Chapter 3: Limits and Continuity

Part A: Limits

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An Example

Consider $f(x) = 2x + 5$. What happens if we take values of x that approach 0? Here are some calculations.

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Suppose ϵ is some positive number and we need $f(x) = 2x + 5$ to be within ϵ of 5. Now,

$$|(2x+5) - 5| < \epsilon \iff |2x| < \epsilon \iff |x| < \epsilon/2.$$

Thus, if $|x| < \epsilon/2$, we are guaranteed that $|f(x) - 5| < \epsilon$.

Definition of Limit

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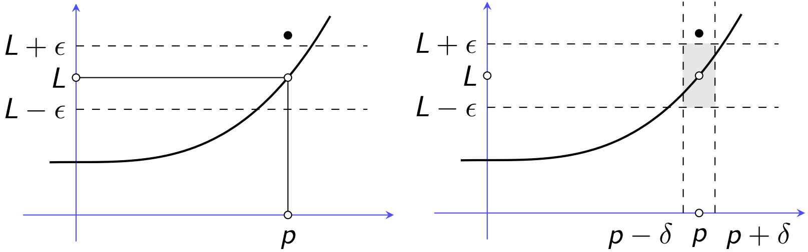
We may also write '$f(x) \to L$ as $x \to p$' for $\lim_{x \to p} f(x) = L$.

Visualising Limits

The two stages in a limit process.



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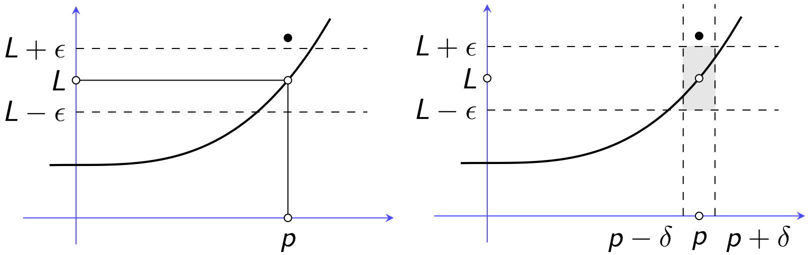

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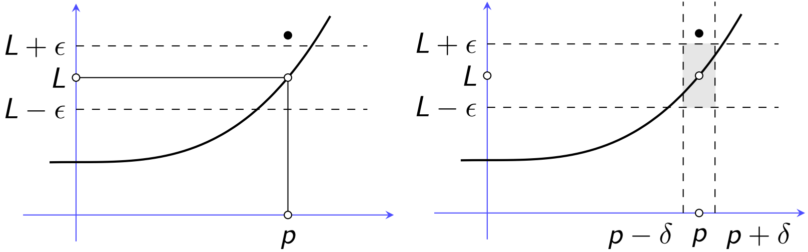

In the first stage, we have a requirement to make the output $f(x)$ lie between $L-\epsilon$ and $L+\epsilon$.
In the second stage, we meet the requirement by finding a δ such that input being between $p-\delta$ and $p+\delta$ guarantees that the output is between $L-\epsilon$ and $L+\epsilon$ (except perhaps at p itself).

Uniqueness of Limit

Theorem 1

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Then there are $\delta_L, \delta_M > 0$ such that

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Let $\delta = \min\{\delta_L, \delta_M\}$ and $x_0 \in (a - \delta, a + \delta)$.

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Let $\delta = \min\{\delta_L, \delta_M\}$ and $x_0 \in (a - \delta, a + \delta)$.
Then $|f(x_0) - L| < \epsilon$ and $|f(x_0) - M| < \epsilon$. Hence,

$$|M - L| \leq |M - f(x_0)| + |f(x_0) - L| < \epsilon + \epsilon = |M - L|,$$

which gives the impossible statement $|M - L| < |M - L|$. □

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We start by considering an $\epsilon > 0$. We need to find a $\delta > 0$ such that $|x - a| < \delta \implies |x - a| < \epsilon$. Clearly $\delta = \epsilon$ will work.

