Worksheet 8.3 Cylindrical crossed-field diode

This Mathcad 14 worksheet is designed to accompany the author's book "Microwave and RF Vacuum Electronic Power Sources", Cambridge University Press (2018). The section, equation, and figure numbers refer to the corresponding sections, equations, and figures in the book. Data input fields are highlighted in yellow and output fields are highlighted in green.

This sheet implements the equations in
A. H. Falkner, "Double stream flow in the smooth-bore magnetron,"
Proceedings of the Institution of Electrical Engineers, vol. 120, pp. 959-961, 1973.
(See Section 8.5)

Find the anode radius as a function of Ec and J by finding the value of R where dR/dτ = 0.

$$
\mathrm{Ra}(\mathrm{Ec},\mathrm{J}) := \left|\begin{array}{l}
R \leftarrow \begin{pmatrix}1\\0\end{pmatrix} \\
D(\tau,R) \leftarrow \begin{bmatrix} R_1 \\ \dfrac{1}{\left(R_0\right)^3} - R_0 + \dfrac{Ec}{R_0} + \dfrac{J\cdot\tau}{R_0} \end{bmatrix} \\
Y \leftarrow \mathrm{AdamsBDF}(R,0,4,100,D) \\
y \leftarrow Y^{\langle 1\rangle} \\
dy \leftarrow Y^{\langle 2\rangle} \\
\text{for } n \in 1..100 \\
\quad \left| \begin{array}{l} Ra \leftarrow y_n \\ (\text{break}) \text{ if } dy_n < 0 \\ Ra \leftarrow 0 \text{ if } n = 100 \end{array}\right. \\
\text{return } Ra
\end{array}\right.
$$

$$\frac{dR_0}{d\tau} = R_1$$

$$\frac{dR_1}{d\tau} = f(Ec, J, R, \tau)$$

Equation 8.84 is expressed as a pair of simultaneous first-order equations.

This calculation is for a diode that is cut off so that equal and opposite currents J_c flow.

Find normalised cathode current J_c for given R_a and Ec

$$
J_c\left(R_a, Ec\right) := \left|\begin{array}{l}
J \leftarrow 0.05 \\
Jc \leftarrow \mathrm{root}\left(\mathrm{Ra}(Ec,J) - R_a, J\right) \\
\text{return } Jc
\end{array}\right.
$$

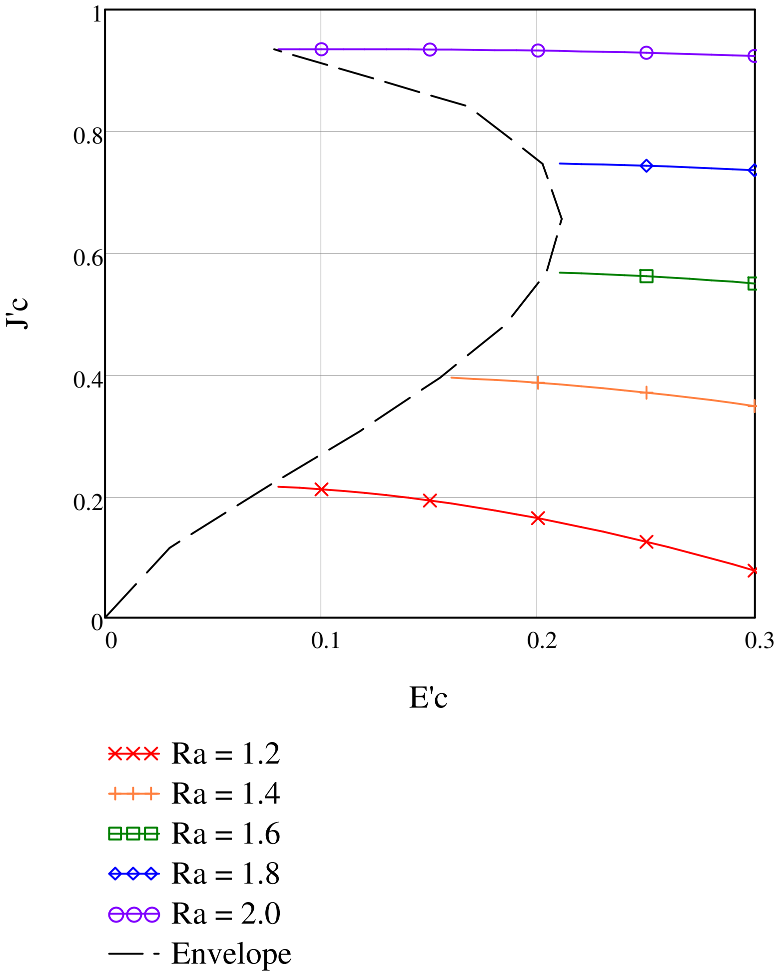
Find the envelope of the minimum value of Ec for a given Ra

