Worksheet 11.2 Computation of the plasma frequency reduction factor

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

This sheet is based on

Brewer, G. R. (1956). "Some effects of magnetic field strength on space-charge-wave propagation." Proceedings of the IRE **44**(7): 896-903.

Note that there are several mistakes in this paper.

and

MacKenzie, L. A. (1962). Space charge waves in finite magnetic fields. Fourth International Conference on Microwave Tubes. The Hague, The Netherlands: 663-668.

Note: For consistency with the usage in this book γ has been replaced by β and Ω is expressed in terms of the ratio m of the magnetic field to the Brillouin field. The ratio a/b is denoted by A.

Define τb. Note that τb can be imaginary if m > 1

$$\tau b(\beta b, m, p) := \beta b \cdot \left[\frac{\frac{1}{p^2} - 1}{\frac{1}{p^2 - 2\cdot(m^2 - 1)} - 1} \right]^{\frac{1}{2}}$$

Equation 11.63

$$p = \frac{\omega_q}{\omega_0}$$

Equation 11.61

$$\frac{{\omega_0}^2}{{\omega_p}^2} = p^2 - 2\cdot(m^2 - 1)$$

Equation 11.65

Define a function based on the right-hand side of Equation 11.62

$$f1(\beta b, A) := \frac{1}{\beta b} \cdot \frac{I1(\beta b)\cdot K0(A\cdot\beta b) + I0(A\cdot\beta b)\cdot K1(\beta b)}{I0(A\cdot\beta b)\cdot K0(\beta b) - I0(\beta b)\cdot K0(A\cdot\beta b)}$$

Define a function based on LHS of Equation 11.62. Note that there is a mistake in Brewer's version of the equation.

$$f2(\beta b, m, p) := \frac{\left(\frac{1}{p^2} - 1\right)}{\tau b(\beta b, m, p)} \cdot \frac{I1(\tau b(\beta b, m, p))}{I0(\tau b(\beta b, m, p))}$$

The solution is the value of p for which f1 = f2.
Since f2 varies very rapidly with p it is necessary to use the inverse functions to ensure a stable solution

$$f(\beta b, A, m, p) := \frac{1}{f1(\beta b, A)} - \frac{1}{f2(\beta b, m, p)}$$

A guessed value of p is used to seed the solution which is then found as a function of βb, A and m using the Mathcad *root* function

