frac{U_1}{R}$$

$$DR(R, U) := \begin{bmatrix} U_1 \\ \frac{k_1}{R} \cdot \left[ \frac{1 + U_0}{\sqrt{2 \cdot U_0 + (U_0)^2}} \right] - \frac{U_1}{R} \end{bmatrix}$$

In the non-relativistic approximation

$$DNR(R, U) := \begin{bmatrix} U_1 \\ \frac{k_1}{R} \cdot \left( \frac{1}{\sqrt{2 \cdot U_0}} \right) - \frac{U_1}{R} \end{bmatrix}$$

These equations are difficult to integrate numerically because of the singularity in the integrand close to the cathode when  $U = 0$ . However, when  $R$  tends to 1,  $\gamma$  tends to zero and  $U$  can be approximated from equation (5.73) by

$$U = a \cdot \gamma^{\frac{4}{3}} \quad \text{so that} \quad R = \exp \left[ \left( \frac{U}{a} \right)^{\frac{3}{4}} \right] \quad \text{or, approximately} \quad R = 1 + \left( \frac{U}{a} \right)^{\frac{4}{3}} \quad \text{whence} \quad \frac{dU}{dR} = \frac{4}{3} \cdot a^{\frac{3}{4}} \cdot U^{\frac{1}{4}}$$

The solution is started from a point a little way from the cathode using this approximation to determine the initial conditions

$$\text{Let} \quad U_0 := 10^{-6} \quad R_1 := \exp \left[ \left( \frac{U_0}{a} \right)^{\frac{3}{4}} \right] = 1.000 \quad R_{\max} := 5 \quad R1 := 1, 1.5 \dots R_{\max}$$

When the cathode lies outside the anode the initial conditions are

$$U := \begin{pmatrix} U_0 \\ \frac{3}{4} \cdot a^{\frac{1}{4}} \cdot U_0^{\frac{1}{4}} \\ \frac{1}{3} \end{pmatrix} \quad U = \begin{pmatrix} 1 \times 10^{-6} \\ 0.042 \end{pmatrix}$$

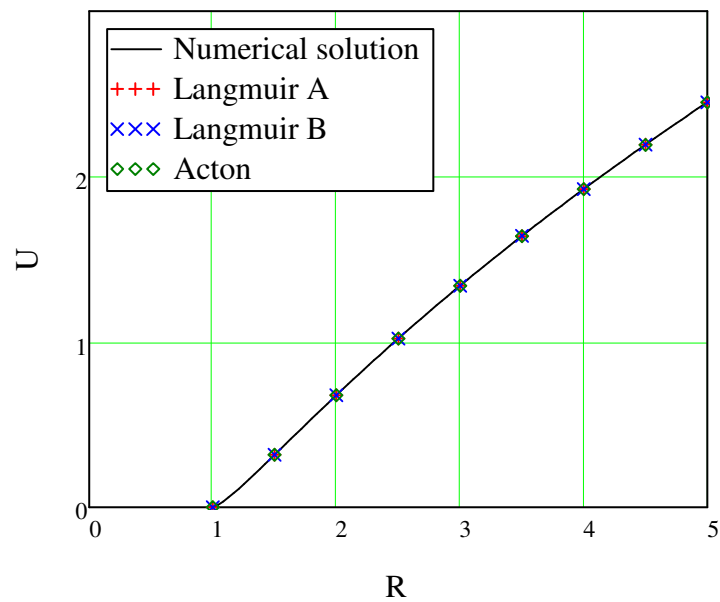
The solutions to the differential equations are

