WS 11.3 Interaction between an unmodulated beam and a gap

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This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

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This worksheet is designed for investigation of the interaction between an unmodulated electron beam, represented by ND discs of charge in one electron wavelength, and the RF field of a gap centred at z = 0 (see Section 11.8). Normalised values ore used for position ($\theta = \beta_e \cdot z$) and time ($\phi = \omega \cdot t$).

Input data

Beam voltage	$V_0 := 25 \cdot kV$	Normalised tunnel radius	$\gamma a := 1.0$	Bunch centre starting position	$\theta_0 := -2\pi$
Beam current	$I_0 := 4.4 \cdot A$	Normalised beam radius	$\beta b := 0.629$	Initial time	$\phi_0 := \theta_0$
Frequency	$f := 1 \cdot GHz$	Normalised gap length	$\beta gap := 1.0$	Final position	$\theta_f := 2\pi$
Space-charge (0 or 1)	$SCF := 0$	Number of discs	$ND := 12$	Final time	$\phi_f := 5\theta_f$

Normalised gap voltage ($X = MV_g/V_0$) $X := 1.2$

Phase of gap voltage ($\Phi = 0$ for accelerating field) $\Phi_1 := 90 \cdot deg$

Constants

Charge/mass ratio of the electron $\eta := 1.759\cdot 10^{11}\cdot\frac{C}{kg}$ $\quad \mu S := 10^{-6}\cdot S \quad \mu P := \mu A\cdot V^{-1.5} \quad Perv := I_0\cdot V_0^{-1.5}$

Beam parameters

Electron velocity $u_0 := c\cdot\left[1-\frac{1}{\left(1+\frac{\eta\cdot V_0}{c^2}\right)^2}\right]^{0.5}$ $\quad Rel := \frac{1}{\sqrt{1-\frac{u_0^2}{c^2}}}$ $\quad u_0 = 9.049\times 10^7\frac{m}{s}$

Propagation constants $\omega := 2\cdot\pi\cdot f \quad \beta_e := \frac{\omega}{u_0} \quad \lambda_e := \frac{2\cdot\pi}{\beta_e} \quad \gamma_e(\beta_e) := \sqrt{\beta_e^2-\frac{\omega^2}{c^2}}$

Physical dimensions $a := \frac{\gamma a}{\gamma_e(\beta_e)} \quad b := \frac{\beta b}{\beta_e} \quad gap := \frac{\beta gap}{\beta_e}$ $\quad a = 15.11\cdot mm \quad b = 9.06\cdot mm \quad gap = 14.40\cdot mm$

Small-signal parameters

Gap coupling factor $M(\beta_e) := \frac{2\cdot I1(\gamma_e(\beta_e)\cdot b)}{(\gamma_e(\beta_e)\cdot b)\cdot I0(\gamma_e(\beta_e)\cdot a)}\cdot\left(\frac{\sin(0.5\cdot\beta_e\cdot gap)}{0.5\cdot\beta_e\cdot gap}\right)$ Equations 11.36 and 11.38 $\quad M(\beta_e) = 0.792$

Beam loading susceptance $G_b(\beta_e) := -\frac{1}{Rel\cdot(Rel+1)}\cdot\frac{I_0}{V_0}\cdot M(\beta_e)\cdot\beta_e\cdot\frac{d}{d\beta_e}M(\beta_e)$ Equation 11.23 $\quad G_b(\beta_e) = 24.6\cdot\mu S$

Plasma frequency $\omega_{p0} := \sqrt{\frac{\eta\cdot I_0}{\varepsilon_0\cdot\pi\cdot b^2\cdot u_0}} \quad \omega_p := \omega_{p0}\cdot\sqrt{\frac{1}{Rel^3}}$ Equation 11.57 et seq. $\quad \frac{\omega_p}{\omega} = 0.287$

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Find the reduced plasma frequency (see Worksheet 11.2)

$$\tau b(\beta b, m, p) := \beta b \cdot \left[\frac{\frac{1}{p^2} - 1}{\frac{1}{p^2 - 2\cdot\left(m^2 - 1\right)} - 1} \right]^{\frac{1}{2}}$$

$$f11(\beta b, A) := \frac{1}{\beta b} \cdot \frac{I1(\beta b)\cdot K0(A\cdot\beta b) + I0(A\cdot\beta b)\cdot K1(\beta b)}{I0(\beta b)\cdot K0(A\cdot\beta b) - I0(A\cdot\beta b)\cdot K0(\beta b)}$$

where $A = \frac{a}{b}$

$$f2(\beta b, m, p) := \frac{\left(1 - \frac{1}{p^2}\right)}{\tau b(\beta b, m, p)} \cdot \frac{I1(\tau b(\beta b, m, p))}{I0(\tau b(\beta b, m, p))}$$

$$f0(\beta b, A, m, p) := \frac{1}{f11(\beta b, A)} - \frac{1}{f2(\beta b, m, p)}$$