$$
E_c\left(R_a\right) := \left|\begin{array}{ll}
E_c \leftarrow 0 \text{ if } R_a = 1 & \\
\text{for } i \in 0..250 & \text{otherwise} \\
\quad \left| \begin{array}{l} E_i \leftarrow 0.25 - 0.001\cdot i \\ (\text{break}) \text{ on error } J_1 \leftarrow J_c\left(R_a, E_i\right) \\ E_c \leftarrow E_i \end{array}\right. & \\
\text{return } E_c &
\end{array}\right.
$$

Calculate J for values of Ec decreasing from 0.25 until a point is reached where there is no solution

$$
J_1\left(R_a\right) := \left|\begin{array}{ll}
0 & \text{if } R_a = 1 \\
J_c\left(R_a, E_c\left(R_a\right)\right) & \text{otherwise}
\end{array}\right.
$$

$Ec := 0, 0.01 .. 0.3$ $RRa := 1, 1.1 .. 2$



Compare Figure 8.12 which is a copy of the figure in Falkner's paper.

Note that it takes a long time to compute this figure so it may be disabled if it is not required.

As the cathode current is increased for fixed anode radius the electric field at the surface of the cathode falls. Thus the points on the envelope curve represent space-charge limitation.

Langmuir's equations for space-charge limited flow.

$\gamma(R) := \ln(R)$ Equation 5.69

$\beta A(R) := \gamma(R) - 0.4\cdot\gamma(R)^2 + 0.0916667\cdot\gamma(R)^3 - 0.014242\cdot\gamma(R)^4 + 0.001679\cdot\gamma(R)^5 - 0.000161\cdot\gamma(R)^6$ Equation 5.71

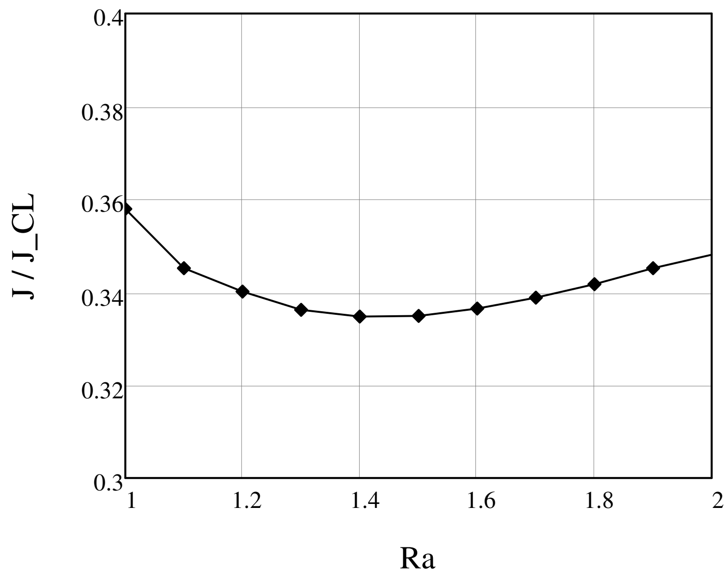
$J_{CL}(R) := \frac{4}{9}\cdot\left(\frac{R}{\beta A(R)}\right)^2\cdot\left(1 - \frac{1}{R^2}\right)^3$ $J'_{CL} = \frac{4}{9}\cdot\frac{R_a^2}{\beta^2}\left(1 - \frac{1}{R_a^2}\right)^3$ Equation 8.86