$$p1 := 0.9 \qquad p(\beta b, A, m) := root(f(\beta b, A, m, p1), p1)$$

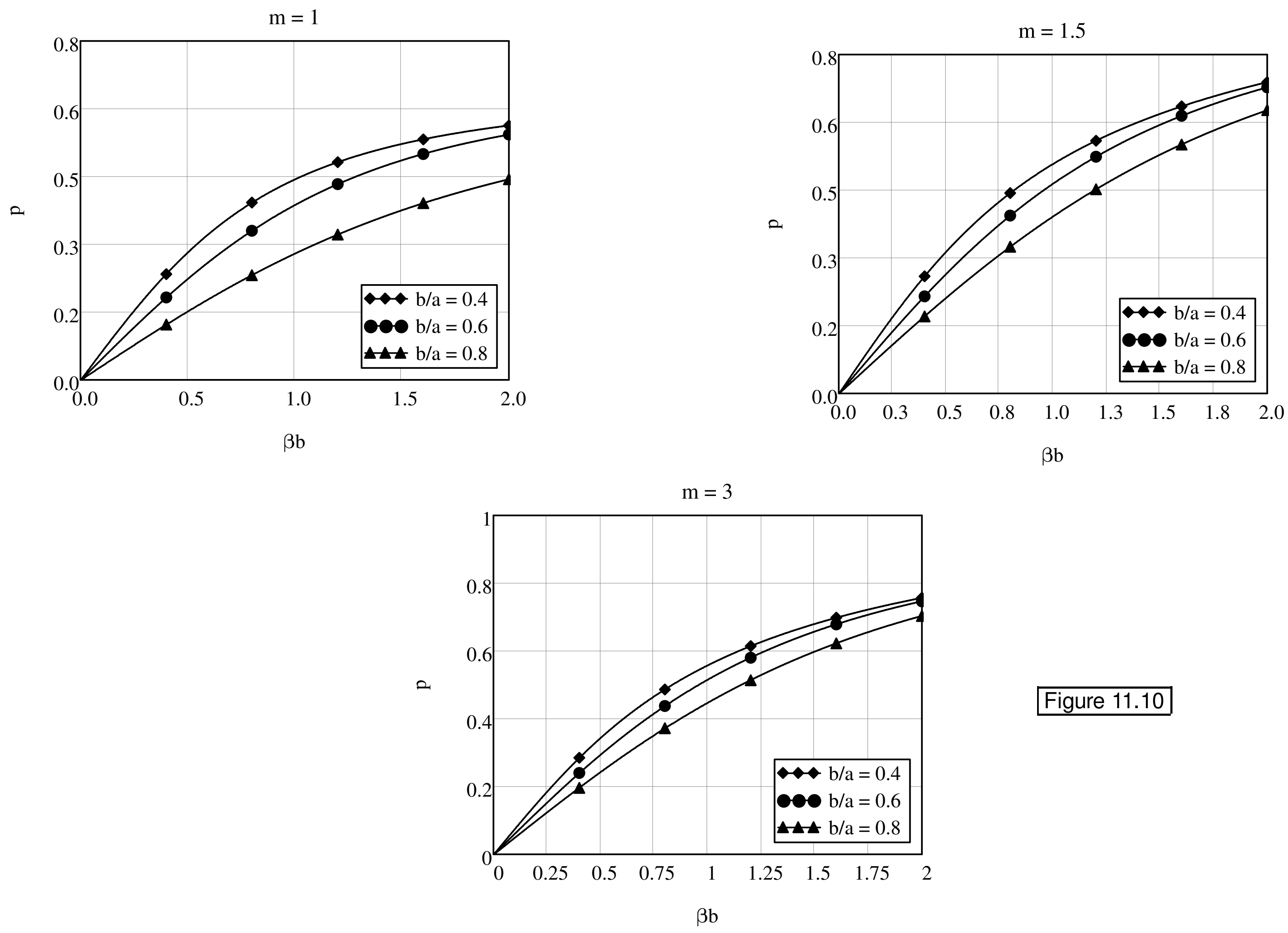

m = 1
0.8
0.6
0.5
0.3
0.2
0.0
p
0.0
0.5
1.0
1.5
2.0
βb
b/a = 0.4
b/a = 0.6
b/a = 0.8
m = 1.5
0.8
0.6
0.5
0.3
0.2
0.0
p
0.0
0.3
0.5
0.8
1.0
1.3
1.5
1.8
2.0
βb
b/a = 0.4
b/a = 0.6
b/a = 0.8
m = 3
1
0.8
0.6
0.4
0.2
0
p
0
0.25
0.5
0.75
1
1.25
1.5
1.75
2
βb
b/a = 0.4
b/a = 0.6
b/a = 0.8

Figure 11.10

Calculation of the ratio of the electromagnetic to the kinetic power flow

See; Louisell, W. H. and J. R. Pierce (1955). "Power flow in electron beam devices."
Proceedings of the IRE **43**(4): 425-427.

Assume that ω_q/ω =0.1 as a typical value and define τ assuming confined flow

$$\omega q\omega := 0.1 \qquad \tau(\beta b, A) := \tau b(\beta b, A, p(\beta b, A, 10))$$

Calculate the ratio of the electromagnetic power within the beam to the kinetic power by integration of the Poynting vector over the beam.

$$I_1(\beta b, A) := \int_0^1 \tau(\beta b, A)^2 \cdot r \cdot I1(\tau(\beta b, A)\cdot r)^2\, dr$$

$$P_1(\beta b, A) := \left(\frac{\omega q\omega}{2}\right)\cdot p(\beta b, A, 10)^2 \cdot \frac{(|\tau b(\beta b, A, p(\beta b, A, 10))|)^2}{(|I1(\tau b(\beta b, A, p(\beta b, A, 10)))|)^2}\cdot I_1(\beta b, A)$$