$p1 := 0.9$ $\quad$ $p(\beta b, A, m) := root(f0(\beta b, A, m, p1), p1)$

$$\omega_q := p\left(\gamma_e\left(\beta_e\right)\cdot b, \frac{a}{b}, 100\right)\cdot\omega_p$$

(Set m = 100 to represent confined flow)

$$\lambda_q := \frac{\omega}{\omega_q}\cdot\lambda_e$$

$$\left(\frac{\omega_q}{\omega}\right) = 0.100$$

Large-signal disc model

The beam is modelled as ND identical rigid discs. The motions of the electrons at the central planes of the discs are tracked. We define their initial positions and velocities using the disc thickness ΔL. As the Mathcad ODE solver rkfixed does not accept variables with dimensions the dimensionless variables: $\theta = \beta_e\cdot z$ and $\phi = \omega t$ are used.

Disc thickness $\quad \Delta L := \frac{\lambda_e}{ND}$

Normalised disk thickness $\quad \theta_d := \beta_e\cdot\Delta L$

Normalised disk starting positions and velocities relative to the gap centre.

$$\theta := \begin{array}{|l} \text{for } j \in 0, 2 .. (2\cdot ND - 1) \\ \quad \begin{array}{|l} \theta_j \leftarrow j\cdot 0.5\cdot\theta_d + \theta_0 - \pi \\ \theta_{j+1} \leftarrow 1 \end{array} \\ \theta \end{array}$$

Disk charge $\quad Q := \frac{2\cdot\pi\, I_0}{\omega\cdot ND}$

Check $\quad f\cdot Q\cdot ND = 4.400\,A \quad I_0 = 4.400\,A$

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Space-Charge Field

Based on the equations given in
J.R. Hechtel, "The effect of potential beam energy on the performance of linear beam devices",
IEEE Transactions on Electron Devices ED-17, pp.999-1009, Nov. 1970

The first ten zeros of the Bessel function $J_0(z)$.

$$\mu B := \frac{1}{a}\cdot(2.405\ \ 5.520\ \ 8.654\ \ 11.791\ \ 14.931\ \ 18.071\ \ 21.212\ \ 24.352\ \ 27.494\ \ 30.635)^T$$

Calculate the axial electric field of a disc having a charge of 1 C at 100 points up to one electronic wavelength from the centre of the disc

$$ESn := \left| \begin{array}{l} \text{for } n \in 0..100 \\ \quad \left| \begin{array}{l} \theta_n \leftarrow 0.02\cdot n\cdot\pi \\ z_n \leftarrow \dfrac{\theta_n}{\beta_e} \\ ES_n \leftarrow \left[\dfrac{-4}{\varepsilon_0\cdot(\pi\cdot b^2\cdot\Delta L)}\right]\cdot\displaystyle\sum_{m=0}^{9}\left[\frac{1}{\mu B_m}\cdot\left(\frac{J1(\mu B_m\cdot b)}{\mu B_m\cdot a\cdot J1(\mu B_m\cdot a)}\right)^2\cdot\exp(-\mu B_m\cdot z_n)\cdot\sinh\left(\frac{\mu B_m\cdot\Delta L}{2}\right)\right] \text{ if } \theta_n \geq 0.5\cdot\theta_d \\ ES_n \leftarrow \left[\dfrac{-4}{\varepsilon_0\cdot(\pi\cdot b^2\cdot\Delta L)}\right]\cdot\displaystyle\sum_{m=0}^{9}\left[\frac{1}{\mu B_m}\cdot\left(\frac{J1(\mu B_m\cdot b)}{\mu B_m\cdot a\cdot J1(\mu B_m\cdot a)}\right)^2\cdot\left(\exp\left(-\mu B_m\cdot\frac{\Delta L}{2}\right)\cdot\sinh(\mu B_m\cdot z_n)\right)\right] \text{ otherwise} \end{array}\right. \\ ES \end{array}\right.$$

