WS 5.3 Spherical space-charge limited diode

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This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

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Physical constants

Charge/mass ratio of the electron $\eta := 1.759 \cdot 10^{11} \cdot C \cdot kg^{-1}$

Rest energy of the electron (eV) $V_R := \dfrac{c^2}{\eta} = 510.947 \cdot kV$

Normalised variables

$$k_2 = \frac{-I}{4 \cdot \pi \cdot \varepsilon_0 \cdot c \cdot V_R}$$ Equation 5.82

$$a_2 = \left(\frac{81}{32} \cdot k_2^{\,2}\right)^{\frac{1}{3}}$$ Equation 5.87

$a_2 \equiv 1$

Langmuir's equations

$$\gamma(R) := \ln(R)$$ Equation 5.83

$$\alpha(R) := \gamma(R) - 0.3\cdot\gamma(R)^2 + 0.075\cdot\gamma(R)^3 - 0.0143182\cdot\gamma(R)^4 + 0.0021609\cdot\gamma(R)^5 - 0.00026791\cdot\gamma(R)^6$$ Equation 5.85

$$UL(R, a_2) := a_2\cdot(|\alpha(R)|)^{\frac{4}{3}}$$ Equation 5.86

Acton's equations

$$UA(R, a_2) := a_2\cdot(|\gamma(R)|)^{\frac{4}{3}}\cdot\left(1 - \frac{2}{5}\cdot\gamma(R) + \frac{3}{25}\cdot\gamma(R)^2\right)$$ Equation 5.88

$$UAR(R, a_2) := a_2\cdot(|\gamma(R)|)^{\frac{4}{3}}\cdot\left[1 - \frac{2}{5}\cdot\gamma(R) + \frac{417}{25\cdot 144}\cdot(|\gamma(R)|)^2\right] + a_2{}^2\cdot(|\gamma(R)|)^{\frac{8}{3}}\cdot\left(-\frac{1}{14} + \frac{5}{7\cdot 72}\cdot\gamma(R)\right)$$ Equation 5.8*

The differential equation for relativistic flow is (5.81)

$$\frac{d^2U}{dR^2} + \frac{2}{R}\cdot\frac{dU}{dR} = \frac{k_2}{R^2}\cdot\frac{1+U}{\sqrt{2U+U^2}}$$

This can be expressed as a pair of simultaneous first order equations

$$\frac{d}{dR}U = U_1$$

$$\frac{d}{dR}U_1 = \frac{k_2}{R^2}\cdot\left(\frac{1+U}{\sqrt{2\cdot U + U^2}}\right) - \frac{2\cdot U_1}{R}$$

Where $R = \frac{r}{r_0}$ $U = \frac{V}{V_R}$

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This equation is difficult to integrate numerically because of the singularity in the integrand close to R = 1 when U = 0.
However, for small values of U, R is close to unity, dU/dR is close to zero and the equation can be approximated by

$$\frac{d^2U}{dR^2} = k_2 \frac{1}{\sqrt{2U}}$$

which can be integrated to give

$$R1(U) = 1 + \left(\frac{1}{\sqrt{2\cdot k_2\cdot\sqrt{2}}}\cdot\frac{4}{3}\cdot U^{\frac{3}{4}}\right)$$

This approximation can be used to start the solution from a point a little way from the cathode to avoid a divide by zero error when U = 0

