Homework #1 Odd Numbered Answers

1. From Wikipedia, there are many very readable accounts of the life and technical achievements of J.C. Maxwell, and H.R. Hertz, G. Marconi, and N. Tesla. Their backgrounds, personalities, professional recognitions, and financial successes from their works, varied great. "History of Radio," is another interesting article on Wikipedia on wireless communications. This article also referenced a PBS documentary program by Ken Burns entitle, " Empire of the Air: The Men Who Made Radio," 1992. Besides the book, "History of Wireless," by T.K. Sarkar, et al, J. Wiley, 2006, many other books on early workers in this area appeared in the references at the end of the article,"History of Radio."

3. **a.** For $R = 1$ G bits/sec. , the new power requirement becomes $P_T = 5.68 \times 10^{-5}$ watts.

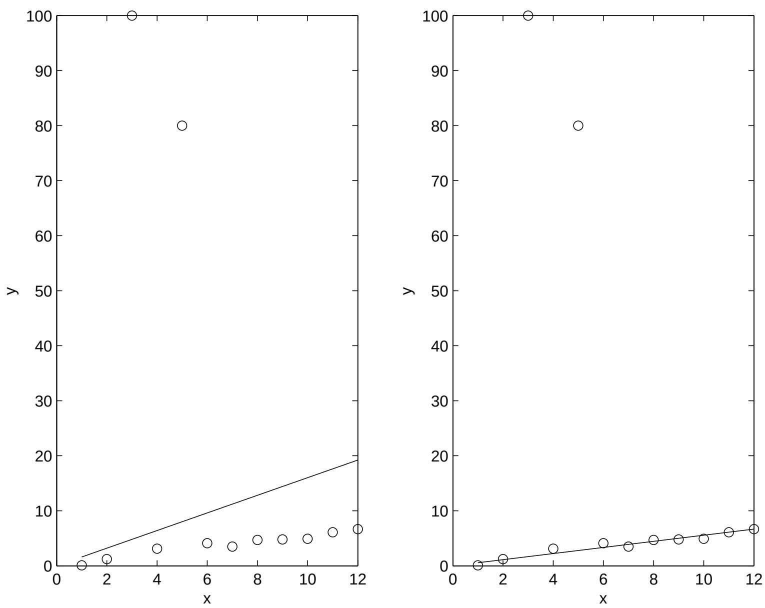
   b. For $R = 100$ K bits/sec., the new power requirement becomes $P_T = 5.54 \times 10^{-5}$ watts. If this power is too demanding, we can get closer to the base-station with a range of $r = 1$ km. Then $P_T = 2.22 \times 10^{-6}$ watts. If this power is still considered to be too high, perhaps we can attach an external higher gain antenna with $G_T = 2$. Then $P_T = 1.11 \times 10^{-6}$ watts.

   c. Under the new conditions, the power requirement becomes $P_T = 1,680$ watts. This power requirement is definitely not reasonable for a small UAV. If $G_T = 100$, then $P_T = 168$ watts, perhaps may still be too demanding.

   d. For a low-altitude satellite, the power becomes $P_T = 3.65 \times 10^{-5}$ watts. By using a lower gain $G_T = 1,000$, the power becomes $P_T = 3.65 \times 10^{-4}$ watts, which is still very modest.

5. Using the Matlab fminsearch.m function to minimize the residual of $||\mathbf{y} - a\mathbf{s}||_{l1}$, with an initial starting point of 0.4, we obtained $\hat{a}_0^{AE} = 0.5545312500000005$ with an AE error of $\epsilon_{AE}(\hat{a}_0^{AE}) = 4.724562499999993$. Using an an initial starting point of 0.6, we obtained $\hat{a}_0^{AE} = 0.5545605468749999$ with an AE error of $\epsilon_{AE}(\hat{a}_0^{AE}) = 4.724484374999999$. We can conclude that regardless of the initial starting points, both solutions are extremely close. Furthermore, $\epsilon_{AE}(\tilde{a}_1) = 11.208$ and $\epsilon_{AE}(\tilde{a}_2) = 5.188$. Clearly, both $\epsilon_{AE}(\tilde{a}_1)$ and $\epsilon_{AE}(\tilde{a}_2)$ are greater than $\epsilon_{AE}(\hat{a}_0^{AE})$, since $\epsilon_{AE}(\hat{a}_0^{AE})$ is the minimum AE error.

7. Using (6-15) for $\hat{x}$ and $\mathbf{y2}$, the explicit solution of the optimum LSE solution $\hat{a}_0^{LS} = 1.601464615384616$. Using the Matlab fminsearch.m function to minimize the residual of $||\mathbf{y2} - a\mathbf{s}||_{l1}$, with an initial starting point of 0.53, we obtained $\hat{a}_0^{AE} = 0.5558271484375$. The following left figure, plots x vs y2 and a straight line of slope $\hat{a}_0^{LS}$. The following right figure, plots x vs y2 and a straight line of slope $\hat{a}_0^{AE}$. In both figures, we notice two outliers at $x = 3$ and $x = 5$. However, in the right figure, most of the values of $\mathbf{y2}$ are quite close to the straight line with the $\hat{a}_0^{AE}$ slope, whilte in the left figure most of the values of $\mathbf{y2}$ are quite far from the straight line with the $\hat{a}_0^{LS}$ slope. This example shows the robust property of linear estimation based on the LAE Criterion over that based on the LSE Criterion.



9. From (6-35) - (6-37), we obtained for $n = 1$, $\hat{\mathbf{a}}_n = [0.4]$ and $\epsilon_{MS}(\hat{\mathbf{a}}_n) = 0.68$. For $n = 2$, $\hat{\mathbf{a}}_n = [0.011905,\ 0.39524]^T$ and $\epsilon_{MS}(\hat{\mathbf{a}}_n) = 0.67976$. For $n = 3$, $\hat{\mathbf{a}}_n = [0.0048193,\ 0.01,\ 0.39518]^T$ and $\epsilon_{MS}(\hat{\mathbf{a}}_n) = 0.67972$. For $n = 4$, $\hat{\mathbf{a}}_n = [0.0016389,\ 0.0040404,\ 0.0099773,\ 0.39517]^T$ and $\epsilon_{MS}(\hat{\mathbf{a}}_n) = 0.67972$. Thus, we note that all the terms in $a_1 X(1) + a_2 X(2) + \ldots + a_{n-1} X(n-1)$ do contribute (although not much) in reducing the MMSE values of $\epsilon_{MS}(\hat{\mathbf{a}}_n)$ for increasing values of n.

Chapter 2
Review of Probability and Random Processes
Odd Numbered Homework Solutions

1. Bernoulli: $p, q \geq 0, p + q = 1$.

$$f_X(x) = q\delta(x) + p\delta(x-1), \quad F_X(x) = qu(x) + pu(x-1), \quad \mu_X = p, \quad \sigma_X^2 = pq.$$

Binomial: $p, q \geq 0, q + q = 1, n \geq 1$.

$$P_n(k) = \binom{n}{k} p^k q^{n-k}, 0 \leq k \leq n, \quad f_X(x) = \sum_{k=0}^{n} P_n(k)\delta(x-k), \quad F_X(x) = \sum_{k=0}^{n} P_n(k)u(x-k),$$

$$\mu_X = np, \quad \sigma_X^2 = npq.$$

Poisson: $b > 0$.

$$P_k = \frac{b^k}{k!}e^{-b}, k = 0, 1, 2, \ldots, \quad f_X(x) = \sum_{k=0}^{\infty} P_k\delta(x-k), \quad F_X(x) = \sum_{k=0}^{\infty} P_k u(x-k),$$

$$\mu_X = b, \quad \sigma_X^2 = b.$$

Cauchy: $a > 0$.

$$f_X(x) = \frac{a}{\pi(x^2+a^2)}, |-\infty < x < \infty, \quad F_X(x) = \frac{1}{\pi}\tan^{-1}\Big(\frac{x}{a}\Big) + \frac{1}{2}, -\infty < x < \infty.$$

The mean and variance of the Cauchy random variable are undefined.

Exponential: $\lambda > 0$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad \mu_X = \frac{1}{\lambda}, \quad \sigma_x^2 = \frac{1}{\lambda^2}.$$

Gaussian:

$$f_X(x) = \tfrac{1}{\sqrt{2\pi}\sigma_X}\exp\Big(\frac{-(x-\mu_X)^2}{2\sigma_X^2}\Big), -\infty < x < \infty, \quad F_X(x) = \Phi\Big(\frac{x-\mu_X}{\sigma_X}\Big), -\infty < x < \infty.$$

$$\mu_X = \mathrm{E}[X], \quad \sigma_X^2 = \mathrm{Var}[X]$$

Laplace: $b > 0$.

$$f_X(x) = \frac{b}{2}\exp(-b|x|), |-\infty < x < \infty, \quad \mu_X = 0, \quad \sigma_X^2 = \frac{2}{b^2},$$

$$F_X(x) = \begin{cases} \frac{1}{2}e^{bx}, & -\infty < x < 0, \\ 1 - \frac{1}{2}e^{-bx}, & 0 \leq x < \infty. \end{cases}$$

Raleigh: $b > 0$.

$$f_X(x) = \begin{cases} \frac{2x}{b}\exp\left(\frac{-x^2}{b}\right), & x \geq 0 \\ 0, & \text{otherwise,} \end{cases} \quad F_X(x) = \begin{cases} 1 - \exp\left(\frac{-x^2}{b}\right), & x \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_X = \sqrt{\tfrac{\pi b}{4}}, \quad \sigma_X^2 = (4-\pi)\tfrac{b}{4}$$

3. **a.**

$$\begin{aligned} 1 &= c\int_0^1\int_0^1 (x+y)\,dx\,dy = c\int_0^1 \left(\tfrac{1}{2}x^2 + xy\right)\Big|_0^1\,dy = c\int_0^1 \left(\tfrac{1}{2}+y\right)dy \\ &= c\left(\tfrac{1}{2}y + \tfrac{1}{2}y^2\right)\Big|_0^1 = c \Longleftrightarrow c = 1. \end{aligned}$$

b.

$$F_{XY}(x,y)=\begin{cases}0, & x\le 0,\ -\infty<y<\infty,\\ 0, & y\le 0,\ -\infty<x<\infty,\\ \frac{xy(x+y)}{2}, & 0<x<1,\ 0<y<1,\\ \frac{x(x+1)}{2}, & 0<1<1,\ 1\le y,\\ \frac{y(y+1)}{2}, & 1\le x,\ 0<y<1,\\ 1, & 1\le x,\ 1\le y.\end{cases}$$

c.

$$F_X(x)=F_{XY}(x,1)\ =\tfrac{x(x+1)}{2},\ 0<x<1$$
$$\Longleftrightarrow f_X(x)=\tfrac{d}{dx}F_X(x)=\begin{cases}x+\frac{1}{2}, & 0<x<1,\\ 0, & \text{elsewhere.}\end{cases}$$

Similarly,

$$f_Y(y)=\begin{cases}y+\frac{1}{2}, & 0<y<1,\\ 0, & \text{elsewhere.}\end{cases}$$

5. **a.** $[\ x_1\ \ x_2\ \ x_3\ ]\Lambda^{-1}[\ x_1\ \ x_2\ \ x_3\ ]^T = 2x_1^2 - x_1x_2 + x_2^2 - 2x_1x_3 + 4x_3^3$. Denote $\Lambda^{-1} = [\sigma_{ij}]$. Then $\sigma_{11}=2, \sigma_{22}=1, \sigma_{33}=4, \sigma_{23}=\sigma_{32}=0, \sigma_{12}=\sigma_{21}=-1/2$, and $\sigma_{13}=\sigma_{31}=-1$. Then

$$\Lambda^{-1}=\begin{bmatrix}2 & -1/2 & -1\\ -1/2 & 1 & 0\\ -1 & 0 & 4\end{bmatrix},\qquad \Lambda=\begin{bmatrix}2/3 & 1/3 & 1/6\\ 1/3 & 7/6 & 1/12\\ 1/6 & 1/12 & 7/24\end{bmatrix}.$$

b. Since $|\Lambda|=1/6$ by direct computation, $C=((2\pi)^{3/2}|\Lambda|^{1/2})^{-1}=\sqrt{6}/(2\pi)^{3/2}\simeq 0.1555$.

c. $[\ Y_1\ \ Y_2\ \ Y_3\ ]^T = B[\ X_1\ \ X_2\ \ X_3\ ]^T$ yields,
$Y_1 = X_1 - X_2/4 - X_3/2;\quad Y_2 = X_2 - 2X_3/7;\quad Y_3 = X_3.$

d. $[\ X_1\ \ X_2\ \ X_3\ ] = B^{-1}[\ Y_1\ \ Y_2\ \ Y_3\ ]$ yields,
$X_1 = Y_1 + Y_2/4 + 4Y_3/7;\quad X_2 = Y_2 + 2Y_3/7;\quad X_3 = Y_3.$

e. The Jacobian of transformation J yields $|J|=|B|=1$.
Then $-\frac{1}{2}(2x_1^2 - x_1x_2 + x_2^2 - 2x_1x_3 + 4x_3^2) = -\frac{1}{2}(2y_1^2 + 7y_2^2/8 + 24y_3^2/7)$ and $f_Y(y)=f_X(x)J(X|Y)=f_X(x)$. Thus, $f_Y(y)=(6/(2\pi)^3)^{1/2}\exp(-\frac{1}{2}(2y_1^2+7y_2^2/8+24y_3^2/7)) = [(2/2\pi)^{1/2}\exp(-y_1^2)]\times[(1/2\pi\cdot 8/7)^{1/2}\exp(-7y_2^2/16)]\times[(1/2\pi\cdot 7/24)^{1/2}\exp(-12y_3^2/7)]$.

f. Since the 3-dimensional pdf factors as the product of three 1-dimensional pdfs, the three r.v.'s are mutually independent. By inspection: $\mu_{Y_1}=\mu_{Y_2}=\mu_{Y_3}=0$ and $\sigma_{Y_1}^2=1/2, \sigma_{Y_2}^2=8/7$, and $\sigma_{Y_3}^2=7/24$.