Calculate the radial current in a cut-off diode normalised to the Child-Langmuir current. When R tends to unity the ratio must tend to that for a planar diode.

$$JJ\_JCL(R) := \begin{cases} 0.358 & \text{if } R = 1 \\ \frac{J_1(R)}{J_{CL}(R)} & \text{otherwise} \end{cases}$$

$R_a := 1, 1.1 .. 2$

Figure 8.13



Investigate the effect of different choices of parameters on the space-charge hub in a cut-off diode

The normalised cathode field and cathode current may be set to the values for space-charge limitation or other values as desired.

Normalised radius of the hub $R_b := 1.6$

For first order trajectories the minimum cathode field and the space-charge limited cathode current are

$E_c(R_b) = 0.20400$ $J_1(R_b) = 0.56817$

Normalised cathode field $Ec := 0.20400$ Corresponding normalised cathode current $J_c(R_b, Ec) = 0.568$

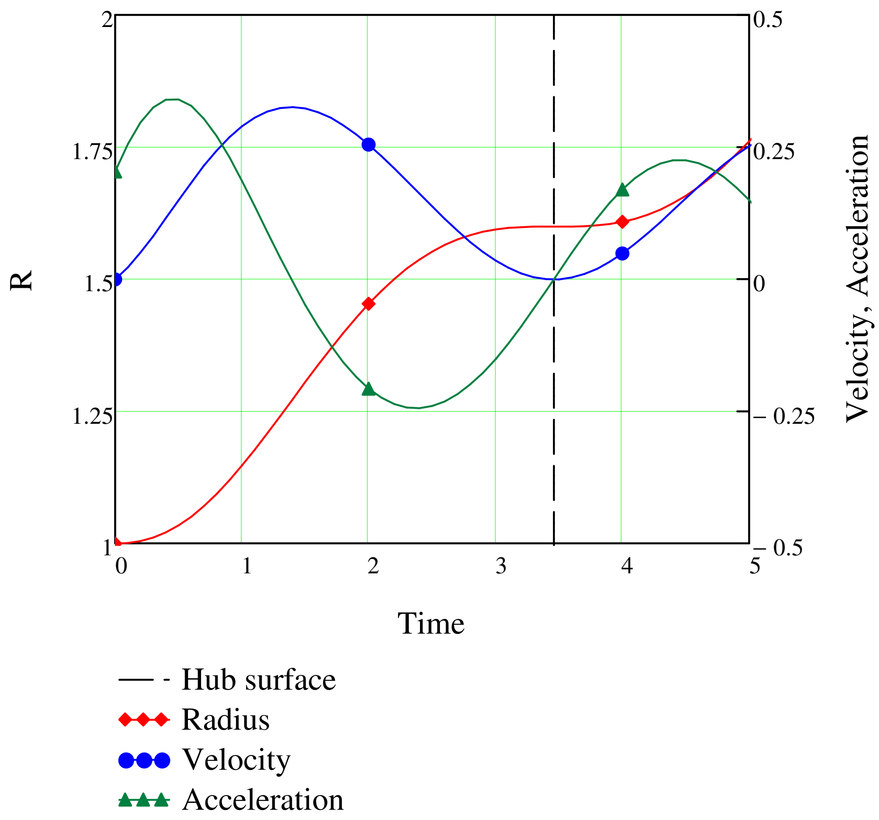
Normalised cathode current $Jc := 0.56817$

