WS 15.3 Magnetron model

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

This sheet implements the models of a magnetron oscillator discussed in Section 15.6.

Part I of this sheet uses the rigid spoke or guiding centre models of the spoke with either zero order or first order models of the hub to compute the DC current, the output power and efficiency for given values of the DC voltage, RF voltage and magnetic field. The output power is computed in two different ways which must give the same result for consistency. Taking the anode current and the magnetic field as given, self-consistent values are found for the DC and RF voltages leading to performance characteristics as given in Table 15.3.

Part II of the sheet computes and displays the performance chart, Rieke diagram and frequency pushing for the magnetron using the model selcted in Part I

Part III of the sheet uses the trajectory model to find the properties of the spokes using the DC and RF voltages determined in Part I. These can be compared with the properties of the rigid and guiding centre spoke models. Note that these results are for the specific DC and RF voltages and they do not represent self-consistent solutions for all three models. Thus the comparison is illustrative and not definitive.

Data input for 4J50 and 4J52 magnetrons as given in Collins, 'Microwave Magnetrons' pp. 780-784 and McDowell (1998)

Frequency	$f_0 := 9.375 \cdot GHz$	Cathode radius	$r_c := 0.5 \cdot 0.209 \cdot in$	Anode radius	$r_a := 0.5 \cdot 0.319 \cdot in$
Number of vanes	$N_v := 16$	Gap between vanes	$w := 0.033 \cdot in$	Anode length	$L_a := 0.25 \cdot in$
Anode admittance	$Y_0 := 0.341 \cdot S$	Unloaded Q	$Q_U := 900$	Circuit efficiency	$\eta_c := 0.7$

Specify the anode current and the magnetic field.
The values of the voltages are used as starting values for the solution and may be adjusted if necessary to improve the convergence.

Anode current	$I_0 := 27 \cdot A$	Magnetic field	$B_z := 4900 \cdot G$
Anode voltage	$V_a := 15.2 \cdot kV$	RF voltage	$V_1 := 8 \cdot kV$
Voltage reflection coefficient	$\rho := 0$	Reflection phase .	$\phi := 0$

Select hub and spoke models

0 = rigid spoke	$SPOKE := 0$	0 = zero order hub	$HUB := 0$	Trajectory model Number of electrons/vane	$nv := 24$
1 = guiding centre		1 = first order hub			

Note: Models other than a rigid spoke with a zero order hub may have convergence problems so that results can only be found for a few parameter values

The section below can be collapsed to allow data and results to be viewed on the same screen

Part I - Calculation of magnetron characteristics using the DC voltage, RF voltage and magnetic field

Define the charge to mass ratio of the electron $\eta := 1.759 \cdot 10^{11} \cdot C \cdot kg^{-1}$

Calculate the mode number of the pi mode, the cavity pitch, the vane thickness and the normalised anode radius.

$n := 0.5 \cdot N_v \qquad p := \dfrac{2 \cdot \pi \cdot r_a}{N_v} \qquad t := p - w \qquad R_a := \dfrac{r_a}{r_c}$

$n = 8 \qquad p = 1.59 \cdot mm \qquad t = 0.75 \cdot mm$

Calculate the synchronous frequency and the loaded and external Q factors

$R_a = 1.526$

$\omega_0 := 2 \cdot \pi \cdot f_0 \qquad \omega_s := \dfrac{2\omega_0}{N_v} \qquad Q_L := (1 - \eta_c) \cdot Q_U \qquad Q_E := \left(\dfrac{1}{Q_L} - \dfrac{1}{Q_U}\right)^{-1}$

$Q_L = 270 \qquad Q_E = 386$

Find the characteristic voltage and field and the threshold voltage

$$V_0 := \frac{1}{2 \cdot \eta} \cdot r_a^{\,2} \cdot \omega_s^{\,2}$$

Equation 15.22

$V_0 = 2.53 \cdot kV$

$$B_0 := \frac{2 \cdot \omega_s}{\eta} \cdot \left(1 - \frac{r_c^{\,2}}{r_a^{\,2}}\right)^{-1}$$

Equation 15.23

$B_0 = 0.147\,T$

$$V_T(B_z) := V_0 \cdot \left(\frac{2 \cdot B_z}{B_0} - 1\right)$$

Equation 15.40

$V_T(B_z) = 14.37 \cdot kV$

Find the hub radius as a function of the anode voltage and the magnetic field.

$$\mathrm{Vab}(r_b, B_z) := \frac{\eta}{4}\cdot B_z^{\,2}\cdot r_b^{\,2}\cdot\left(1-\frac{r_c^{\,4}}{r_b^{\,4}}\right)\cdot\ln\left(\frac{r_a}{r_b}\right)+\frac{\eta}{8}\cdot B_z^{\,2}\cdot r_b^{\,2}\cdot\left(1-\frac{r_c^{\,2}}{r_b^{\,2}}\right)^2$$

Equation 15.2

$$r_b(V_a, B_z) := \left| \begin{array}{l} b \leftarrow r_c \\ r_b \leftarrow \mathrm{root}(\mathrm{Vab}(b, B_z) - V_a, b) \\ \text{return } r_b \end{array} \right.$$

$$R_b(V_a, B_z) := \frac{r_b(V_a, B_z)}{r_c}$$

$r_b(V_a, B_z) = 3.06\cdot\mathrm{mm}$

$R_b(V_a, B_z) = 1.155$

Hub voltage and angular velocity. Ratio of the angular velocity to the synchronous angular velocity

$$V_b(V_a, B_z) := \frac{\eta}{8}\cdot B_z^{\,2}\cdot r_b(V_a, B_z)^2\cdot\left[1-\left(\frac{r_c}{r_b(V_a, B_z)}\right)^2\right]^2$$

Equation 15.1

$V_b(V_a, B_z) = 3.10\cdot\mathrm{kV}$

$$\omega_b(V_a, B_z) := \frac{\sqrt{2\cdot\eta\cdot V_b(V_a, B_z)}}{r_b(V_a, B_z)}$$

Equation 15.4

$\frac{\omega_b(V_a, B_z)}{\omega_s} = 1.463$

Define the DC potential and the radial DC electric field

$$V0(r, V_a, B_z) := \left| \begin{array}{ll} r_b \leftarrow r_b(V_a, B_z) & \\ V \leftarrow \frac{\eta}{8}\cdot B_z^{\,2}\cdot r^2\cdot\left[1-\left(\frac{r_c}{r}\right)^2\right]^2 \quad \text{if } r \le r_b & \text{Equation 15.1} \\ V \leftarrow \frac{\eta}{4}\cdot B_z^{\,2}\cdot r_b^{\,2}\cdot\left[1-\left(\frac{r_c}{r_b}\right)^4\right]\cdot\ln\left(\frac{r}{r_b}\right)+\frac{\eta}{8}\cdot B_z^{\,2}\cdot r_b^{\,2}\cdot\left[1-\left(\frac{r_c}{r_b}\right)^2\right]^2 \quad \text{otherwise} & \text{Equation 15.2} \\ \text{return } V & \end{array} \right.$$

$$E0\left(r, V_a, B_z\right) := \begin{array}{|l} r_b \leftarrow r_b\left(V_a, B_z\right) \\ E \leftarrow -\dfrac{\eta}{4} \cdot B_z^{\,2} \cdot r \cdot \left[1 - \left(\dfrac{r_c}{r}\right)^4\right] \ \text{ if } \ r \le r_b \\ E \leftarrow -\dfrac{\eta}{4} \cdot B_z^{\,2} \cdot \dfrac{r_b^{\,2}}{r} \cdot \left[1 - \left(\dfrac{r_c}{r_b}\right)^4\right] \ \text{ otherwise} \\ \text{return } E \end{array}$$

