WS 4.1 Capacitance of a cylindrical grid

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This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

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This sheet models a section of the grid of width p between the centres of adjacent strips. The grid comprises an array of circular loops surrounded by a uniform dielectric layer and a conducting metal cylinder. The sheet calculates and displays the potential map using a finite difference calculation on a square mesh and calculates the capacitance per unit length of one strip divided by ε_0

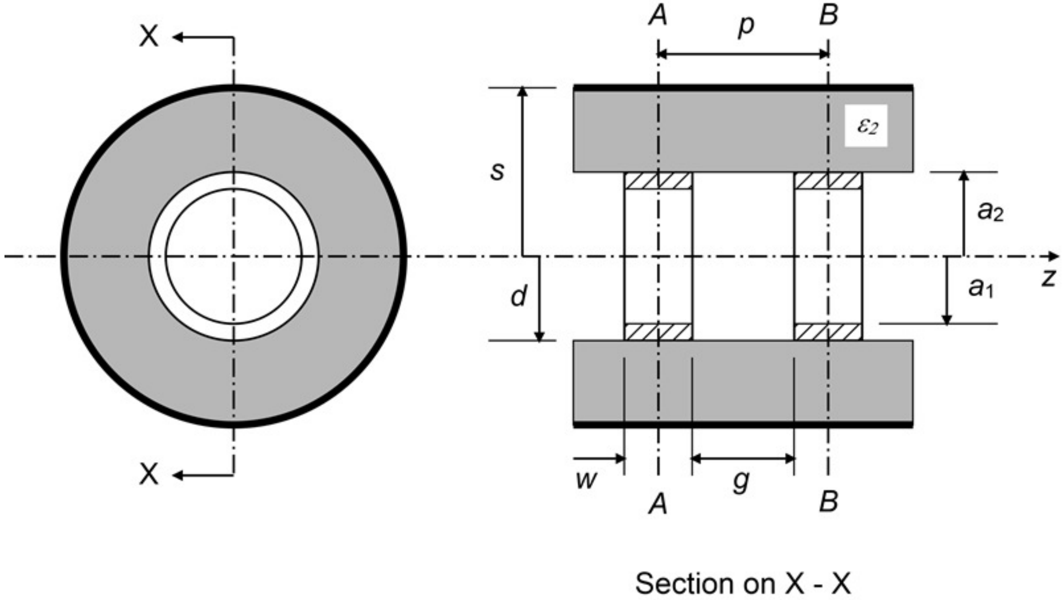

Figure 4.18

Structure dimensions

$a_1 := 10 \cdot mm$ $s := 14 \cdot mm$ $p := 10.0 \cdot mm$ $w := 8 \cdot mm$ $t := 1 \cdot mm$ $\varepsilon_r := 2.5$ $a_2 = a_1 + t$

Choose mesh size for FD calculations $mesh := 0.25 \cdot mm$

$NA := \frac{a_1}{mesh}$ $NS := \frac{s}{mesh}$ $NP := \frac{p}{mesh}$ $NW := \frac{w}{mesh}$ $NT := \frac{t}{mesh}$

$NA = 40$ $NS = 56$ $NP = 40$ $NW = 32$ $NT = 4$ These numbers should be as close to integers as possible

$Na1 := round(NA)$ $Ns := round(NS)$ $Np := round(NP)$ $Nw := round(NW)$ $Nt := round(NT)$ $Na2 := Na1 + Nt$ $Nw2 := \frac{Nw}{2}$ $Np2 := \frac{Np}{2}$

$Na1 = 40$ $Ns = 56$ $Np = 40$ $Nw2 = 16$ $Na2 = 44$ NB: Np and Nw must be even numbers

Mesh point index on the surface of the dielectric. This is normally equal to Na2 but can be increased to model the effect of a gap between the dielectric and the rings. $Nd := Na2$

Right hand boundary condition $Psym := 2$

Voltages on left-hand and right-hand strips $V_1 := 100$ $V_2 := 100$

Maximum number of iterations $Niter := 5000$

The zero, π/2 and π modes can be modelled as follows

Zero mode:	V1 = 100	V2 = 100	Psym = 2
π/2 mode	V1 = 100	V2 = 0	Psym = 0
π mode	V1 = 100	V2 = -100	Psym = 2

Calculation of voltages at the mesh points using a finite difference solution of Laplace's equation

The region below can be collapsed to allow the input data and the results to be on the screen simulatneously.