$$U1NR := \text{AdamsBDF}(U, R_1, R_{\max}, 95, \text{DNR})$$

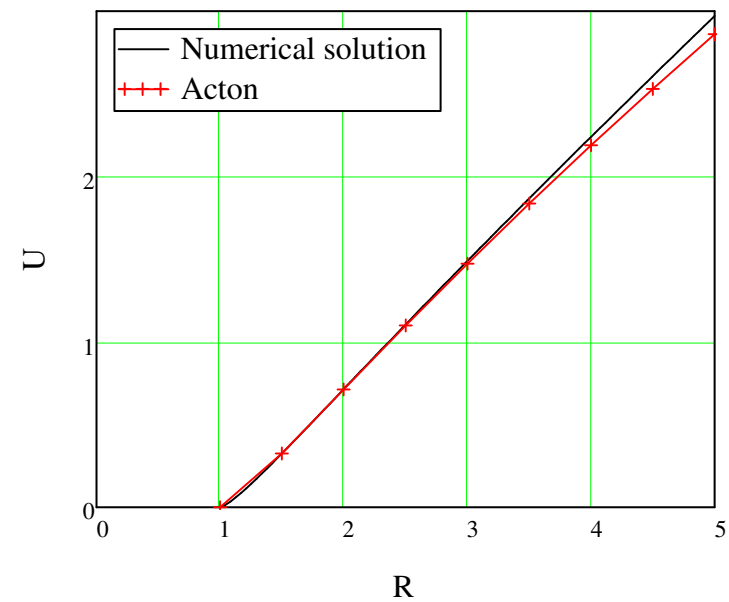
$$U1R := \text{AdamsBDF}(U, R_1, R_{\max}, 95, \text{DR})$$

The results are in matrices in which the first column is the independent variable, the second column is U and the third column its differential

Non-relativistic



Relativistic



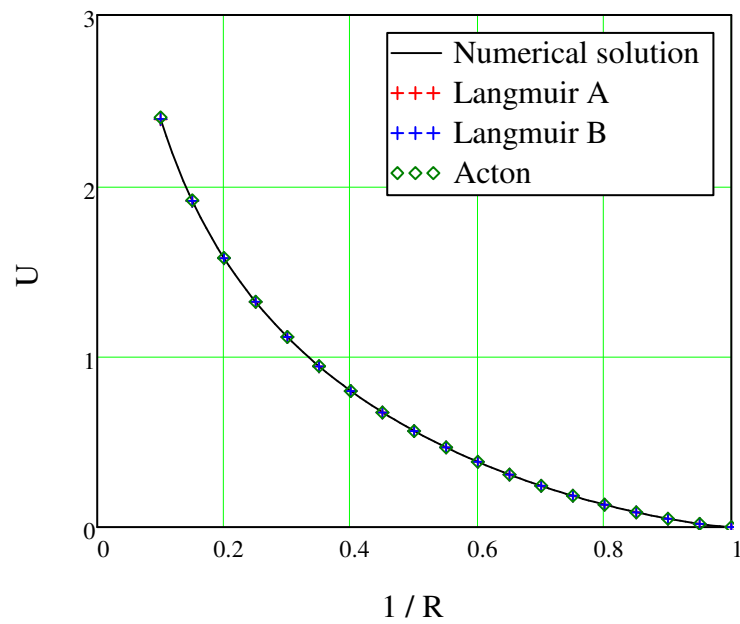
When the anode lies outside the cathode the initial conditions are

$$U := \begin{pmatrix} U_0 \\ -\sqrt{2 \cdot k_1 \cdot \sqrt{2 \cdot U_0}} \end{pmatrix} \quad U = \begin{pmatrix} 1 \times 10^{-6} \\ -0.042 \end{pmatrix} \quad R_2 := \frac{1}{R_1} \quad R_{2\min} := 0.1 \quad R_2 := 1, 0.95 \dots R_{2\min}$$

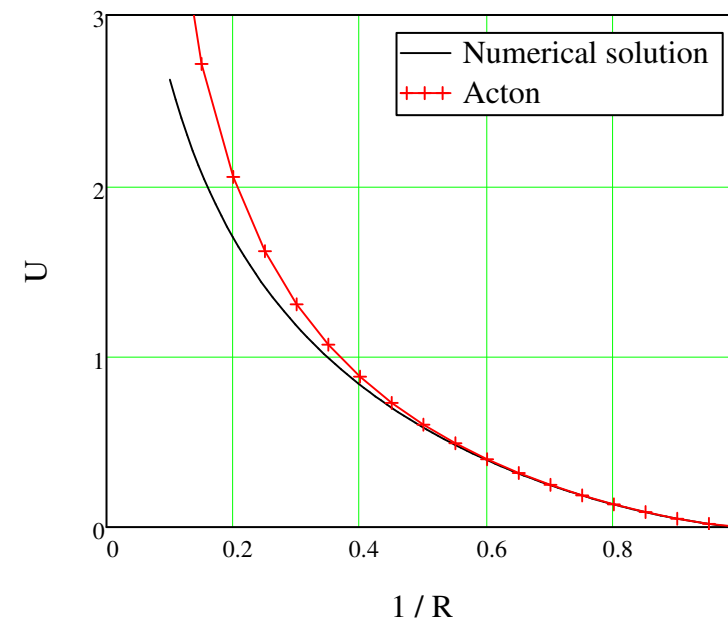
$$U2NR := \text{AdamsBDF}(U, R_2, R_{2\min}, 100, \text{DNR})$$