7. **a.** $\mu_Z = E\{Z\} = \mu_X + \mu_Y\ = \lambda_X + \lambda_Y$.
b. $\sigma_Z^2 = Var\{Z\} = \sigma_X^2 + \sigma_Y^2 = \lambda_X + \lambda_Y$.
c.

$$\phi_Z(t)=E\{e^{t(X+Y)}\}=E\{e^{tX}\}E\{e^{tY}\}=e^{-\lambda_X}e^{e^t\lambda_X}e^{-\lambda_Y}e^{e^t\lambda_Y}=e^{-(\lambda_X+\lambda_Y)}e^{e^t(\lambda_X+\lambda_Y)}.$$
$\Updownarrow$
Z is a Poisson rv with $\mu_Z=\lambda_X+\lambda_Y$ and $\sigma_Z^2=\lambda_X+\lambda_Y$.

d.

$$\begin{aligned}P(X+Y=k) &= P(\{X=0,\,Y=k\}\text{ or }\{X=1,\,Y=k-1\}\text{ or }\ldots\{X=k,\,Y=0\})\\ &= P(X=0,\,Y=k)+P(X=1,\,Y=k-1)+\ldots+P(X=k,\,Y=0)\\ &= P(X=0)\,P(Y=k)+P(X=1)\,P(Y=k-1)+\ldots+P(X=k)\,P(Y=0)\\ &= \frac{e^{-\lambda_X}\lambda_X^0}{0!}\frac{e^{-\lambda_Y}\lambda_Y^k}{k!}+\frac{e^{-\lambda_X}\lambda_X^1}{1!}\frac{e^{-\lambda_Y}\lambda_Y^{k-1}}{(k-1)!}+\ldots+\frac{e^{-\lambda_X}\lambda_X^k}{k!}\frac{e^{-\lambda_Y}\lambda_Y^0}{0!}\\ &= e^{-(\lambda_X+\lambda_Y)}\left[\frac{\lambda_X^0}{0!}\frac{\lambda_Y^k}{k!}+\frac{\lambda_X^1}{1!}\frac{\lambda_Y^{k-1}}{(k-1)!}\cdots+\frac{\lambda_X^k}{k!}\frac{\lambda_Y^0}{0!}\right].\end{aligned}$$

But the sum in the bracket $=(\lambda_X+\lambda_Y)^k/k!$. Thus,
$P(X+Y=k)=\ e^{-(\lambda_X+\lambda_Y)}(\lambda_X+\lambda_Y)^k/k!\,,\ k=0,1,\ldots.$
$\Updownarrow$
Z is a Poisson rv with $\mu_Z=\lambda_X+\lambda_Y$ and $\sigma_Z^2=\lambda_X+\lambda_Y$.

e. Use the moment generating function method to attack this problem.

$$\phi_Z(t) = E\{e^{t(2X+3Y)}\} = E\{e^{t2X}\}E\{e^{t3Y)}\} = e^{-\lambda_X}e^{e^{2t}\lambda_X}e^{-\lambda_Y}e^{e^{3t}\lambda_Y} = e^{-(\lambda_X+\lambda_Y)}e^{e^{2t}\lambda_X+e^{3t}\lambda_Y}.$$

No. Z is not a Poisson rv , since its $\phi_Z(t)$ is not of the form $e^{-\lambda}e^{e^t\lambda}$.

9.

$$\begin{aligned} S(\omega) &= R(0) + R(1)[e^{\imath\omega} + e^{-\imath\omega}] + R(2)[e^{\imath 2\omega} + e^{-\imath 2\omega}] \\ &= 1 + \cos(\omega) + \frac{1}{2}\cos(2\omega), \quad -\pi \le \omega < \pi. \end{aligned}$$

$$\begin{aligned} R(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega)e^{\imath\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1+\cos(\omega)+\frac{1}{2}\cos(2\omega)]e^{\imath\omega n}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[e^{\imath\omega n} + \frac{1}{2}(e^{\imath\omega(n+1)} + e^{\imath\omega(n-1)}) \\ &\qquad + \frac{1}{4}(e^{\imath\omega(n+2)} + e^{\imath\omega(n-2)})]\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\{\cos(\omega n) + \frac{1}{2}[\cos(\omega(n+1)) + \cos(\omega(n-1))] \\ &\qquad + \frac{1}{4}[\cos(\omega(n+2)) + \cos(\omega(n-2))]\}\,d\omega \\ &= \frac{1}{2\pi}\left\{\frac{\sin(\omega n}{n}) + \frac{1}{2}\left(\frac{\sin(\omega(n+1))}{n+1} + \frac{\sin(\omega(n-1))}{n-1}\right)\right. \\ &\qquad \left.\left. + \frac{1}{4}\left(\frac{\sin(\omega(n+2))}{n+2} + \frac{\sin(\omega(n-2))}{n-2}\right)\right\}\right|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left\{\frac{2\sin(\pi n)}{n} + \frac{\sin(\pi(n+1))}{n+1} + \frac{\sin(\pi(n-1))}{n-1}\right. \\ &\qquad \left. + \frac{1}{2}\frac{\sin(\pi(n+2))}{(n+2)} + \frac{1}{2}\frac{\sin(\pi(n-2))}{(n-2)}\right\}. \end{aligned}$$

But

$$\frac{\sin(\pi m)}{m} = \begin{cases} \pi, & m = 0 \\ 0, & m \neq 0. \end{cases}$$

Thus, $R(n)$ in indeed yields

$$R(n) = \begin{cases} 1 & , n = 0, \\ 1/2 & , n = 1, \\ 1/2 & , n = -1, \\ 1/4 & , n = 2, \\ 1/4 & , n = -2, \\ 0 & , \text{all other integral } n. \end{cases}$$

11. **a.**

$$E\{X(t,\Theta)\} = \int_0^{2\pi} A\,cos(\omega t + \theta)\frac{d\theta}{2\pi} = 0.$$

b.

$$\begin{aligned} R_X(t, t+\tau) &= E\{X(t,\theta)X(t+\tau,\theta)\} = \int_0^{2\pi} A\,cos(\omega t+\theta)A\,cos(\omega(t+\tau)+\theta)\frac{d\theta}{2\pi} \\ &= \frac{A^2}{2}\int_0^{2\pi}[cos(\omega\tau) + cos(2\omega t + 2\theta)]\frac{d\theta}{2\pi} = \frac{A^2}{2}cos(\omega\tau), \end{aligned}$$

where we used the identity $cos(u)cos(v) = (1/2)cos(u+v) + (1/2)cos(u-v)$.

c.

$$< X(t,\theta) >= lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\,cos(\omega t+\theta)dt = 0.$$

This integral is zero since there is as much area above the abscissa as below due to the symmetry of the problem. Of course, if we do the integration, we will find the same result.

d.

$$< X(t,\theta)X(t+\tau,\theta) >= lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\,cos(\omega t+\theta)A\,cos(\omega(t+\tau)+\theta)dt$$

$$= \frac{A^2}{2}lim_{T\to\infty}\frac{1}{2T}\{\int_{-T}^{T} cos(\omega\tau)dt + \int_{-T}^{T} cos(2\omega t+\omega\tau+2\theta)dt\} = \frac{A^2}{2}cos(\omega\tau).$$

e. Yes, both the ensemble and time averaged means and autocorrelations are the same for this random process. Of course, we can not claim this process is ergodic having shown only these two averages are equivalent.

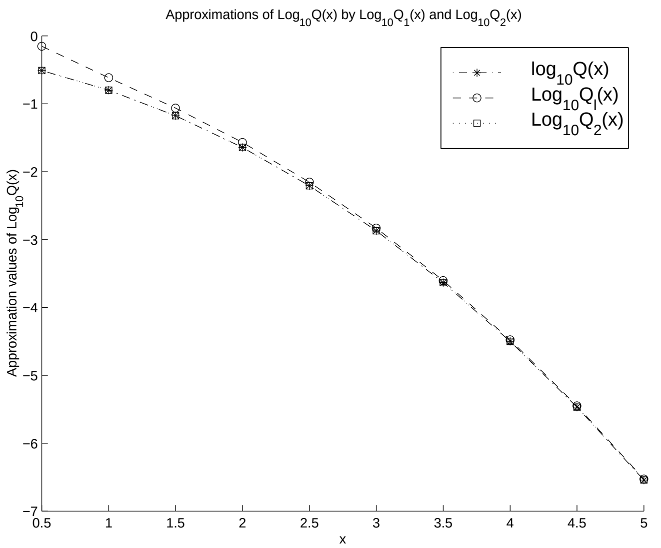
Chapter 3
Hypothesis Testing
Odd Numbered Homework Solutions

1. **a.** $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = 2\frac{1}{\sqrt{2\pi}\cdot\frac{1}{\sqrt{2}}} \int_0^x e^{\frac{-t^2}{2\cdot\frac{1}{2}}} dt = 1 - 2 \cdot Q(x/\frac{1}{\sqrt{2}})$.

   $Q(x) = \frac{1}{2} - \frac{1}{2} \cdot erf(\frac{x}{\sqrt{2}}) = \frac{1}{2} \cdot erfc(\frac{x}{\sqrt{2}})$.

   b&c.

x	Q(x)	Q1(x)	Q2(x)	logQ(x)	logQ1(x)	logQ2(x)
0.5000	0.3085	0.7041	0.3085	-0.5107	-0.1523	-0.5107
1.0000	0.1587	0.2420	0.1587	-0.7995	-0.6162	-0.7995
1.5000	0.0668	0.0863	0.0668	-1.1752	-1.0638	-1.1752
2.0000	0.0228	0.0270	0.0228	-1.6430	-1.5687	-1.6430
2.5000	0.0062	0.0070	0.0062	-2.2069	-2.1542	-2.2069
3.0000	0.0013	0.0015	0.0013	-2.8697	-2.8305	-2.8697
3.5000	0.0002	0.0002	0.0002	-3.6333	-3.6032	-3.6333
4.0000	0.0000	0.0000	0.0000	-4.4993	-4.4755	-4.4991
4.5000	0.0000	0.0000	0.0000	-5.4688	-5.4495	-5.4684
5.0000	0.0000	0.0000	0.0000	-6.5426	-6.5267	-6.5420



3. Under H_0, $p_0(x) = (1/2)e^{-x/2}, x \geq 0$; under H_1, $p_1(x) = (x^2/16)e^{-x/2}, x \geq 0$. Then $\Lambda(x) = 2x^2, x \geq 0$. Thus, $R_1 = \{x : \Lambda(x) \geq \Lambda_0 = 1\} = \{x : x^2 \geq 8\} = \{x : x \geq 2\sqrt{2}\}$ and $R_0 = \{x : 0 \leq x \leq 2\sqrt{2}\}$. Then $P_{FA} = P(x \in R_1|H_0) = \int_{2\sqrt{2}}^{\infty}(1/2)exp(-x/2)dx = exp(-\sqrt{2}) = 0.243$ and $P_D(x \in R_1|H_1) = \int_{2\sqrt{2}}^{\infty}(x^2/16)exp(-x/2)dx = (1/16)exp(-\sqrt{2})(32 + 16\sqrt{2}) = 0.830$.

5. We are given

$$p_N(n) = \begin{cases} e^{-n}, & 0 \leq n < \infty, \\ 0, & n < 0. \end{cases}$$

   Then

$$p_0(x) = \begin{cases} e^{-x}, & 0 \leq x < \infty, \\ 0, & x < 0, \end{cases}$$
$$p_1(x) = \begin{cases} e^{-(x-1)}, & 1 \leq x < \infty, \\ 0, & x < 1. \end{cases}$$

   Thus,

$$\Lambda(x) = \frac{p_1(x)}{p_0(x)} = \begin{cases} e, & 1 \leq x < \infty, \\ 0, & 0 \leq x < 1. \end{cases}$$

   We note, $\Lambda(x)$ is not defined (and need not be defined) for $x < 0$, since $p_0(x)$ and $p_1(x)$ are zero for $x < 0$.

If $P(\{x : \Lambda(x) = \Lambda_0\}) = 0$, then we can arbitrarily associate the equality of $\Lambda(x) = \Lambda_0$ with R_1 and have

$$\begin{aligned} R_1 &= \{x : \Lambda(x) \geq \Lambda_0\} \\ R_0 &= \{x : \Lambda(x) < \Lambda_0\}. \end{aligned}$$

If $P(\{x : \Lambda(x) = \Lambda_0\}) > 0$ (which is the case for this problem), we need to associate the equality with R_1 and R_0 in a "fairer" manner consistent with other constraints.