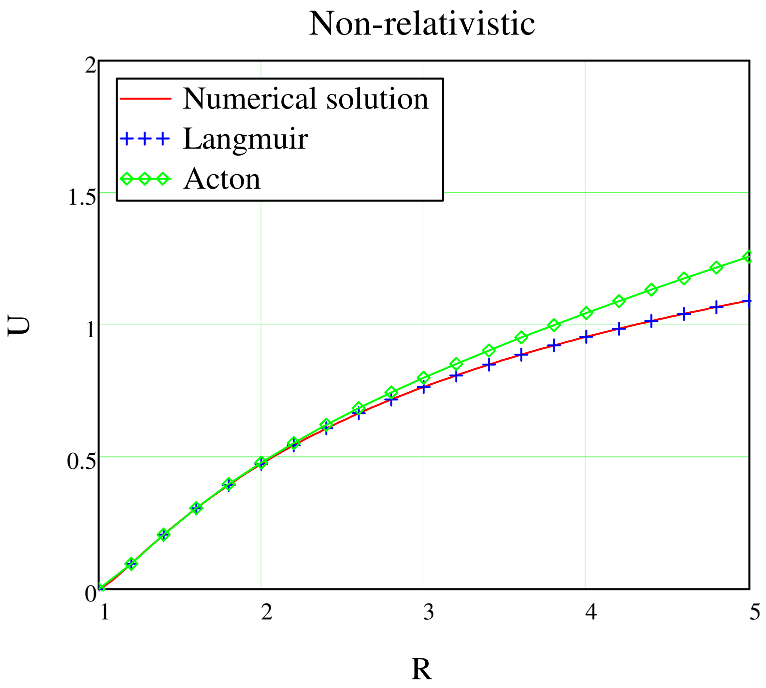
When the cathode lies inside the anode

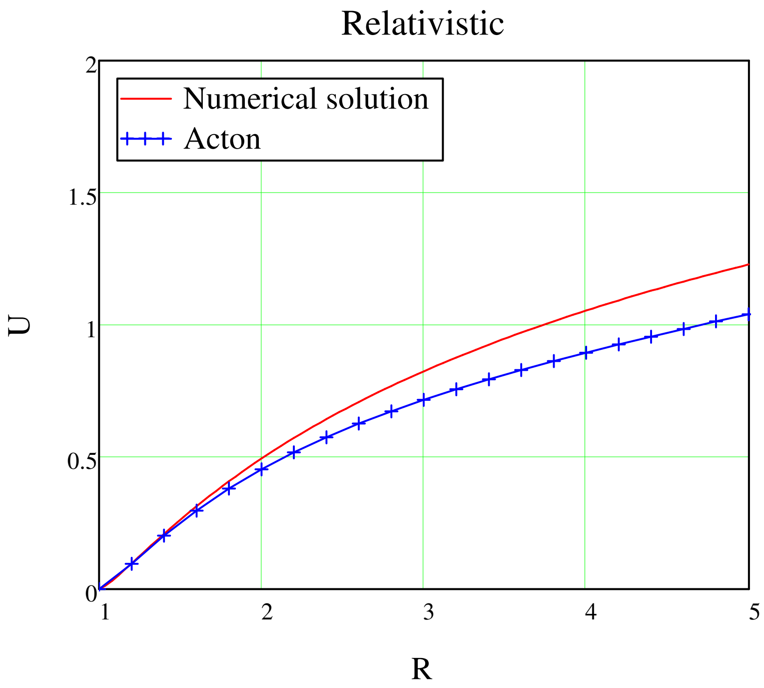
Let $U0 := 10^{-6}$ $\quad R_{max} := 5$ $\quad R2 := 1, 1.2 .. R_{max}$

$$U1(a) := \left| \begin{array}{l} k_2 \leftarrow \sqrt{\frac{32}{81}\cdot a^3} \\ R1 \leftarrow 1 + \left(\frac{1}{\sqrt{2\cdot k_2\cdot\sqrt{2}}}\cdot\frac{4}{3}\cdot U0^{\frac{3}{4}}\right) \\ D1(R,U) \leftarrow \begin{bmatrix} U_1 \\ \frac{k_2}{R^2}\cdot\left(\frac{1}{\sqrt{2\cdot U_0}}\right) - \frac{2\cdot U_1}{R} \end{bmatrix} \\ U \leftarrow \begin{pmatrix} U0 \\ \sqrt{2\cdot k_2\cdot\sqrt{2\cdot U0}} \end{pmatrix} \\ U1 \leftarrow \text{AdamsBDF}(U, R1, R_{max}, 100, D1) \\ \text{return } U1 \end{array} \right.$$

$$U1R(a) := \left| \begin{array}{l} k_2 \leftarrow \sqrt{\frac{32}{81}\cdot a^3} \\ R1 \leftarrow 1 + \left(\frac{1}{\sqrt{2\cdot k_2\cdot\sqrt{2}}}\cdot\frac{4}{3}\cdot U0^{\frac{3}{4}}\right) \\ D1R(R,U) \leftarrow \begin{bmatrix} U_1 \\ \frac{k_2}{R^2}\cdot\left[\frac{1 + U_0}{\sqrt{2\cdot U_0 + (U_0)^2}}\right] - \frac{2\cdot U_1}{R} \end{bmatrix} \\ U \leftarrow \begin{pmatrix} U0 \\ \sqrt{2\cdot k_2\cdot\sqrt{2\cdot U0}} \end{pmatrix} \\ U1R \leftarrow \text{AdamsBDF}(U, R1, R_{max}, 100, D1R) \\ \text{return } U1R \end{array} \right.$$

The results are in matrices in which the first column is the independent variable, the second column is U and the third column its differential