Equation 15.3

Define the radial and azimuthal RF electric fields in terms of the normalised radius R = r/rc

$$E_r\left(R, \theta, V_a, V_1, B_z\right) := \begin{array}{|l} \left[-\dfrac{n \cdot V_1}{R \cdot r_c} \cdot \dfrac{\left(R^n + R^{-n}\right)}{\left(R_a^{\,n} - R_a^{\,-n}\right)} \cdot \cos(n \cdot \theta) + E0\left(R \cdot r_c, V_a, B_z\right)\right] \ \text{ if } \ R \ge 1 \wedge R \le R_a \\ 0 \ \text{ otherwise} \end{array}$$

Equation 15.18

$$E_\theta\left(R, \theta, V_1\right) := \begin{array}{|l} \left[\dfrac{n \cdot V_1}{R \cdot r_c} \cdot \dfrac{\left(R^n - R^{-n}\right)}{\left(R_a^{\,n} - R_a^{\,-n}\right)} \cdot \sin(n \cdot \theta)\right] \ \text{ if } \ R \ge 1 \wedge R \le R_a \\ 0 \ \text{ otherwise} \end{array}$$

Equation 15.19

RIGID SPOKE MODEL (SPOKE = 0)

Find Welch's conduction angle, and the spoke phase in terms of the threshold voltage, the anode voltage and the r.f. wave voltage

Conduction angle

$$\theta_{c0}(V_a, V_1, B_z) := \left| \begin{array}{l} \theta \leftarrow \frac{1}{n} \cdot \mathrm{acos}\left(\frac{V_T(B_z) - V_a}{V_1}\right) \\ \theta \leftarrow \frac{\pi}{n} \ \text{ if } \mathrm{Im}(\theta) \neq 0 \end{array} \right.$$

Equation 15.105

Spoke phase = conduction angle / 2

$$\theta_{s0}(V_a, V_1, B_z) := 0.5 \cdot \theta_{c0}(V_a, V_1, B_z)$$

$$n \cdot \theta_{c0}(V_a, V_1, B_z) = 96 \cdot \mathrm{deg}$$

$$n \cdot \theta_{s0}(V_a, V_1, B_z) = 48 \cdot \mathrm{deg}$$

Note: This angle is the physical position of the spoke in the rotating co-ordinate system in which it is stationary with respect to the angle at which the wave voltage is maximum. It is assumed that electrons can only reach the anode if the local anode potential exceeds the threshold potential. The angle defined by Vaughan is the phase angle with respect to the zero of the r.f. voltage. Thus his phase is $\theta = \left(n \cdot \theta_s - \frac{\pi}{2}\right)$

Find the tangential and radial velocities at radius r and angle θ using the guiding centre approximation

$$v_\theta(R, \theta, V_a, V_1, B_z) := -\frac{E_r(R, \theta, V_a, V_1, B_z)}{B_z}$$

Equation 15.111

$$v_r(R, \theta, V_1, B_z) := \frac{E_\theta(R, \theta, V_1)}{B_z}$$

Equation 15.112

Energy of electrons striking the anode

$$V_{ia0}(V_a, V_1, B_z) := \left[\frac{1}{2 \cdot \eta} \cdot \left(v_r(R_a, \theta_{s0}(V_a, V_1, B_z), V_1, B_z)^2 + v_\theta(R_a, \theta_{s0}(V_a, V_1, B_z), V_a, V_1, B_z)^2\right)\right]$$

Equation 15.126

$$\frac{V_{ia0}(V_a, V_1, B_z)}{V_a} = 46.08 \cdot \%$$

Vaughan's estimate of the kinetic energy dissipated on the cathode divided by the anode current

$$V_{ic0}(V_a, V_1, B_z) := \frac{0.04 \cdot V_a}{\sin(n \cdot \theta_{s0}(V_a, V_1, B_z))}$$

Equation 15.127

$$\frac{V_{ic0}(V_a, V_1, B_z)}{V_a} = 5.384 \cdot \%$$

GUIDING CENTRE MODEL (SPOKE = 1)

Note that Riyopoulos uses rectangular coordinates but that circular coordinates are used here.

$$VV1(V_1, r) := V_1 \cdot \left[\left(\frac{r}{r_c}\right)^n - \left(\frac{r}{r_c}\right)^{-n}\right] \cdot \left[\left(\frac{r_a}{r_c}\right)^n - \left(\frac{r_a}{r_c}\right)^{-n}\right]^{-1}$$

Equation 15.18

Conduction angle

$$\theta_{c1}(V_a, V_1, B_z) := \left| \begin{array}{l} R_b \leftarrow \dfrac{r_b(V_a, B_z)}{r_c} \\ ff(R) \leftarrow \dfrac{v_\theta\left(R, \frac{\pi}{n}, V_a, V_1, B_z\right)}{(R \cdot r_c \cdot \omega_s)} - 1 \\ R1 \leftarrow 0.5 \cdot (R_a + R_b) \\ r_e \leftarrow r_b(V_a, B_z) \text{ on error root}(ff(R1), R1) \cdot r_c \\ f \leftarrow \dfrac{1}{VV1(V_1, r_b(V_a, B_z))} \cdot \left[\begin{array}{l} (V0(r_e, V_a, B_z) - V0(r_b(V_a, B_z), V_a, B_z) + VV1(V_1, r_e) \cdot \cos(\pi)) \ldots \\ + \dfrac{\omega_s \cdot \eta \cdot B_z}{2 \cdot \eta} \cdot \left(r_b(V_a, B_z)^2 - r_e^{\,2}\right) \end{array}\right] \\ \theta \leftarrow \dfrac{\mathrm{acos}(f)}{n} \\ \text{return } \theta \end{array} \right.$$

Equation 15.123

$$n \cdot \theta_{c1}(V_a, V_1, B_z) = 169 \cdot deg$$

Spoke phase

$$\theta_{s1}(V_a,V_1,B_z) := \left| \begin{array}{l} R_b \leftarrow \dfrac{r_b(V_a,B_z)}{r_c} \\ f \leftarrow \dfrac{1}{VV1(V_1,r_a)} \cdot \left[\begin{array}{l} V0(r_b(V_a,B_z),V_a,B_z) - V0(r_a,V_a,B_z) + VV1(V_1,r_b(V_a,B_z)) \cdot \cos(0.5n \cdot \theta_{c1}(V_a,V_1,B_z)) \ldots \\ + \dfrac{\omega_s \cdot \eta \cdot B_z}{2 \cdot \eta} \cdot \left[{r_a}^2 - (r_b(V_a,B_z))^2 \right] \end{array} \right] \\ \theta \leftarrow \dfrac{\mathrm{acos}(f)}{n} \\ \text{return } \theta \end{array} \right.$$

$$n \cdot \theta_{s1}(V_a,V_1,B_z) = 85.4 \cdot deg$$

Energy of electrons striking the anode

$$V_{ia1}(V_a,V_1,B_z) := \left[\frac{1}{2 \cdot \eta} \cdot \left(v_r(R_a,\theta_{s1}(V_a,V_1,B_z),V_1,B_z)^2 + v_\theta(R_a,\theta_{s1}(V_a,V_1,B_z),V_a,V_1,B_z)^2 \right) \right] \quad \boxed{\text{Equation 15.126}}$$

$$\frac{V_{ia1}(V_a,V_1,B_z)}{V_a} = 30.47 \cdot \%$$

Vaughan's estimate of the kinetic energy dissipated on the cathode divided by the anode current

$$V_{ic1}(V_a,V_1,B_z) := \frac{0.04 \cdot V_a}{\sin(n \cdot \theta_{s1}(V_a,V_1,B_z))} \quad \boxed{\text{Equation 15.127}}$$

$$\frac{V_{ic1}(V_a,V_1,B_z)}{V_a} = 4.013 \cdot \%$$

Anode dissipation

$$V_{ia}(V_a,V_1,B_z) := \left| \begin{array}{l} Via \leftarrow V_{ia0}(V_a,V_1,B_z) \ \text{ if } \ SPOKE = 0 \\ Via \leftarrow V_{ia1}(V_a,V_1,B_z) \ \text{ if } \ SPOKE = 1 \\ Via \leftarrow 0 \ \text{ otherwise} \end{array} \right.$$