If we set $0 < \Lambda_0 < e$, then

$$\begin{aligned} R_1 &= \{1 \leq x < \infty\} = \{x : e = \Lambda(x) > \Lambda_0\} \\ R_0 &= \{0 \leq x < 1\} = \{x : 0 = \Lambda(x) < \Lambda_0\} \end{aligned}$$

For this case,

$$P_{FA} = \int_{R_1} p_0(x)\, dx = \int_1^\infty e^{-x}\, dx = e^{-1} \simeq 0.37 \neq 0.1,$$

as required. Thus $0 < \Lambda_0 < e$ is not possible.

If we set $e < \Lambda_0 < \infty$, then

$$\begin{aligned} \{1 \leq x < \infty : e = \Lambda(x) < \Lambda_0\} &\subset R_0 \\ \{0 \leq x < 1 : 0 = \Lambda(x) < \Lambda_0\} &\subset R_0. \end{aligned}$$

Thus,

$$\begin{aligned} R_0 &= \{0 \leq x < \infty\}, \\ R_1 &= \emptyset. \end{aligned}$$

For this case,

$$P_{FA} = \int_{R_1} p_0(x)\, dx = 0 \neq 0.1,$$

as required. Thus, $e < \Lambda_0 < \infty$ is not possible.

Thus, we must have $\Lambda_0 = e$. In this case R_1 cannot contain any $x \in \{0 \leq x < 1\}$, since these x's satisfy $0 = \Lambda(x) < \Lambda_0 = e$. Then R_1 includes only those $x \in [1, \infty)$ such that

$$P_{FA} = \int_{R_1} p_0(x)\, dx = 0.1. \qquad (1)$$

a. Clearly, there are many possible R_1 that can satisfy (1). Two examples are given in (b) and (c). Thus, R_1 and R_0 are not unique.

b. Suppose we pick $R_1 = \{x : 1 \leq x_1 \leq x\}$. The we need

$$P_{FA} = \int_{x_1}^\infty e^{-x}\, dx = e^{-x_1} = 0.1.$$

Thus, $x_1 = -\ln 0.1 = \ln 10 \simeq 2.3$ and

$$P_{D_1} = \int_{R_1} p_1(x)\, dx = \int_1^\infty e^{-(x-1)}\, dx = e \int_{x_1}^\infty e^{-x}\, dx = 0.1e.$$

This shows,

$$\begin{aligned} R_1 &= \{x : x_1 = \ln 10 \leq x\}, \\ R_0 &= \{x : 0 \leq x < x_1 = \ln 10\}. \end{aligned}$$

c. Suppose we pick $R_1 = \{x : 1 \leq x \leq x_2\}$. Then

$$P_{FA} = \int_{R_1} p_0(x)\, dx = \int_1^{x_2} e^{-x}\, dx = e^{-1} - e^{-x_2} = 0.1.$$

Thus, $x_2 = -\ln(e^{-1} - 0.1) \simeq 1.32$ and

$$P_{D_2} = \int_{R_1} p_1(x)\,dx = \int_1^{x_2} e^{-(x-1)}\,dx = e\int_1^{x_2} e^{-x}\,dx = 0.1e.$$

This shows,

$$\begin{aligned} R_1 &= \{x : 1 \leq x \leq x_2 = -\ln(e^{-1} - 0.1)\}, \\ R_0 &= \{x : x_2 < x\} \cup \{x : 0 \leq x < 1\}. \end{aligned}$$

d. Consider any $R_1 \subset [1, \infty)$ such that

$$P_{FA} = \int_{R_1} p_0(x)\,dx = 0.1.$$

For $x \in R_1$, $\Lambda(x) = p_1(x)/p_0(x) = \Lambda_0 = e$ or $p_1(x) = p_0(x)e$. Thus,

$$P_D = \int_{R_1} p_1(x)\,dx = e\int_{R_1} p_0(x)\,dx = 0.1e.$$

Indeed, it is consistent that $P_{D_1} = P_{D_2} = P_D = 0.1e$.

7. Often we denote as P_I, the probability of error of type I, for the probability of declaring H_1 given H_0 is true, and P_{II}, for the probability of error of type II, for the probability of declaring H_0 given H_1 is true. Thus,$P_{FA} = P_I = P(\text{Declare } H_1|H_0) = P(X \leq 5|p = 0.05) = \sum_{k=0}^{5} C_k^{200}(0.05)^k(.95)^{200-k}$. We can use the Poisson approximation to the binomial probability using $\lambda = np = 200 \times 0.05 = 10$. Thus, $P_{FA} \approx \sum_{k=0}^{5} \exp(-10)\frac{10^k}{k!} = 0.067$ or $P(\text{Declare } H_0|H_0) = 1 - P_{FA} = 1 - 0.067 = 0.933$. Similarly,$P_M = P_{II} = P(\text{Declare } H_0|H_1) = P(X > 5|p = 0.02) = \sum_{k=6}^{\infty} C_k^{200}(0.02)^k(.98)^{200-k}$. Now, $\lambda = np = 200 \times 0.02 = 4$ and $P_M \approx 1 - \sum_{k=0}^{5} \exp(-4)\frac{4^k}{k!} = 0.215$.
In plain English, if one uses the existing manufacturing process (i.e., H_0), then one has a probability of 0.933 of having declaring more than 5 defective items in a sample of 200 items, while if one uses the new manufacturing process (i.e., H_1), then one has a much lower probability of 0.215 of declaring more than 5 defective items in a sample of 200 items. The manufacturing manager will be much happier with the new manufacturing process.

9. **a.** From the LR Test, we know $R_1 = \{x : x > x_0\}$ and $R_0 = \{x : x < x_0\}$ where $x_0 = \frac{1}{2} + \ln\Lambda_0 = \frac{1}{2} + \ln\frac{\pi}{1-\pi}$. Then

$$\begin{aligned} \bar{C}(\pi,\pi) &= \bar{C}(\pi) = \pi\int_{x_0}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx + (1-\pi)\int_{-\infty}^{x_0} e^{-(x-1)^2/2}\,dx \\ &= \pi Q(x_0) + (1-\pi)(1 - Q(x_0 - 1)). \end{aligned}$$

b.

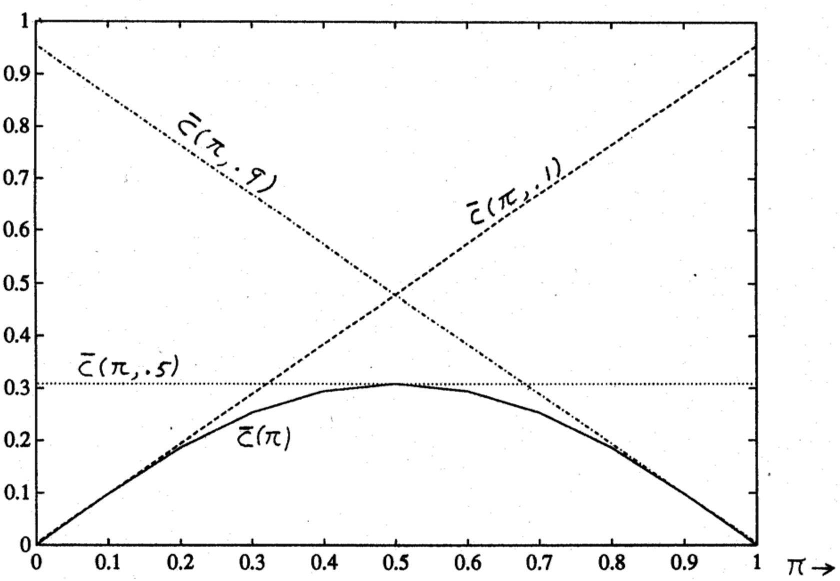
π	$\bar{C}(\pi)$	$\bar{C}(\pi, 0.1)$	$\bar{C}(\pi, 0.5)$	$\bar{C}(\pi, 0.9)$
1E−3	1E−3	4.4477E−3	0.30854	0.95422
0.01	0.01	0.013013	0.30854	0.94566
0.1	0.098664	0.098664	0.30854	0.86001
0.2	0.18616	0.19383	0.30854	0.76484
0.3	0.253	0.289	0.30854	0.66967
0.4	0.2945	0.38417	0.30854	0.5745
0.5	0.30854	0.47933	0.30854	0.47933
0.6	0.2945	0.5745	0.30854	0.38417
0.7	0.253	0.66967	0.30854	0.289
0.8	0.18616	0.76484	0.30854	0.19383
0.9	0.098664	0.866001	0.30854	0.098664
0.99	0.01	0.94566	0.30854	0.013013
0.999	1E−3	0.95422	0.30854	4.4477E−3

Table 1. $\bar{C}(\pi)$ and $\bar{C}(\pi, \pi_0)$ vs. π.

c. $\bar{C}(\pi, \pi_0) = \pi Q(x_1) + (1-\pi)(1 - Q(x_1 - 1))$, where $x_1 = \frac{1}{2} + \ln(\frac{\pi_0}{1-\pi_0})$.

d. See Table 1.

e. Plots of $\bar{C}(\pi)$ and $\bar{C}(\pi, \pi_0)$ vs. π.



11. From the solution of Problem 6, we know

$$\Lambda(x) = \tfrac{p_0(x)}{p_1(x)} = (m_1/m_0)\tfrac{m_0^2+x^2}{m_1^2+x^2}\,,\ -\infty < x < \infty\,,\ m_1 > m_0\,,$$
$$\Lambda(x) \ge \Lambda_0 \Leftrightarrow \tfrac{m_0^2+x^2}{m_1^2+x^2} \ge (m_0/m_1)\Lambda_0 = \gamma\,.$$

If $\gamma \le 0$, then $R_1 = \Re$ and $R_0 = \emptyset$. If $\gamma > 0$, then $R_1 = \{x : |x| \ge \sqrt{\gamma}\}$ and $R_0 = \{x : |x| \le \sqrt{\gamma}\}$. This shows that R_1 and R_0 are functions of the parameter m_1. Furthermore,

$$P_{FA} = \int_{-\infty}^{-\sqrt{\gamma}} p_0(x)dx \ + \int_{\sqrt{\gamma}}^{\infty} p_0(x)dx \ = 1 - \int_{-\sqrt{\gamma}}^{\sqrt{\gamma}} p_0(x)dx$$
$$= 1 - \tfrac{2m}{\pi}\int_0^{\sqrt{\gamma}} \tfrac{dx}{m_0^2+x^2} = 1 - \tfrac{2}{\pi}\tan^{-1}\left(\tfrac{\sqrt{\gamma}}{m_0}\right)$$
$$\Leftrightarrow m_0 \tan\left((\pi/2)(1-P_{FA})\right) = \sqrt{\gamma}\,.$$

This shows that P_{FA} is indeed a function of the parameter m_1, and thus this LR test is not UMP.

13. **a.**

$$L_m(x_m) = \frac{\theta_1^{x_m}(1-\theta_1)^{m-x_m}}{\theta_0^{x_m}(1-\theta_0)^{m-x_m}}\,.$$

b.

$$\log(B) < x_n \log(\theta_1/\theta_0) + (m - x_m)\log((1-\theta_1)/(1-\theta_0)) < \log(A)\,,\ b + c\,m \ < \ x_m \ < a + c\,m\,,$$
$$b = \tfrac{\log(B)}{\log(\theta_1(1-\theta_0)/(\theta_0(1-\theta_1)))}\,,\ a = \tfrac{\log(A)}{\log(\theta_1(1-\theta_0)/(\theta_0(1-\theta_1)))}\,,\ c = \tfrac{\log((1-\theta_1)/(1-\theta_0))}{\log(\theta_1(1-\theta_0)/(\theta_0(1-\theta_1)))}\,.$$

c.

$$A \simeq \frac{\beta}{\alpha} = \frac{0.95}{0.05} = 19\,,\ B \simeq \frac{1-\beta}{1-\alpha} = \frac{0.05}{0.95} = 0.0526\,.$$

$$\bar{n}_{H_0} \simeq \tfrac{\log(B)(1-\alpha)+\log(A)\beta}{E\{z|H_0\}}\,,\ \bar{n}_{H_1} \simeq \tfrac{\log(B)\beta+\log(A)(1-\beta)}{E\{z|H_1\}}\,,$$
$$E\{z|H_0\} = E_{H_0}\left\{\log\left[\tfrac{p_{X_i}(x_i|H_1)}{p_{X_i}(x_i|H_0)}\right]\right\} = E_{H_0}\left\{\log\left[\tfrac{\theta_1^{x_i}(1-\theta_1)^{1-x_i}}{\theta_0^{x_i}(1-\theta_0)^{1-x_i}}\right]\right\}$$
$$= \theta_0\log\left(\tfrac{\theta_1}{\theta_0}\right) + (1-\theta_0)\log\left(\tfrac{1-\theta_1}{1-\theta_0}\right) = \theta_0\left[\log\left(\tfrac{\theta_1}{\theta_0}\right) - \log\left(\tfrac{1-\theta_1}{1-\theta_0}\right)\right] + \log\left(\tfrac{1-\theta_1}{1-\theta_0}\right)$$
$$= \theta_0\log\left(\tfrac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right) + \log\left(\tfrac{1-\theta_1}{1-\theta_0}\right) = 0.2\log\left(\tfrac{0.6\times 0.8}{0.2\times 0.4}\right) + \log\left(\tfrac{0.4}{0.8}\right) = -0.3348\,,$$
$$\bar{n}_{H_0} \simeq \tfrac{-2.79778+0.14722}{-0.3346} = 7.92\,,$$
$$E\{z|H_1\} = E_{H_1}\left\{\log\left[\tfrac{p_{X_i}(x_i|H_1)}{p_{X_i}(x_i|H_0)}\right]\right\} = 0.3820;\ \ \bar{n}_{H_1} \simeq \tfrac{-1.4725+2.7972}{0.3820} = 6.94\,.$$

d. Under the H_0 simulation, the 10 L_n values are: $\{4.6875e-002, 4.2235e-002, 3.1250e-002, 3.1250e-002, 3.5156e-002, 3.1250e-002, 3.1250e-002, 4.6875e-002, 4.6875e-002, 4.6875e-002\}$. The associated n values are: $\{7, 33.5, 5, 10, 5, 5, 7, 7, 7\}$. Thus, the average number of terms of these realizations is $\bar{n} = 9.10$. The theoretical value is given by $\bar{n}_{H_0} = 7.92$. We note, simulation result of $\bar{n}$ is fairly close to the theoretical value of $\bar{n}_{H_0}$.