Task: Let $f(x) = c$ be a constant function. Show that $\lim_{x \to p} f(x) = c$.

Example of $y = x^2$

Consider the limit of $y = x^2$ at $x = 2$. A natural guess is that $x^2 \to 2^2 = 4$ as $x \to 2$. We test this for some values of $\epsilon > 0$.

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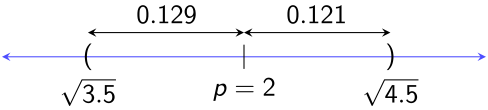
We note that the function maps $(\sqrt{3.5},\ \sqrt{4.5})$ into $(3.5,\ 4.5)$. The interval $(\sqrt{3.5},\ \sqrt{4.5})$ contains 2 but is not centered on it.



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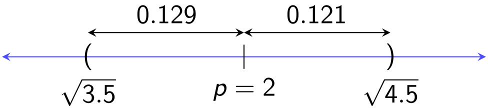

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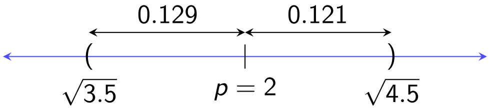

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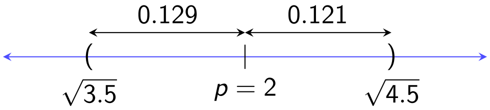

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Generally, for any $\epsilon > 0$, take $\delta = \min\{2 - \sqrt{4 - \epsilon},\ \sqrt{4 + \epsilon} - 2\}$.

Characterisations of Limit

Theorem 2

$$\lim_{x \to p} f(x) = L \iff \lim_{x \to p} (f(x) - L) = 0 \iff \lim_{h \to 0} f(p + h) = L.$$

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The first two are identical. The first can be converted to the third, and conversely, by the substitution $x = p + h$. □

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Now note that $||f(x)| - 0| = |f(x) - 0|$. □

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Proof. The triangle inequality gives $||f(x)| - |M|| \leq |f(x) - M|$.

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Proof. The triangle inequality gives $||f(x)| - |M|| \leq |f(x) - M|$.
Let $\epsilon > 0$. Since $\lim_{x \to p} f(x) = M$, there is a $\delta > 0$ such that
$0 < |x - p| < \delta \implies |f(x) - M| < \epsilon$. The same δ works for $|f(x)|$
since $|f(x) - M| < \epsilon$ implies $||f(x)| - |M|| \leq |f(x) - M| < \epsilon$. □

Non-existence of Limit: Example 1

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There is a $\delta > 0$ such that $0 < |x| < \delta \implies |\operatorname{sgn}(x) - L| < 1$.

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Consider the signum function, $\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$

Suppose $\lim_{x\to 0} \mathrm{sgn}(x) = L$. Consider $\epsilon = 1$.

There is a $\delta > 0$ such that $0 < |x| < \delta \implies |\mathrm{sgn}(x) - L| < 1$.

Then $|\mathrm{sgn}(\delta/2) - L| < 1$ and $|\mathrm{sgn}(-\delta/2) - L| < 1$.

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Therefore, by triangle inequality,

$$\begin{aligned} |\operatorname{sgn}(\delta/2) - \operatorname{sgn}(-\delta/2)| &\leq |\operatorname{sgn}(\delta/2) - L| + |\operatorname{sgn}(-\delta/2) - L| \\ &< 1 + 1 = 2. \end{aligned}$$

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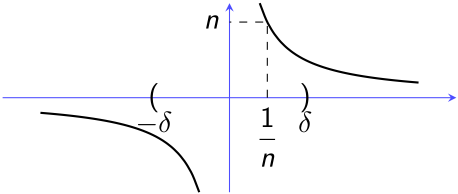
This equality contradicts the previous inequality. So $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

Non-existence of Limit: Example 2

Define $f: \mathbb{R} \to \mathbb{R}$ by $f(0) = 0$ and $f(x) = 1/x$ when $x \neq 0$.



