WS 13.4 Large Signal Model of Klystron Bunching

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

Disk model

The bunch is represented by a set of rigid disks of equal dimensions whose charges are equal. The motion of ND disks of charge is tracked with time as the independent variable using Runge-Kutta integration. Dimensionless variables $\theta = \beta_e \cdot z$ and $\phi = \omega \cdot t$ are used. The results of the calculation are transferred into the space domain at nmax equally spaced reference planes. Good results are normally obtained with: ND = 24, nmax = 100. The initial position of the bunch should be such that no electrons have entered the field of the first gap. The final time should be great enough to ensure that all the electrons have moved clear of the output gap.

The space-charge calculation is based on the field of a disc of charge in a conducting tunnel found using the quasi-static approximation. This works well for low beam voltages. For high beam voltages the space-charge is reduced by the factor SCF which is imported from WS 13.1. If SCF = 0 the model is run without space-charge. For simplicity the space-charge force is assumed to be periodic in space. It is automatically zero until the beam reaches $\theta = 0$. The fields acting on the discs are averaged across them. Electrons are treated as positive, the beam current is positive and the r.f. current is positive at the bunch centre.

The gap voltages in this model can be set freely without consideration of physical limitations in order to explore the creation of bunches in a klystron. The model allows an idealised pre-bunched beam to be used to explore bunch compression and optimum power extraction.

Beam parameters and frequency

Anode voltage $V_a := 25 \cdot kV$

Beam current $I_0 := 3.953 \cdot 1 \cdot A$

Frequency $f := 1.3 \cdot GHz$

Beam radius $b := 3.3 \cdot mm$

Tunnel radius $a := 5.5 \cdot mm$

Number of harmonics used to define an ideal pre-bunched beam
(NB = 0 if the beam is unmodulated) $NB := 1.2$

Number of beam current harmonics calculated $NH := 4$

Parameters of the model

Number of discs $ND := 24$

Space-charge force parameter imported from WS 13.1 $SCF := 0.915$

Bunch centre initial position $\theta_0 := -2\pi$

Final position $\theta_f := 4 \cdot \pi$

$PEf := 1.024$

Initial time $\phi_0 := \theta_0$

Final time $\phi_f := 2 \cdot \theta_f$

Good results are normally obtained with: ND = 24, nmax = 30, $\theta_0 = -2\pi$, $\theta_f = 4\pi$.

Number of integration steps $nmax := 100$

Distance between reference planes $\Delta\theta := \frac{\pi}{100}$

Gap parameters

Number of gaps $NCAV := 2$

Gap positions in electronic wavelengths	Gap lengths normalised to β_e	Normalised gap voltages	Relative gap phases	Cavity harmonic numbers
$\theta g := 2\pi\cdot\begin{pmatrix} 0 \\ 0 \\ 0.95 \\ 1.925 \\ 4.25 \\ 4.4 \end{pmatrix}$	$\beta eg := \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	$X := \begin{pmatrix} 0 \\ 0.305 \\ 0.9 \\ 1.14 \\ 0.54 \\ 0 \end{pmatrix}$	$\Phi r := \begin{pmatrix} 0 \\ 90 \\ -175 \\ -180 \\ -130 \\ 58 \end{pmatrix}\cdot deg$	$nh := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- NB. The first element of each vector is not used. The cavity count starts from 1
- Phases are relative to the phase of an electron travelling with constant velocity.
- The gap voltage is positive giving maximum accelerating field at the relative phase specified.
- $X = \dfrac{MV_g}{V_a}$
- Only phases within +/- 90 deg of the relative phase of the current can be achieved with passive cavities.
- When the cavity phase is equal to the phase of the current then the field is maximum retarding at the bunch centre.
- If the gap phase is less than the phase of the current (i.e. leading) then the cavity has inductive reactance and is tuned to a frequency above the signal frequency.

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The detailed calculations can be hidden to allow the data and results to be viewed on the screen simultaneously

Method for investigating initial bunch formation

1. WS 13.1 is run with the proposed beam data and the value of SCF found and inserted above. This corrects the space-charge calculation for the difference between the axial and radial propagation constants.
2. The model is run without prebunching (NB = 0) and PEf = 1.0
3. The gap positions, the amplitudes and phases of the gap voltages and the harmonic numbers (nh) are adjusted progresively to produce the desired bunch at the end of the initial bunching section. The objective is to gather as many electrons as possible into a phase range of 180 deg without crossing trajectories. I have not yet found a satisfactory figure of merit for this.

Method for investigating bunch compression and power extraction.

1. WS 13.1 is run with the proposed beam data and the value of SCF found and inserted above. This corrects the space-charge calculation for the difference between the axial and radial propagation constants.
2. This sheet is run with the proposed prebunching (NB) and the cavity voltages set to zero. The value of PEfactor output below is inserted as the value of PEf above. The steps are repeated if necessary until PEfactor = 1. This corrects the initial electron velocity for the additional space-charge potential depression in the bunched beam.
3. The phase set to 90 deg in cavity 1 and the gap voltage is adjusted to maximise the chosen figure of merit (normally F2 or F3). The position of the optimum (θ2 or θ3) is inserted as the position of the ouput gap.
4. The phase in gap 2 is set initially to -180 deg and the voltage adjusted until either the efficiency is maximised, or the velocity of one of the electrons is just not negative. The amplitude and phase of the gap voltage are adjusted to achieve the best possible efficiency.

Define the charge/mass ratio of the electron. Note that the primary electric constant and the velocity of light are already defined in Mathcad.

$$\eta := 1.759 \cdot 10^{11} \cdot \frac{C}{kg} \qquad \varepsilon_0 = 8.854 \times 10^{-12} \cdot \frac{F}{m} \qquad c = 2.998 \times 10^{8} \frac{m}{s} \qquad \mu Perv := \mu A \cdot V^{-1.5} \qquad dB := 1$$

Calculate tube constants and small-signal parameters

Calculate the beam voltage and velocity allowing for space-charge potential depression and relativity

$\boxed{\text{Equation 7.8}}$

$$V_0 := \begin{array}{|l} V_0 \leftarrow V_a \\ \text{for } n \in 0..3 \\ \quad \begin{array}{|l} u_n \leftarrow c \cdot \left[1 - \frac{1}{\left(1 + \frac{\eta \cdot V_n}{c^2}\right)^2}\right]^{0.5} \\ V_{n+1} \leftarrow V_0 - \frac{I_0}{2 \cdot \pi \cdot \varepsilon_0 \cdot u_n} \cdot \left(\frac{1}{2} + \ln\left(\frac{a}{b}\right)\right) \end{array} \\ \text{return } V_{n+1} \end{array}$$