Under the H_1 simulation, the 10 L_n values are: $\{4.5562e+001, 2.7000e+001, 3.0375e+001, 2.0250e+001, 2.0250e+001, 2.7000e+001, /, 2.7000e+001, 2.5629e+001, 2.5629e+001\}$. The associated n values are: $\{10, 3, 8, 6, 6, 3, /, 3, 3, 16, 15\}$ Thus, the average number of terms of these realizations is $\bar{n}_{H_1} = 7.79$. The theoretical value is given by $\bar{n}_{H_1} = 6.94$. We note, in realization 7, even after considering up to $n = 35$, the L_n value still did not exceed the threshold value of $A = 19$. Thus, by omitting that outlier, only 9 terms were used in the evaluation of $\bar{n}_{H_1}$. We also note, simulation result of $\bar{n}$ is fairly close to the theoretical value of $\bar{n}_{H_1}$.

15. There are three typos on p. 90. In Table 3.3, 1., for $n = 20$, α_A should $= 0.13159$ and 2., for $n = 10$, α_A should $= 0.005909$. 3. Five lines below Table 3.3, α should $= -\Phi(\gamma_0/\sqrt{n})$.

The answer to this problem is tabulated below in Table 1.

n	α_{A}	β_{G} and β_{ST}
$n = 10$		
	$\alpha_{\mathrm{A}} = 0.17188$	
		$\beta_{\mathrm{G}} = 0.95680 > \beta_{\mathrm{ST}} = 0.87913$
	$\alpha_{\mathrm{A}} = 0.010742$	
		$\beta_{\mathrm{G}} = 0.64135 > \beta_{\mathrm{ST}} = 0.37581$
$n = 20$		
	$\alpha_{\mathrm{A}} = 0.13159$	
		$\beta_{\mathrm{G}} = 0.99591 > \beta_{\mathrm{ST}} = 0.96786$
	$\alpha_{\mathrm{A}} = 0.005909$	
		$\beta_{\mathrm{G}} = 0.89368 > \beta_{\mathrm{ST}} = 0.62965$

Table 1: Comparisons of probability of detection $P_{\mathrm{D}}(\text{Gaussian}) = \beta_{\mathrm{G}}$ to $P_{\mathrm{D}}(\text{SignTest}) = \beta_{\mathrm{ST}}$ for $n = 10$, $\alpha_{\mathrm{A}} = 0.17188$ and 0.010742, and for $n = 20$, $\alpha_{\mathrm{A}} = 0.13159$ and 0.005909, with $p = 0.8$.

Chapter 4
Detection of Known Binary Deterministic Signals in Gaussian Noises
Odd Numbered Homework Solutions

1.

$$\|\underline{s}_0\|^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14 = \|\underline{s}_1\|^2$$

$$\Lambda_0 = \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})} = \frac{0.25 \times (2-0)}{0.75 \times (1-0)} = \frac{0.50}{0.75} = \frac{2}{3}\ ; \quad \sigma^2 = 1.$$

$$\gamma = \sigma^2 \ln \Lambda_0 + \frac{1}{2}\|\underline{s}_1\|^2 - \frac{1}{2}\|\underline{s}_0\|^2 = -0.406.$$

$$\underline{x}^T(\underline{s}_1 - \underline{s}_0) \geq \gamma \Rightarrow H_1. \qquad -2[x_1, x_2, x_3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \geq \gamma \Rightarrow H_1.$$

$$\begin{aligned} \eta_1 &= x_1 + 2x_2 + 3x_3 \leq \frac{0.406}{2} = 0.203 \Rightarrow H_1. \\ \mu_{\eta_1|H_0} &= \mathrm{E}\{\eta_1|H_0\} = \|\underline{s}_0\|^2 = 14 \\ \mu_{\eta_1|H_1} &= -14 \\ \sigma^2_{\eta_1} &= 1^2 + 2^2 + 3^2 = 14. \end{aligned}$$

$$R_1 = \{\underline{x} : \eta_1 = \underline{x}^T\underline{s}_0 \leq 0.203\}\ , \quad R_0 = \{\underline{x} : \eta_1 \geq 0.203\}.$$

$$\begin{aligned} P_{\mathrm{FA}} &= P\{\eta_1 \leq 0.203|H_0\} = \frac{1}{\sqrt{2\pi \times 14}}\int_{-\infty}^{0.203} e^{-(\eta_1 - 14)^2/28}\, d\eta_1 \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(0.203-14)/\sqrt{14}} e^{-t^2/2}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-13.8/3.74} e^{-t^2/2}\, dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{3.69}^{\infty} e^{-t^2/2}\, dt = Q(3.69) = 0.000112. \end{aligned}$$

$$\begin{aligned} P_{\mathrm{D}} &= P\{\eta_1 \leq 0.203|H_1\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0.203} e^{-(\eta_1 + 14)^2/28}\, d\eta_1 \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{14.203/3.74} e^{-t^2/2}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{3.80} e^{-t^2/2}\, dt \\ &= 1 - Q(3.80) = 0.999928. \end{aligned}$$

Since $\underline{s}_0 = [2, -\sqrt{6}, 2]^T = -\underline{s}_1$ has the same norm as $\underline{s}_0 = [1, 2, 3]^T = -\underline{s}_1$, all the previous results on R_0, R_1, P_{FA}, and P_D apply.

3. **a.** From theory, we know for a binary detection of deterministic waveform in WGN, the sufficient statistic is given by $\eta = \int_0^T x(t)[s_1(t) - s_0(t)]dt = 10\int_0^T x(t)\sin(2\pi f_0 t)dt$. Futhermore, the threshold constant for η is given by $\gamma = (N_0/2)ln(\Lambda_0) + 0.5\int_0^T [s_1^2(t) - s_0^2(t)]dt = 50\int_0^T \sin^2(2\pi f_0 t)dt = 25$. Then $R_1 = \{x(t) : \eta \geq \gamma\}$ and $R_o = \{x(t) : \eta \leq \gamma\}$.

b. Since η is a Gaussian r.v., we need to find its means and variances under H_0 and H_1. $\mu_0 = E\{\eta|H_0\} = 0$. $\mu_1 = E\{\eta|H_1\} = 100\int_o^T \sin^2(2\pi f_0)tdt = 50$. $\sigma^2_\eta = E\{(\eta - \mu_0)^2|H_0\} = 100\int_0^T \sin^2(2\pi f_0 t)dt = 50$. Then $P_{FA} = Q(\gamma/\sigma_\eta) = Q(3.5355) = 2.035E-4$. $P_D = Q((\gamma - \mu_1)/\sigma_\eta) = 0.9998$.

c. Now, $\eta' = \int_0^T x(t)s'(t)dt$. Then $\mu_0 = E\{\eta'|H_0\} = 0$.

$\mu_1 = E\{\eta'|H_1\} = 100 \times 2 \times 10^5 \int_0^{0.5\times 10^{-5}} \sin(2\pi f_0 t)dt = 200/\pi = 63.6620$.

$\sigma^2_{\eta'} = 100\int_0^T (s'(t))^2 dt = 100 \times 1 = 100$.

$P'_{FA} = Q(\gamma/\sigma_{\eta'}) = Q(25/10) = Q(2.5) = 6.2E - 3.$
$P'_D = Q((25 - 63.662)/10) = Q(-3.8662) = 0.9999.$
We note, $P'_D > P_D$ (which is good), but $P'_{FA} > P_{FA}$ (which is bad.) Thus, it is hard to compare the performance of the receiver using $s'(t)$ instead of $s(t)$ explicitly if we allow both probability of false alarm and probability of detection to vary as compared to the original optimum case.

d. In order to find a new γ', we set $P_{FA} = P'_{FA}(\gamma') = 2.035E-4$. Then, we must have $3.5355 = \gamma'/10$, or $\gamma' = 35.355$. Then $P'_D(\gamma') = Q((\gamma'-63.662)/10) = Q(-2.8307) = 0.9977$. We note, this $P'_D(\gamma') < P_D$. In other words, having used the sub-optimum $s'(t)$ instead of the optimum $s(t)$ at the receiver, for the same fixed probability of false alarm, the probability of detection went down from 0.9998 to 0.9977. The degradation of the probability of detection is very small. However, the reduction of complexity of the receiver is fairly significant.

5. Under H_1, the output signal is given by

$$s_o(t_o) = \begin{cases} 0, & t_o < 0, \\ \int_o^{t_o} s(\tau)h(t_o - \tau)d\tau, & 0 \leq t_o \leq T, \\ \int_o^T s(\tau)h(t_o - \tau)d\tau, & T \leq t_o. \end{cases}$$

Under H_0, $s_o(t_o) = 0$. The output noise under both hypothesis is given by

$$\tilde{N}(t_o) = \int_o^{t_o} N(\tau)h(t-\tau)d\tau.$$

Since the mean of $\tilde{N}(.)$ is zero, its variance is given by

$$\sigma_o^2(t_o) = \begin{cases} 0, & t_o < 0, \\ (N_o/2)\int_o^{t_o} h^2(t_o - \tau)d\tau, & 0 \leq t_o \leq T, \\ (N_o/2)\int_o^T h^2(t_o - \tau)d\tau, & T \leq t_o. \end{cases}$$

The output SNR can be defined by $SNR(t_o) = s_o^2(t_o)/\sigma_o^2(t_o)$. For $t_o < 0$, $SNR(t_o)$ is not defined. Under H_0, for $0 < t_o$, then $SNR(t_o) = 0$. From Schwarz Inequality, we have $(\int_a^b f(t)h(t)dt)^2 \leq \int_a^b f^2(t)dt \int_a^b h^2(t)dt$, with equality if and only if $f(t) = ch(t)$ for some constant c. Then under H_1, we have

$$Max_{\{h(.)\}}\{SNR(t_o)\} = (2/N_0)\int_0^{t_o} s^2(\tau)d\tau, \ \ 0 \leq t_o \leq T,$$

$$Max_{\{h(.)\}}\{SNR(t_o)\} = (2/N_0)\int_0^{T} s^2(\tau)d\tau, \ \ T \leq t_o.$$

Thus,

$$Max_{\{0\leq t_o;\, h(.)\}}\{SNR(t_o)\} = (2/N_0)\int_0^{T} s^2(\tau)d\tau.$$

7. Let $K(t) = h(t)$ and $f(t) = exp(i\omega_0 t)$. Then denote $t - s = u$ and $ds = -du$. Thus,

$$\int_{-\infty}^{\infty} K(t-s)f(s)ds = -\int_{\infty}^{-\infty} h(u)\exp(i\,\omega_0(t-u))du = \left\{ \int_{-\infty}^{\infty} h(u)\exp(-i\,\omega_0 u)du \right\} \exp(i\,\omega_0 t) = H(\omega_0)\ \exp(i\,\omega_0 t)\,.$$

This show that the eigenvalue $\lambda = H(\omega_0)$ and the eigenfunction $f(t) = exp(i\,\omega_0 t)$.

9. **a.**

$$\underline{R} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} \qquad |\underline{R} - \lambda \underline{I}| = 0$$

$$(8-\lambda)(5-\lambda) - 4 = 0 = 40 - 13\lambda + \lambda^2 - 4 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4) \Rightarrow \lambda_1 = 9, \lambda_2 = 4.$$

$$\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \theta_{11} \\ \theta_{12} \end{bmatrix} = 9 \begin{bmatrix} \theta_{11} \\ \theta_{12} \end{bmatrix} \Rightarrow \begin{array}{c} 8\theta_{11} + 2\theta_{12} = 9\theta_{11} \\ 2\theta_{11} + 5\theta_{12} = 9\theta_{12} \end{array} \Rightarrow \begin{array}{c} -\theta_{11} + 2\theta_{12} = 0 \\ 2\theta_{11} - 4\theta_{12} = 0 \end{array}$$

$$\theta_{11} = 2\theta_{12}, 4\theta_{12}^2 + \theta_{12}^2 = 1 \Rightarrow 5\theta_{12}^2 = 1 \Rightarrow \theta_{12} = 1/\sqrt{5},\ \theta_{11} = 2/\sqrt{5}.$$

$$\begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \theta_{21} \\ \theta_{22} \end{bmatrix} = 4 \begin{bmatrix} \theta_{21} \\ \theta_{22} \end{bmatrix} \Rightarrow \begin{array}{c} 8\theta_{21} + 2\theta_{22} = 4\theta_{21} \\ 2\theta_{21} + 5\theta_{22} = 4\theta_{22} \end{array} \Rightarrow \begin{array}{c} 2\theta_{21} + \theta_{22} = 0 \\ 2\theta_{21} + \theta_{22} = 0 \end{array}$$

$$\underline{\theta}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{5}}, \qquad \underline{\theta}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \frac{1}{\sqrt{5}}$$

b. $\underline{R}^{-\frac{1}{2}} = \underline{T}\underline{D}^{-\frac{1}{2}}\underline{T}^T$

$$\underline{T} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \frac{1}{\sqrt{5}}, \qquad \underline{D} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \underline{D}^{-\frac{1}{2}} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

c. Whitening filter.