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$$
\begin{aligned}
VM := \;& \text{for } i \in 0..Np \\
& \quad \text{for } j \in 0..Ns \\
& \quad\quad V_{i,j} \leftarrow 0 \\
& \text{for } n \in 1..Niter \\
& \quad V_{0,0} \leftarrow \frac{4 \cdot V_{0,1} + 2 \cdot V_{1,0}}{6} \\
& \quad V_{Np,0} \leftarrow \frac{4 \cdot V_{Np,1} + Psym \cdot V_{Np-1,0}}{6} \\
& \quad \text{for } i \in 1..Np-1 \\
& \quad\quad V_{i,0} \leftarrow \frac{V_{i-1,0} + 4 \cdot V_{i,1} + V_{i+1,0}}{6} \\
& \quad \text{for } j \in 1..(Na1-1) \\
& \quad\quad V_{0,j} \leftarrow \left[2\,V_{1,j} + \left(1 + \frac{1}{2 \cdot j}\right)V_{0,j+1} + \left(1 - \frac{1}{2 \cdot j}\right)V_{0,j-1}\right] \cdot 0.25 \\
& \quad\quad V_{Np,j} \leftarrow \left[Psym\,V_{Np-1,j} + \left(1 + \frac{1}{2 \cdot j}\right)V_{Np,j+1} + \left(1 - \frac{1}{2 \cdot j}\right)V_{Np,j-1}\right] \cdot 0.25 \\
& \quad\quad \text{for } i \in 1..Np-1 \\
& \quad\quad\quad V_{i,j} \leftarrow \left[V_{i+1,j} + \left(1 + \frac{1}{2 \cdot j}\right)V_{i,j+1} + V_{i-1,j} + \left(1 - \frac{1}{2 \cdot j}\right)V_{i,j-1}\right] \cdot 0.25 \\
& \quad \text{for } j \in Na1..Na2 \\
& \quad\quad \text{for } i \in 0..Np \\
& \quad\quad\quad V_{i,j} \leftarrow V_1 \;\; \text{if } i \le Nw2 \\
& \quad\quad\quad V_{i,j} \leftarrow V_2 \;\; \text{if } i \ge Np - Nw2 \\
& \quad\quad\quad \text{for } i \in (Nw2+1)..(Np-Nw2-1) \\
& \quad\quad\quad\quad V_{i,j} \leftarrow \left[V_{i+1,j} + \left(1 + \frac{1}{2 \cdot j}\right)V_{i,j+1} + V_{i-1,j} + \left(1 - \frac{1}{2 \cdot j}\right)V_{i,j-1}\right] \cdot 0.25 \;\; \text{if } j \ne Nd
\end{aligned}
$$

Array indices are i for z and j for r.
Initially set all the mesh voltages to zero

Voltages on the axis.

Voltages at mesh points inside the rings

Voltages at mesh points in the thickness of the rings

$$V_{i,j} \leftarrow \frac{\left(1+\frac{1}{2\cdot j}\right)\varepsilon_r\cdot V_{i,j+1} + \left(1-\frac{1}{2\cdot j}\right)V_{i,j-1} + \frac{\left(1+\varepsilon_r\right)}{2}\cdot\left(V_{i+1,j}+V_{i-1,j}\right)}{2\cdot\left(\varepsilon_r+1\right)+\frac{1}{2\cdot j}\cdot\left(\varepsilon_r-1\right)} \quad \text{otherwise}$$