Cathode dissipation

$$V_{ic}(V_a, V_1, B_z) := \begin{array}{|l} Vic \leftarrow V_{ic0}(V_a, V_1, B_z) \ \text{ if } \ SPOKE = 0 \\ Vic \leftarrow V_{ic1}(V_a, V_1, B_z) \ \text{ if } \ SPOKE = 1 \\ Vic \leftarrow 0 \ \text{ otherwise} \end{array}$$

Electronic efficiency

$$\eta_e(V_a, V_1, B_z) := 1 - \left(\frac{V_{ia}(V_a, V_1, B_z) + V_{ic}(V_a, V_1, B_z)}{V_a} \right)$$

$$\eta_e(V_a, V_1, B_z) = 48.5 \cdot \%$$

Conduction angle

$$\theta_c(V_a, V_1, B_z) := \begin{array}{|l} \theta c \leftarrow \theta_{c0}(V_a, V_1, B_z) \ \text{ if } \ SPOKE = 0 \\ \theta c \leftarrow \theta_{c1}(V_a, V_1, B_z) \ \text{ if } \ SPOKE = 1 \\ \theta c \leftarrow 0 \ \text{ otherwise} \end{array}$$

$$n \cdot \theta_c(V_a, V_1, B_z) = 96.0 \cdot deg$$

Spoke phase

$$\theta_s(V_a, V_1, B_z) := \begin{array}{|l} \theta s \leftarrow \theta_{s0}(V_a, V_1, B_z) \ \text{ if } \ SPOKE = 0 \\ \theta s \leftarrow \theta_{s1}(V_a, V_1, B_z) \ \text{ if } \ SPOKE = 1 \\ \theta s \leftarrow 0 \ \text{ otherwise} \end{array}$$

$$n \cdot \theta_s(V_a, V_1, B_z) = 48.0 \cdot deg$$

Returned current fraction

$$\alpha(V_a, V_1, B_z) := 1 - \frac{n \cdot \theta_c(V_a, V_1, B_z)}{2 \cdot \pi}$$

Equation 15.106

$$\alpha(V_a, V_1, B_z) = 0.733$$

Calculate the d.c. anode current for a zero order hub (HUB = 0)

The current is calculated at the base of the spoke which is taken to be on the surface of the hub.

Charge density

Equation 15.96

$$\rho_h(V_a,B_z) := \rho \leftarrow \frac{\varepsilon_0\cdot\eta}{2}\cdot B_z^{\,2}\cdot\left[1+\left(\frac{r_c}{r_b(V_a,B_z)}\right)^4\right]$$

DC anode current

Equation 15.98

$$I_{a0}(V_a,V_1,B_z,\theta_c) := \left|\begin{array}{l} R_b \leftarrow \dfrac{r_b(V_a,B_z)}{r_c} \\ I \leftarrow \dfrac{n\cdot\rho_h(V_a,B_z)\cdot L_a\cdot V_1}{B_z}\cdot\left(\dfrac{R_b^{\,n}-R_b^{\,-n}}{R_a^{\,n}-R_a^{\,-n}}\right)\cdot\left(1-\cos(n\cdot\theta_c)\right) \\ I \leftarrow 0 \ \text{ if } \ V_1 < V_T(B_z) - V_a \\ \text{return } I \end{array}\right.$$

Find the d.c. current for a first order hub (HUB = 1) based on the Child-Langmuir current and current division

Child-Langmuir current

$$\gamma(R) := \ln(R)$$

Equation 5.69

$$\beta A(R) := \gamma(R) - 0.4\cdot\gamma(R)^2 + 0.0916667\cdot\gamma(R)^3 - 0.014242\cdot\gamma(R)^4 + 0.001679\cdot\gamma(R)^5 - 0.000161\cdot\gamma(R)^6$$

Equation 5.71

$$I_{CL}(V_a,B_z) := \frac{8\cdot\pi\cdot\varepsilon_0\cdot L_a\cdot\sqrt{2\cdot\eta}}{9}\cdot\frac{V_b(V_a,B_z)^{1.5}}{r_b(V_a,B_z)\cdot\beta A\left(r_b(V_a,B_z)\cdot r_c^{\,-1}\right)^2}$$

Equation 5.70

DC anode current

$$I_{a1}(V_a,V_1,B_z,\alpha) := I\alpha \leftarrow \frac{1-\alpha}{1+\alpha}\cdot 0.35\cdot I_{CL}(V_a,B_z)$$

Equation 15.102

DC anode current for HUB and SPOKE selected

$$I_{DC}(V_a, V_1, B_z) := \begin{array}{|l} I_{dc} \leftarrow I_{a0}(V_a, V_1, B_z, \theta_c(V_a, V_1, B_z)) \ \text{ if } \ HUB = 0 \\ I_{dc} \leftarrow I_{a1}(V_a, V_1, B_z, \alpha(V_a, V_1, B_z)) \ \text{ if } \ HUB = 1 \\ I_{dc} \leftarrow 0 \ \text{ otherwise} \end{array}$$

$$I_{DC}(V_a, V_1, B_z) = 25.8\ A$$

Calculate the RF power delivered to the anode

Using the efficiency calculated from the power dissipated

$$P_{RF}(V_a, V_1, B_z) := V_a \cdot I_{DC}(V_a, V_1, B_z) \cdot \eta_e(V_a, V_1, B_z)$$

$$P_{RF}(V_a, V_1, B_z) = 190.6 \cdot kW$$

Calculate the r.f. voltage between the vanes (V_g) in terms of the amplitude of the synchronous voltage wave on the anode.

$$V_g(V_1) := \pi \cdot V_1 \cdot \left(\mathrm{sinc}\left(\frac{\pi \cdot w}{2 \cdot p}\right)\right)^{-1}$$

$$\frac{V_g(V_1)}{V_a} = 1.858$$

Power delivered to the load $\quad P_E(V_1, \rho, \phi) := \frac{1}{2} \cdot V_g(V_1)^2 \cdot \frac{Y_0}{Q_E} \cdot \mathrm{Re}\left(\frac{1 + \rho \cdot \exp(j \cdot \phi)}{1 - \rho \cdot \exp(j \cdot \phi)}\right) \qquad G_E := \frac{Y_0}{Q_E}$

Power dissipated in the anode $\quad P_U(V_1) := \frac{1}{2} \cdot V_g(V_1)^2 \cdot \frac{Y_0}{Q_U} \qquad G_U := \frac{Y_0}{Q_U}$

Total power delivered to the anode $\quad P_a(V_1, \rho, \phi) := P_E(V_1, \rho, \phi) + P_U(V_1)$

$$P_a(V_1, 0, 0) = 503.9 \cdot kW$$

Note: P_a is not equal to P_{RF}. For each combination of hub and spoke models it is necessary to find values of V_a and V_1 that give a self-consistent solution.

Calculate the frequency pulling

$$\Delta f(\rho,\phi) := \frac{f_0}{2 \cdot Q_E} \cdot \mathrm{Im}\left(\frac{1 + \rho \cdot \exp(j \cdot \phi)}{1 - \rho \cdot \exp(j \cdot \phi)}\right)$$

$$f_p := |2 \cdot \Delta f(0.2, 2 \cdot 0.588)| = 10.1 \cdot \mathrm{MHz}$$

Compare equation 15.93

$$0.417 \cdot \frac{f_0}{Q_E} = 10.1 \cdot \mathrm{MHz}$$

Find self-consistent values of the parameters for a given anode current and the hub and spoke models selected. The results can be confirmed by inserting the values of the anode voltage and the RF voltage computed into the data fields at the top of the sheet.