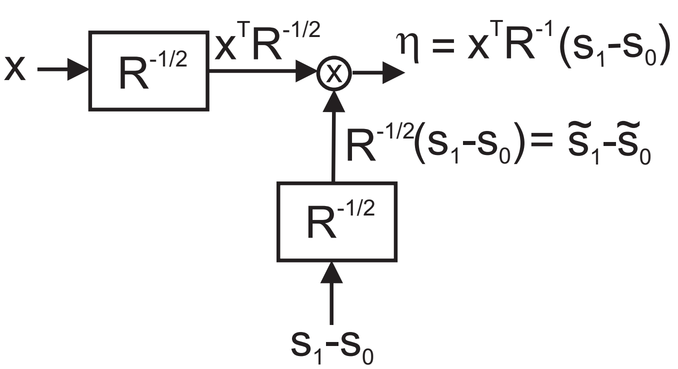
$$\begin{aligned} \underline{R}^{-\frac{1}{2}} &= \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & -1 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} \frac{4}{3} + \frac{1}{2} & \frac{2}{3} - 1 \\ \frac{2}{3} - 1 & \frac{1}{3} + 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \frac{8+3}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1+6}{3} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \frac{11}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{7}{3} \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 11 & -2 \\ -2 & 14 \end{bmatrix} \end{aligned}$$

Check.

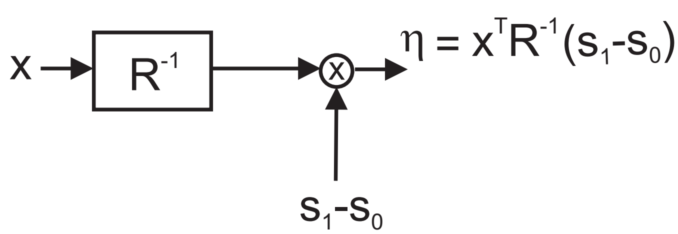
$$\begin{aligned} \underline{R}^{-1} &= \frac{1}{25} \begin{bmatrix} \frac{121}{36} + \frac{1}{9} & -\frac{11}{18} - \frac{7}{9} \\ -\frac{11}{18} - \frac{7}{9} & \frac{1}{9} + \frac{49}{9} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} \frac{125}{36} & -\frac{25}{18} \\ -\frac{25}{18} & \frac{50}{9} \end{bmatrix} \\ &= \frac{1}{9 \times 25} \begin{bmatrix} \frac{125}{4} & -\frac{25}{2} \\ -\frac{25}{2} & 50 \end{bmatrix} = \frac{1}{900} \begin{bmatrix} 125 & -50 \\ -50 & 200 \end{bmatrix} \end{aligned}$$

$$\underline{R}\underline{R}^{-1} = \frac{1}{900} \begin{bmatrix} 1000 - 100 & -400 + 400 \\ 250 - 250 & -100 + 1000 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d.



Whitening filter approach
to LR Receiver for CGN



Conventional approach
to LR Receiver for CGN

$$
\begin{aligned}
\eta &= \underline{x}^T \underline{R}^{-\frac{1}{2}} (\tilde{\underline{s}}_1 - \tilde{\underline{s}}_0) = \underline{x}^T \underline{R}^{-1} (\underline{s}_1 - \underline{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} 0 \\
&= \underline{x}^T \frac{1}{900} \begin{bmatrix} 125 & -50 \\ -50 & 200 \end{bmatrix} \begin{bmatrix} -60 \\ 60 \end{bmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \\
&= \frac{5}{3} \underline{x}^T \begin{bmatrix} -7 \\ 10 \end{bmatrix} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \\
&= (-7x_1 + 10x_2) \underset{H_0}{\overset{H_1}{\gtrless}} 0.
\end{aligned}
$$

$$
\begin{aligned}
R_1 &= \{10x_2 - 7x_1 \geq 0\} \\
R_0 &= \{10x_2 - 7x_1 \leq 0\}.
\end{aligned}
$$

11.

The integral equation of interest has the form

$$\int_0^T R(t-t')q(t')dt' = \lambda q(t), \qquad 0 \le \mathrm{t} \le \mathrm{T}. \tag{1}$$

The power spectral density function of $R(t)$ is given by

$$S(\omega) = F\{R(t)\}.$$

Since the integral in the left hand side of (1) is a convolution, thus by using Fourier transform, this integral equation can be converted to

$$S(\omega)Q(j\omega) = \lambda Q(j\omega). \tag{2}$$

Since S(ω) is an even function and if it is assumed to be a rational function, we can write it as

$$S(\omega) = N(-\omega^2)/D(-\omega^2),$$

and (2) becomes

$$N(-\omega^2)Q(j\omega) = \lambda D(-\omega^2)Q(j\omega), \tag{3}$$

where

$$N(-\omega^2) = \sum_{k=0}^{m} b_k(-\omega^2)^k \text{ and } D(-\omega^2) = \sum_{k=0}^{n} a_k(-\omega^2)^k. \tag{4}$$

We also have

$$-\omega^2 \Leftrightarrow d^2(e^{j\omega t})/dt^2 = (-\omega^2)e^{j\omega t}, \tag{5}$$

thus from operational calculus (3) and (4) become

$$\sum_{k=0}^{m} b_k \frac{d^{2k}}{dt^{2k}} q(t) = \lambda \sum_{k=0}^{n} a_k \frac{d^{2k}}{dt^{2k}} q(t). \tag{6}$$

We know that $q(t)$ cannot contain any impulse function, and therefore the solutions to the above differential equation are also solutions to the integral equation.

Let us consider the specific kernel of

$$R(t-t') = \alpha \exp(-\beta|t-t'|) \,. \qquad (7)$$

Then (1) becomes

$$\int_0^T \alpha \exp(-\beta|t-t'|)q(t')dt' = \lambda q(t), \qquad 0 \le t \le T. \qquad (8)$$

The Fourier transform of the kernel of (7) is given by

$$S(\omega) = \frac{2\alpha\beta}{\omega^2+\beta^2}, \; -\infty < \omega < \infty. \qquad (9)$$

Then the integral equation of (8) can be converted to the following homogeneous differential equation of

$$2\alpha\beta q(t) = -\lambda q''(t) + \lambda\beta^2 q(t), \qquad (10)$$

which can be rewritten as

$$q''(t) + \beta(2\alpha/\lambda - \beta)q(t) = 0\,. \qquad (11)$$

The solution to this differential equation is given by

$$q(t) = A\exp(\gamma t) + B\exp(-\gamma t)\,, \qquad (12)$$

where

$$\gamma = (\beta^2 - 2\alpha\beta/\lambda)^{1/2}\,. \qquad (13)$$

Substituting (12) into (1), and after some manipulation, we obtain

$$A(\beta-\gamma) + B(\beta+\gamma) = 0, \qquad (14a)$$

$$A(\beta+\gamma)\exp(\gamma T) + B(\beta-\gamma)\exp(-\gamma T) = 0\,. \qquad (14b)$$

For a solution to exist, the determinant of the matrix of the coefficients of these two equations must be 0. Therefore

$$\exp(\gamma T) = (\beta-\gamma)/(\beta+\gamma)\;, \qquad (15)$$

where γ is a complex number.

To proceed, let

$$\gamma = j\omega \Rightarrow \omega = (2\alpha\beta / \lambda - \beta^2)^{1/2}, \qquad (16)$$

then (15) becomes

$$\cos(\omega T) + j\sin(\omega T) = (\beta - j\omega)^2 / (\beta^2 + \omega^2). \qquad (17)$$

From the real and imaginary components, we obtain

$$\cos(\omega T) = \lambda(\beta^2 - \omega^2) / 2\alpha\beta, \qquad (18a)$$

$$\sin(\omega T) = -\lambda\omega / \alpha, \qquad (18b)$$

when combined with (16) result in

$$\tan(\omega T) = 2\beta\omega / (\omega^2 - \beta^2). \qquad (19)$$

Let $\beta = \sin(x), \omega = \cos(x)$, then with $x = \omega T / 2$, (19) becomes

$$\tan(\omega T / 2) = (\beta T / 2) / (\omega T / 2). \qquad (20)$$

If we let $\omega = \sin(x), \beta = \cos(x)$, and $x = -\omega T / 2$, then (19) becomes

$$-\cot(\omega T / 2) = (\beta T / 2) / (\omega T / 2). \qquad (21)$$

Let $\xi_k = \omega_k T / 2$, then from (20), we obtain

$$\tan(\xi_k) = \frac{\beta T / 2}{\xi_k}, \qquad (22)$$

and from (21), we obtain

$$-\cot(\xi_k) = \frac{\beta T / 2}{\xi_k}. \qquad (23)$$

The solutions to (22) are the intersections of the tangent function to a hyperbola and the solutions of (23) are the intersections of the -cotangent function to a hyperbola. Both of these equations have countably infinite number of solutions and thus countably infinite number of eigenvalues.

The hyperbola is monotonically decreasing in (0, ∞) and the tangent function and negative cotangent function have period of π and are monotonically increasing in each period. In addition, they have discontinuities at multiples of π /2. Thus, the hyperbola intersects these two functions alternatively, and the locations of the intersections increase with increasing $\omega T/2$. In order to have a clearer picture of the solutions, these three functions are plotted in Fig. 1 with α = 10 and β = 5.5. (Without loss of generality, we picked T = 1). From Fig. 1, we can find the first three intersections corresponding to the three largest eigenvalues. From Fig. 1, intersections occur at ξ_k values of about 1.1, for k = 1, of about 2.4, for k = 2, and about 3.7, for k = 3. Upon solving (22) and (23) more precisely (e.g., using Newton's Method), $\xi_1 = \omega_1 T/2$ = 1.16888, $\xi_2 = \omega_2 T/2$ = 2.41995, and $\xi_3 = \omega_3 T/2$ = 3.77161. We note the plot on page 368 of McDonough and Whalen (2nd edition) are incorrect.

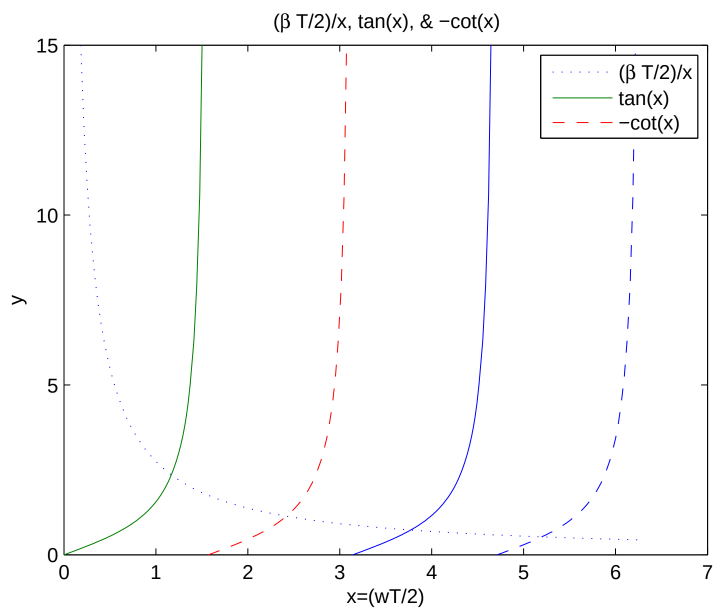


Figure 1. Intersections of $(\beta T/2)/x$ with tan (x) and –cot (x) for $\alpha = 10$ and $\beta = 5.5$.

We note from (16), we can obtain the eigenvalues

$$\lambda_k = 2\alpha\beta / (\omega_k^2 + \beta^2), \quad k = 1, 2, \ldots \tag{24}$$

For α = 10 and β = 5.5, (24) yields the three largest eigenvalues by the differential equation approach to be

$$\lambda_1^{(DF)} = 3.07992,\ \lambda_2^{(DF)} = 2.04939, \text{ and } \lambda_3^{(DF)} = 1.26219. \tag{25}$$

The subscript of (*DF*) indicates these eigenvalues were obtained based on the differential equation method. Shortly, we will evaluate these three largest eigenvalues by the matrix eigenvalue method.

Substitute the above value of $\lambda_k^{(DF)}$ in (25) into (16) and then into (15), we obtain

$$q(t) = a\cos(\omega t) + b\sin(\omega t),$$

where *a* and *b* are arbitrary constants. To find the values of *a* and *b*, we substitute this $q(t)$ into the original integral equation and obtain

$$\beta a - \omega b = 0, \tag{26a}$$

$$\left[-\beta\cos(\omega T) + \omega\sin(\omega T)\right]a + \left[-\beta\sin(\omega T) - \omega\cos(\omega T)\right]b = 0, \tag{26b}$$

which yields

$$[\tan(\omega T) - 2\beta\omega/(\omega^2 - \beta^2)]a = 0. \tag{27}$$

From (19), this equation is already satisfied for any ω_k, which is the solution to (22) and 23).