Normalised positions at which ES was calculated

$$\theta n := \left| \begin{array}{l} \text{for } n \in 0..100 \\ \quad \theta_n \leftarrow 0.02\cdot n\cdot\pi \\ \theta \end{array}\right.$$

Define a continuous function for the field by linear interpolation

$$Es(\theta) := \text{sign}(\theta)\cdot\text{linterp}(\theta n, ESn, |\theta|)$$

$$ES(\theta) := \left| \begin{array}{ll} Es(\theta + 2\cdot\pi) & \text{if } \theta < -\pi \\ Es(\theta - 2\cdot\pi) & \text{if } \theta > \pi \\ Es(\theta) & \text{otherwise} \end{array}\right.$$

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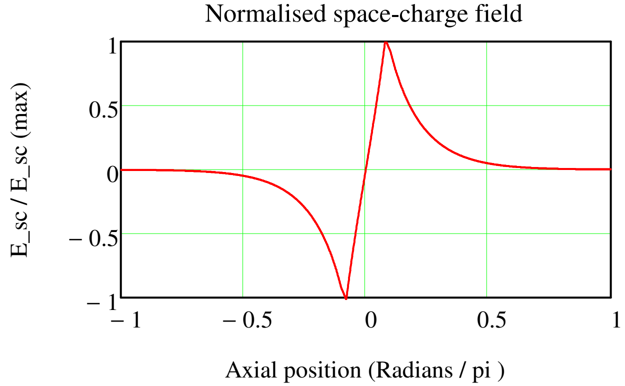
$\theta 1 := (-\pi), (-0.999 \cdot \pi) .. \pi$



The space-charge field of adjacent bunches is included by assuming that the field is periodic in z. This is not correct but tests with an initially unmodulated beam and three wavelengths of electrons give almost identical results for the trajectories and the current harmonics except well beyond the first bunch and at microperveance greater than 2.

The field of the gap

The electric field for unit gap voltage is found from the Fourier Transform of the field in the gap (assumed to be constant). The average of the field over the beam is used. Linear interpolation on the values calculated at regular intervals is used to provide a fast look-up function.

$$\begin{aligned} En := \; & \text{for } n \in 0..100 \\ & \quad \theta_n \leftarrow 0.02 \cdot n \cdot \pi \\ & \quad \gamma(\beta) \leftarrow \sqrt{\beta^2 - \omega^2 c^{-2}} \\ & \quad E_n \leftarrow -\frac{V}{\pi} \cdot \int_0^{\frac{20\cdot\pi}{gap}} \frac{2 \cdot I1(\gamma(\beta) \cdot b)}{(\gamma(\beta) \cdot b) \cdot I0(\gamma(\beta) \cdot a)} \cdot \left(\frac{\sin(0.5 \cdot \beta \cdot gap)}{0.5 \cdot \beta \cdot gap} \right) \cdot \cos\left(\frac{\beta}{\beta_e} \cdot \theta_n \right) d\beta \\ & E \end{aligned}$$

$Egap(\theta) := linterp(\theta n, En, |\theta|)$

Equation 3.79

CHECK the gap coupling factor $\frac{-2}{\beta_e \cdot V} \cdot \int_0^{2\pi} Egap(\theta) \cdot \cos(\theta) \, d\theta = 0.792$ $M(\beta_e) = 0.792$

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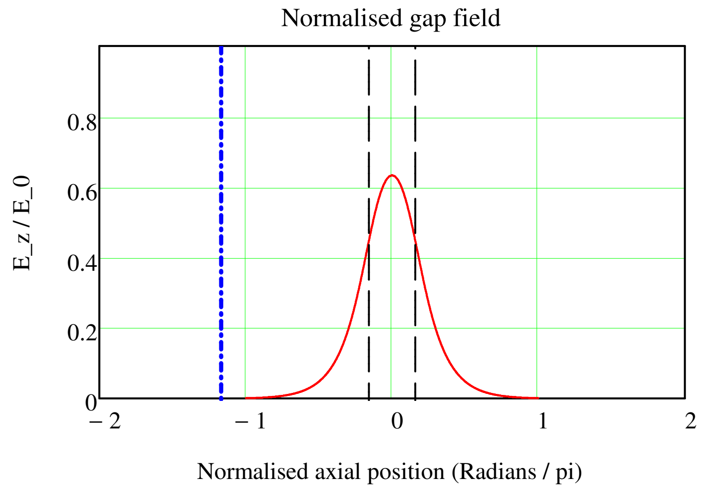
Electric field of the gap as function of normalised position and time