Given $\quad I_0 = I_{DC}(V_a, V_1, B_z) \quad V_1 > V_T(B_z) - V_a \quad P_a(V_1, \rho, \phi) = P_{RF}(V_a, V_1, B_z)$

$$X(I_0, B_z, \rho, \phi) := \mathrm{Find}(V_a, V_1)$$

$$Va(I_0, B_z, \rho, \phi) := X(I_0, B_z, \rho, \phi)_0 \qquad Va(I_0, B_z, \rho, \phi) = 16.51 \cdot \mathrm{kV}$$

$$V1(I_0, B_z, \rho, \phi) := X(I_0, B_z, \rho, \phi)_1 \qquad V1(I_0, B_z, \rho, \phi) = 5.99 \cdot \mathrm{kV}$$

CHECK the calculations for the given anode current

$$I_{DC}(Va(I_0, B_z, 0, 0), V1(I_0, B_z, 0, 0), B_z) = 27\,\mathrm{A}$$

CHECK self-consistency of the result

$$\frac{P_a(V1(I_0, B_z, \rho, \phi), \rho, \phi)}{P_{RF}(Va(I_0, B_z, \rho, \phi), V1(I_0, B_z, \rho, \phi), B_z)} = 1.000$$

Redefine DC and RF anode voltages

$$V_a := Va(I_0, B_z, \rho, \phi) \qquad V_1 := V1(I_0, B_z, \rho, \phi)$$

Results (see Table 15.3)

Characteristic voltage	$V_0 = 2.53 \cdot kV$	Characteristic field	$B_0 = 0.147\,T$	Threshold voltage	$V_T(B_z) = 14.37 \cdot kV$
Cathode radius	$r_c = 2.65 \cdot mm$	Hub radius	$r_b(V_a, B_z) = 3.11 \cdot mm$	Anode radius	$r_a = 4.05 \cdot mm$
Conduction angle	$n \cdot \theta_c(V_a, V_1, B_z) = 111 \cdot deg$	Spoke phase	$n \cdot \theta_s(V_a, V_1, B_z) = 55 \cdot deg$	Electronic efficiency	$\eta_e(V_a, V_1, B_z) = 63.5 \cdot \%$

$HUB = 0$ $SPOKE = 0$

Anode voltage	$V_a = 16.51 \cdot kV$	Anode dissipation	$\dfrac{V_{ia}(V_a, V_1, B_z)}{V_a} = 32 \cdot \%$
V_a / V_T	$\dfrac{V_a}{V_T(B_z)} = 1.15$	Cathode dissipation	$\dfrac{V_{ic}(V_a, V_1, B_z)}{V_a} = 5 \cdot \%$
Anode current	$I_0 = 27\,A$	Circuit loss	$(1 - \eta_c) \cdot \eta_e(V_a, V_1, B_z) = 19\,\%$
RF power	$P_E(V_1, \rho, \phi) = 198 \cdot kW$	Returned current fraction	$\alpha(V_a, V_1, B_z) = 0.69$
RF efficiency	$\eta_c \cdot \eta_e(V_a, V_1, B_z) = 44 \cdot \%$	RF voltage/Anode voltage	$\dfrac{V_1}{V_a} = 0.36$
DC input power	$V_a \cdot I_0 = 446\,kW$		

Part II - Performance graphs

Plot the Gauss lines

$$VG := \left| \begin{array}{l} \text{for } m \in 4..7 \\ \quad \left| \begin{array}{l} BG_m \leftarrow m \cdot 1000 \cdot G \\ \text{for } n \in 2,4..32 \\ \quad VG_{m,n} \leftarrow \dfrac{Va\left(n \cdot A, BG_m, 0, 0\right)}{kV} \end{array} \right. \\ VG \end{array} \right.$$

$nG := 2,4..32$

Solve block to find the lines of constant power

Note: IV is the guessed value of the current. The solve block won't accept mixed units so voltage VI has been used as proxy for I_0

$IV := 20 \cdot A \cdot \Omega$

Given

$$P_L = P_E\left(V1\left(\frac{IV}{\Omega}, B_z, \rho, \phi\right), \rho, \phi\right) \qquad V_a = Va\left(\frac{IV}{\Omega}, B_z, \rho, \phi\right) \qquad XX\left(P_L, B_z, \rho, \phi\right) := \mathrm{Find}\left(IV, V_a\right)$$

$$X1 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad \left| \begin{array}{l} B_i \leftarrow 0.4 \cdot T + 0.05 \cdot i \cdot T \\ X_i \leftarrow XX\left(50 \cdot kW, B_i, 0, 0\right) \end{array} \right. \\ \text{return } X \end{array} \right. \qquad I01 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad I_i \leftarrow \left(X1_i\right)_0 \cdot \Omega^{-1} \\ \text{return } I \end{array} \right. \qquad V01 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad V_i \leftarrow \left(X1_i\right)_1 \cdot kV^{-1} \\ \text{return } V \end{array} \right.$$

$$X2 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad \left| \begin{array}{l} B_i \leftarrow 0.4 \cdot T + 0.05 \cdot i \cdot T \\ X_i \leftarrow XX\left(150 \cdot kW, B_i, 0, 0\right) \end{array} \right. \\ \text{return } X \end{array} \right. \qquad I02 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad I_i \leftarrow \left(X2_i\right)_0 \cdot \Omega^{-1} \\ \text{return } I \end{array} \right. \qquad V02 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad V_i \leftarrow \left(X2_i\right)_1 \cdot kV^{-1} \\ \text{return } V \end{array} \right.$$

$$X3 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad \left| \begin{array}{l} B_i \leftarrow 0.4 \cdot T + 0.05 \cdot i \cdot T \\ X_i \leftarrow XX\left(250 \cdot kW, B_i, 0, 0\right) \end{array} \right. \\ \text{return } X \end{array} \right. \qquad I03 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad I_i \leftarrow \left(X3_i\right)_0 \cdot \Omega^{-1} \\ \text{return } I \end{array} \right. \qquad V03 := \left| \begin{array}{l} \text{for } i \in 0..6 \\ \quad V_i \leftarrow \left(X3_i\right)_1 \cdot kV^{-1} \\ \text{return } V \end{array} \right.$$

Find the power and frequency contours on the Rieke diagram

$$\rho\rho(PL,\phi 1) := \left| \begin{array}{l} \rho \leftarrow -1 \\ f(\rho) \leftarrow P_E(V1(Iplot, Bplot, \rho, \phi 1), \rho, \phi 1)\cdot PL^{-1} - 1 \\ \rho\rho \leftarrow root(f(\rho), \rho) \\ \text{return } \rho\rho \end{array} \right.$$

$$\Delta\rho(\Delta F,\phi 1) := \left| \begin{array}{l} \rho \leftarrow 0 \\ \rho\rho \leftarrow root\left(\Delta f(\rho, \phi 1)\cdot \Delta F^{-1} - 1, \rho\right) \\ \text{return } \rho\rho \end{array} \right.$$