$\boxed{\text{Equation 1.4}}$

$$u_0 := c \cdot \left[1 - \frac{1}{\left(1 + \frac{\eta \cdot V_0}{c^2 \cdot PEf}\right)^2}\right]^{0.5} \qquad Rel := \frac{1}{\sqrt{1 - \frac{u_0^2}{c^2}}} \qquad P_{DC} := I_0 \cdot V_a$$

$V_0 = 24.2 \cdot kV$ $\qquad u_0 = 8.814 \times 10^{7} \cdot m \cdot s^{-1}$ $\qquad Rel = 1.046$

$$\frac{I_0}{V_a^{1.5}} = 1.00 \cdot \mu Perv \qquad G_0 := \left|\frac{I_0}{V_a}\right| = 0.158 \cdot \frac{1}{k\Omega}$$

Electronic propagation constant

$$\omega := 2 \cdot \pi \cdot f \qquad \beta_e := \frac{\omega}{u_0} \qquad \lambda_e := \frac{2 \cdot \pi}{\beta_e} \qquad \gamma_e := \sqrt{\beta_e^2 - \frac{\omega^2}{c^2}}$$

$\gamma_e \cdot b = 0.3$ $\qquad \gamma_e \cdot a = 0.487$

Define an idealised pre-bunched beam

Define the theoretical waveform for an optimally-bunched beam with *n* harmonics normalised to unit beam current. Note *n* does not have to be an integer.

Equation 11.175

$$I_n(\phi, n) := \frac{2\cdot\pi\cdot(1+\cos(\phi))^n}{\int_{-\pi}^{\pi}(1+\cos(\phi))^n\,d\phi}$$

Define a function used to find the initial positions of the discs in a pre-bunched beam.

$$f1(\alpha) := \frac{1}{2\cdot\pi}\cdot\int_{-\pi}^{\alpha} I_n(\phi, NB)\,d\phi$$

Find the amplitudes of the harmonic currents in a pre-bunched beam.

Equation 11.177

$$Ih(n) := \left|\begin{array}{l} \text{for } nn \in 0..n \\ \quad Ih_{nn} \leftarrow \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} I_n(\phi, n)\cdot\cos(nn\cdot\phi)\,d\phi \\ Ih_0 \leftarrow 0.5\cdot Ih_0 \\ Ih \end{array}\right.$$

Beam current as a function of normalised time $I(\phi) := I_0\cdot I_n(\phi, NB)$

$$Ih(NB) = \begin{pmatrix} 1.000 \\ 1.091 \end{pmatrix}$$

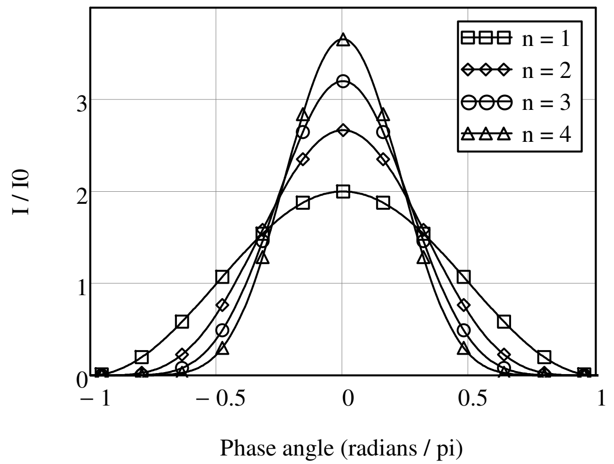


Figure 11.30

Calculate the gap lengths coupling factors and voltages

$$gap := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad gap_n \leftarrow \dfrac{\beta eg_n}{\beta_e} \\ \text{return } gap \end{array} \right.$$

$$Mg := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad \left| \begin{array}{l} \beta \leftarrow \beta_e \cdot nh_n \\ \gamma \leftarrow \sqrt{\beta^2 - \dfrac{\omega^2}{c^2}} \\ M_n \leftarrow \dfrac{2 \cdot I1(\gamma \cdot b)}{(\gamma \cdot b) \cdot I0(\gamma \cdot a)} \cdot \mathrm{sinc}\left(\dfrac{\beta \cdot gap_n}{2}\right) \end{array} \right. \\ \text{return } M \end{array} \right.$$

$$Vg := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Vg_n \leftarrow \dfrac{X_n \cdot V_a}{Mg_n} \\ \text{return } Vg \end{array} \right.$$

$$gap = \begin{pmatrix} 0.00 \\ 5.40 \\ 5.40 \end{pmatrix} \cdot mm \qquad Mg = \begin{pmatrix} 0.000 \\ 0.943 \\ 0.943 \end{pmatrix} \qquad Vg = \begin{pmatrix} 0.000 \\ 8.082 \\ 23.850 \end{pmatrix} \cdot kV$$

The Bunch is modelled as ND rigid discs. The motions of the electrons at the disc centres are followed. We define their initial positions and velocities using the disc thickness ΔL. As the Mathcad ODE solver rkfixed does not accept variables with dimensions the dimensionless variables: $\theta = \beta_e z$ and $\phi = \omega t$ are used.

Calculate the normalised disk thickness (q_d)

$$\theta_d := \frac{2\cdot\pi}{ND}$$

Define disk charge

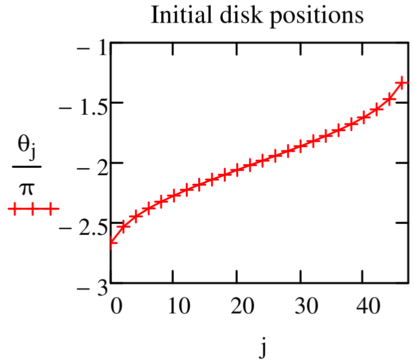
$$Q := \frac{2\cdot\pi\, I_0}{\omega\cdot ND}$$

Define normalised disk starting positions and velocities for the prebunched current specified.

$$\alpha := 0$$

$$\theta := \left| \begin{array}{l} \text{for } j \in 0, 2 .. 2\cdot(ND-1) \\ \quad \left| \begin{array}{l} \theta_j \leftarrow \text{root}\left(f1(\alpha) - \dfrac{j+1}{2ND}, \alpha\right) + \theta_0 \\ \theta_{j+1} \leftarrow 1 \end{array} \right. \\ \theta \end{array} \right.$$

$$j := 0, 2 .. 2\cdot ND - 1$$



Check the D.C. beam current calculated from the disks is the same as that previously computed.