Calculate the ratio of the electromagnetic power outside the beam to the kinetic power by integration of the Poynting vector over the space between the beam edge and the tunnel wall

$$N(\beta b, A, r) := K0(A\cdot\beta b)\cdot I1(\beta b\cdot r) + I0(A\cdot\beta b)\cdot K1(\beta b\cdot r)$$

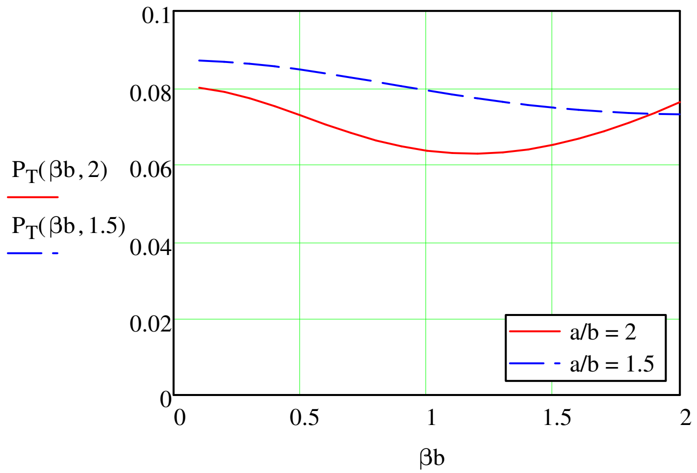
$$D(\beta b, A) := I0(\beta b)\cdot K0(\beta b\cdot A) - I0(\beta b\cdot A)\cdot K0(\beta b)$$

$$I_2(\beta b, A) := \int_1^A \beta b^2 \cdot r \cdot N(\beta b, A, r)^2\, dr$$

$$P_2(\beta b, A) := \left(\frac{\omega q\omega}{2}\right)\cdot p(\beta b, A, 10)^2 \cdot \left(\left|\frac{\tau(\beta b, A)\cdot I0(\tau(\beta b, A))}{\beta b\cdot I1(\tau(\beta b, A))\cdot D(\beta b, A)}\right|\right)^2 \cdot I_2(\beta b, A)$$

Calculate the ratio of the total electromagnetic power to the kinetic power

$$P_T(\beta b, A) := P_1(\beta b, A) + P_2(\beta b, A) \qquad \beta b := 0, 0.1 .. 2$$



Note: This graph agrees quantitatively with the approximate analysis of Louisell and Pierce which gives the ratio as ω_q/ω

Calculation of normalised body and surface currents using corrected formulae

Use Ω to denote Ω/ωp $\qquad \Omega(m) := \sqrt{\frac{\left(m^2 - 1\right)}{2}} \qquad m := 1, 1.01 .. 5 \qquad A := 100 \qquad \beta b := 0.5$

Surface current

$$I_S(\beta b, A, m) := \frac{-(1 - 0.1 \cdot p(\beta b, A, m)) \cdot \tau b(\beta b, m, p(\beta b, A, m)) \cdot I1(\tau b(\beta b, m, p(\beta b, A, m)))}{\left(p(\beta b, A, m)^2 - 4 \cdot \Omega(m)^2\right) \cdot \beta b^2}$$

Body current

$$I_{B1}(\beta b, A, m) := -\left[\frac{0.1}{p(\beta b, A, m)} - (1 - 0.1 \cdot p(\beta b, A, m)) \cdot \left(\frac{\tau b(\beta b, m, p(\beta b, A, m))^2}{\beta b^2} - 1\right)\right]$$

$$I_B(\beta b, A, m) := I_{B1}(\beta b, A, m) \cdot \frac{I1(\tau b(\beta b, m, p(\beta b, A, m)))}{\tau b(\beta b, m, p(\beta b, A, m))}$$

