Worksheet 11.1 Calculation of coupling factors and beam loading for gridded and ungridded gaps

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

1. Gridded Gaps

The coupling factor and beam loading conductance are calculated in terms of the gap transit angle $\theta = \beta_e g$

$M(\theta) := \mathrm{sinc}(0.5 \cdot \theta)$ Equation 11.7

$Gg(\theta) := -\frac{1}{4} \cdot \theta \cdot \frac{d}{d\theta} M(\theta)^2$ Equation 11.22

$Gb(\theta) := 0.5 \cdot \mathrm{sinc}(0.5 \cdot \theta) \cdot (\mathrm{sinc}(0.5 \cdot \theta) - \cos(0.5 \cdot \theta))$

Equation 11.20

$Bb(\theta) := 0.5 \cdot \frac{\cos(0.5 \cdot \theta)}{0.5 \cdot \theta} \cdot (\mathrm{sinc}(0.5 \cdot \theta) - \cos(0.5 \cdot \theta))$

$\theta := 0, 0.1 .. 4 \cdot \pi$

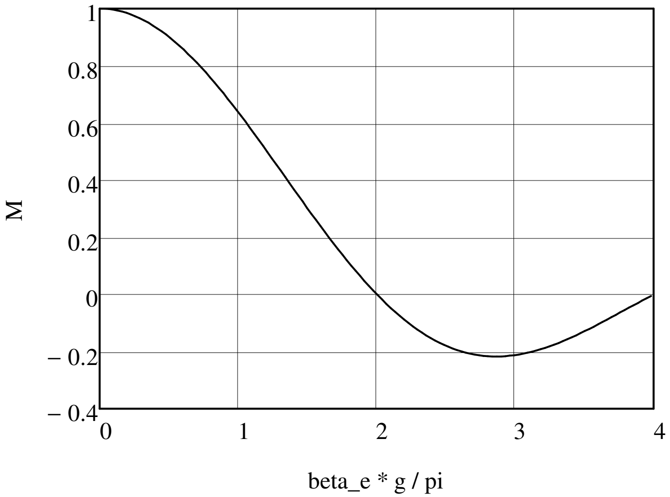

Figure 11.3

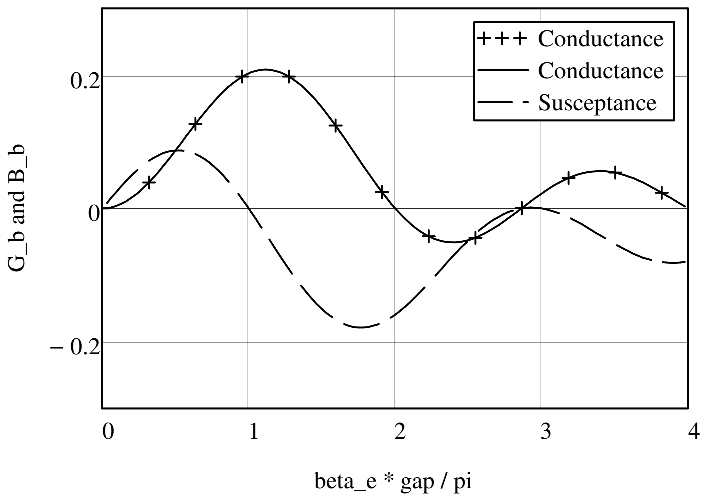

Figure 11.5

2. Coupling in an ungridded gap

Radial Coupling Factor

See Garland M. Branch: Electron Beam Coupling in Interaction Gaps of Cylindrical Symmetry
IRE Trans ED Vol.ED-8, pp.193-207 (1961)

a. Brillouin Beam: interaction concentrated on beam surface.

$$\mu_B(A, \gamma b) := \frac{I0(\gamma b)}{I0(A \cdot \gamma b)}$$

Equation 11.40

where $A = \frac{a}{b}$

b. Confined flow: Coupling averaged over area of beam

$$\mu_c(A, \gamma b) := \frac{2 \cdot I1(\gamma b)}{\gamma b \cdot I0(A \cdot \gamma b)}$$