Calculate frequency pushing

$$\Delta fp1\left(I_0, B_z\right) := \left| \begin{array}{l} V_a \leftarrow Va\left(I_0, B_z, 0, 0\right) \\ V_1 \leftarrow V1\left(I_0, B_z, 0, 0\right) \\ I_{1r} \leftarrow 2I_0\cdot\dfrac{V_1}{V_g\left(V_1\right)} \\ \phi_s \leftarrow n\cdot\theta_s\left(V_a, V_1, B_z\right) - \dfrac{\pi}{2} \\ I_{1\theta} \leftarrow \dfrac{V_g\left(V_1\right)\cdot\left(G_U + G_E\right) - I_{1r}\cdot\sin\left(\phi_s\right)}{\cos\left(\phi_s\right)} \\ \tan\phi \leftarrow \dfrac{\left(I_{1\theta}\cdot\sin\left(\phi_s\right) - I_{1r}\cdot\cos\left(\phi_s\right)\right)}{V_g\left(V_1\right)\cdot\left(G_U + G_E\right)} \\ \Delta f \leftarrow \dfrac{f_0\cdot\tan\phi}{2\cdot Q_L\cdot MHz} \\ \text{return } \Delta f \end{array} \right.$$

Calculate the efficiency

$$\eta G := \left| \begin{array}{l} \text{for } m \in 4..7 \\ \quad \left| \begin{array}{l} BG_m \leftarrow m\cdot 1000\cdot G \\ \text{for } n \in 2, 4..32 \\ \quad \eta G_{m,n} \leftarrow \eta_c\cdot\eta_e\left(Va\left(n\cdot A, BG_m, 0, 0\right), V1\left(n\cdot A, BG_m, 0, 0\right), BG_m\right) \end{array} \right. \\ \eta G \end{array} \right.$$

$Iplot \equiv 8.674 \cdot A$ $Bplot \equiv 0.63 \cdot T$ $\phi 1 := -0.7 \cdot \pi, -0.65 \cdot \pi .. 0.7\pi$

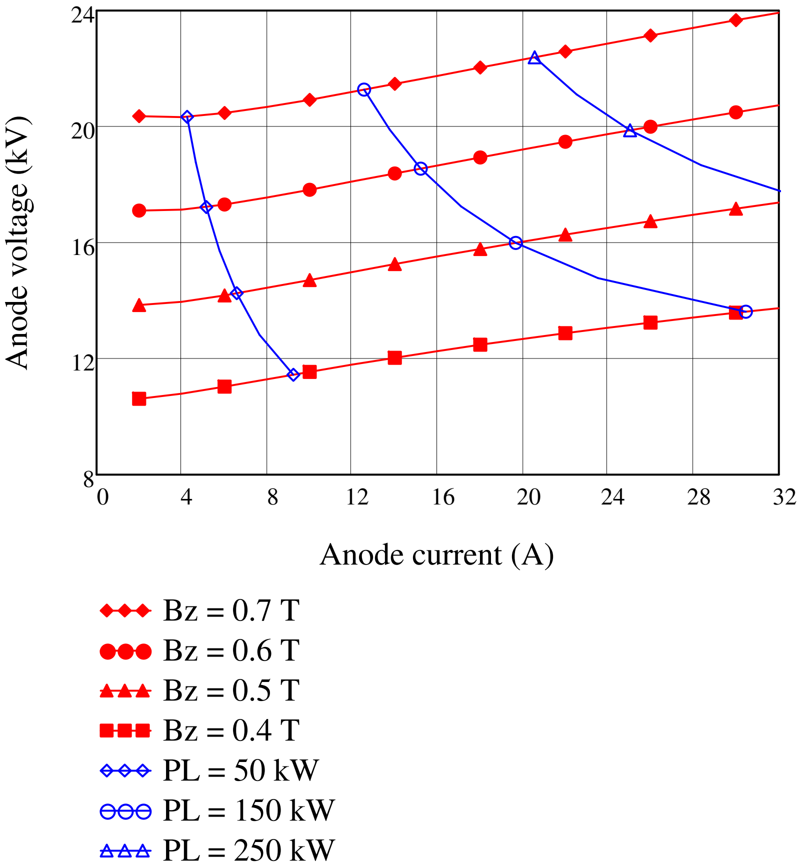


Figure 15.37

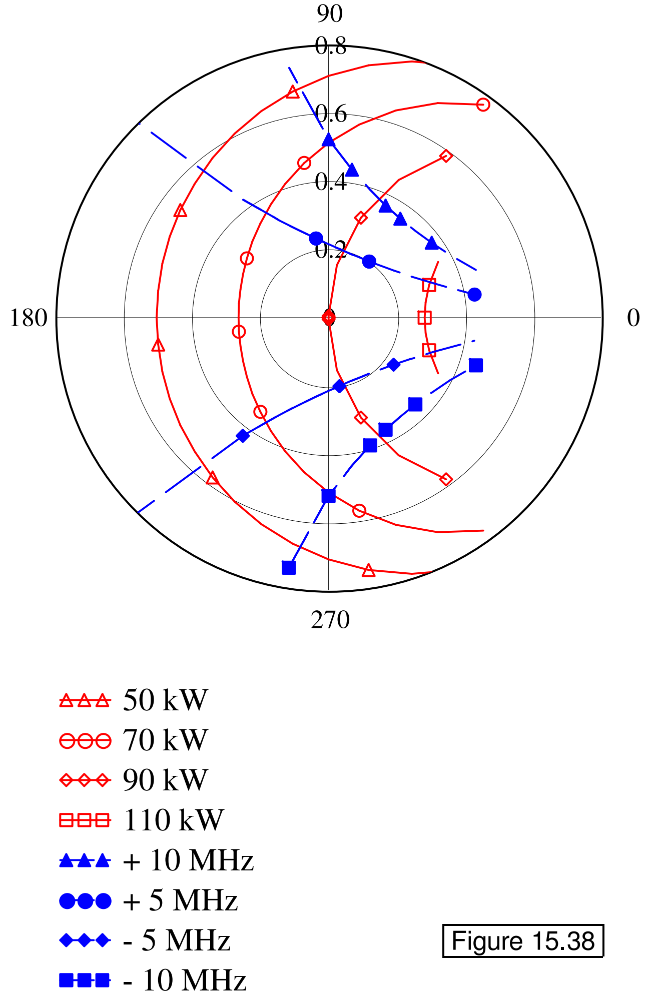
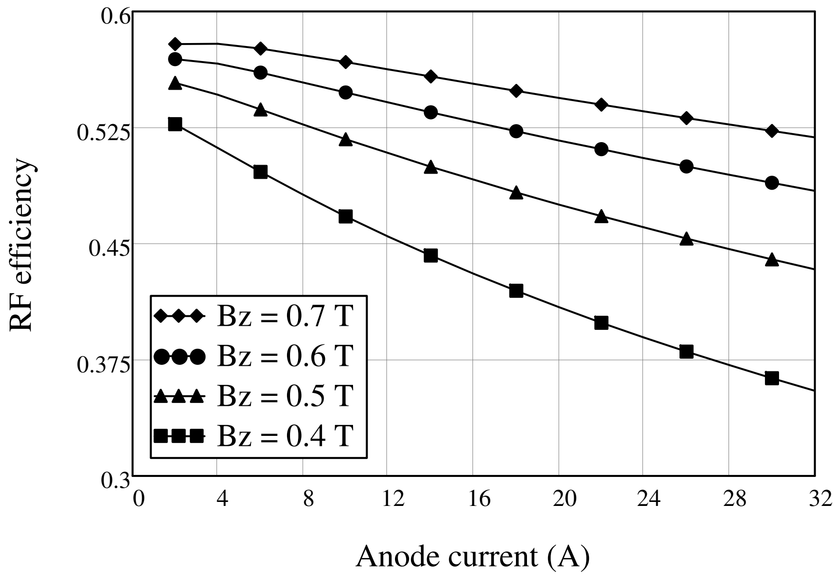
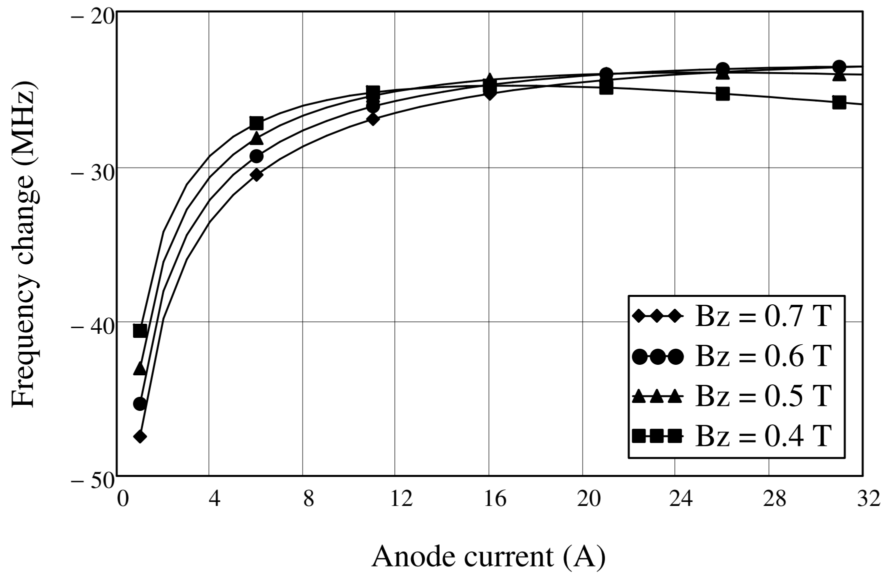