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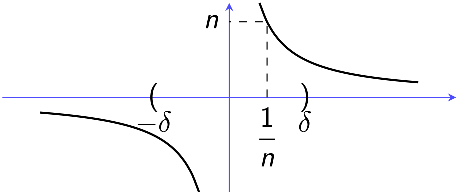

Suppose $\lim_{x \to 0} f(x) = L$ and consider $\epsilon = 1/2$.

Now take any $\delta > 0$. By the Archimedean property, $(-\delta, \delta)$ contains points of the form $1/n$ and $1/(n+1)$ with $n \in \mathbb{N}$.

Non-existence of Limit: Example 2

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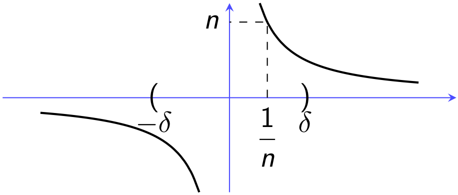

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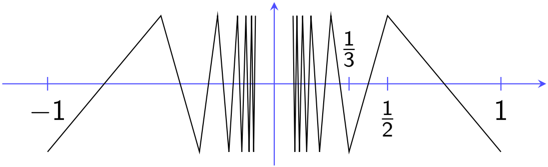
Now take any $\delta > 0$. By the Archimedean property, $(-\delta, \delta)$ contains points of the form $1/n$ and $1/(n+1)$ with $n \in \mathbb{N}$.

Then $f(1/(n+1)) - f(1/n) = 1$ and so it is impossible that both $f(1/(n+1))$ and $f(1/n)$ are within a distance $\epsilon = 1/2$ of L.

Non-existence of Limit: Example 3

Let $S\colon [-1,1] \to \mathbb{R}$ be defined by $S(1/n) = (-1)^n$ for each $n \in \mathbb{N}$ and let its graph be a straight line on each interval between these points. Further, let $S(0) = 0$.



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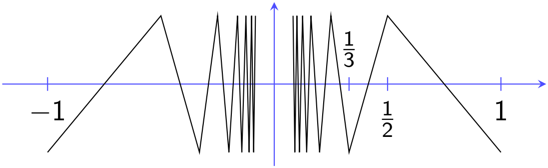

In any $(-\delta, \delta)$ interval, S takes both the values ± 1 and so we can argue as in the previous two examples to show that $\lim_{x \to 0} S(x)$ does not exist.

Limit and function value

Let $f(x) = 0$ when $x \neq 0$ and $f(0) = 1$. We will show that $\lim_{x \to 0} f(x) = 0$.

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Then

$$0 < |x - 0| < \delta \implies x \neq 0 \implies f(x) = 0 \implies |f(x) - 0| = 0 < \epsilon.$$

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So the limit exists at $x = 0$ but does not equal $f(0)$.

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Functions with zero limit

Lemma 5

Let f, g be real functions with $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = 0$. Then

1. $\lim_{x\to p} c\, f(x) = 0 \quad (c \in \mathbb{R})$,
2. $\lim_{x\to p} (f(x) + g(x)) = 0$,
3. $\lim_{x\to p} f(x)g(x) = 0$,
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Functions with zero limit

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Proof.

1. This is trivial if $c = 0$. Suppose $c \neq 0$.

Functions with zero limit

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1. This is trivial if $c = 0$. Suppose $c \neq 0$. For $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x)| < \epsilon/|c|$.

Functions with zero limit

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Proof.

1. This is trivial if $c = 0$. Suppose $c \neq 0$. For $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x)| < \epsilon/|c|$. Now, $0 < |x - p| < \delta$ implies $|cf(x) - 0| = |c||f(x)| < |c|\dfrac{\epsilon}{|c|} = \epsilon$.

Functions with zero limit

(*proof continued*)

2. Take any $\epsilon > 0$.

Functions with zero limit

(*proof continued*)

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Functions with zero limit

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Functions with zero limit

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   Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$0 < |x - p| < \delta \implies |f(x) + g(x) - 0| \leq |f(x)| + |g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Functions with zero limit

(*proof continued*)

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Functions with zero limit

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$$0 < |x - p| < \delta \implies |f(x)g(x)| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

Functions with zero limit

(*proof continued*)

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Functions with zero limit

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   There is a $\delta_1 > 0$ such that $0 < |x - p| < \delta_1$ implies $\frac{1}{2} < h(x) < \frac{3}{2}$.