$$\sum_{j=0}^{ND-1} (f\cdot Q) = 3.953\,A$$

$$I_0 = 3.953\,A$$

The Space-Charge Field is found from the equations given in
J.R. Hechtel, "The effect of potential beam energy on the performance of linear beam devices", *IEEE Transactions on Electron Devices* ED-17, pp.999-1009, Nov. 1970.
The calculations are for a disc of charge of radius *b* and length *ΔL* with uniform charge density ρ_0

Compute the space-charge lookup function ES(θ)

Define the first ten zeros of $J_0(z)$.

$$\mu B := \frac{1}{a}\cdot(2.405\ \ 5.520\ \ 8.654\ \ 11.791\ \ 14.931\ \ 18.071\ \ 21.212\ \ 24.352\ \ 27.494\ \ 30.635)^T$$

$$\Delta L := \frac{\theta_d}{\beta_e} \qquad \rho_0 := \frac{1}{\pi\cdot b^2\cdot\Delta L}$$

ρ_0 is calculated for a disk charge of +1C.
Thus the electric field must be multiplied by the charge of the source disk

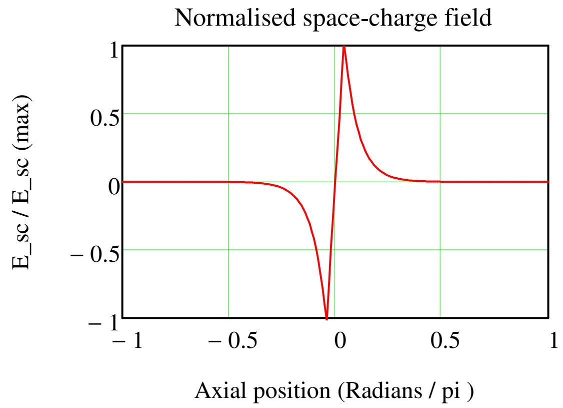
$$\begin{aligned} ESn := \ & \text{for } n \in 0..100 \\ & \quad \theta_n \leftarrow 0.02\cdot n\cdot\pi \\ & \quad z_n \leftarrow \frac{\theta_n}{\beta_e} \\ & \quad ES_n \leftarrow \left(\frac{4\cdot\rho_0}{\varepsilon_0}\right)\cdot\sum_{m=0}^{9}\left[\frac{1}{\mu B_m}\cdot\left(\frac{J1(\mu B_m\cdot b)}{\mu B_m\cdot a\cdot J1(\mu B_m\cdot a)}\right)^2\cdot\exp(-\mu B_m\cdot z_n)\cdot\sinh\left(\frac{\mu B_m\cdot\Delta L}{2}\right)\right] \text{ if } \theta_n \ge 0.5\cdot\theta_d \\ & \quad ES_n \leftarrow \left(\frac{4\cdot\rho_0}{\varepsilon_0}\right)\cdot\sum_{m=0}^{9}\left[\frac{1}{\mu B_m}\cdot\left(\frac{J1(\mu B_m\cdot b)}{\mu B_m\cdot a\cdot J1(\mu B_m\cdot a)}\right)^2\cdot\left(\exp\left(-\mu B_m\cdot\frac{\Delta L}{2}\right)\cdot\sinh(\mu B_m\cdot z_n)\right)\right] \text{ otherwise} \\ & ES \end{aligned}$$

$$\begin{aligned} \theta n := \ & \text{for } n \in 0..100 \\ & \quad \theta_n \leftarrow 0.02\cdot n\cdot\pi \\ & \theta \end{aligned}$$

$$Es(\theta) := \mathrm{sign}(\theta)\cdot\mathrm{linterp}(\theta n, ESn, |\theta|)$$

$$ES(\theta) := \begin{cases} Es(\theta + 2\cdot\pi) & \text{if } \theta < -\pi \\ Es(\theta - 2\cdot\pi) & \text{if } \theta > \pi \\ Es(\theta) & \text{otherwise} \end{cases}$$

$$\theta 1 := (-\pi), (-0.99\cdot\pi) .. \pi$$



The space-charge field of adjacent bunches is included by assuming that the field is periodic in z. This is not correct but tests with an initially unmodulated beam and three wavelengths of electrons give almost identical results for the trajectories and the current harmonics except well beyond the first bunch and at microperveance greater than 2.

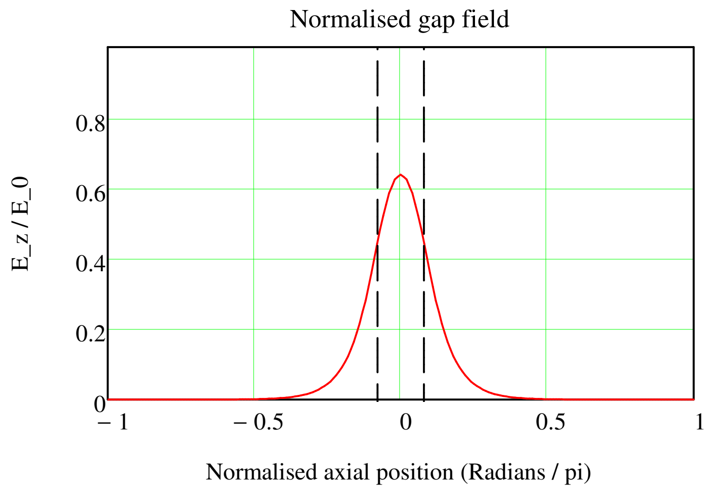
The Interaction Field on the axis is found from the Fourier Transform of the field in the gap (assumed to be constant). The field in the gap is assumed to be constant. The average of the field over the beam is used. Linear interpolation on the values calculated at regular intervals is used to provide a fast look-up function.