Figure 15.38

$II_0 := 1 \cdot A, 2 \cdot A .. 32 \cdot A$





Note: These curves show that the efficiency increases with magnetic field and decreases with increasing current.

However, the maximum of efficiency shown in the experimental performance chart is not reproduced correctly.

Figure 15.40

Part III - Comparison of spoke models

Guiding centre trajectories

$$rr_b := r_b(V_a, B_z)$$

$$f(r,\theta_0) := \frac{1}{VV1(V_1, r)} \cdot \left[V0(r_b(V_a, B_z), V_a, B_z) - V0(r, V_a, B_z) \ldots + VV1(V_1, rr_b) \cdot \cos(n \cdot \theta_0) + \frac{\omega_s \cdot \eta \cdot B_z}{2 \cdot \eta} \cdot \left(r^2 - rr_b^{\,2}\right) \right]$$

Equation 15.123

Calculate θ' as a function of r for a range of initial angles including the conduction angle

$$\theta0(r) := \frac{1}{deg} \cdot acos(f(r,0)) \qquad \theta1(r) := \frac{1}{deg} \cdot acos\left(f\left(r, \frac{\pi}{6n}\right)\right) \qquad \theta2(r) := \frac{1}{deg} \cdot acos\left(f\left(r, \frac{\pi}{3n}\right)\right) \qquad \theta3(r) := \frac{1}{deg} \cdot acos\left(f\left(r, \frac{\pi}{2n}\right)\right)$$

$$\theta4(r) := \frac{1}{deg} \cdot acos\left(f\left(r, \frac{2\pi}{3n}\right)\right) \qquad \theta5(r) := \frac{1}{deg} \cdot acos\left(f\left(r, \frac{5\pi}{6n}\right)\right) \qquad \theta6(r) := \frac{1}{deg} \cdot acos\left(f\left(r, \frac{\pi}{n}\right)\right) \qquad \theta7(r,\theta_c) := \frac{1}{deg} \cdot acos(f(r,\theta_c))$$

TRAJECTORY MODEL

Derivations of the working equations in a frame of reference rotating with angular velocity ω_s. Note that the spoke space-charge is ignored

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta'}{dt}\right)^2 - 2r\omega_s\frac{d\theta'}{dt} - r\omega_s^2 = -\frac{e}{m_0}E_r - r\frac{d\theta'}{dt}\frac{e}{m_0}B_z - r\omega_s\frac{e}{m_0}B_z$$

$$\frac{d^2(r/r_c)}{d(\omega_s t)^2} = \frac{r}{r_c}\left(\frac{d\theta'}{d(\omega_s t)}\right)^2 + 2\frac{r}{r_c}\frac{d\theta'}{d(\omega_s t)} + \frac{r}{r_c} - \frac{e}{m_0\omega_s^2 r_c}E_r - \frac{r}{r_c}\frac{\omega_c}{\omega_s}\frac{d\theta'}{d(\omega_s t)} - \frac{r}{r_c}\frac{\omega_c}{\omega_s}$$

$$\frac{dR}{d\tau} = \dot{R} \quad where \quad R = r/r_c \quad and \quad \tau = \omega_s t$$

$$\frac{d\dot{R}}{d\tau} = R\left\{\dot{\theta}'^2 + \left(2 - \frac{\omega_c}{\omega_s}\right)\dot{\theta}' + 1 - \frac{\omega_c}{\omega_s}\right\} - \frac{e}{m_0\omega_s^2 r_c}E_r$$

$$= R\left(\dot{\theta}' + 1\right)\left\{\dot{\theta}' + 1 - \frac{\omega_c}{\omega_s}\right\} - \frac{e}{m_0\omega_s^2 r_c}E_r$$

$$\frac{d}{dt}\left(r^2\frac{d\theta'}{dt} + r^2\omega_s\right) = -\frac{e}{m_0}rE_\theta + r\omega_c\frac{dr}{dt}$$

$$2r\frac{dr}{dt}\frac{d\theta'}{dt} + r^2\frac{d^2\theta'}{dt^2} + 2r\omega_s\frac{dr}{dt} = -\frac{e}{m_0}rE_\theta + r\omega_c\frac{dr}{dt}$$

$$\frac{d^2\theta'}{d(\omega_s t)^2} + = -\frac{1}{r}\left\{\frac{e}{m_0\omega_s^2}E_\theta + \left(2 - \frac{\omega_c}{\omega_s}\right)\frac{dr}{d(\omega_s t)} + 2\frac{dr}{d(\omega_s t)}\frac{d\theta'}{d(\omega_s t)}\right\}$$

$$\frac{d\theta'}{d\tau} = \dot{\theta}'$$

$$\frac{d\dot{\theta}'}{d\tau} = -\frac{e}{m_0\omega_s^2 R\, r_c}E_\theta - \frac{1}{R}\left(2 + 2\dot{\theta} - \frac{\omega_c}{\omega_s}\right)\frac{dR}{d\tau}$$

Define constants $\omega c\_\omega s := \dfrac{\eta \cdot B_z}{\omega_s}$ $\quad \omega b := \omega_b(V_a, B_z)$ $\quad RR_b := \dfrac{r_b(V_a, B_z)}{r_c}$

Number of electrons $n_e := nv \cdot N_v$

$\omega c\_\omega s = 11.7$

$\dfrac{\omega b}{\omega_s} = 1.597$

$RR_b = 1.173$

Initial conditions

$$R1 := \left| \begin{array}{l} \text{for } i \in 0, 4 .. 4 \cdot (n_e - 1) \\ \quad \left| \begin{array}{l} R_i \leftarrow RR_b \\ R_{i+1} \leftarrow 0 \\ R_{i+2} \leftarrow \left(\dfrac{i}{2 \cdot n_e} - 1 \right) \cdot \pi \\ R_{i+3} \leftarrow \dfrac{\omega b}{\omega_s} - 1 \end{array} \right. \\ \text{return } R \end{array} \right.$$