$$V_g := \frac{X \cdot V_0}{M(\beta_e)}$$

$$E_z(\theta, \phi) := \mathrm{Re}\left[\frac{V_g}{V} \cdot \mathrm{Egap}(\theta)\,\exp\left[j \cdot (\phi - \Phi_1)\right]\right]$$

$$V_g = 37.883 \cdot \mathrm{kV}$$



The blue chain-dotted line is the initial position of the first electron in the bunch. It should lie outside the range of the gap field.

Turn space-charge force off until the electrons reach the gap centre to avoid non-physical dispersion of the initial group of electrons.

$$SC(\theta) := \begin{cases} SCF & \text{if } \theta \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

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Equations of motion

The Coefficients of the Differential Equations for the motions of the electrons are defined.

$$D(\phi,\theta) := \left| \begin{array}{l} \text{for } j \in 0, 2 .. 2\cdot(ND-1) \\ \quad \left| \begin{array}{l} D_j \leftarrow \theta_{j+1} \\ D_{j+1} \leftarrow -\dfrac{\eta}{\omega\cdot u_0}\cdot\left[1-\left(\dfrac{u_0}{c}\cdot\theta_{j+1}\right)^2\right]^{1.5}\cdot\left(E_z(\theta_j,\phi) + SC(\theta_j)\cdot Q\cdot\sum_{i=0}^{ND-1} ES(\theta_j - \theta_{2\cdot i})\right) \end{array}\right. \\ D \end{array}\right.$$

Definitions of normalised variables

$\phi = \omega\cdot t$ $\qquad \theta = \beta_e\cdot z \qquad \theta' = \dfrac{v}{u_0}$

$\dfrac{d}{dt}z = v \qquad \dfrac{d}{d\phi}\theta = \dfrac{v}{u_0}$

$\dfrac{d}{dt}v = -\eta\cdot E \qquad \dfrac{d}{d\phi}\dfrac{v}{u_0} = \dfrac{\eta\cdot E}{\omega\cdot u_0}$

The Equations are Solved using with *nmax* time steps starting from ϕ_0 which is defined in such a way that the centre electron would cross the gap centre at t = 0 if it travelled with a constant velocity u_0. The final time is ϕ_f

Number of integration steps $\quad nmax := 100$

The variable tol specifies the tolerance on the solution of the differential equations. 10E-6 works well normally but much smaller values may be needed at low drive levels $\quad tol := 10^{-6}$

$$Z := \text{AdamsBDF}(\theta, \phi_0, \phi_f, nmax, D, tol)$$

The results are in a single table (Z) in which the first column (0) is the time and the other columns are the positions and velocities of the electrons in the same order as before at each value of n.

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Extract the vector of normalised time, the matrices containing the normalised positions and velocities of the disks, and the vector of the final velocities of the electrons

$$\phi n := \left| \begin{array}{l} \text{for } n \in 0..nmax \\ \quad \phi_n \leftarrow Z_{n,0} \\ \phi \end{array} \right.$$

$$\theta n := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } n \in 0..nmax \\ \quad\quad \theta_{n,j} \leftarrow Z_{n,2\cdot j+1} \\ \theta \end{array} \right.$$

$$un := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } n \in 0..nmax \\ \quad\quad u_{n,j} \leftarrow Z_{n,2\cdot j+2} \\ u \end{array} \right.$$

$$umax := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad u_j \leftarrow Z_{nmax,2\cdot j+2} \\ u \end{array} \right.$$

Total kinetic energy of the electrons at each time step.

$$KE := \left| \begin{array}{l} \text{for } n \in 0..nmax \\ \quad KE_n \leftarrow \dfrac{Q\cdot c^2}{\eta}\cdot \displaystyle\sum_{j=0}^{ND-1} \left[\dfrac{1}{\sqrt{1-\dfrac{(un_{n,j}\cdot u_0)^2}{c^2}}} - 1 \right] \\ KE \end{array} \right.$$