Convection current

$$I_C(\beta b, A, m) := I_S(\beta b, A, m) + I_B(\beta b, A, m)$$

Displacement currents for 0 < r < b and b < r < a

$$I_{D1}(\beta b, A, m) := \frac{I1(\tau b(\beta b, m, p(\beta b, A, m)))}{\tau b(\beta b, m, p(\beta b, A, m))}$$

$$I_{D2}(\beta b, A, m) := \left[\frac{1 - \beta b \cdot (K0(A \cdot \beta b) \cdot I1(\beta b) + I0(A \cdot \beta b) \cdot K1(\beta b))}{I0(\beta b) \cdot K0(A \cdot \beta b) - I0(A \cdot \beta b) \cdot K0(\beta b)}\right] \cdot \frac{I0(\tau b(\beta b, m, p(\beta b, A, m)))}{\beta b^2}$$

Equation 11.69

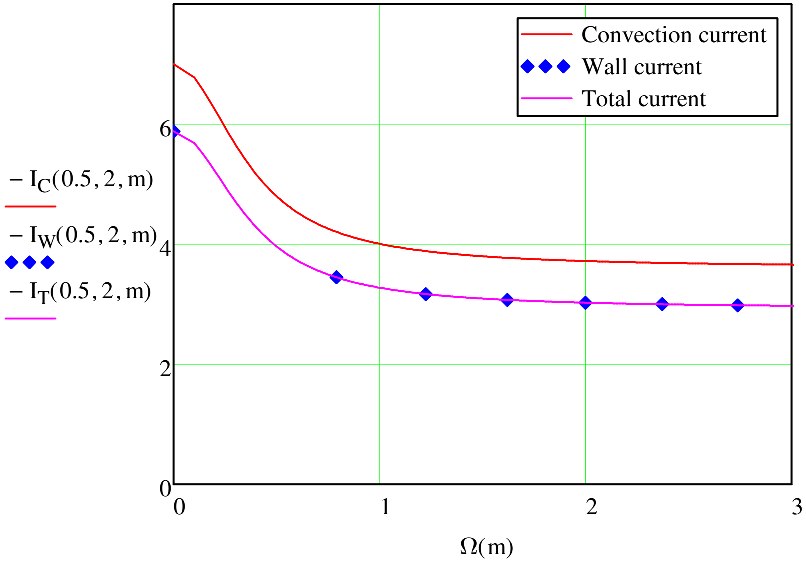
Total current as sum of convection and displacement currents

$$I_T(\beta b, A, m) := I_C(\beta b, A, m) + I_{D1}(\beta b, A, m) + I_{D2}(\beta b, A, m)$$

Wall current computed directly from the azimuthal magnetic field at r = a

$$I_W(\beta b, A, m) := \frac{I0(\tau b(\beta b, m, p(\beta b, A, m)))}{\left(\beta b^2\right)\cdot(I0(\beta b)\cdot K0(A\cdot\beta b) - I0(A\cdot\beta b)\cdot K0(\beta b))}$$

$m := 1, 1.01 .. 5$ $A := 2$ $\beta b := 0.5$



The normalised a.c. body and surface currents are

$$i_b(\beta b, m) := \frac{I_B(\beta b, A, m)}{I_C(\beta b, A, m)} \qquad i_s(\beta b, m) := \frac{I_S(\beta b, A, m)}{I_C(\beta b, A, m)} \qquad A = 2 \qquad \beta b = 0.5$$