$$\mathrm{En} := \left| \begin{array}{l} \gamma(\beta) \leftarrow \sqrt{\beta^2 - \dfrac{\omega^2}{c^2}} \\ \text{for } ng \in 1 .. NCAV \\ \quad \text{for } n \in 0 .. 100 \\ \quad \left| \begin{array}{l} \theta_n \leftarrow 0.02\cdot n\cdot\pi \\ E_{n,ng} \leftarrow \dfrac{V}{\pi}\cdot\displaystyle\int_0^{\frac{20\cdot\pi\cdot\beta_e}{\beta eg_{ng}}} \dfrac{2\cdot I1(\gamma(\beta)\cdot b)}{(\gamma(\beta)\cdot b)\cdot I0(\gamma(\beta)\cdot a)}\cdot \mathrm{sinc}\left(\dfrac{\beta\cdot gap_{ng}}{2}\right)\cdot\cos\left(\dfrac{\beta}{\beta_e}\cdot\theta_n\right) d\beta \end{array} \right. \\ \text{return } E \end{array} \right.$$

Equation 11.30

$$\mathrm{Egap}(\theta) := \left| \begin{array}{l} \text{for } ng \in 1 .. NCAV \\ \quad E_{ng} \leftarrow \mathrm{linterp}\left(\theta n, En^{\langle ng \rangle}, |\theta|\right) \\ \text{return } E \end{array} \right.$$

Plot the normalised gap field



CHECK small-signal coupling factors by direct integration of the field.

$$M_g := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad M_n \leftarrow \dfrac{-1}{\beta_e \cdot V} \cdot \displaystyle\int_{-2\pi}^{2\pi} Egap(\theta)_n \cdot \cos(nh_n \cdot \theta)\, d\theta \\ \text{return } M \end{array} \right.$$

$$Mg = \begin{pmatrix} 0.000 \\ 0.943 \\ 0.943 \end{pmatrix} \qquad Mg = \begin{pmatrix} 0.000 \\ 0.943 \\ 0.943 \end{pmatrix}$$

Absolute phase of the gap field

$$\Phi g := \Phi r + \theta g$$

Superimpose the electric fields of the gaps

$$E_z(\theta,\phi) := \mathrm{Re}\left[\sum_{n=1}^{NCAV}\left[\frac{Vg_n}{V}\cdot Egap\left(\theta - \theta g_n\right)_n \exp\left[j\cdot nh_n\cdot\left(\phi - \Phi g_n\right)\right]\right]\right]$$

Turn space-charge force off until the electrons reach the origin.

$$SC(\theta) := \left| \begin{array}{ll} SCF & \text{if } \theta \geq 0 \\ 0 & \text{otherwise} \end{array} \right.$$

The Coefficients of the Differential Equations for the motions of the electrons are defined.

The rows represent, in order, the position in radians and the normalised velocity of the electrons.

$$D(\phi,\theta) := \left| \begin{array}{l} \text{for } j \in 0,2..2\cdot(ND-1) \\ \quad \left| \begin{array}{l} D_j \leftarrow \theta_{j+1} \\ D_{j+1} \leftarrow \dfrac{\eta}{\omega\cdot u_0}\cdot\left[1-\left(\dfrac{u_0}{c}\cdot\theta_{j+1}\right)^2\right]^{1.5}\cdot\left[E_z(\theta_j,\phi)+SC(\theta_j)\cdot Q\cdot\sum_{i=0}^{ND-1}\left(ES(\theta_j-\theta_{2\cdot i})\right)\right] \end{array}\right. \\ D \end{array}\right.$$

Definitions of normalised variables

$$\phi = \omega\cdot t \qquad \theta = \beta_e\cdot z \qquad \theta' = \frac{v}{u_0}$$

$$\frac{d}{dt}z = v \qquad \frac{d}{d\phi}\theta = \frac{v}{u_0}$$

$$\frac{d}{dt}v = \eta\cdot E \qquad \frac{d}{d\phi}\frac{v}{u_0} = \frac{\eta\cdot E}{\omega\cdot u_0}$$

The Equations are Solved using with *nmax* time steps starting from ϕ_0 which is defined in such a way that the centre electron would cross the gap centre at t = 0 if it travelled with a constant velocity u_0. The final time is t_f

$$n := 0..nmax \qquad Z := \text{rkfixed}(\theta,\phi_0,\phi_f,nmax,D) \qquad \Delta\phi := \frac{\phi_f-\phi_0}{nmax}$$

The results are in a single table (Z) in which the first column (0) is the time and the other columns (1-12) are the positions and velocities of the electrons in the same order as before at each value of n.

Extract the vector of phase, the matrices containing the normalised positions and velocities of the disks and the vector of the final velocities of the electrons

$$\phi n := \left| \begin{array}{l} \text{for } n \in 0..nmax \\ \quad \phi_n \leftarrow Z_{n,0} \\ \phi \end{array}\right. \qquad \theta n := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } n \in 0..nmax \\ \quad\quad \theta_{n,j} \leftarrow Z_{n,2\cdot j+1} \\ \theta \end{array}\right. \qquad un := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } n \in 0..nmax \\ \quad\quad u_{n,j} \leftarrow Z_{n,2\cdot j+2} \\ u \end{array}\right. \qquad umax := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad u_j \leftarrow Z_{nmax,2\cdot j+2} \\ u \end{array}\right.$$

The Kinetic Energy of the bunch at each time stepis calculated using the relativistically correct formulae by summing the energies of the disks. The figure becomes unstable when siginificant cross-overs occur

$$KE := \left| \begin{array}{l} \text{for } n \in 0..nmax \\ \quad KE_n \leftarrow \dfrac{Q \cdot c^2}{\eta} \cdot \displaystyle\sum_{j=0}^{ND-1} \left[\dfrac{1}{\sqrt{1 - \dfrac{(un_{n,j} \cdot u_0)^2}{c^2}}} - 1 \right] \\ KE \end{array} \right.$$

Compare the frequency times the intial KE with the DC beam power

$KE_0 \cdot f = 93.4 \cdot kW$ $\quad$ $I_0 \cdot V_0 = 95.6 \cdot kW$

When the beam is prebunched the electron energy is reduced from V0 by the potential energy stored in the bunch. To estimate this the model is run with all the gap voltages set to zero. The normalised final kinetic energy is taken as a measure of the additional potential energy. The initial electron energy is reduced by the factor PEf given by

$$PEfactor := \frac{KE_{nmax} \cdot f}{I_0 \cdot V_0}$$

Check for electrons brought to rest and reflected

$$VELOCITY := \left| \begin{array}{l} V \leftarrow \text{"Positive"} \\ \text{for } j \in 0..(ND-1) \\ \quad \text{for } n \in 0..nmax \\ \quad\quad \text{if } un_{n,j} \le 0 \\ \quad\quad\quad \left| \begin{array}{l} V \leftarrow \text{"Negative"} \\ \text{break} \end{array} \right. \\ \text{return } V \end{array} \right.$$

Define a set of equally-spaced planes in θ and compute the phases and velocities at which the electrons cross them using linear interpolation.