Definitions of the variables

$$R_0 = \frac{r}{r_c}$$

$$R_1 = \frac{d}{d\tau} R_0$$

$$R_2 = \theta$$

$$R_3 = \frac{d}{d\tau} \theta$$

Set up the differential equations of motion for the electrons

$$D(\tau, R) := \left| \begin{array}{l} \text{for } n \in 0, 4 .. 4(n_e - 1) \\ \quad \left| \begin{array}{l} D_n \leftarrow R_{n+1} \\ D_{n+1} \leftarrow R_n \cdot (R_{n+3} + 1) \cdot (R_{n+3} + 1 - \omega c\_\omega s) - \dfrac{\eta}{\omega_s^{\,2} \cdot r_c} \cdot E_r(R_n, R_{n+2}, V_a, V_1, B_z) \\ D_{n+2} \leftarrow R_{n+3} \\ D_{n+3} \leftarrow \dfrac{-1}{R_n} \cdot \dfrac{\eta}{\omega_s^{\,2} \cdot r_c} \cdot E_\theta(R_n, R_{n+2}, V_1) - \dfrac{1}{R_n} \cdot (2 + 2 \cdot R_{n+3} - \omega c\_\omega s) \cdot R_{n+1} \end{array} \right. \\ \text{return } D \end{array} \right.$$

Number of data points $nmax := 200$

Solve the differential equations $Y := \mathrm{rkfixed}(R1, 0, \tau max, nmax, D)$

The first column of the solution matrix Y is the normalised time. The following columns in groups of 4 are the radius, radial velocity, angle and angular velocity of the electrons being tracked.

$$
YY := \left|
\begin{array}{l}
\text{for } n \in 0,4..4\cdot(n_e - 1) \\
\quad \left|
\begin{array}{l}
intc \leftarrow 0 \\
inta \leftarrow 0 \\
\text{for } i \in 1..nmax \\
\quad \left|
\begin{array}{l}
ii \leftarrow i \\
\text{break if } Y_{i,n+1} \le 1 \lor Y_{i,n+1} \ge 0.99R_a
\end{array}
\right. \\
\text{for } i \in ii..nmax \\
\quad \left|
\begin{array}{l}
Y_{i,n+1} \leftarrow 1 \text{ if } Y_{ii,n+1} \le 1 \\
Y_{i,n+1} \leftarrow R_a \text{ if } Y_{ii,n+1} \ge 0.99R_a \\
Y_{i,n+2} \leftarrow Y_{ii,n+2} \\
Y_{i,n+3} \leftarrow Y_{ii,n+3} \\
Y_{i,n+4} \leftarrow Y_{ii,n+4}
\end{array}
\right.
\end{array}
\right. \\
\text{return } Y
\end{array}
\right.
$$

The matrix YY is derived from the matrix Y. When a trajectory is intercepted by either the cathode or the anode all subsequent values of Y are set to the values at the point of interception

Unpack the solution matrix to give matrices of the variables at each time step

$$
R_e := \left|
\begin{array}{l}
\text{for } n \in 0,4..4\cdot(n_e - 1) \\
\quad \left|
\begin{array}{l}
nn \leftarrow \frac{n}{4} \\
\text{for } i \in 0..nmax \\
\quad R_{i,nn} \leftarrow YY_{i,n+1}
\end{array}
\right. \\
\text{return } R
\end{array}
\right.
\qquad
dR := \left|
\begin{array}{l}
\text{for } n \in 0,4..4\cdot(n_e - 1) \\
\quad \left|
\begin{array}{l}
nn \leftarrow \frac{n}{4} \\
\text{for } i \in 0..nmax \\
\quad dR_{i,nn} \leftarrow YY_{i,n+2}
\end{array}
\right. \\
\text{return } dR
\end{array}
\right.
$$

$$
\theta_e := \left|
\begin{array}{l}
\text{for } n \in 0,4..4\cdot(n_e - 1) \\
\quad \left|
\begin{array}{l}
nn \leftarrow \frac{n}{4} \\
\text{for } i \in 0..nmax \\
\quad \theta_{i,nn} \leftarrow YY_{i,n+3}
\end{array}
\right. \\
\text{return } \theta
\end{array}
\right.
\qquad
d\theta := \left|
\begin{array}{l}
\text{for } n \in 0,4..4\cdot(n_e - 1) \\
\quad \left|
\begin{array}{l}
nn \leftarrow \frac{n}{4} \\
\text{for } i \in 0..nmax \\
\quad d\theta_{i,nn} \leftarrow YY_{i,n+4}
\end{array}
\right. \\
\text{return } d\theta
\end{array}
\right.
$$

Note: Occasionally an error "Not enough memory" occurs here. If that happens set the cursor on the variable identified and press F9.

Find the fraction of trajectories intercepted on the anode and the cathode. Check that $F_a + F_c = 1$ (see below)

$$F_a := \left| \begin{array}{l} n_a \leftarrow 0 \\ \text{for } n \in 0..n_e - 1 \\ \quad n_a \leftarrow n_a + 1 \text{ if } R_{e_{nmax,n}} = R_a \\ \text{return } \dfrac{n_a}{n_e} \end{array} \right.$$

$$F_c := \left| \begin{array}{l} n_a \leftarrow 0 \\ \text{for } n \in 0..n_e - 1 \\ \quad n_a \leftarrow n_a + 1 \text{ if } R_{e_{nmax,n}} = 1 \\ \text{return } \dfrac{n_a}{n_e} \end{array} \right.$$

$F_a = 0.479$ $F_c = 0.521$

$F_a + F_c = 1$

Conduction angle $\theta_{c2} := \frac{\pi}{n} \cdot F_a$ $n \cdot \theta_{c2} = 86.3 \cdot deg$

Alternative definition which assumes that only electrons whose initial radial velocity is negative must return to the cathode

$$F1_a := \left| \begin{array}{l} n_a \leftarrow 0 \\ \text{for } n \in 0..n_e - 1 \\ \quad n_a \leftarrow n_a + 1 \text{ if } R_{e_{nmax,n}} = R_a \wedge dR_{1,n} \geq 0 \\ \text{return } \dfrac{n_a}{n_e} \end{array} \right.$$

$F1_c := 1 - F1_a$ $F1_a = 0.438$ $F1_c = 0.563$

Conduction angle $\theta 1_{c2} := \frac{\pi}{n} \cdot F1_a$ $n \cdot \theta 1_{c2} = 78.8 \cdot deg$

Calculate the total impact energies of the electrons striking the anode and the cathode divided by the number of electrons striking the anode. This procedure means that the power dissipated on the anode and the cathode is obtained by multiplying the energies in electron volts by the dc anode current. Note: This model may over-estimate the number of trajectories reaching the anode because they include some which start in the accelerating phase and would be returned to the cathode if a first order hub were assumed.

$$V_{ia2} := \left| \begin{array}{l} V_i \leftarrow 0 \\ \text{for } n \in 0..n_e - 1 \\ \quad V_i \leftarrow V_i + \dfrac{1}{2 \cdot \eta} \cdot \left[\left(r_c \cdot \omega_s \cdot dR_{nmax,n} \right)^2 + \left[r_a \cdot \omega_s \cdot \left(d\theta_{nmax,n} + 1 \right) \right]^2 \right] \text{ if } R_{e_{nmax,n}} = R_a \\ \text{return } \dfrac{V_i}{n_e \cdot F_a} \end{array} \right.$$

$$V_{ic2} := \begin{array}{|l} V_i \leftarrow 0 \\ \text{for } n \in 0 .. n_e - 1 \\ \quad V_i \leftarrow V_i + \frac{1}{2\cdot\eta}\cdot\left[\left(r_c\cdot\omega_s\cdot dR_{nmax,n}\right)^2 + \left[r_c\cdot\omega_s\cdot\left(d\theta_{nmax,n} + 1\right)\right]^2\right] \text{ if } R_{e_{nmax,n}} = 1 \\ \text{return } \frac{V_i}{n_e\cdot F_a} \end{array}$$

$$\frac{V_{ia2}}{V_a} = 18\cdot\%$$

$$\frac{V_{ic2}}{V_a} = 8\cdot\%$$

Define quantities for plotting

$$VA1(\theta) := \left(1 + \frac{V_1}{V_a}\cdot\cos(n\cdot\theta)\right)\cdot 2$$