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When the anode lies inside the cathode

$$R3_{min} := \frac{1}{R_{max}} \qquad R3 := 1, 0.95 .. R3_{min}$$

$$U2(a) := \left| \begin{array}{l} k_2 \leftarrow \sqrt{\frac{32}{81} \cdot a^3} \\ R_3 \leftarrow 1 + \left(\frac{1}{\sqrt{2 \cdot k_2 \cdot \sqrt{2}}} \cdot \frac{4}{3} \cdot U0 \right)^{\frac{3}{4}} \\ D2(R, U) \leftarrow \begin{bmatrix} U_1 \\ \frac{k_2}{R^2} \cdot \left(\frac{1}{\sqrt{2 \cdot U_0}} \right) - \frac{2 \cdot U_1}{R} \end{bmatrix} \\ U \leftarrow \begin{pmatrix} U0 \\ -\sqrt{2 \cdot k_2 \cdot \sqrt{2 \cdot U0}} \end{pmatrix} \\ U2 \leftarrow \mathrm{AdamsBDF}(U, R_3, R3_{min}, 100, D2) \\ \text{return } U2 \end{array} \right.$$

$$U2R(a) := \left| \begin{array}{l} k_2 \leftarrow \sqrt{\frac{32}{81} \cdot a^3} \\ R_3 \leftarrow 1 + \left(\frac{1}{\sqrt{2 \cdot k_2 \cdot \sqrt{2}}} \cdot \frac{4}{3} \cdot U0 \right)^{\frac{3}{4}} \\ D2R(R, U) \leftarrow \begin{bmatrix} U_1 \\ \frac{k_2}{R^2} \cdot \left[\frac{1 + U_0}{\sqrt{2 \cdot U_0 + (U_0)^2}} \right] - \frac{2 \cdot U_1}{R} \end{bmatrix} \\ U \leftarrow \begin{pmatrix} U0 \\ -\sqrt{2 \cdot k_2 \cdot \sqrt{2 \cdot U0}} \end{pmatrix} \\ U2R \leftarrow \mathrm{AdamsBDF}(U, R_3, R3_{min}, 100, D2R) \\ \text{return } U2R \end{array} \right.$$

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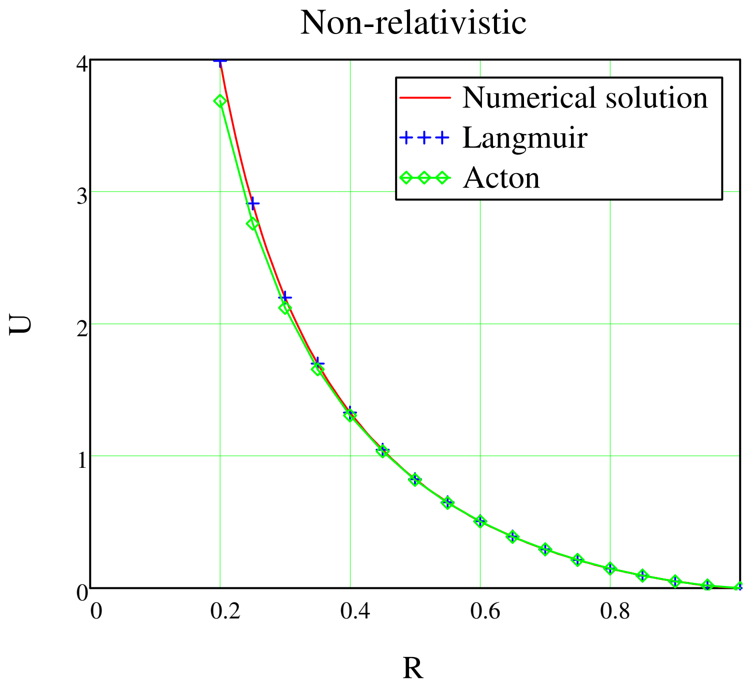
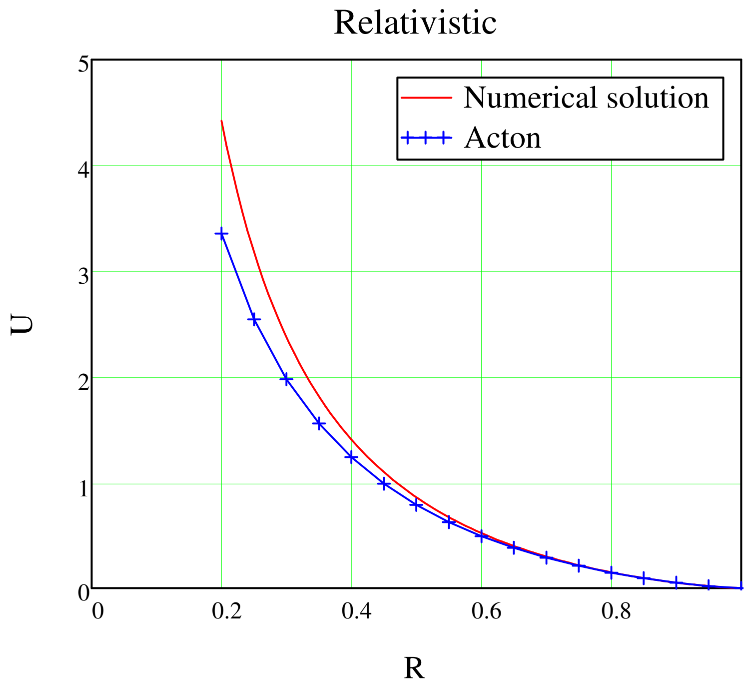