Check that the frequency times the intial KE is equal to the DC beam power

$KE_0\cdot f = 110.0\cdot kW$

$P_{DC} := V_0\cdot I_0 = 110.0\cdot kW$

Final total kinetic power in the beam

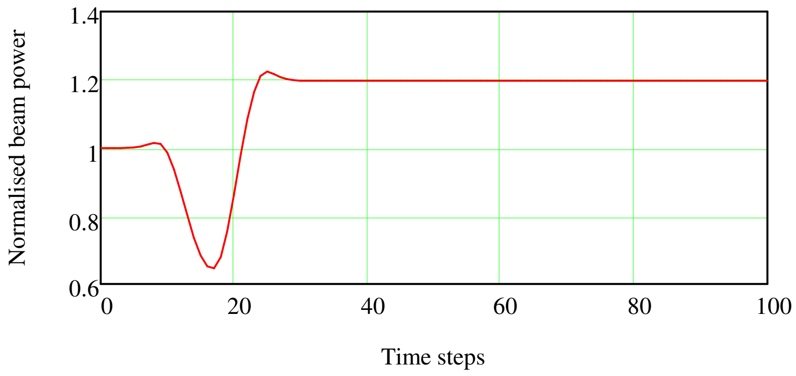
$(KE_{nmax})\cdot f = 131.734\cdot kW$

$n := 0..nmax$

Beam loading conductance

$$Gbd := \frac{2\cdot (KE_{nmax} - KE_0)\cdot f}{V_g^{\,2}}$$

$Gbd = 30.3\cdot \mu S$



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Define a set of equally-spaced planes in θ and compute the times and velocities at which the electrons cross them using linear interpolation.

Reference plane interval $\Delta\theta := 0.01\cdot\pi$

Number of reference planes $NP := \dfrac{\theta_f - \theta_0}{\Delta\theta}$ $NP = 400$

Normalised positions

$$\theta p := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad \theta_p \leftarrow \dfrac{p}{NP}\cdot\left(\theta_f - \theta_0\right) + \theta_0 \\ \text{return } \theta \end{array} \right.$$

Normalised times

$$\phi p := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } p \in 0..NP \\ \quad\quad \text{for } n \in 1..nmax \\ \quad\quad\quad \left| \begin{array}{l} flag \leftarrow 0 \\ flag \leftarrow 1 \ \text{ if } \ \theta n_{n,j} > \theta p_p \\ \phi p_{p,j} \leftarrow \phi n_{n-1} + \dfrac{\theta p_p - \theta n_{n-1,j}}{\theta n_{n,j} - \theta n_{n-1,j}}\cdot\left(\phi n_n - \phi n_{n-1}\right) - \theta p_p \ \text{ if } \ flag = 1 \\ (\text{break}) \ \text{ if } \ flag = 1 \end{array} \right. \\ \text{return } \phi p \end{array} \right.$$

Normalised velocities

$$up := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } p \in 0..NP \\ \quad\quad \text{for } n \in 1..nmax \\ \quad\quad\quad \left| \begin{array}{l} flag \leftarrow 0 \\ flag \leftarrow 1 \ \text{ if } \ \theta n_{n,j} > \theta p_p \\ up_{p,j} \leftarrow un_{n-1,j} + \dfrac{\theta p_p - \theta n_{n-1,j}}{\theta n_{n,j} - \theta n_{n-1,j}}\cdot\left(un_{n,j} - un_{n-1,j}\right) \ \text{ if } \ flag = 1 \\ (\text{break}) \ \text{ if } \ flag = 1 \end{array} \right. \\ \text{return } up \end{array} \right.$$

Electron energies (eV) at the final plane

$$Vpf := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad V_j \leftarrow \dfrac{c^2}{\eta\cdot V_0}\cdot\left[\dfrac{1}{\sqrt{1 - \dfrac{\left[u_0\cdot\left(up_{NP,j}\right)\right]^2}{c^2}}} - 1\right] \\ \text{return } V \end{array} \right.$$

Calculate the complex current harmonics at each plane by superimposing the Fourier components of the currents of the discs.
For simplicity each disc is treated as having constant length.