Functions with zero limit

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Functions with zero limit

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   Let $\delta = \min\{\delta_1, \delta_2\}$. Then
   $$0 < |x - p| < \delta \implies \left|\frac{f(x}{h(x)}\right| < \frac{\epsilon/2}{1/2} = \epsilon.$$

Algebra of Limits

Theorem 6

Let f, g be real functions such that $\lim_{x \to p} f(x) = M$ and $\lim_{x \to p} g(x) = N$. Then

1. $\lim_{x \to p} c\, f(x) = cM \quad (c \in \mathbb{R})$,
2. $\lim_{x \to p} (f(x) + g(x)) = M + N$,
3. $\lim_{x \to p} (f(x) - g(x)) = M - N$,
4. $\lim_{x \to p} f(x) g(x) = MN$,
5. $\lim_{x \to p} \dfrac{f(x)}{g(x)} = \dfrac{M}{N} \quad (N \neq 0)$.

Algebra of Limits

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5. $\lim_{x\to p} \dfrac{f(x)}{g(x)} = \dfrac{M}{N} \quad (N \neq 0)$.

We shall use $\lim_{x\to p} F(x) = K \iff \lim_{x\to p}(F(x) - K) = 0$ to reduce these to the previous lemma.

Algebra of Limits

Proof.

1. $$\lim_{x \to p} \left(c\, f(x) - c\, M \right) = \lim_{x \to p} c \left(f(x) - M \right) = 0.$$

(By part 1 of the Lemma)

Algebra of Limits

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Proof.

1. $\lim_{x\to p}\Big(c\,f(x) - c\,M\Big) = \lim_{x\to p} c\,\Big(f(x) - M\Big) = 0.$

   (By part 1 of the Lemma)

2. $$\lim_{x\to p}\Big((f(x)+g(x)) - (M+N)\Big) = \lim_{x\to p}\Big((f(x)-M) + (g(x)-N)\Big) = 0$$

   (By part 2 of the Lemma)

Algebra of Limits

Proof.

1. $$\lim_{x\to p}\Big(c\,f(x) - c\,M\Big) = \lim_{x\to p} c\,\Big(f(x) - M\Big) = 0.$$
   (By part 1 of the Lemma)

2. $$\lim_{x\to p}\Big((f(x)+g(x)) - (M+N)\Big)$$
   $$= \lim_{x\to p}\Big((f(x)-M) + (g(x)-N)\Big) = 0$$
   (By part 2 of the Lemma)

3. Combine parts 1 and 2 of this theorem, using $c = -1$.

Algebra of Limits

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(*proof continued*)

4. We use part 3 of the Lemma and parts 1, 2, 3 of this theorem:

$$\begin{aligned}\lim_{x\to p}\Big(f(x)g(x)-MN\Big) &= \lim_{x\to p}\Big((f(x)-M)(g(x)-N) \\ &\quad + Mg(x)+Nf(x)-2MN\Big) \\ &= \lim_{x\to p}((f(x)-M)(g(x)-N)) \\ &\quad + \lim_{x\to p}(Mg(x)) + \lim_{x\to p}(Nf(x)) - \lim_{x\to p} 2MN \\ &= 0+MN+NM-2MN=0.\end{aligned}$$

Algebra of Limits

(*proof continued*)

5. Due to part 4 of this theorem, it is enough to prove that $\lim_{x \to p} \frac{1}{g(x)} = \frac{1}{N}$:

Algebra of Limits

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(*proof continued*)

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$$\lim_{x\to p}\left(\frac{1}{g(x)} - \frac{1}{N}\right) = \lim_{x\to p}\frac{N-g(x)}{g(x)}$$
$$= \lim_{x\to p}\frac{1-g(x)/N}{g(x)/N}$$
$$= 0. \quad \text{(Part 4 of the Lemma)}$$

□

Examples

1. Calculate $\lim_{x \to 2} (x^2 + 9)$:

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   By (2) of Algebra of Limits, we have

$$\lim_{x\to 2}(x^2+9) = \lim_{x\to 2} x^2 + \lim_{x\to 2} 9 = \lim_{x\to 2} x^2 + 9.$$

Examples

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   By (1) of Algebra of Limits, we have $\lim_{x\to 2}(7x)^9 = 7^9 \lim_{x\to 2} x^9$.