Number of planes at which results are stored $\quad NP := \frac{\theta_f - \theta_0}{\Delta\theta} \qquad NP = 600$

$$\theta p := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad \theta_p \leftarrow \theta_0 + p \cdot \Delta\theta \\ \text{return } \theta \end{array} \right.$$

$$\phi p := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } p \in 0..NP \\ \quad\quad \text{for } n \in 1..nmax \\ \quad\quad\quad \left| \begin{array}{l} flag \leftarrow 0 \\ flag \leftarrow 1 \text{ if } \theta n_{n,j} > \theta p_p \\ \phi p_{p,j} \leftarrow \phi n_{n-1} + \dfrac{\theta p_p - \theta n_{n-1,j}}{\theta n_{n,j} - \theta n_{n-1,j}} \cdot \left(\phi n_n - \phi n_{n-1}\right) \text{ if } flag = 1 \\ (\text{break}) \text{ if } flag = 1 \end{array} \right. \\ \text{return } \phi p \end{array} \right.$$

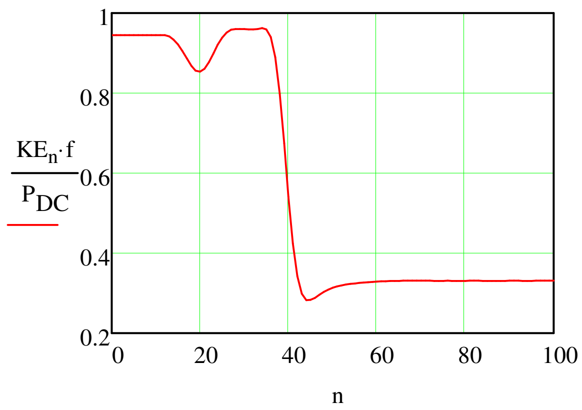
$$up := \left| \begin{array}{l} \text{for } j \in 0..(ND-1) \\ \quad \text{for } p \in 0..NP \\ \quad\quad \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad \left| \begin{array}{l} flag \leftarrow 0 \\ flag \leftarrow 1 \text{ if } \theta n_{n,j} > \theta p_p \\ up_{p,j} \leftarrow un_{n-1,j} + \dfrac{\theta p_p - \theta n_{n-1,j}}{\theta n_{n,j} - \theta n_{n-1,j}} \cdot \left(un_{n,j} - un_{n-1,j}\right) \text{ if } flag = 1 \\ (\text{break}) \text{ if } flag = 1 \end{array} \right. \\ up_{p,j} \leftarrow 10^{-6} \text{ if } up_{p,j} = 0 \end{array} \right. \\ \text{return } up \end{array} \right.$$

Phase relative to an electron with velocity u_0

$$\phi r := \left| \begin{array}{l} \text{for } j \in 0..ND-1 \\ \quad \text{for } p \in 0..NP \\ \quad\quad \phi r_{p,j} \leftarrow \phi p_{p,j} - \theta p_p \\ \text{return } \phi r \end{array} \right.$$

Calculate the **total kinetic energy** of the electrons as they cross each plane.

$$KEp := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad KE_p \leftarrow \dfrac{Q \cdot c^2}{\eta} \cdot \displaystyle\sum_{j=0}^{ND-1} \left[\dfrac{1}{\sqrt{1 - \dfrac{\left(up_{p,j} \cdot u_0\right)^2}{c^2}}} - 1 \right] \\ KE \end{array} \right.$$



Calculate the complex current harmonics at each plane by superimposing the Fourier components of the discs. For simplicity each disc is treated as having constant charge and length.

$$Ip := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad \left| \begin{array}{l} Ip_{p,0} \leftarrow \dfrac{Q}{\Delta L} \cdot \dfrac{1}{2 \cdot \pi} \cdot \displaystyle\sum_{j=0}^{ND-1} \theta_d \\ \text{for } h \in 1..NH \\ \quad Ip_{p,h} \leftarrow \dfrac{Q}{\Delta L} \cdot \dfrac{2}{\pi \cdot h} \cdot \displaystyle\sum_{j=0}^{ND-1} \left(up_{p,j} \cdot \sin\left(\dfrac{h \cdot \theta_d}{2 \cdot up_{p,j}}\right) \cdot \exp\left(j \cdot h \cdot \phi p_{p,j}\right)\right) \end{array} \right. \\ \text{return } Ip \cdot u_0 \end{array} \right.$$

Instantaneous current = $\dfrac{Q_j \cdot up_{p,j}}{\Delta L}$ Pulse phase duration = $\dfrac{\theta_d}{up_{p,j}}$

Relative phases of the RF beam current harmonics

$$argrI1 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad argI1_p \leftarrow \arg\left(Ip_{p,1} \cdot \exp\left(-j \cdot \theta p_p\right)\right) \\ \text{return } argI1 \end{array} \right.$$

$$argrI2 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad argI2_p \leftarrow \arg\left(Ip_{p,2} \cdot \exp\left(-2 \cdot j \cdot \theta p_p\right)\right) \\ \text{return } argI2 \end{array} \right.$$

Find the variation of current with time at plane p by Fourier synthesis

$$IP(p,\phi) := \mathrm{Re}\left[\sum_{h=0}^{NH}\left(Ip_{p,h}\cdot\exp(j\cdot h\cdot\phi)\right)\right]$$

Calculate the variation of current with time at plane p using compressible discs

$$IPd := \left|\begin{array}{l} \text{for } p \in 0..NP \\ \quad \left|\begin{array}{l} \text{for } j \in 0..ND-2 \\ \quad IP_{p,j} \leftarrow \left|\dfrac{Q\cdot\omega}{\phi r_{p,j} - \phi r_{p,j+1} + 10^{-6}}\right| \\ IP_{p,ND-1} \leftarrow \left|\dfrac{Q\cdot\omega}{\phi r_{p,ND-1} - \left(\phi r_{p,0} - 2\cdot\pi\right)}\right| \\ IP_{p,ND} \leftarrow IP_{p,0} \\ IP_{p,ND+1} \leftarrow IP_{p,1} \\ IP_{p,ND+2} \leftarrow IP_{p,2} \end{array}\right. \\ \text{return } IP \end{array}\right.$$

$$\phi rd := \left|\begin{array}{l} \text{for } p \in 0..NP \\ \quad \left|\begin{array}{l} \text{for } j \in 0..ND-2 \\ \quad \phi_{p,j} \leftarrow 0.5\cdot\left(\phi r_{p,j} + \phi r_{p,j+1}\right) \\ \phi_{p,ND-1} \leftarrow 0.5\cdot\left(\phi r_{p,ND-1} + \phi r_{p,0} - 2\cdot\pi\right) \\ \phi_{p,ND} \leftarrow \phi_{p,0} - 2\cdot\pi \\ \phi_{p,ND+1} \leftarrow \phi_{p,1} - 2\cdot\pi \\ \phi_{p,ND+2} \leftarrow \phi_{p,2} - 2\cdot\pi \end{array}\right. \\ \text{return } \phi \end{array}\right.$$