The parameter *a* can be arbitrary, and the eigenfunctions we are looking for are

$$q_k(t) = a_k\left[\cos(\omega_k t) + (\beta/\omega_k)\sin(\omega_k t)\right], \quad k = 1, 2, \ldots, \tag{28}$$

with a_k satisfying (29)

$$\frac{1}{{a_k}^2} = \int_0^T \left[\cos(\omega_k t) + \left(\frac{\beta}{\omega_k}\right)\sin(\omega_k t)\right]^2 dt\ ,\ k = 1, 2, \ldots, \tag{29}$$

so that these eigenfunctions will form an orthonormal set. The three associated eigenfunctions corresponding to the eigenvalues in (25) are given by

$$q_1(t) = 0.483657\left[\cos(2.33777t) + (2.35267)\sin(2.33777t)\right], \tag{30}$$

$$q_2(t) = 0.850800\left[\cos(4.83990t) + (1.13639)\sin(4.83990t)\right], \tag{31}$$

$$q_3(t) = 1.07616\left[\cos(7.54321t) + (0.729132)\sin(7.54321t)\right]. \tag{32}$$

Plots of these three eigenfunctions (given by (30) – (32)) obtained by the differential equation method are shown in Fig. 2.

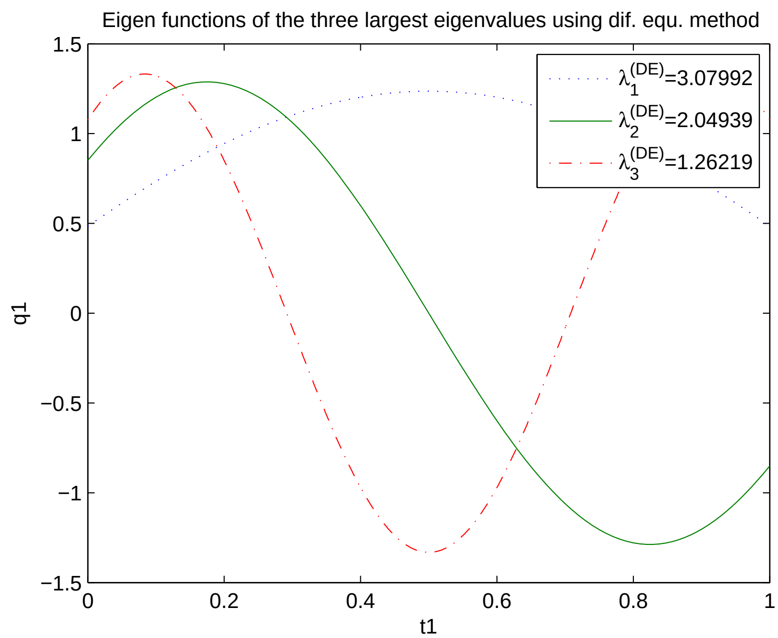


Figure 2. Eigenfunctions associated with the three largest eigenvalues obtained by the differential equation method.

From the plots in Figure 2, we can see the eigenfunction associated with the largest eigenvalue 3.08 has no zero-crossing, one for eigenfunction associated with eigenvalue 2.05, and two for eigenfunction associated with eigenvalue 1.26. The number of zero-crossings increases with decreasing eigenvalues.

Now, consider the solution of this problem using the discretized matrix eigenvalue method as considered in Problem 10. The three associated eigenvalues are now given by

$$\lambda_1^{(ME)} = 3.08064\,,\ \lambda_2^{(ME)} = 2.05085\,,\ \text{and}\ \ \lambda_3^{(ME)} = 1.26363. \tag{33}$$

The three largest eigenvalues obtained from the matrix eigenvalue method shown in (33) are extremely close to those three largest eigenvalues obtained from the differential equation method shown in (25). Clearly, there are some numerical errors in the computations of these eigenvalues in both methods. Figure 3 shows the plots of the three eigenvectors corresponding to the eigenvalues of (33) evaluated based on the matrix eigenvalue method.

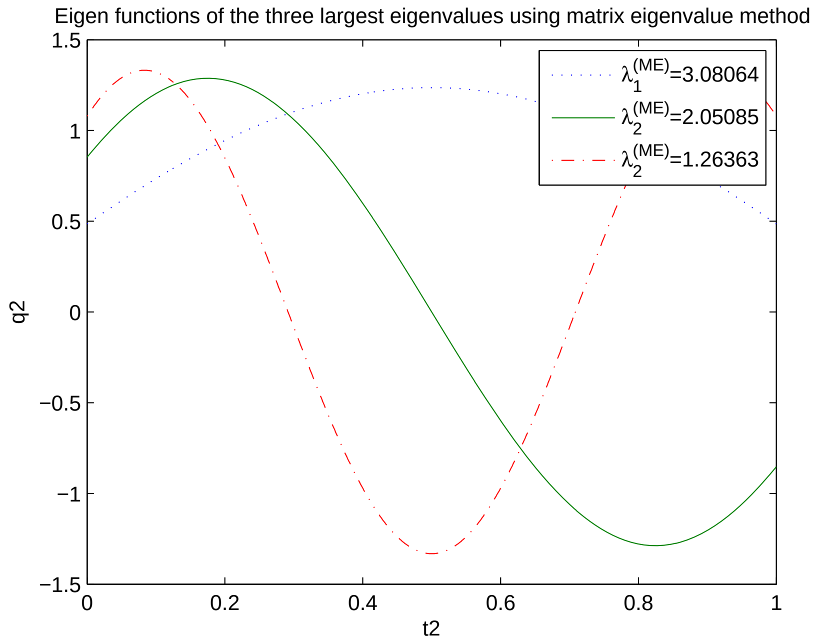


Figure 3: Eigenfunctions associated with the three largest eigenvalues obtained with discretized matrix eigenvalue method.

Figure 4 plots the two eigenfunctions corresponding to $\lambda_1^{(DF)}$ and $\lambda_1^{(ME)}$. Similarly, Figure 5 plots the two eigenfunctions corresponding to $\lambda_2^{(DF)}$ and $\lambda_2^{(ME)}$ and Figure 6 plots the two eigenfunctions corresponding to $\lambda_3^{(DF)}$ and $\lambda_3^{(ME)}$. We note, in Figures 4, 5, and 6, values of the eigenfunctions obtained from the differential equation method are essentially identical to those obtained from the matrix eigenvalue method.

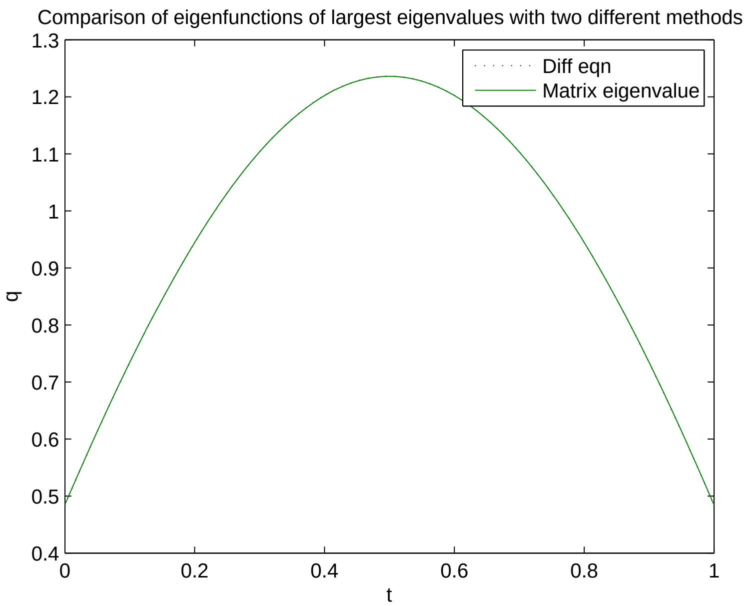


Figure 4: Eigenfunctions of two largest eigenvalues with two different methods.

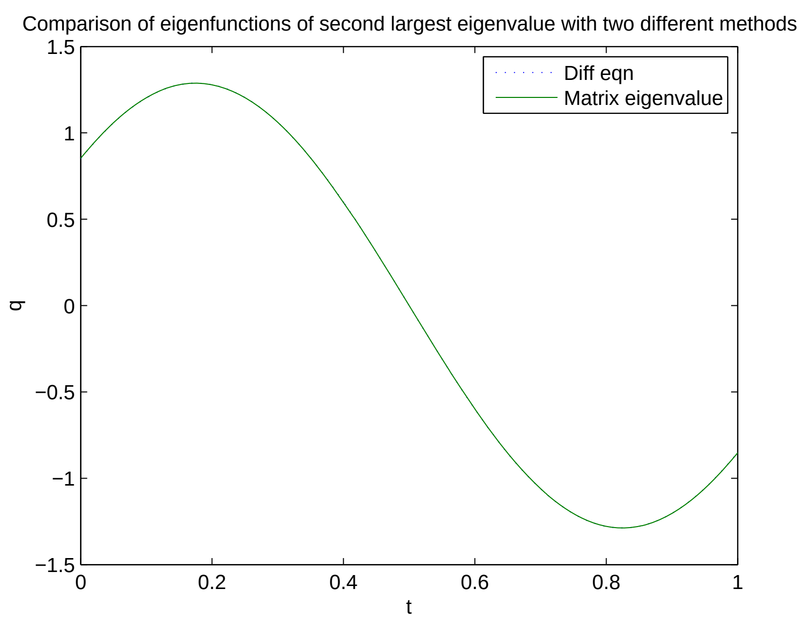


Figure 5: Eigenfunctions for second largest eigenvalues with two different methods.

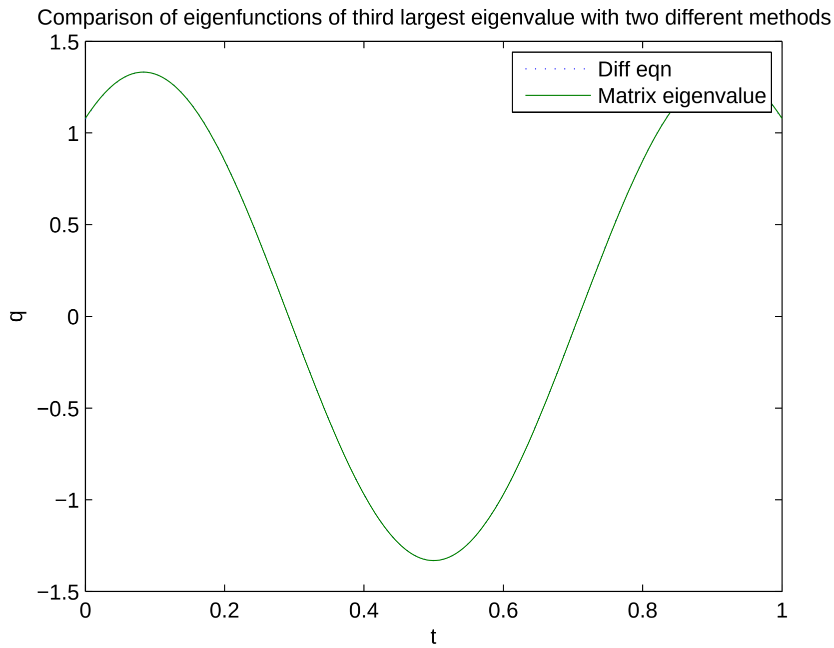


Figure 6. Eigenfunctions for third largest eigenvalues with two different methods.

KY wants to acknowledge the contributions of [16; pp. 361-370] and various students (and particularly S. Lee) in a class project dealing with this problem.

13. **a.** The optimum correlation function $\{g(t),\ 0 \le t \le T\}$ for the CGN problem is obtained from the solution of

$$\int_o^T R(t-t')g(t')dt' = s(t),\ 0 \le t \le T. \tag{1}$$

The CGN matched filter impulse response function $h(t)$ is given by

$$h(t) = \begin{cases} g(T-t), & 0 \le t \le T, \\ 0, & elsewhere. \end{cases} \tag{2}$$

An approximation for the solution of $g(t)$ in (1) for large T can be obtained from

$$\int_{-\infty}^{\infty} R(t-t')g(t')dt' = s(t),\ 0 \le t \le T. \tag{3}$$

Denote $N(\omega) = \mathcal{F}\{R(t)\} = \int_{-\infty}^{\infty} R(t)exp(-j\omega t)dt$, $S(\omega) = \mathcal{F}\{s(t)\}$, $G(\omega) = \mathcal{F}\{g(t)\}$, and $H(\omega) = \mathcal{F}\{h(t)\}$. Apply the Fourier transform to the convolutional equation in (3). Then

$$N(\omega)G(\omega) = S(\omega),\ -\infty < \omega < \infty. \tag{4}$$

Apply the Fourier transform to (2). Then

$$H(\omega) = G^*(\omega)e^{-j\omega T}.\ -\infty < \omega < \infty. \tag{5}$$

Combine (4) with (5) to obtain

$$H(\omega) = \frac{S^*(\omega)}{N^*(\omega)}e^{-j\omega T} = \frac{S^*(\omega)}{N(\omega)}e^{-j\omega T},\ -\infty < \omega < \infty. \tag{6}$$

b. If the input is a CGN process of power spectral density

$$N(\omega) = \frac{K}{\omega^2 + \omega_0^2} = \frac{K^{1/2}}{j\omega + \omega_0} \times \frac{K^{1/2}}{-j\omega + \omega_0},$$

then the causal whitening filter $H_W(\omega)$ satisfies

$$N(\omega)|H_W(\omega)|^2 = C,\ \text{forany}\ C > 0.$$

Thus,

$$|H_W(\omega)|^2 = \frac{C}{N(\omega)} = \frac{C(\omega^2 + \omega_0^2)}{K}$$

and

$$H_W(\omega) = C_0(j\omega + \omega_0),\ C_0 = (C/K)^{1/2}.$$

c. The Fourier transform of the input signal is $S(\omega)$. Then the signal component of the output of the matched filter in the frequency domain is given by $(|S(\omega)|^2/N(\omega))exp(-j\omega T)$. Thus the signal component of the output of the matched filter in the time domain $s_0(t)$ is given by

$$s_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{N(\omega)} e^{jw(t-T)}d\omega.$$

d. From (4.211), we have

$$P_D = Q\left(\frac{\gamma_0^c - \mu_1}{\sigma_1}\right),$$

where from (4.206)

$$\gamma_0^c = \frac{1}{2}\sum_{i=0}^{\infty}\frac{s_i^2}{\lambda_i} + \ln(\Lambda_0),$$

from (4.208)

$$\mu_1 = \sum_{i=0}^{\infty}\frac{s_i^2}{\lambda_i},$$

and from (4.209)

$$\sigma_1 = \sqrt{\sum_{i=0}^{\infty} \frac{s_i^2}{\lambda_i}}\,.$$

Then

$$P_D = Q\left(-\frac{1}{2}\sqrt{\sum_{i=0}^{\infty} \frac{s_i^2}{\lambda_i}} + \frac{\ln(\Lambda_0)}{\sqrt{\sum_{i=0}^{\infty} \frac{s_i^2}{\lambda_i}}} \right).$$