Number of current harmonics $NH := 6$

$$
Ip := \left|\begin{array}{l}
\text{for } p \in 0..NP \\
\quad \left|\begin{array}{l}
Ip_{p,0} \leftarrow \dfrac{Q}{2\cdot\pi\cdot\Delta L}\cdot\displaystyle\sum_{j=0}^{ND-1}\theta_d \\
\text{for } h \in 1..NH \\
\quad Ip_{p,h} \leftarrow \dfrac{2Q}{\pi\cdot h\cdot\Delta L}\cdot\displaystyle\sum_{j=0}^{ND-1}\left(\sin\left(\dfrac{h\cdot\theta_d}{2}\right)\cdot\exp\left(j\cdot h\cdot\phi p_{p,j}\right)\right)
\end{array}\right. \\
\text{return } Ip\cdot u_0
\end{array}\right.
$$

Instantaneous current = $\dfrac{Q_j\cdot up_{p,j}}{\Delta L}$

Pulse phase duration = $\dfrac{\theta_d}{up_{p,j}}$

Find the position of the first maximum of the fundamental component of the RF beam current

$$
pmax := \left|\begin{array}{l}
\text{for } p \in 200..NP \\
\quad \left|\begin{array}{l}
Ip1_p \leftarrow \left|Ip_{p,1}\right| \\
(\text{break}) \ \text{ if } Ip1_p < Ip1_{p-1}
\end{array}\right. \\
\text{return } p
\end{array}\right.
$$

$\theta pmax := \theta p_{pmax}$

$Ipmax := \left|Ip_{pmax,1}\right|$

Plotting ranges for graphs $j := 0..ND-1$ $x := 0, 0.1..1.5$ $\phi := \phi_0..1.2\phi_f$ $pp := 0..NP$

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Small-signal interaction with space-charge (X = 0.1)

Reduced plasma frequency $\frac{\omega_q}{\omega} = 0.100$

Gap coupling factor $M(\beta_e) = 0.792$

Bunching length $\frac{\lambda_q}{4 \cdot \lambda_e} = 2.50$

Maximum I1 / I0 $\frac{1}{2} \cdot \frac{\omega}{\omega_q} \cdot X = 5.988$

Disc model results

Bunching length $\frac{\theta pmax}{2 \cdot \pi} = 0.34$

Maximum I1 / I0 $\frac{Ipmax}{I_0} = 1.182$

Figure 11.25

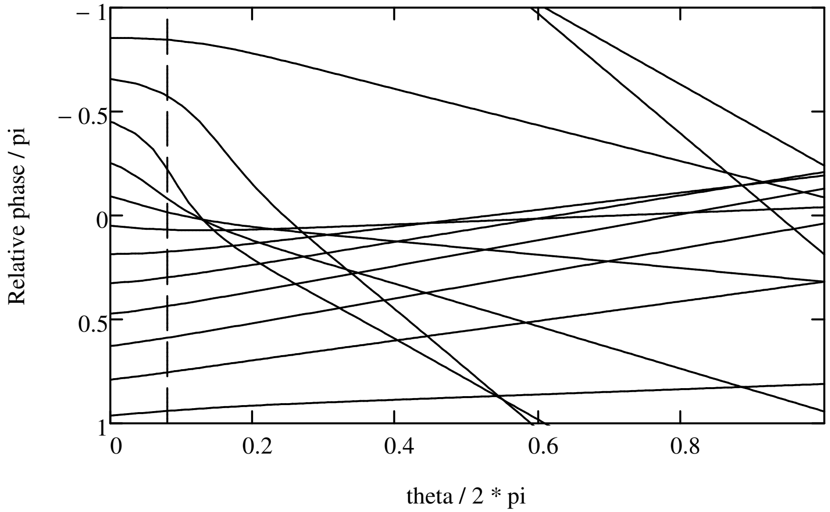
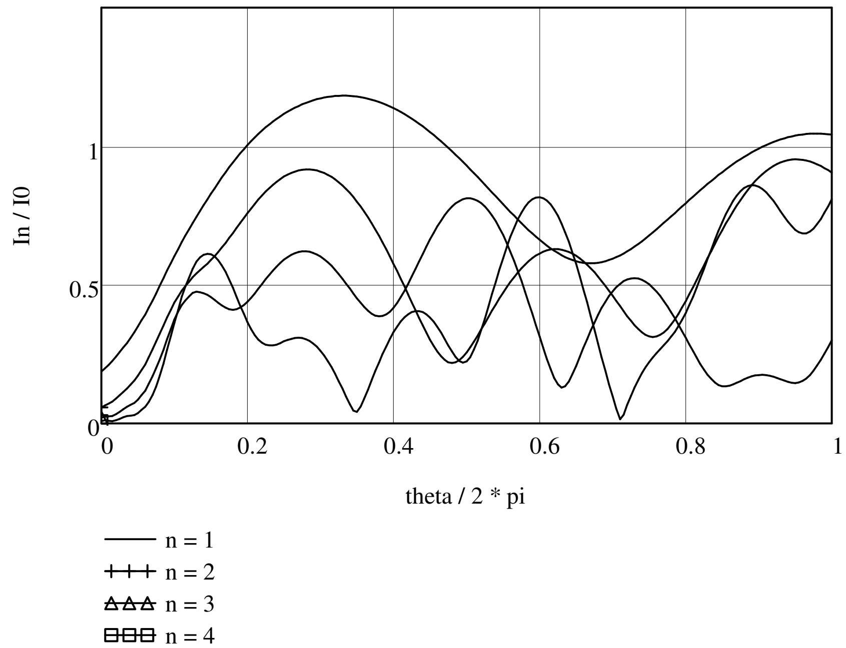