for $j \in Na2+1..Ns-1$

Voltages at mesh points between the rings and the shield

$$V_{0,j} \leftarrow \begin{cases} \left[2\,V_{1,j} + \left(1+\frac{1}{2\cdot j}\right)V_{0,j+1} + \left(1-\frac{1}{2\cdot j}\right)V_{0,j-1}\right]\cdot 0.25 & \text{if } j \neq Nd \\ \dfrac{\left(1+\varepsilon_r\right)\cdot V_{1,j} + \left(1+\frac{1}{2\cdot j}\right)\varepsilon_r\cdot V_{0,j+1} + \left(1-\frac{1}{2\cdot j}\right)V_{0,j-1}}{2\cdot\left(\varepsilon_r+1\right)+\frac{1}{2\cdot j}\cdot\left(\varepsilon_r-1\right)} & \text{otherwise} \end{cases}$$

$$V_{Np,j} \leftarrow \begin{cases} \left[Psym\,V_{Np-1,j} + \left(1+\frac{1}{2\cdot j}\right)V_{Np,j+1} + \left(1-\frac{1}{2\cdot j}\right)V_{Np,j-1}\right]\cdot 0.25 & \text{if } j \neq Nd \\ \dfrac{\frac{\left(1+\varepsilon_r\right)\cdot Psym}{2}\cdot V_{Np-1,j} + \left(1+\frac{1}{2\cdot j}\right)\varepsilon_r\cdot V_{Np,j+1} + \left(1-\frac{1}{2\cdot j}\right)V_{Np,j-1}}{2\cdot\left(\varepsilon_r+1\right)+\frac{1}{2\cdot j}\cdot\left(\varepsilon_r-1\right)} & \text{otherwise} \end{cases}$$

for $i \in 1..Np-1$

$$V_{i,j} \leftarrow \left[V_{i+1,j} + \left(1+\frac{1}{2\cdot j}\right)V_{i,j+1} + V_{i-1,j} + \left(1-\frac{1}{2\cdot j}\right)V_{i,j-1}\right]\cdot 0.25 \quad \text{if } j \neq Nd$$

$$V_{i,j} \leftarrow \frac{\left(1+\frac{1}{2\cdot j}\right)\varepsilon_r\cdot V_{i,j+1} + \left(1-\frac{1}{2\cdot j}\right)V_{i,j-1} + \frac{\left(1+\varepsilon_r\right)}{2}\cdot\left(V_{i+1,j}+V_{i-1,j}\right)}{2\cdot\left(\varepsilon_r+1\right)+\frac{1}{2\cdot j}\cdot\left(\varepsilon_r-1\right)} \quad \text{otherwise}$$

for $i \in 0..Np$

$V_{i,Ns} \leftarrow 0$

$VM \leftarrow V$

return VM

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Working equations

On the axis
$$V_{i,0} = \frac{1}{6}\left(V_{i+1,0} + 4V_{i,j+1} + V_{i-1,0}\right)$$

In regions of uniform permittivity
$$V_{i,j} = \frac{1}{4}\left[V_{i+1,j} + \left(1+\frac{1}{2j}\right)V_{i,j+1} + V_{i-1,j} + \left(1-\frac{1}{2j}\right)V_{i,j-1}\right]$$

On the surface of the dielectric
$$V_{i,j} = \frac{\left(1+\frac{1}{2j}\right)\varepsilon_r V_{i,j+1} + \left(1-\frac{1}{2j}\right)V_{i,j-1} + \left(\frac{1+\varepsilon_r}{2}\right)\left(V_{i+1,j} + V_{i-1,j}\right)}{2\left(\varepsilon_r+1\right) + \frac{1}{2j}\left(\varepsilon_r - 1\right)}$$

Calculation of the charge per unit length on the surfaces of the solution region

$$Q1 := \left[\sum_{i=0}^{Np2}\left(2\cdot VM_{i,Ns-1}\right) - VM_{0,Ns-1} - VM_{Np2,Ns-1}\right]\cdot\pi\cdot\varepsilon_r\cdot\left(Ns - \frac{1}{2}\right)$$

$Q1 = 1.3\times 10^{5}$

$$Q2 := \frac{\pi}{4}\cdot\sum_{j=0}^{Nd-1}\left[\left(VM_{Np2-1,j} - VM_{Np2+1,j} + VM_{Np2-1,j+1} - VM_{Np2+1,j+1}\right)\cdot(2\cdot j + 1)\right]$$