$$VT := \frac{V_T(B_z)}{V_a}\cdot 2$$

$$\theta := -\pi, -0.99\cdot\pi .. \pi$$

$$\theta77(r) := \theta7\left(r, \theta_{c1}\left(V_a, V_1, B_z\right)\right)$$

Rigid spoke coordinates

$$\phi s := \begin{pmatrix} 0 \\ n\cdot\theta_{s0}\left(V_a, V_1, B_z\right)\cdot deg^{-1} \\ n\cdot\theta_{c0}\left(V_a, V_1, B_z\right)\cdot deg^{-1} \end{pmatrix}$$

$$Rs := \begin{pmatrix} 1 \\ \frac{r_a}{rr_b} \\ 1 \end{pmatrix}$$

$$\theta s := n\cdot\theta_s\left(V_a, V_1, B_z\right)$$

$$RR(r) := \frac{r}{rr_b}$$

Fractions of trajectories landing on the anode and the cathode

$F_a = 0.479$ $F_c = 0.521$ $F_a + F_c = 1$

The solution must be continued until the sum of F_a and F_c is 1. If this is not so then the computation time τmax must be increased until all the electrons have landed on an electrode. It is not always possible to find a solution at low currents.

$\tau max \equiv 5$

Plot the trajectories of the electrons, the Brillouin hub, the spoke phase, the threshold voltage and the sum of the dc and rf anode voltages. Note that the voltages are normalised to a dc anode voltage of 2 for clarity.

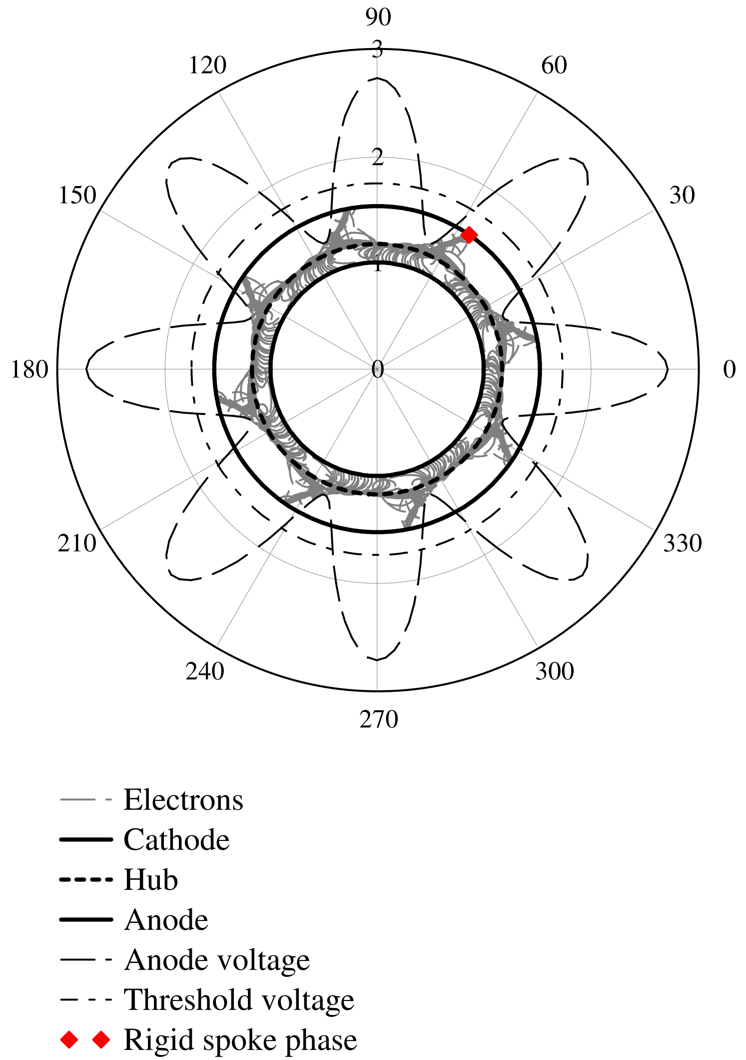


Figure 15.35

Plot the trajectories and the spoke boundaries based on the guiding centre and the rigid spoke approximations. Note that this plot is in Cartesian coordinates for convenience so that the direction of motion of the spokes is reversed compared with the polar plot.

$r := rr_b, 1.01 \cdot rr_b .. r_a$

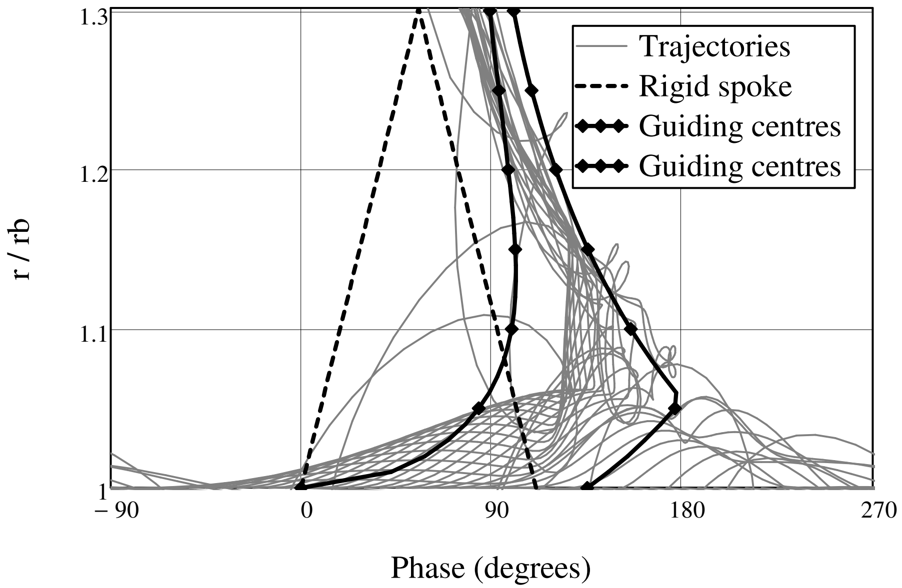


Figure 15.36

Plot the guiding centre trajectories

$r := r_c, 1.0002 \cdot r_c .. r_a$

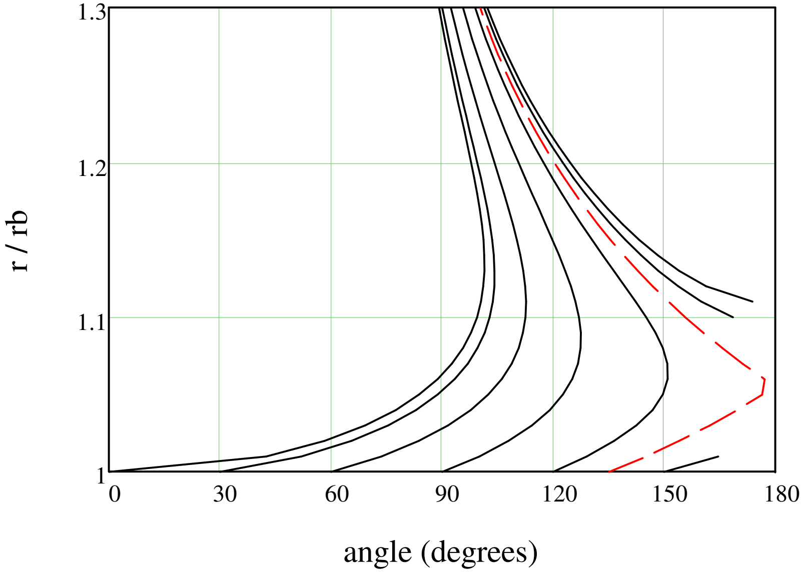


Figure 15.34