Acceleration $d2R(R,\tau) := \frac{1}{R^3} - R + \frac{Ec}{R} + \frac{Jc \cdot \tau}{R}$ Equation 8.84

Transit time for zero radial acceleration $\tau_b := \frac{1}{Jc} \cdot \left({R_b}^2 - \frac{1}{{R_b}^2} - Ec \right)$ $\tau_b = 3.459$

Solve the differential equation for the electron motion. The columns of the solution matrix Y are, in order: τ, R, dR/dτ

$$Y := \left| \begin{array}{l} R \leftarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ D(\tau, R) \leftarrow \begin{bmatrix} R_1 \\ \frac{1}{(R_0)^3} - R_0 + \frac{Ec}{R_0} + \frac{Jc \cdot \tau}{R_0} \end{bmatrix} \\ R1 \leftarrow \mathrm{rkfixed}(R, 0, 10, 100, D) \\ \text{return } R1 \end{array} \right.$$


R
Velocity, Acceleration
Time
2
1.75
1.5
1.25
1
0.5
0.25
0
− 0.25
− 0.5
0
1
2
3
4
5
Hub surface
Radius
Velocity
Acceleration

Electric field as a function of radius

$$E_r = \frac{\eta}{4} B_z^2 \frac{r_c r_b}{r_c}\left(1 - \frac{1}{R^4}\right)$$

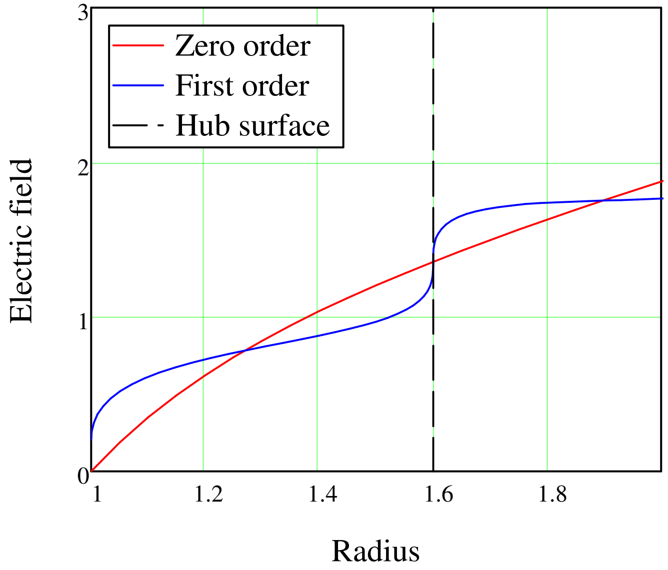
$$\frac{\eta E_r}{r_c \omega_L^2} = E_r' = R\left(1 - \frac{1}{R^4}\right)$$

$$E_r' = \frac{J_+' \tau + E_c'}{R}$$

Zero order hub $\mathrm{Er0}(R) := R - \frac{1}{R^3}$ Derived from Equation 8.69

First order hub $\mathrm{Er1} := \begin{array}{|l} \text{for } n \in 0..100 \\ \quad \mathrm{Er}_n \leftarrow \dfrac{\mathrm{Jc}\cdot\left(Y^{\langle 0\rangle}\right)_n + \mathrm{Ec}}{\left(Y^{\langle 1\rangle}\right)_n} \\ \text{return } \mathrm{Er} \end{array}$ From equation 8.82

Plotting ranges $R1 := 1.0, 1.05..2.2$ $y1 := 0..3$



Normalised charge density as a function of radius

The normalised charge density is represented by $\dfrac{{\omega_p}^2}{{\omega_L}^2}$