$Q2 = -0$

$$Q3 := \frac{\pi\cdot\varepsilon_r}{4}\cdot\sum_{j=Nd}^{Ns-1}\left[\left(VM_{Np2-1,j} - VM_{Np2+1,j} + VM_{Np2-1,j+1} - VM_{Np2+1,j+1}\right)\cdot(2\cdot j + 1)\right]$$

$Q3 = -0$

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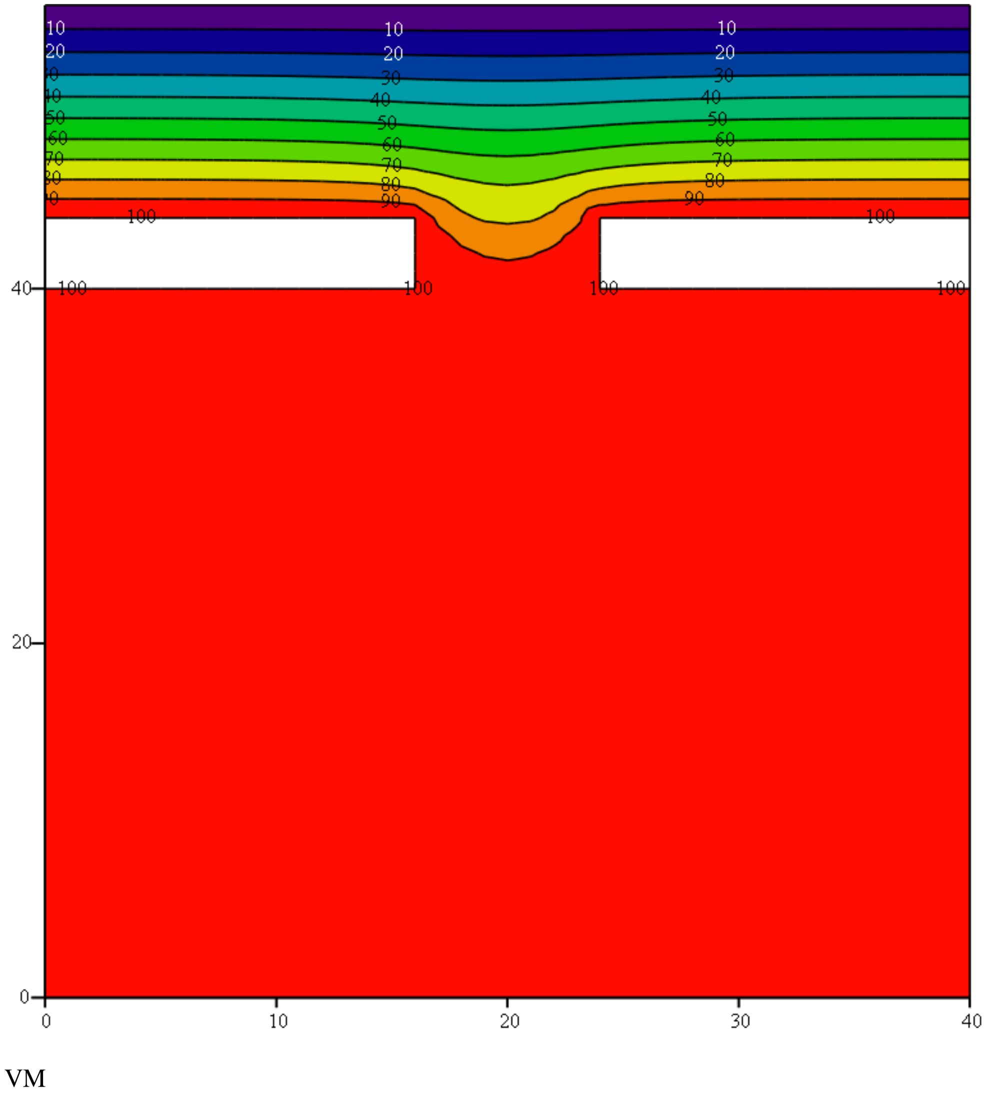

NB Care should be taken that the routine has converged completely.