Calculation of the induced curent in each cavity

Find the serial number of the plane at the centre of each cavity and plane numbers at the edges of the gap field. The gap field is assumed to be zero at 10g from the gap centre. Note that an array element error will be reported if pc2 > NP for the last cavity.

$$pc := \mathrm{round}\left(\frac{\theta g - \theta_0}{\Delta\theta}\right) \qquad pc1 := \mathrm{round}\left(\frac{\theta g - \theta_0 - 10\cdot\beta eg}{\Delta\theta}\right) \qquad pc2 := \mathrm{round}\left(\frac{\theta g - \theta_0 + 10\cdot\beta eg}{\Delta\theta}\right)$$

Check that field of cavity number *cn* is zero at planes *pc*1 and *pc*2.

Cavity number $cn := 2$ $\quad pc1_{cn} = 231 \quad pc_{cn} = 390 \quad pc2_{cn} = 549.00 \quad \theta c1 := \left(pc1_{cn} - pc_{cn}\right)\cdot\Delta\theta \quad \theta c2 := \left(pc2_{cn} - pc_{cn}\right)\cdot\Delta\theta$

Induced current in each cavity

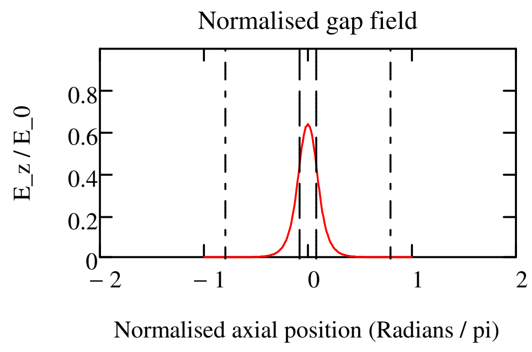
$$Iind := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad Iind_n \leftarrow \dfrac{-\Delta\theta}{\beta_e\cdot V}\cdot\left[\displaystyle\sum_{p = pc1_n}^{pc2_n}\left(Ip_{p,nh_n}\cdot Egap\left(\theta p_p - \theta g_n\right)_n\right)\right] \\ \text{return } Iind \end{array}\right.$$

Equation 11.173
The integral is approximated by a sum

RF output power

$$P_{out} := \mathrm{Re}\left[\frac{1}{2}\cdot Iind_{NCAV}\cdot\overline{\left(Vg_{NCAV}\cdot\exp\left(j\cdot\Phi g_{NCAV}\right)\right)}\right]$$

$$Energy\_balance := 1 - \frac{P_{out} + KE_{nmax}\cdot f}{I_0\cdot V_0}$$



Bunching figures of merit for final bunching and the planes at which their values are maximum

$$F1 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad F1_p \leftarrow \left| \begin{array}{ll} \left[\left| \dfrac{Ip_{p,1}}{2I_0} \right| \cdot \min\left[\left(up^T \right)^{\langle p \rangle} \right] \right] & \text{if } \theta p_p > 0 \\ 0 & \text{otherwise} \end{array} \right. \\ \text{return } F1 \end{array} \right.$$

$$popt1 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad (\text{break}) \text{ if } F1_p = \max(F1) \\ \text{return } p \end{array} \right.$$

$$\theta 1 := \frac{\theta p_{popt1}}{2 \cdot \pi}$$

Equation 13.31

$$F2 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad \left| \begin{array}{l} \Delta u \leftarrow \sqrt{\displaystyle\sum_{j=0}^{ND-1} \left(up_{p,j} - 1 \right)^2} \\ F2_p \leftarrow \left| \begin{array}{ll} \left[\left| \dfrac{Ip_{p,1}}{2I_0} \right| \cdot (1 - \Delta u) \right] & \text{if } \theta p_p > 0 \\ 0 & \text{otherwise} \end{array} \right. \end{array} \right. \\ \text{return } F2 \end{array} \right.$$

$$popt2 := \left| \begin{array}{l} \text{for } p \in 0..NP \\ \quad (\text{break}) \text{ if } F2_p = \max(F2) \\ \text{return } p \end{array} \right.$$

$$\theta 2 := \frac{\theta p_{popt2}}{2 \cdot \pi}$$

Equation 13.32

Plotting data

$j := 0..ND - 1$ $\quad jj := 0..ND$ $\quad x := 0, 0.1..1.5$ $\quad \phi := \phi_0..1.2\phi_f$ $\quad \phi_g := -1..1$

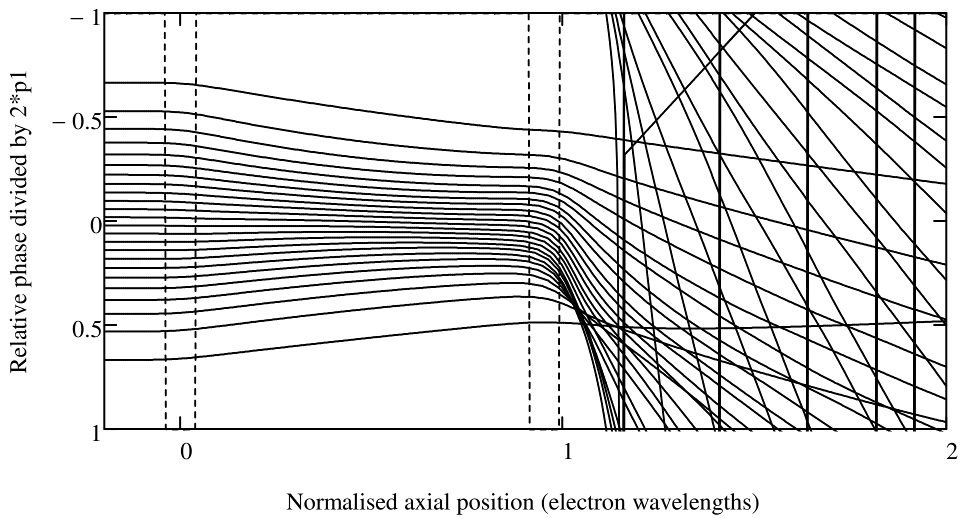
$z2 := \dfrac{\theta g - 0.5 \cdot \beta eg}{2 \cdot \pi}$ $\quad z1 := \dfrac{\theta g + 0.5 \cdot \beta eg}{2 \cdot \pi}$ $\quad \phi 1 := -\pi, -0.99 \cdot \pi..\pi$ $\quad nc := 1..NCAV$ $\quad p1 := 0..NP$