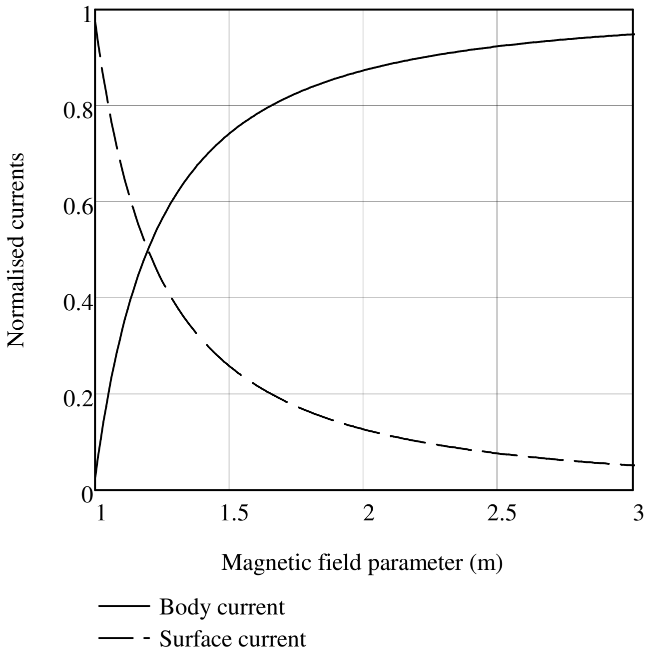


Figure 11.12

Define the ratio of the wall current to the convection current

$$q(\beta b, A, m) := \frac{I_W(\beta b, A, m)}{I_C(\beta b, A, m)} \qquad \beta b1 := 0, 0.1 .. 2$$