Figure 11.26



Note: For this figure SCF = 1 and $\theta_f = 20 \cdot \pi$. The sign of phase is reversed so that accelerated electrons move upwards

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Applegate diagram

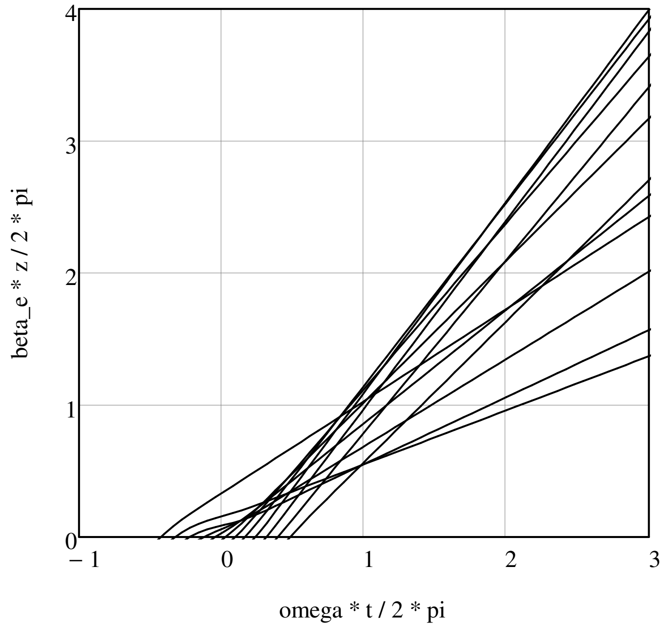

Figure 11.24
Note: for this figure X = 0.2, Φ = 90 deg and SCF = 0

Beam perveance $Perv = 1.11 \cdot \mu P$

Normalised gap voltage $\frac{V_g}{V_0} = 1.515$

Note: Beam loading conductance

The text of the last paragraph of section 11.8.2 (p.423) is wrong. The beam loading conductance calculated using the disc model without space-charge is close to that from small-signal theory. However, when space-charge is included the results are erratic because the conversion of some energy into potential energy in the bunched beam varies with position. The best results of the disc model are obtained with $\theta_f = 2 \cdot \pi$.

Small-signal theory $G_b(\beta_e) = 24.6 \cdot \mu S$

Using the disc model $Gbd = 30.3 \cdot \mu S$

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Approximate models of the beam/gap interaction

Calculate the exit energy as a function of ϕ using piecewise constant velocity and iteration to find the exit velocity. The results are compared with those from the disk model.

1. Using the small-signal value of M

$$\mathrm{Vs0}(\phi, X, \Phi) := 1 + X\cdot\cos(\phi - \Phi)$$

$\boxed{\text{Equation 11.8}}$

2. Effective value of M calculated as the mean of the values of M calculated at the initial and final velocities.
This is the version used in the book for fig. 11.31(a))

$$\beta e(V_e) := \frac{\omega}{c}\cdot\left[1 - \frac{1}{\left(1 + \frac{\eta\cdot V_e}{c^2}\right)^2}\right]^{-0.5}$$

$$\mathrm{Vss2}(\phi, X, \Phi, V_s) := 1 + \frac{1}{2}\cdot\left(1 + \frac{M(\beta e(V_0\cdot V_s))}{M(\beta_e)}\right)\cdot X\cdot\cos(\phi - \Phi)$$