$$U2R := \text{AdamsBDF}(U, R_2, R_{2\min}, 100, \text{DR})$$

Non-relativistic



Relativistic



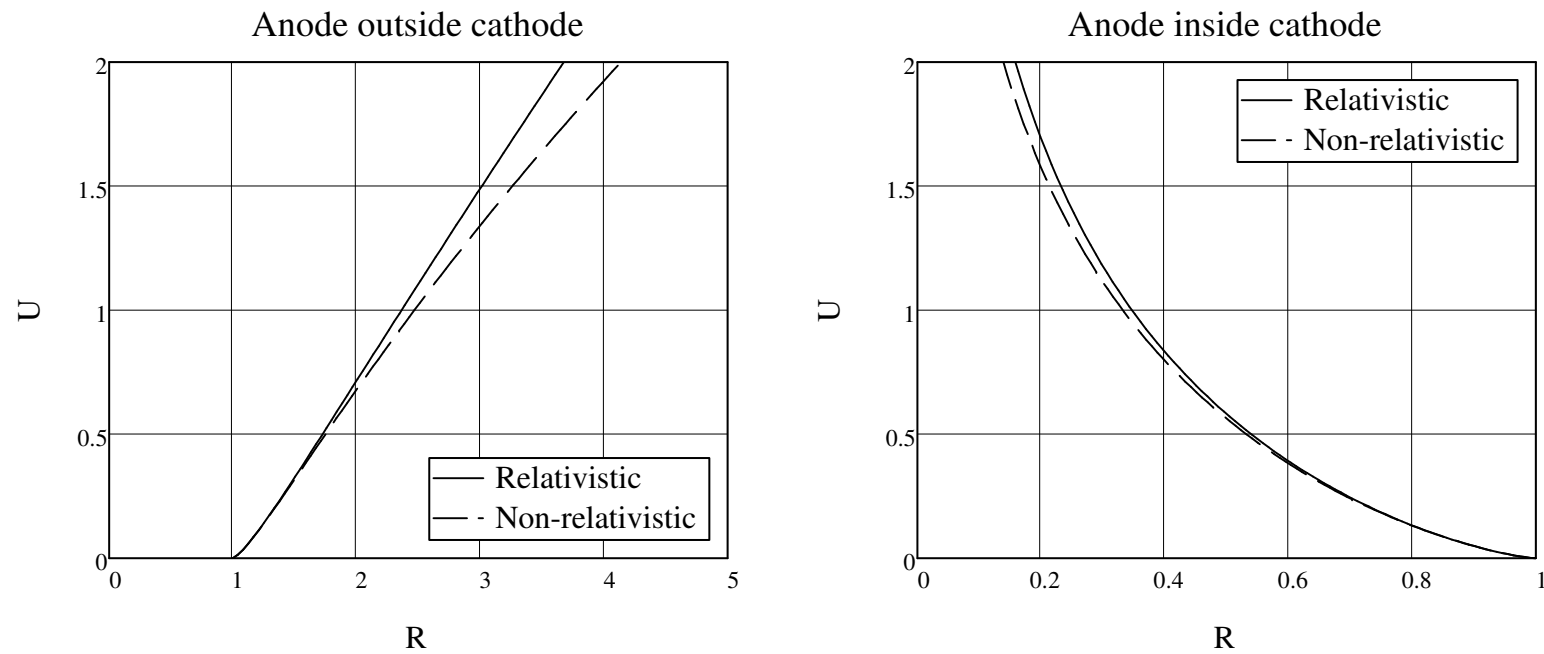


Figure 5.9(a) and (b)