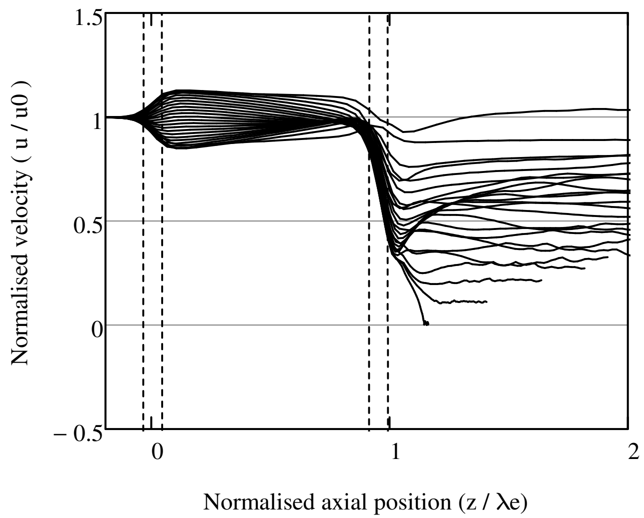
$$\phi g(\theta) := \left| \begin{array}{l} \text{for } n \in 1..NCAV \\ \quad \phi g \leftarrow 1 \text{ if } \left| \theta - \theta g_n \right| \le \dfrac{\beta eg_n}{2} \\ \text{return } 2\phi g - 1 \end{array} \right.$$

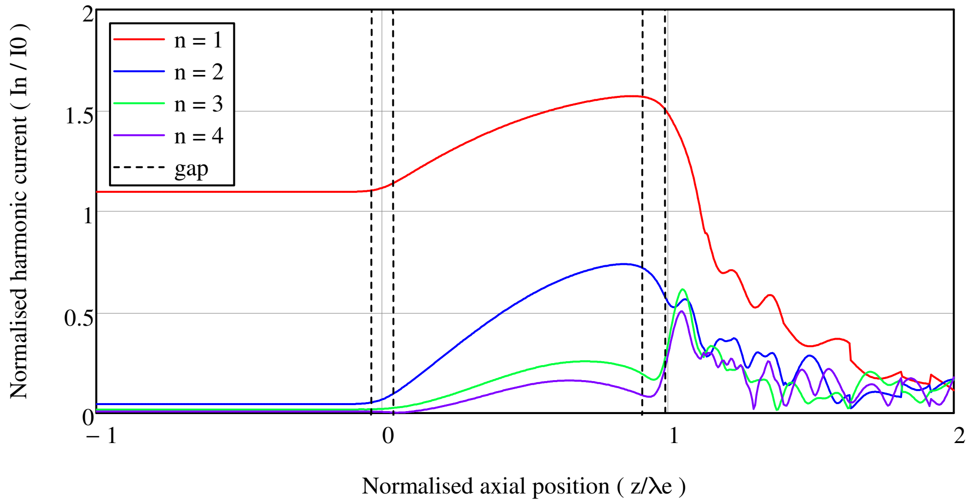
$\max(F1) = 0.730$ | $\max(F2) = 0.6462$ | $PEfactor = 0.342$ | $\frac{I_0}{V_a^{1.5}} = 1.00 \cdot \mu Perv$

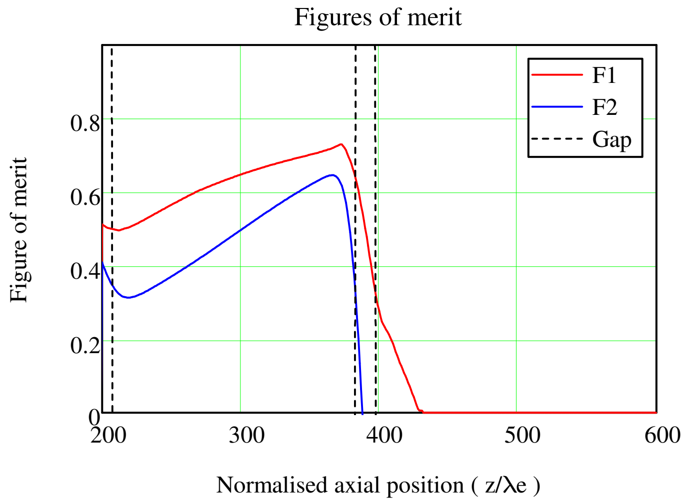
$\theta 1 = 0.860$ | $\theta 2 = 0.830$

VELOCITY = "Positive" | $\frac{P_{out}}{P_{DC}} = 63.7 \cdot \%$ | $P_{out} = 62.9 \cdot kW$ | Energy_balance = $0.002 \cdot \%$ | $\gamma_e \cdot b = 0.3$ | $\gamma_e \cdot a = 0.487$