$\boxed{\text{Equation 11.168}}$

$V_s := 0.2$

$\mathrm{fn2}(V_s, \phi, X, \Phi) := V_s - \mathrm{Vss2}(\phi, X, \Phi, V_s)$

$\mathrm{Vs2}(\phi, X, \Phi) := \mathrm{root}(\mathrm{fn2}(V_s, \phi, X, \Phi), V_s)$

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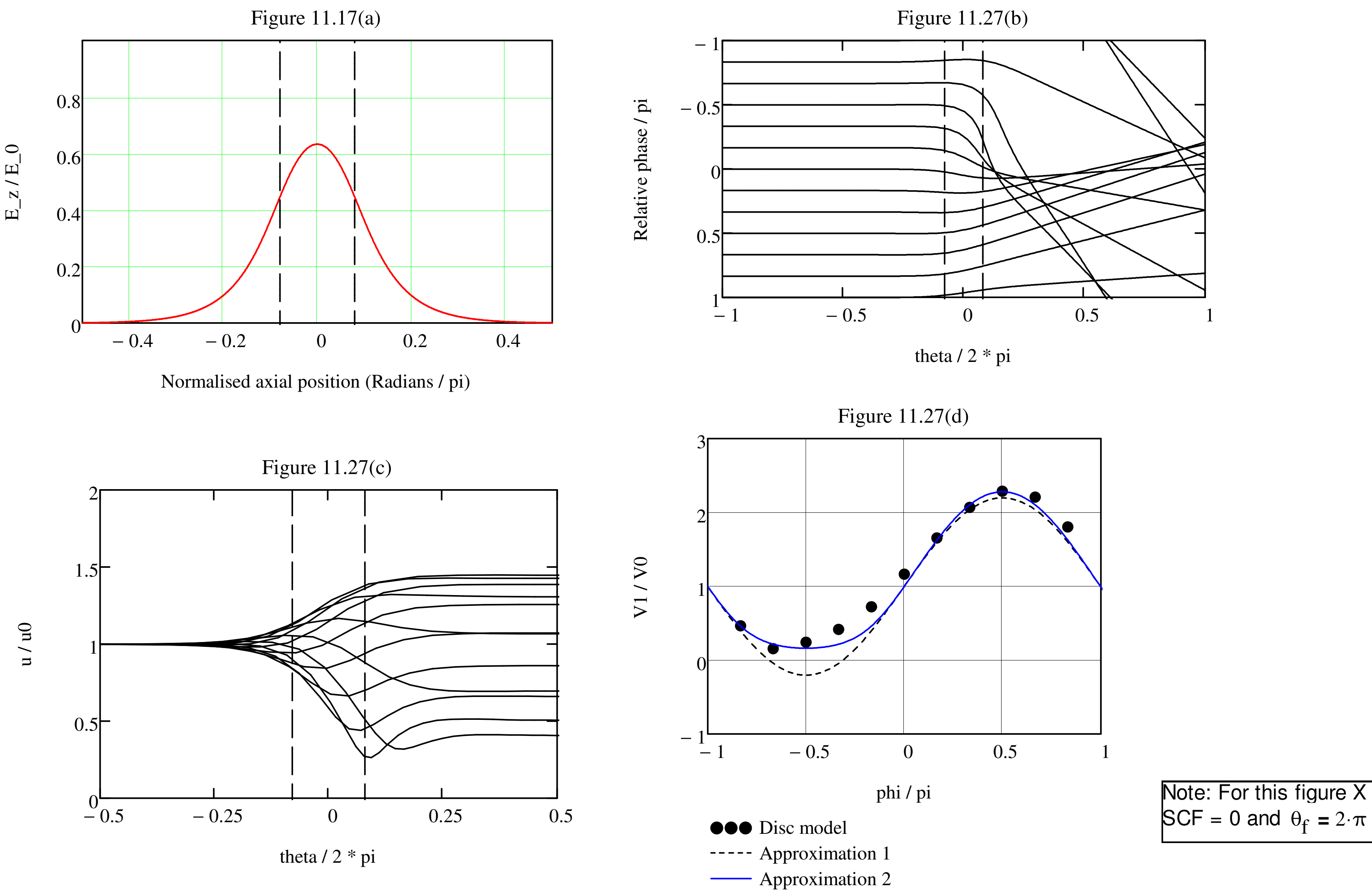

Note: For this figure X = 1.2, SCF = 0 and $\theta_f = 2\cdot\pi$

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