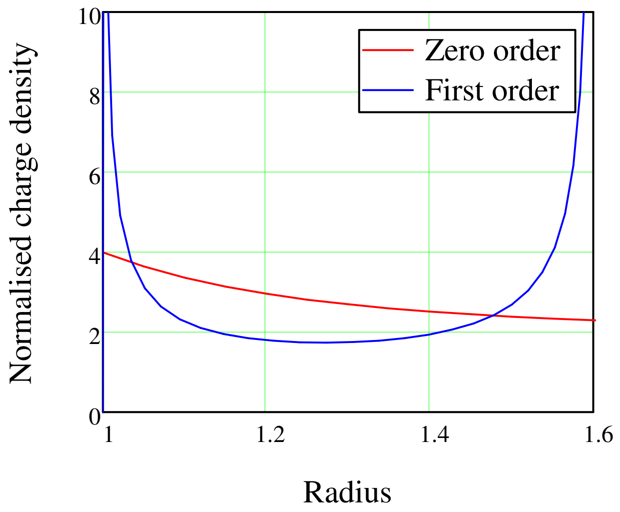
Zero order hub $\quad \rho n_0(R) := 2\cdot\left(1+\dfrac{1}{R^4}\right)$ $\quad \boxed{\text{Equation 8.67}}$

First order hub $\quad \rho 1 := \left| \begin{array}{l} \text{for } n \in 1..99 \\ \quad \rho_n \leftarrow Jc\cdot\left[\left(Y^{\langle 2\rangle}\right)_n\right]^{-1} \\ \text{return } \rho \end{array}\right.$ $\quad \boxed{\text{Derived from Equations 8.78 and 8.83}}$

$$J'_+ = \frac{2\eta}{\varepsilon_0\omega_L^3 r_c}\cdot\frac{1}{2}\rho r_c\omega_L R\frac{dR}{d\tau}$$

$$J'_+ = \frac{\omega_p^2}{\omega_L^2}\cdot R\frac{dR}{d\tau}$$

$$Q_1 = 2\pi\int_1^{R_b}\frac{J'_+}{R\dfrac{dR}{d\tau}}R\,dR = 2\pi J'_+\tau_b$$



Normalised hub charge per unit length

Zero order hub

$$Q_0(R_b) := 2\cdot\pi\cdot\int_1^{R_b} 2\cdot\left(1+\frac{1}{R^4}\right)\cdot R\,dR$$

$$Q_0(R_b) = 13.631$$

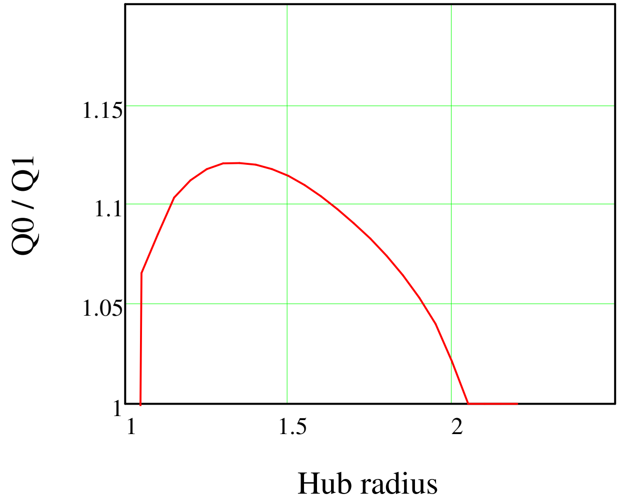
First order hub

$$Q_1(R_b) := 2\cdot\pi\cdot\left(R_b^{\,2} - \frac{1}{R_b^{\,2}} - E_c(R_b)\right)$$

$$Q_1(R_b) = 12.349$$

Ratio of charges

$$\frac{Q_0(R_b)}{Q_1(R_b)} = 1.104$$



Note:

The last sentence of the first paragraph on p. 307 of the book is incorrect. The graph to the left shows the charge in a first order hub is always less than or equal to that in a zero order hub. The flux of the electric field external to the hub is the same in both cases. But, because the flux of the electric field on the surface of the cathode is non-zero in a first order hub, the application of Gauss' Theorem shows that the hub charge must be reduced in that case.