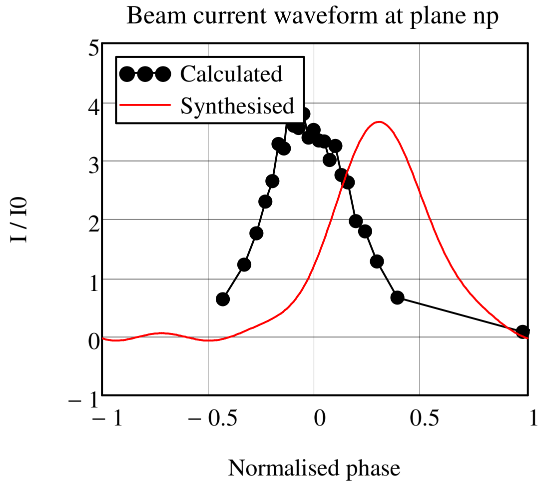


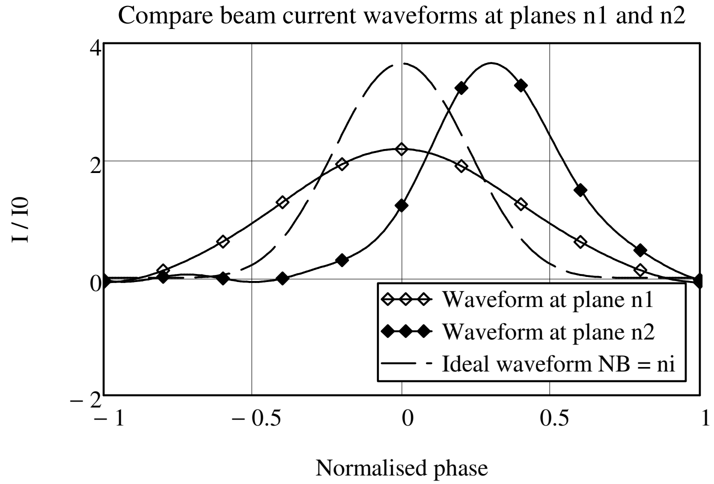


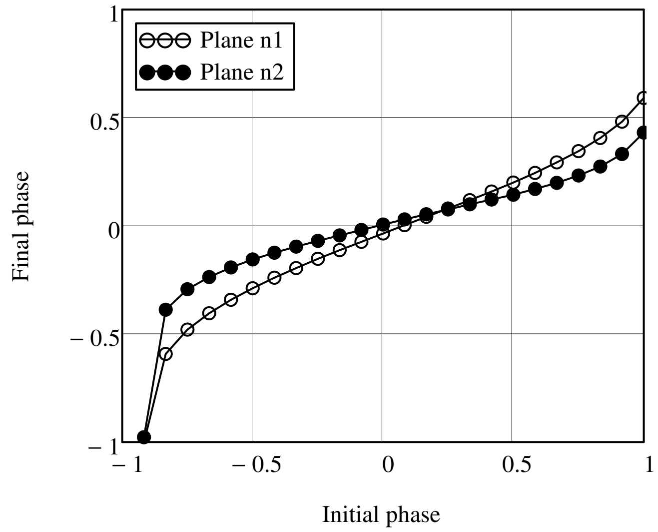


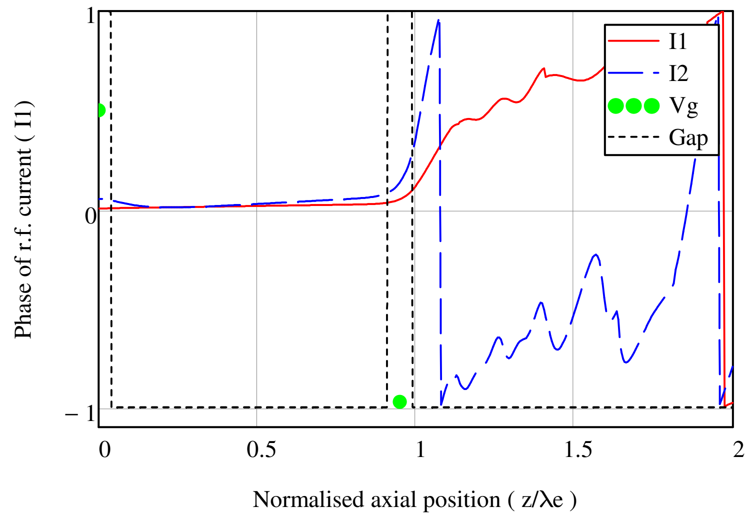


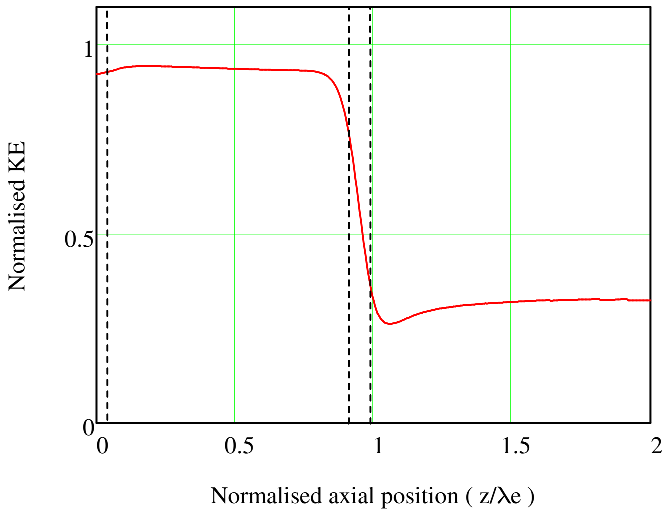
n1 := 200 n2 := 366 ni := 4

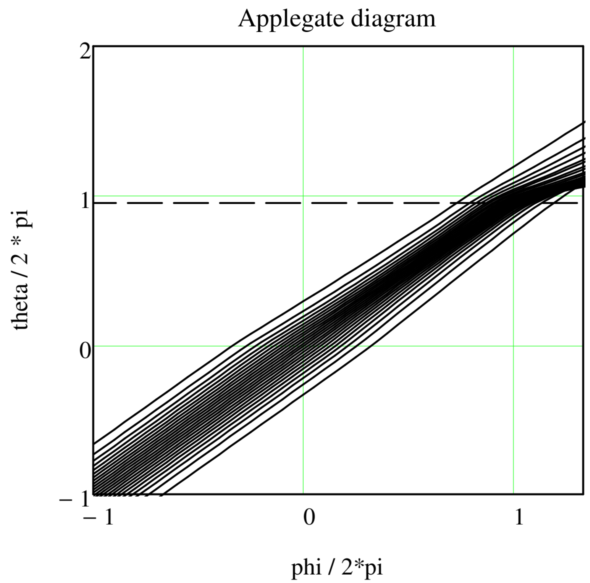





Plane n1
Plane n2
Final phase
Initial phase
1
0.5
0
− 0.5
− 1
0.5
1


I1
I2
Vg
Gap
Phase of r.f. current (I1)
Normalised axial position (z/λe)
1
0
− 1
0.5
1.5
2


Normalised KE
Normalised axial position (z/λe)
1
0.5
0
1.5
2


Applegate diagram
theta / 2 * pi
phi / 2*pi
2
1
0
− 1