Equation 11.38

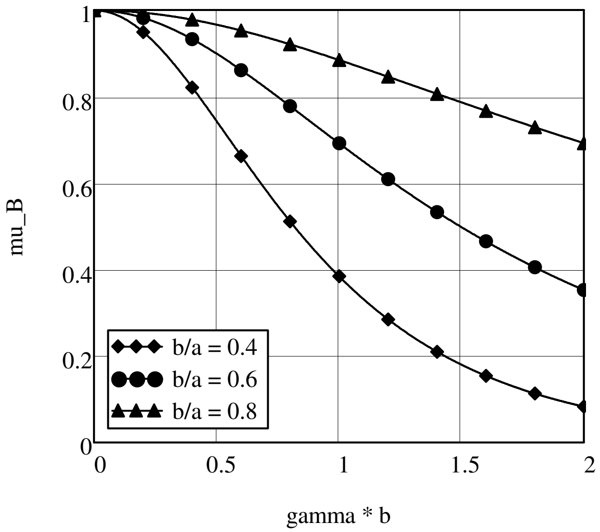

Figure 11.7(a)

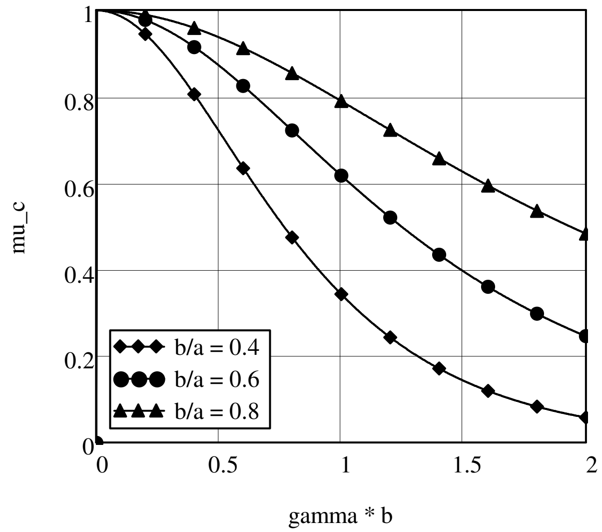

Figure 11.7(b)

Computation of Beam Loading Admittance of an ungridded gap by a beam in confined flow

Define the propagation constant, the coupling coefficient and the normalised gap loading conductance in terms of the transit angle θ (in radians) and the dimensions of the beam and the gap

$\theta = \beta_e \cdot g$ $\quad\quad \mu_d(\theta) := \left(\dfrac{\sin(0.5\cdot\theta)}{0.5\cdot\theta}\right)$ $\quad\quad \beta_e(g,\theta) := \dfrac{\theta}{g}$ $\quad\quad \mu_r(a,b,g,\theta) := \dfrac{2\cdot I1\left(\beta_e(g,\theta)\cdot b\right)}{\beta_e(g,\theta)\cdot b\cdot I0\left(\beta_e(g,\theta)\cdot a\right)}$

$M(a,b,g,\theta) := \mu_r(a,b,g,\theta)\cdot\mu_d(\theta)$ $\quad\quad G_b(a,b,g,\theta) := \dfrac{-1}{4}\cdot\theta\cdot\dfrac{d}{d\theta}M(a,b,g,\theta)^2$

$a := 1$ $\quad\quad b := 0.7$ $\quad\quad Gb(g,\theta) := G_b(a,b,g,\theta)$

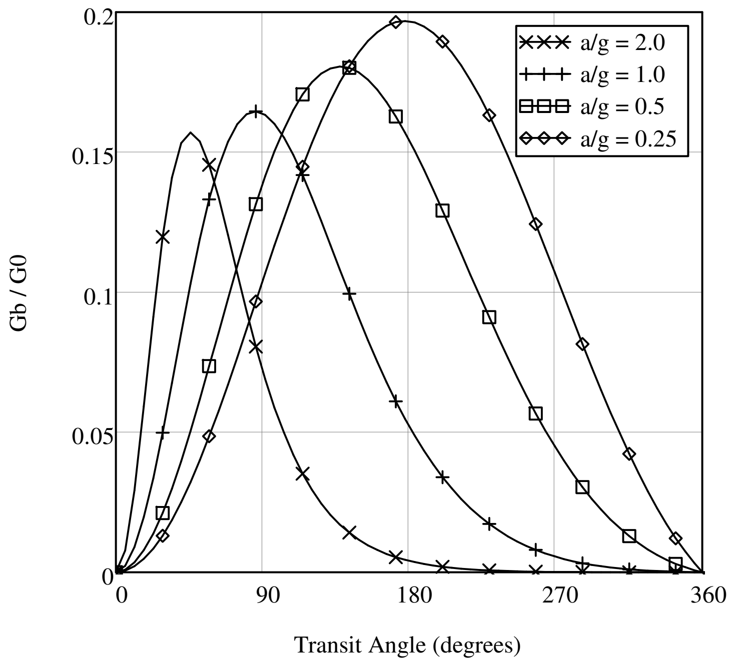

Figure 11.16
(Gb/G0 only)