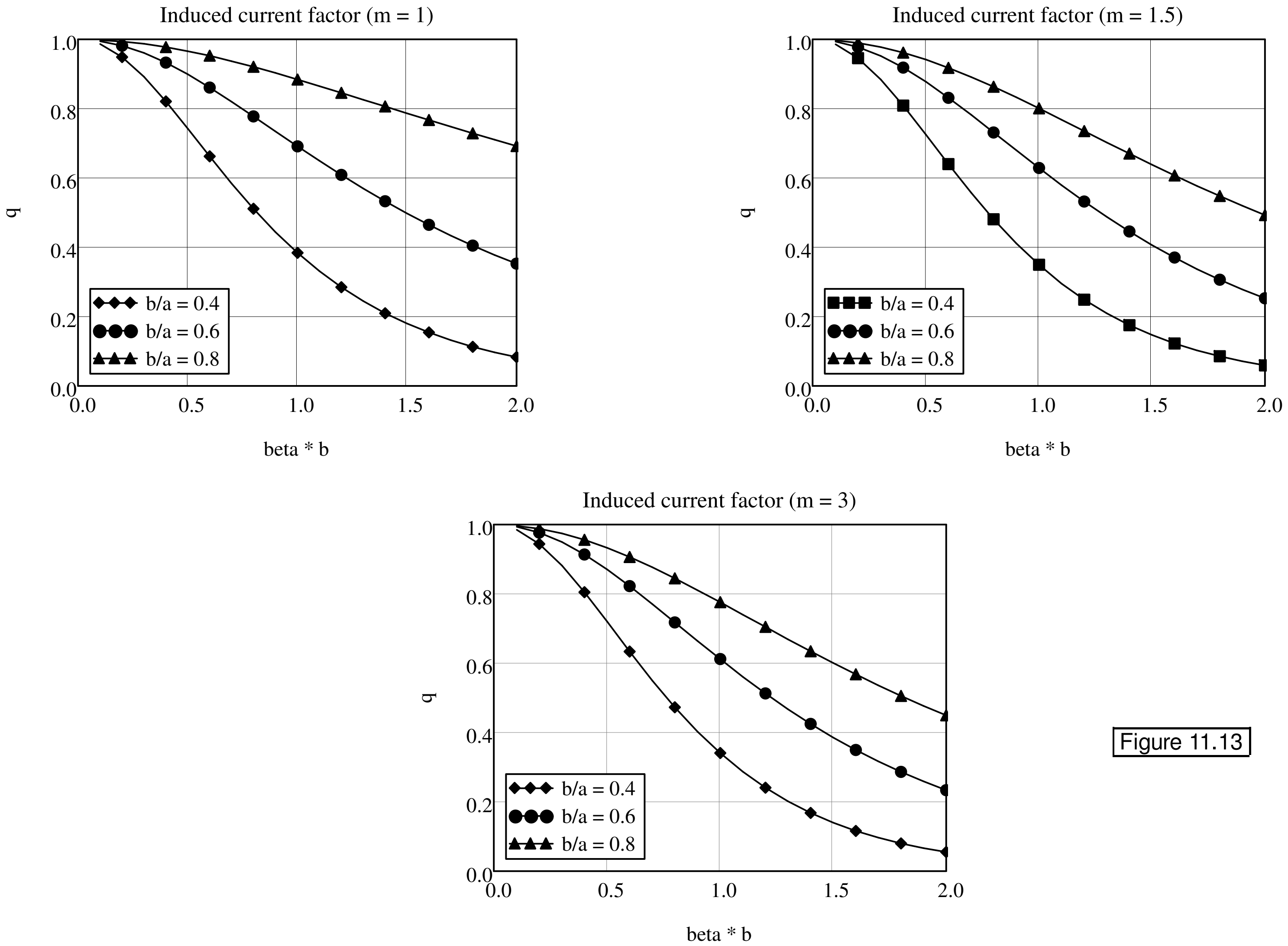

Induced current factor (m = 1)
q
beta * b
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
b/a = 0.4
b/a = 0.6
b/a = 0.8
Induced current factor (m = 1.5)
q
beta * b
b/a = 0.4
b/a = 0.6
b/a = 0.8
Induced current factor (m = 3)
q
beta * b
b/a = 0.4
b/a = 0.6
b/a = 0.8

Figure 11.13

Dispersion diagram for space-charge waves on a beam with radial boundaries

$u_0 := 1$ $\beta_e(\omega) := \frac{\omega}{u_0}$ $\omega_p := 0.2$ $b := 1$ $A := 2$ $m := 2$

$$\beta_f(\omega) := \frac{\left(\omega - p\left(\beta_e(\omega)\cdot b, A, m\right)\cdot\omega_p\right)}{u_0} \qquad \beta_s(\omega) := \frac{\left(\omega + p\left(\beta_e(\omega)\cdot b, A, m\right)\cdot\omega_p\right)}{u_0}$$

$$\beta_1(\omega) := \frac{\omega - \omega_p}{u_0} \qquad \beta_2(\omega) := \frac{\omega + \omega_p}{u_0} \qquad \omega := 0, 0.1 .. 2$$

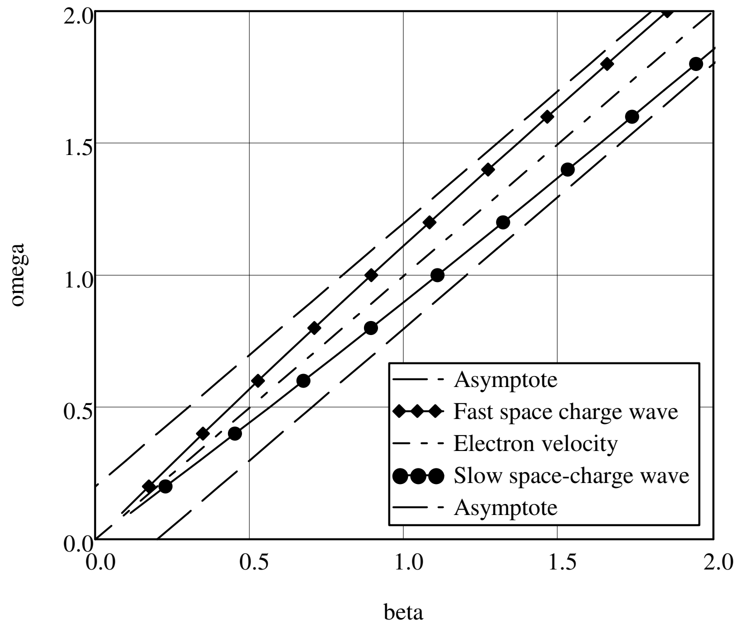


Figure 11.11