From Parseval Theoren

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{s_i^2}{\lambda_i} &= \sum_{i=0}^{\infty} \frac{\text{Power ofsignal}s(t)\text{in}\theta_i \text{ coordinate}}{\text{Av.powerofnoise}N(t) \text{ in}\theta_i \text{ coordinate}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{N(\omega)} d\omega = \infty\,.\end{aligned}$$

Thus, $P_D = Q(-\infty) = 1$.

e. If $N(\omega) = 0$, $\omega \in \{(-b, -a) \cup (a, b)\}$, but $S(\omega) \neq 0$, $\omega \in \{(-b, -a) \cup (a, b)\}$,, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{N(\omega)} d\omega = \infty\,.$$

Thus, $P_D = Q(-\infty) = 1$.

15.

In order to obtain the upper bound of $\left(\mathbf{s}^T\boldsymbol{\Lambda}\mathbf{s}\right)\left(\mathbf{s}^T\boldsymbol{\Lambda}^{-1}\mathbf{s}\right)$ in (4.242), where $\|\mathbf{s}\|^2$ is taken to have unit norm, we can perform the following eigenvalue decomposition. Denote $\boldsymbol{\Lambda} = \mathbf{UDU}^T$ and $\boldsymbol{\Lambda}^{-1} = \mathbf{UD}^{-1}\mathbf{U}^T$. where diag(D) = $[\lambda_1, \ldots, \lambda_n]$, with $\lambda_1 > \lambda_2 > \ldots \lambda_n > 0$. Since $\mathbf{U}$ is an orthogonal matrix, define a new n×1 vector $\mathbf{z} = \mathbf{U}^T\mathbf{s}$ or $\mathbf{s} = \mathbf{U}\,\mathbf{z}$ and $\mathbf{s}^T = \mathbf{u}^T\,\mathbf{U}^T$, with $\|\mathbf{z}\|^2 = 1$.
Thus,

$$\begin{aligned}&\left(\mathbf{s}^T\boldsymbol{\Lambda}^{-1}\mathbf{s}\right)\left(\mathbf{s}^T\boldsymbol{\Lambda}\mathbf{s}\right) = \left(\mathbf{z}^T\mathbf{U}^T\mathbf{U}\mathbf{D}^{-1}\mathbf{U}^T\mathbf{U}\mathbf{z}\right)\left(\mathbf{z}^T\mathbf{U}^T\mathbf{U}\mathbf{D}\mathbf{U}^T\mathbf{U}\mathbf{z}\right)\\ &= \left(\mathbf{z}^T\mathbf{D}^{-1}\mathbf{z}\right)\left(\mathbf{z}^T\mathbf{D}\mathbf{z}\right) = \left(\sum_{\mathrm{i=1}}^{\mathrm{M}} z_i^2/\lambda_i\right)\left(\sum_{\mathrm{i=1}}^{\mathrm{M}} z_i^2\lambda_i\right).\end{aligned} \tag{A1}$$

Define n new variables $y_i = z_i^2 \geq 0$, $i = 1,\ldots,n$, and substitute them into (A1). Then

$$\left(\mathbf{s}^T\boldsymbol{\Lambda}^{-1}\mathbf{s}\right)\left(\mathbf{s}^T\boldsymbol{\Lambda}\mathbf{s}\right) = \left(\sum_{\mathrm{i=1}}^{\mathrm{M}} y_i/\lambda_i\right)\left(\sum_{\mathrm{i=1}}^{\mathrm{M}} y_i\lambda_i\right). \tag{A2}$$

Thus, the upper bound of $\left(\mathbf{s}^T\boldsymbol{\Lambda}\mathbf{s}\right)\left(\mathbf{s}^T\boldsymbol{\Lambda}^{-1}\mathbf{s}\right)$ is given by the solution of the following Theorem.

Theorem. The maximum of (A3) given by

$$\left(\sum_{\mathrm{i=1}}^{\mathrm{n}} y_i/\lambda_i\right)\left(\sum_{\mathrm{i=1}}^{\mathrm{n}} y_i\lambda_i\right), \tag{A3}$$

subject to

$$\sum_{\mathrm{i=1}}^{\mathrm{n}} y_i = 1,\ y_i \geq 0,\ i = 1,\ldots,n, \tag{A4}$$

and

$$\lambda_1 > \lambda_2 > \ldots \lambda_n > 0, \tag{A5}$$

as attained by using

$$\hat{y}_1 = \hat{y}_n = 1/2,\ \hat{y}_i = 0,\ i = 2,\ldots,n-1. \tag{A6}$$

Proof: Consider the nonlinear minimization problem of

$$\mathrm{Min}_{\,y}\ \mathrm{f}(\boldsymbol{y}), \tag{A7}$$

subject to

$$g_i(\boldsymbol{y}) \leq 0,\ i = 1,\ldots, M, \tag{A8}$$

$$h(\boldsymbol{y}) = 0. \tag{A9}$$

The celebrated Karush-Kuhn-Tucker (KKT) [4] necessary conditions for $\mathbf{y} = \hat{\mathbf{y}} = [\hat{y}_1,\ldots,\hat{y}_M]^T$ to be a local minimum solution of (A7) subject to (A8) and (A9), are such that there exist constants μ_I, $i = 1,\ldots, M$, and υ satisfying

1. $\nabla f(\hat{\mathbf{y}}) + \sum_{i=1}^{M} \mu_i \nabla g_i(\hat{\mathbf{y}}) + \upsilon \nabla h(\hat{\mathbf{y}}) = 0,$ (A10)

2. $g_i(\hat{\mathbf{y}}) \le 0,\ i = 1,\ldots,M,$ (A11)

3. $h(\hat{\mathbf{y}}) = 0,\ i = 1,\ldots,M,$ (A12)

4. $\mu_i \ge 0,\ i = 1,\ldots,M,$ (A13)

5. $\mu_i g_i(\hat{\mathbf{y}}) = 0,\ i = 1,\ldots,M.$ (A14)

In order to use the KKT method for our maximization of (A7) with the constraints of (A8) and (A9), denote

$$f(\hat{\mathbf{y}}) = -A(i)B(i), \tag{A15}$$

$$g_i(\hat{\mathbf{y}}) = -y_i\ ,\ i = 1,\ldots,M, \tag{A16}$$

$$h(\hat{\mathbf{y}}) = \sum_{i=1}^{M} y_i - 1, \tag{A17}$$

$$\lambda_1 > \lambda_2 > \ldots \lambda_M > 0\ , \tag{A18}$$

where we define

$$A(i) = \sum_{i=1}^{M} y_i / \lambda_i\ ,\quad B(i) = \sum_{i=1}^{M} y_i \lambda_i\,. \tag{A19}$$

Now, we show the conditions of (1)- (5) of (A10) – (A14) are satisfied for the expressions of (A15) - (A18). From Condition 1, by taking its partial derivative wrt to y_i, $i = 1, \ldots, M$, we have

$$\frac{-1}{\lambda_i} B(j) - \lambda_i A(j) - \mu_i + \upsilon = 0. \tag{A20}$$

Multiply (A20) by y_i yields

$$\frac{-y_i}{\lambda_i} B(j) - \lambda_i y_i A(j) - \mu_i y_i + \upsilon y_i = 0. \tag{A21}$$

From Condition 4 and $y_i \ge 0$, then $-\mu_i y_i = 0$. Thus, (A21) becomes

$$\frac{-y_i}{\lambda_i} B(j) - \lambda_i y_i A(j) + \upsilon y_i = 0. \tag{A22}$$

Summing (A22) over all $i = 1, \ldots, M$, yields

$$\upsilon \sum_{i=1}^{M} y_i = A(i)B(j) + B(i)A(j),$$

and from (A8), we have

$$\upsilon = A(i)B(j) + B(i)A(j) = 2A(j)B(j). \tag{A23}$$

Substitute (A23) into (A22) yields

$$-\frac{y_i}{\lambda_i}B(j)-\lambda_i y_i A(j)+2y_i B(j)A(j)=0,$$

or

$$y_i\left[\frac{1}{\lambda_i}B(j)+\lambda_i A(j)-2B(j)A(j)\right]=0, \qquad \text{(A24)}$$

Thus, (A24) shows either $y_i = 0$ or

$$\frac{1}{\lambda_i}B(j)+\lambda_i A(j)-2B(j)A(j)=0. \qquad \text{(A25)}$$

Multiply (A25) by λ_i , to obtain

$$\lambda_i^2 A(j)-2\lambda_i B(j)A(j)+B(j)=0. \qquad \text{(A26)}$$

The quadratic equation of (A26) in λ_i, has two non-zero solutions given by

$$\lambda_i=\frac{2B(j)A(j)\pm\sqrt{4B(j)^2A(j)^2-4B(j)A(j)}}{2A(j)}. \qquad \text{(A27)}$$

The rest of the (M -2) number of $y_i = 0$. Denote the indices of the two non-zero y_i by a and b. Then the maximization of $A(i)B(i)$ in (A7)) reduces to the maximization of

$$\begin{aligned}
&\left(\frac{y_a}{\lambda_a}+\frac{y_b}{\lambda_b}\right)\left(y_a\lambda_a+y_b\lambda_b\right)\\
&=y_a^2+\frac{\lambda_a}{\lambda_b}y_a y_b+y_b^2+\frac{\lambda_b}{\lambda_a}y_a y_b\\
&=y_a^2+y_b^2+\left(\frac{\lambda_a}{\lambda_b}+\frac{\lambda_b}{\lambda_a}\right)y_a y_b\\
&=\left(y_a+y_b\right)^2+\left(\frac{\lambda_a}{\lambda_b}+\frac{\lambda_b}{\lambda_a}-2\right)y_a y_b.
\end{aligned} \qquad \text{(A28)}$$

By denoting $y_b = 1 - y_a$ in (A28), we obtain

$$\begin{aligned}
&\left(\frac{y_a}{\lambda_a}+\frac{y_b}{\lambda_b}\right)\left(y_a\lambda_a+y_b\lambda_b\right)\\
&=1+\left(\frac{\lambda_a}{\lambda_b}+\frac{\lambda_b}{\lambda_a}-2\right)(1-y_a)y_a\\
&\triangleq H(y_a,\lambda_a,\lambda_b).
\end{aligned} \qquad \text{(A29)}$$

Thus, the maximization in (A7)) reduces to the maximization of $H(y_a, \lambda_a, \lambda_b)$ in (A29). We note $H(y_a, \lambda_a, \lambda_b)$ is a quadratic function of y_a. Taking the partial derivative of $H(y_a, \lambda_a, \lambda_b)$ wrt to y_a yields

$$\frac{\partial H(y_a, \lambda_a, \lambda_b)}{\partial y_a} = -\left(\frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2\right) y_a + \left(\frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2\right)(1 - y_a) = 0,$$

or

$$2\left(\frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2\right) y_a = \left(\frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2\right),$$

or

$$y_a = y_b = 1/2, \tag{A30}$$

since

$$\left(\frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2\right) \neq 0,$$

with $\lambda_a \neq \lambda_b$. The second partial derivative of $H(y_a, \lambda_a, \lambda_b)$ shows

$$\frac{\partial^2 H(y_a, \lambda_a, \lambda_b)}{\partial y_a^2} = -2\left(\frac{\lambda_a}{\lambda_b} + \frac{\lambda_b}{\lambda_a} - 2\right) = -2\left(\frac{(\lambda_a + \lambda_b)^2}{\lambda_a \lambda_b}\right) < 0,$$

since $\lambda_a > 0$ and $\lambda_b > 0$. Thus, the local maximum solution of $y_a = y_b = 1/2$ in (A30) is a global maximum solution of $H(y_a, \lambda_a, \lambda_b)$. By using $y_a = y_b = 1/2$ and denoting $\lambda = \lambda_a / \lambda_b > 1$, with the assumption of $\lambda_a > \lambda_b > 0$, then $H(y_a, \lambda_a, \lambda_b)$ of (A29) can be expressed as

$$H(\lambda) = 1 + (1/4)(\lambda - 2 + 1/\lambda) = 1 + (1/4)\left(\frac{(\lambda + 1)(\lambda - 1)}{\lambda^2}\right) > 0. \tag{A31}$$

This shows $H(\lambda)$ is a positive monotonically increasing function of λ. Since $\lambda = \lambda_a / \lambda_b$ and $\lambda_a > \lambda_b > 0$, the maximum of $H(\lambda)$ is attained by using $\lambda_{\max} = \lambda_1 / \lambda_M$. This means

$$\hat{y}_1 = \hat{y}_M = 1/2, \hat{y}_i = 0, i = 2, ..., M - 1. \tag{A32}$$

Thus, the solution given by (A32) is the only solution that satisfies the KKT Condition 1 of (A15) that provides the local minimum of

$f(\hat{\mathbf{y}}) = -\left(\sum_{i=1}^{M} y_i / \lambda_i\right)\left(\sum_{i=1}^{M} y_i\lambda_i\right)$ or the local maximum of $\left(\sum_{i=1}^{M} y_i / \lambda_i\right)\left(\sum_{i=1}^{M} y_i\lambda_i\right)$. But $H(y_a, \lambda_a, \lambda_b)$ of (A29) is a quadratic function, thus the solution given by (A32) yields the global maximum of (A3) or the upper bound of (A2) and (A1). □

(KY wants to thank discussions with Prof. L. Vandenberghe and various students in a class project of EE230A for various inputs in the solution of this problem.)