Non-relativistic
Numerical solution
Langmuir
Acton
U
R
0
0.2
0.4
0.6
0.8
1
2
3
4


Relativistic
Numerical solution
Acton
U
R
0
0.2
0.4
0.6
0.8
1
2
3
4
5

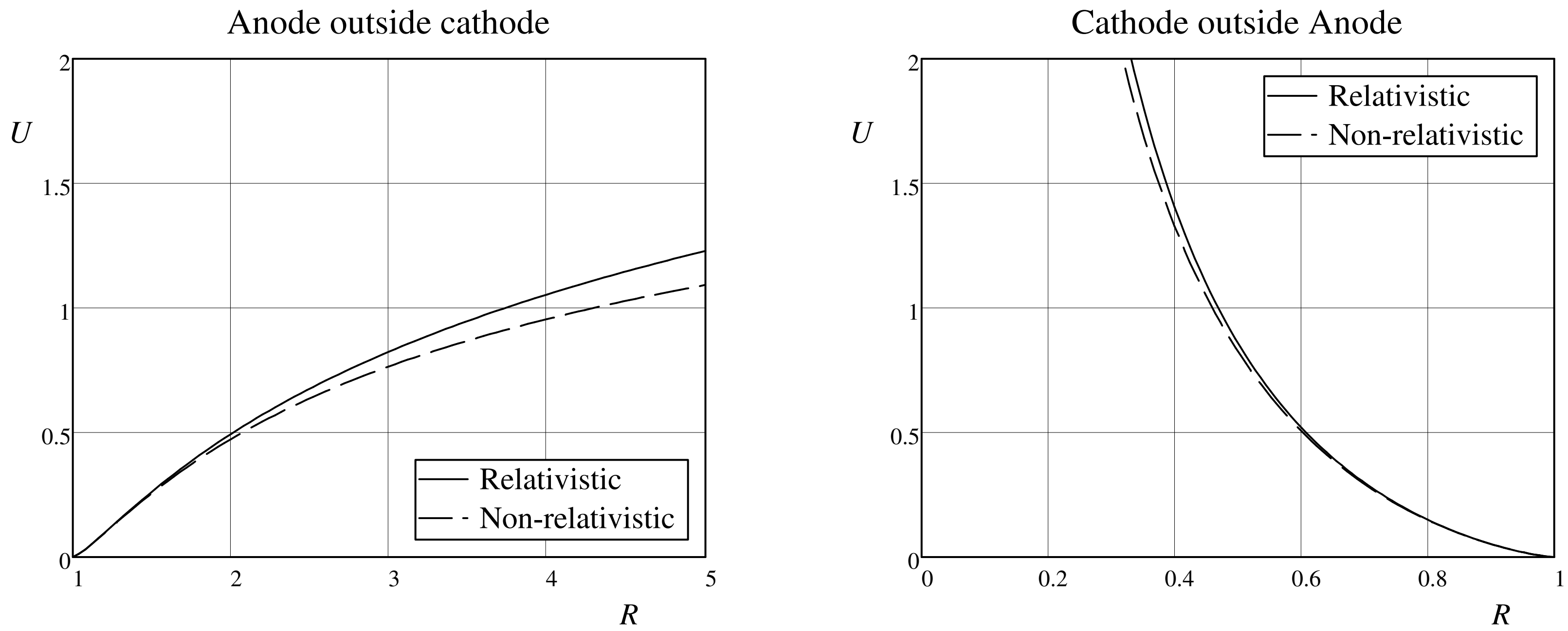


Figure 5.10(a) and (b)

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