Compare fig.6 in
Craig, E.J., The beam loading admittance of gridless klystron gaps, IEEE Trans. Vol.ED-14, No.5, pp.273-278 (1967)

Beam Loading Admittance for a relativistic beam

Based on E. J. Craig, "Relativistic beam-loading admittance," *IEEE Transactions on Electron Devices,* vol. 16, pp. 139-139, 1969 and using his notation. The results are for an ungridded gap assuming uniform field in the gap.

Assume that the normalised reduced plasma frequency is $\omega q\_\omega := 0.05$

Ratio of beam velocity to velocity of light $u0\_c := 0.01$

$$R := \frac{1}{\sqrt{1 - u0\_c^2}} \qquad \gamma = \frac{\beta}{R}$$

a = tunnel radius
b = beam radius
gap = gap length $gap := 2$
beta = beta_e

$$ReI(\beta,\gamma,a,b,gap) := \frac{2\cdot\sin(0.5\cdot\beta\cdot gap)^2\left(I0(\gamma\cdot b)^2 - I1(\gamma\cdot b)^2\right)}{(\beta\cdot gap)^2\cdot I0(\gamma\cdot a)^2}$$

Equation 11.101

$$G2(\beta,a,b,gap) := \frac{1}{2\cdot R\cdot(R+1)\cdot\omega q\_\omega}\cdot\left[ReI\left[(1-\omega q\_\omega)\cdot\beta,\frac{(1-\omega q\_\omega)\cdot\beta}{R},a,b,gap\right] - ReI\left[(1+\omega q\_\omega)\cdot\beta,\frac{(1+\omega q\_\omega)\cdot\beta}{R},a,b,gap\right]\right]$$

Equation 11.100 (Real part)