Examples

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   Hence $\lim_{x\to 2}(7x)^9 = 7^9 2^9 = 14^9$.

Examples

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$$\lim_{x \to 1} \frac{(x-1)^2}{x^2-1} = \lim_{x \to 1} \frac{(x-1)^2}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x-1}{x+1}.$$

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$$\lim_{x\to 1} \frac{(x-1)^2}{x^2-1} = \lim_{x\to 1} \frac{(x-1)^2}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x-1}{x+1}.$$

The cancellation in the last step is allowed because when we calculate $\lim_{x\to 1}$ we work with $x \neq 1$ and hence $x - 1 \neq 0$.

Examples

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Suppose that $f(x) \leq g(x) \leq h(x)$ in an interval $(p-\alpha, p+\alpha)$, with $\alpha > 0$, except perhaps at p. If $\lim_{x \to p} f(x) = \lim_{x \to p} h(x) = L$ then $\lim_{x \to p} g(x) = L$.

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Hence $L-\epsilon < g(x) < L+\epsilon$. Therefore $\lim_{x\to p} g(x) = L$. □

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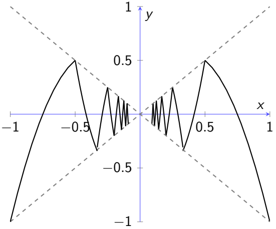

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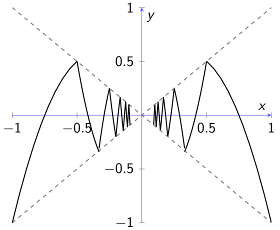

To avoid the $x > 0$ and $x < 0$ cases we work with $|xS(x)|$:

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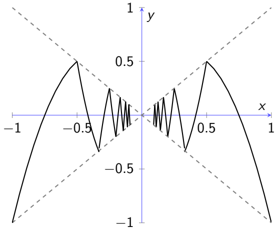

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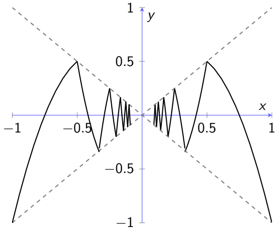

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Hence, by the Sandwich Theorem, $\lim_{x \to a} |\sqrt{x} - \sqrt{a}| = 0$.

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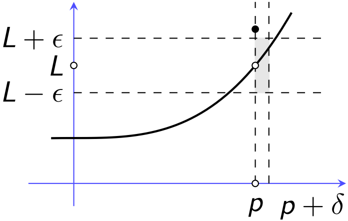
The right-hand limit at p can be considered if there is an $\alpha > 0$ such that $(p, p+\alpha)$ is in the domain of f. The left-hand limit needs an $\alpha > 0$ such that $(p-\alpha, p)$ is in the domain.

Visualising one-sided limits



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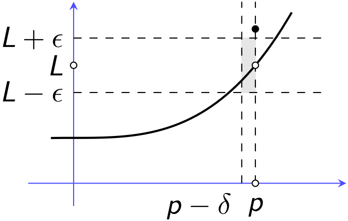

Left-hand Limit

One-sided and two-sided limits

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Theorem 8

$$\lim_{x \to p} f(x) = L \text{ if and only if } \lim_{x \to p+} f(x) = \lim_{x \to p-} f(x) = L.$$

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One-sided and two-sided limits

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Since the one-sided limits are not equal, $\lim_{x \to 0} H(x)$ does not exist.

An Exercise

Confirm that the Algebra of Limits and the Sandwich Theorem also hold for one-sided limits.