Capacitance per unit length of the structure divided by ε_0

$$\text{Capacitance} := \frac{Q1 + Q2 + Q3}{V_1 \cdot Np2} \qquad \text{Capacitance} = 63.09$$

Capacitance per unit length divided by ε_0 of a coaxial line with inner and outer radii (a1 + t/2) and s.

$$C_{00} := \frac{2 \cdot \pi \cdot \varepsilon_r}{\ln\left(\frac{s}{a_1 + 0.5 \cdot t}\right)} \qquad C_{00} = 54.60$$

$$\alpha_C := \frac{\text{Capacitance}}{C_{00}}$$

$$\alpha_C = 1.155$$

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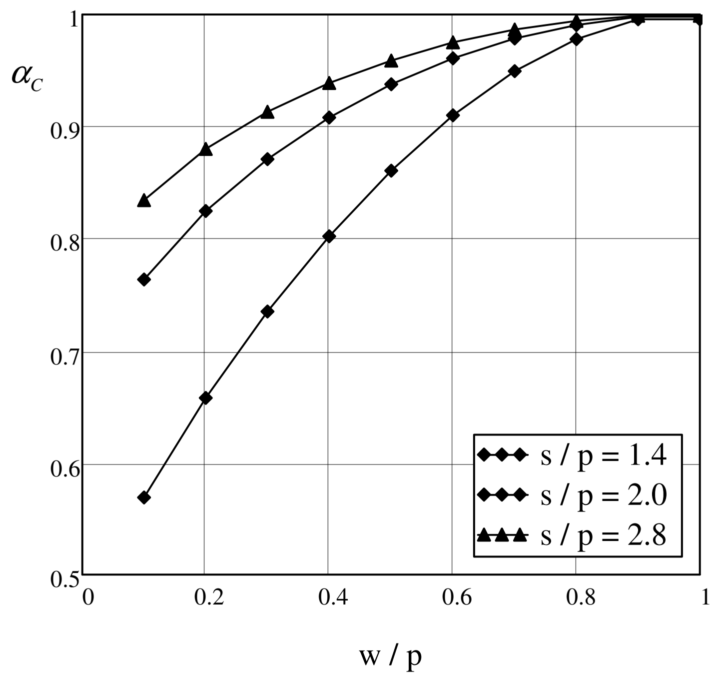
The results of calculations with p = 10mm, a = 10mm, t = 1mm and ε_r = 1 were computed and stored in the matrices below for a range of values of w/p and s/p.

$$wp := \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \\ 1.0 \end{pmatrix} \quad \alpha_{14} := \begin{pmatrix} 0.679 \\ 0.785 \\ 0.877 \\ 0.957 \\ 1.027 \\ 1.086 \\ 1.133 \\ 1.167 \\ 1.188 \\ 1.188 \end{pmatrix} \cdot \frac{\ln\left(\frac{14}{11}\right)}{\ln\left(\frac{14}{10.5}\right)} \quad \alpha_{20} := \begin{pmatrix} 0.823 \\ 0.889 \\ 0.939 \\ 0.979 \\ 1.011 \\ 1.036 \\ 1.055 \\ 1.068 \\ 1.076 \\ 1.076 \end{pmatrix} \cdot \frac{\ln\left(\frac{20}{11}\right)}{\ln\left(\frac{20}{10.5}\right)} \quad \alpha_{28} := \begin{pmatrix} 0.876 \\ 0.924 \\ 0.959 \\ 0.986 \\ 1.007 \\ 1.024 \\ 1.036 \\ 1.044 \\ 1.049 \\ 1.049 \end{pmatrix} \cdot \frac{\ln\left(\frac{28}{11}\right)}{\ln\left(\frac{28}{10.5}\right)}$$

The multiplying factors convert the values of α from those computed at the mean radius of the helix to those at the outer radius.

Figure 4.19



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