Zeroes of the Bessel function J_0

Number of terms in the series $nmax := 3$

```
λ := | for n ∈ 1..nmax
     |   | x ← n·π
     |   | λn ← root(J0(x), x)
     | return λ
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$$r(\beta,a,gap) := \left|\begin{array}{l} \text{for } n \in 1..nmax \\ \quad r_n \leftarrow \dfrac{\lambda_n\cdot gap}{a}\cdot\sqrt{1-\left(\dfrac{a\cdot\beta}{\lambda_n}\right)^2\cdot u0\_c^2} \\ \text{return } r \end{array}\right.$$

Equation 11.103

$$N(\beta, a, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad N_n \leftarrow \left[1 - \left(\dfrac{a\cdot\beta}{\lambda_n}\right)^2 \cdot u0\_c^2\right]^{-1} \\ \text{return } N \end{array} \right.$$

$$x(a, b, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad x_n \leftarrow \left(\dfrac{gap\cdot J0\left(\dfrac{b\cdot\lambda_n}{a}\right)}{a\cdot J1(\lambda_n)}\right)^2 \\ \text{return } x \end{array} \right.$$

Equation 11.106

$$y(a, b, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad x_n \leftarrow \left(\dfrac{gap\cdot J1\left(\dfrac{b\cdot\lambda_n}{a}\right)}{a\cdot J1(\lambda_n)}\right)^2 \\ \text{return } x \end{array} \right.$$

Equation 11.107

$$A(\beta, a, b, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad A_n \leftarrow 2\cdot N(\beta, a, gap)_n \cdot \left(x(a, b, gap)_n + y(a, b, gap)_n\right) \cdot \dfrac{\left(r(\beta, a, gap)_n - 1 + \exp\left(-r(\beta, a, gap)_n\right)\right)}{r(\beta, a, gap)_n} \\ \text{return } A \end{array} \right.$$

Equation 11.104

$$B(\beta, a, b, gap) := \left| \begin{array}{l} \text{for } n \in 1..nmax \\ \quad B_n \leftarrow \left[N(\beta, a, gap)_n \cdot \left(x(a, b, gap)_n + y(a, b, gap)_n\right) \cdot \left[\dfrac{3 - 2\cdot r(\beta, a, gap)_n - \left(3 + r(\beta, a, gap)_n\right)\cdot \exp\left(-r(\beta, a, gap)_n\right)}{\left(r(\beta, a, gap)_n\right)^3}\right] \ldots \right. \\ \quad \left. + \dfrac{2\cdot x(a, b, gap)_n \cdot \left(r(\beta, a, gap)_n - 1 + \exp\left(-r(\beta, a, gap)_n\right)\right)}{\left(r(\beta, a, gap)_n\right)^3} \right] \\ \text{return } B \end{array} \right.$$

Equation 11.105

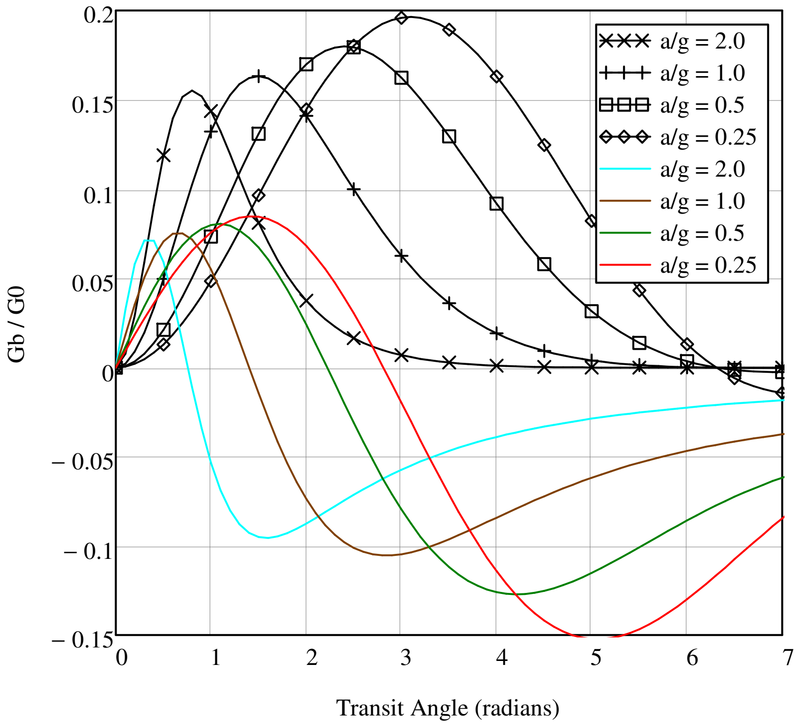
$$\mathrm{ImI}(\beta,\gamma,a,b,gap) := -\left[\frac{(\beta\cdot gap - \sin(\beta\cdot gap))\left(\mathrm{I0}(\gamma\cdot b)^2 - \mathrm{I1}(\gamma\cdot b)^2\right)}{\beta^2\cdot gap^2\cdot \mathrm{I0}(\gamma\cdot a)^2} + 2\cdot\beta\cdot gap\cdot\sum_{n=1}^{nmax}\left[\frac{A(\beta,a,b,gap)_n}{\left[\beta^2\cdot gap^2 + \left(r(\beta,a,gap)_n\right)^2\right]^2} - \frac{B(\beta,a,b,gap)_n}{\left[\beta^2\cdot gap^2 + \left(r(\beta,a,gap)_n\right)^2\right]}\right]\right]$$

Equation 11.102

$$B2(\beta,a,b,gap) := \frac{1}{2\cdot R\cdot(R+1)\cdot \omega q\_\omega}\cdot\left[\mathrm{ImI}\left[(1-\omega q\_\omega)\cdot\beta,\frac{(1-\omega q\_\omega)\cdot\beta}{R},a,b,gap\right] - \mathrm{ImI}\left[(1+\omega q\_\omega)\cdot\beta,\frac{(1+\omega q\_\omega)\cdot\beta}{R},a,b,gap\right]\right]$$

Equation 11.100 (Imaginary part)

$\beta := 0, 0.05 .. 7$



The results are independent of the reduced plasma frequency if $\frac{\omega_q}{\omega} < 0.1.$

$u0\_c = 0.01$

$\omega q\_\omega = 0.05$

Figure 11.16

Computed in the limit of small beam velocity