Chapter 5
M-ary Detection
Odd Numbered Homework Solutions

1. The M-ary MAP decision rule state that we decide for hypothesis H_j, if

$$p(H_j|x) > p(H_i|x), \infty < x < \infty, \text{all}\, j \neq i,\ 1 \leq i,\ j \leq M.$$

From Bayes rule, $p(H_i/x) = p(x|H_i)\pi_i/p(x)$, $1 \leq i \leq M$, and $p(x) = \sum_{k=1}^{M} p(x|H_k)\pi_k > 0$. Thus,we decide for hypothesis H_j, if

$$p(x|H_j)\pi_j > p(x|H_i)\pi_i, \text{all}\, j \neq i,\ 1 \leq i,\ j \leq M.$$

3. List of some M-ary orthogonal transmission system using a carrier frequency includes:

a. M-ary PSK (including conventional QPSK; offset QPSKl $\pi - /4$ QPSK; etc.);

b. M-FSK.

List of some M-ary orthogonal baseband signals modulating a carrier frequency includes:

a. Time-multiplexed PPM signals;

b. Time-multiplexed PWM signals;

c. Time-multiplexed PAM signals;

d. Time-multiplexed PCM signals.

List of some M-ary non-orthogonal transmission system using a carrier frequency includes:

a. M-QAM;

b. M-CPM (including MSK, etc.).

List of some M-ary non-orthogonal baseband signals modulating a carrier frequency includes:

a. Non-time-multiplexed PPM signals;

b. Non-time-multiplexed PWM signals;

c. Non-time-multiplexed PAM signals;

d. Non-time-multiplexed PCM signals.

5. In a M-ary PAM system where the signal set is given by $\{\pm A, \pm 3A, \ldots, (M/2)A\}$. We assume M is an even integer, and denote $s_m = (2m-1)A$, $m = 1, \ldots, M/2$ and $s_{-m} = -(2m-1)A$, $m = 1, \ldots, M/2$. The transmited signal is disturbed by an AWGN noise of zero-mean and variance $\sigma^2 = N_0/2$. Then under the minimum-distance decision rule, the decision regions R_m, for the received value x is given by

$$R_m = \begin{cases} \{x : 2(m-1)A \leqslant x < 2mA\}, \text{ for } m = 1, 2, \ldots, (M/2) - 1, & (1a) \\ \{x : (m-2)A \leqslant x < \infty\}, \text{ for } m = M/2, & (1b) \\ \{x : 2mA < x \leqslant 2(m+1)A\}, \text{ for } m = -1, -2, \ldots, -(M/2) + 1, & (1c) \\ \{x : -\infty < x \leqslant (m+2)A\}, \text{ for } m = -M/2. & (1d) \end{cases} \quad (5.1)$$

For the two extreme symbols in (1b) and (1d), their two error probabilities are given by $2Q(A/\sigma)$, while the other $(M-2)$ error probabilities due to (1a) and (1c) are given by $(M-2)2Q(A/\sigma)$. Thus, the average symbol error proability is given by

$$P_e^s = (1/M)[2Q(A/\sigma) + (M-2)2Q(A/\sigma)] = (2(M-1)/M)Q(A/\sigma). \quad (5.2)$$

The average energy $E_{av} = (1/M)\sum_{m=-M/2}^{M/2} s_m^2 = (M^2-1)A^2/3$ or $A/\sigma = \sqrt{3E_{av}/(\sigma^2(M^2-1)}$. The energy per bit is given by $E_b = E_{av}/log_2 M$. Thus, the average symbol error proability can be expressed as

$$P_e^s = (2(M-1)/M)Q(\sqrt{(6(E_b/N_0)log_2 M)/(M^2-1)}). \quad (5.3)$$

7. Let $P_b(\psi)$ be the probability of bit error given a phase error of ψ, where ψ is the realization of a random variable Ψ with pdf equal to

$$p_\Psi(\psi) = \frac{1}{\sqrt{2\pi\sigma_\Psi^2}} \exp\left(\frac{-\psi^2}{2\sigma_\Psi^2}\right),$$

defined over $\psi \in (-\pi/2, \pi/2]$ (Note that we assume that $Q(\pi/(2\sigma_\Psi)) \approx 0$). Then,

$$P_b(\psi) = Q\left(\sqrt{\frac{2E}{N_0}} \cos(\psi)\right).$$

The average BER is equal to

$$P_b = \int_{-\pi/2}^{\pi/2} P_b(\psi) p_\Psi(\psi) \, \mathrm{d}\psi.$$

If we expand $P_b(\psi)$ into a power series up to the second-order term, we have

$$P_b(\psi) \approx P_b(0) + \left.\frac{\mathrm{d}P_b(\psi)}{\mathrm{d}\psi}\right|_{\psi=0} \psi + \frac{1}{2} \left.\frac{\mathrm{d}^2 P_b(\psi)}{\mathrm{d}\psi^2}\right|_{\psi=0} \psi^2,$$

where

$$\begin{aligned}
P_b(0) &= Q\left(\sqrt{\frac{2E}{N_0}}\right), \\
\left.\frac{\mathrm{d}P_b(\psi)}{\mathrm{d}\psi}\right|_{\psi=0} &= \left.\sqrt{\frac{E}{\pi N_0}} \exp\left(-\frac{E}{N_0}(\cos(\psi))^2\right) \sin(\psi)\right|_{\psi=0} = 0, \\
\left.\frac{\mathrm{d}^2 P_b(\psi)}{\mathrm{d}\psi^2}\right|_{\psi=0} &= \left.\sqrt{\frac{E}{\pi N_0}} \exp\left(-\frac{E}{N_0}(\cos(\psi))^2\right) \left[\frac{2E}{N_0} \cos(\psi)(\sin(\psi))^2 + \cos(\psi)\right]\right|_{\psi=0} = \sqrt{\frac{E}{\pi N_0}} \exp\left(-\frac{E}{N_0}\right).
\end{aligned}$$

With this approximation, we obtain for P_b

$$P_b \approx Q\left(\sqrt{\frac{2E}{N_0}}\right) + \frac{1}{2}\sqrt{\frac{E}{\pi N_0}} \exp\left(-\frac{E}{N_0}\right) \sigma_\Psi^2.$$

9. Consider an uncorrelated random sequence $\{B_n, -\infty < n < \infty\}$, with $E\{B_n\} = \mu$, $\mathrm{E}\{B_n B_{n+m}\} = \mu^2$, and its variance denoted by $\sigma_{B_n}^2$.

a. In $s(t) = \sum_{n=-N}^{N} B_n g(t - nT)$, $-\infty < t < \infty$, first assume B_n's are deterministic and then take its Fourier transform to yield

$$C(f) = G(f) \sum_{n=-N}^{N} B_n e^{-in2\pi fT}, -\infty < f < \infty. \tag{9.1}$$

After taking the time and statistical averages of $|C(f)|^2$, we obtain

$$\begin{aligned}
S_S(f) &= |G(f))|^2 \lim_{N\to\infty} \frac{1}{(2N+1)T} E\left\{\left|\sum_{n=-N}^{N} B_n e^{-in2\pi fT}\right|^2\right\} \\
&= \frac{|G(f))|^2}{T} \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} \sum_{m=-N}^{-N} R(m) e^{-i(m-n)2\pi fT} \qquad (9.2) \\
&= \frac{|G(f))|^2}{T} \sum_{m=-\infty}^{\infty} R(m) e^{-im2\pi fT}.
\end{aligned}$$

b. Since

$$R(m) = E\{B_{n+m} B_n\} = \begin{cases} \mu^2, \, m \neq 0, \\ \sigma_{B_n}^2 + \mu^2, \, m = 0, \end{cases} \tag{9.3}$$

upon substituting (9.3) into (9.2), we obtain

$$S_S(f) = \frac{|G(f))|^2}{T} \left(\sigma_{B_n}^2 + \mu^2 \sum_{m=-\infty}^{\infty} e^{-im2\pi fT}\right). \tag{9.4}$$

By using the Poisson formula

$$\sum_{m=-\infty}^{\infty} e^{-im2\pi fT} = \tfrac{1}{T} \sum_{m=-\infty}^{\infty} \delta(f - m/T)\,, \tag{9.5}$$

in (9.4), we obtain

$$\begin{aligned} S_S(f) &= \frac{|G(f))|^2}{T}\left(\sigma_{B_n}^2 + \frac{\mu^2}{T}\sum_{m=-\infty}^{\infty}\delta(f-\frac{m}{T})\right) \\ &= \frac{|G(f))|^2\sigma_{B_n}^2}{T} + \left(\frac{\mu}{T}\right)^2 \sum_{m=-\infty}^{\infty} |G(\frac{m}{T}))|^2\delta(f-\frac{m}{T})\,. \end{aligned} \tag{9.6}$$

11.
```
% input data
X=[[-.225 0.93]',[-1 2]',[-2.5 .5]',[-3 -1]'];
Y=[[-0.75 0.75]',[1 .8]',[2 -1]',[2.5 1.5]',[3 1]'];
N=4; M =5;
% Solution via CVX
cvx_begin
    variables a(2,1) b u(4,1) v(5,1)

    X'*a - b >= 1 - u ;
    Y'*a - b <= -(1-v);
    %a'*X(:,i) - b >= 1 - u(i), i = 1,2,3,4;
    %a'*Y(:,j) - b <= -(1 - v(j)); j = 1,2,3,4,5;
    u >= 0;
    v >= 0;
    minimize ([ 1 1 1 1]*u + [1 1 1 1 1]*v)
cvx_end
b
a
% Displaying results
linewidth = 0.5;  % for the squares and circles
t_min = min([X(1,:),Y(1,:)]);
t_max = max([X(1,:),Y(1,:)]);
tt = linspace(t_min-1,t_max+1,100);
p = -a(1)*tt/a(2) + b/a(2);
p1 = -a(1)*tt/a(2) + (b+1)/a(2);
p2 = -a(1)*tt/a(2) + (b-1)/a(2);

graph = plot(X(1,:),X(2,:), 'o', Y(1,:), Y(2,:), 'o',-4:.01:4,zeros(1,801),'k',zeros(1,601),(-3:.01
graph = plot(X(1,:),X(2,:), 'o', Y(1,:), Y(2,:), 'o',-4:.01:4,zeros(1,801),'k',zeros(1,601),(-3:.01
set(graph(1),'LineWidth',linewidth);
set(graph(2),'LineWidth',linewidth);
set(graph(1),'MarkerFaceColor',[0 0.5 0]);
hold on;
plot(tt,p, '-r', tt,p1, '--r', tt,p2, '--r');
axis([-4 4 -3 3])
xlabel('x(1)')
ylabel('x(2)')
```

Homework #6 Answers

1. Since X and Y are two independent Gaussian r.v.'s, then

$$p_{X,Y}(x,y) = \left(\frac{e^{-x^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\right)\left(\frac{e^{-y^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\right) = \frac{e^{-(x^2+y^2)/(2\sigma^2)}}{2\pi\sigma^2}.$$

Let $x = r\cos(\theta)$ and $y = \sin(\theta)$. Upon converting from the rectangular coordinate to the polar coordinate, $r^2 = x^2 + y^2$ and $rdrd\theta = dxdy$. Then

$$p_{R,\Theta}(r,\theta)\,|dr\,d\theta| = p_{X,Y}(x,y)\,|dx\,dy| = \left[\frac{e^{-(x^2+y^2)/(2\sigma^2)}}{2\pi\sigma^2} r\,|dr\,d\theta|\right] \Big|_{\substack{x = r\cos(\theta) \\ y = r\sin(\theta)}} = \frac{r\,e^{-r^2/(2\sigma^2)}}{2\pi\sigma^2}|dr\,d\theta|.$$

$$p_R(r)\,dr = \int_0^{2\pi} p_{R,\Theta}(r,\theta)\,d\theta = \frac{r\,dre^{-r^2/(2\sigma^2)}}{2\pi\sigma^2}\int_0^{2\pi} d\theta = \frac{r\,e^{-r^2/(2\sigma^2)}}{\sigma^2}dr,\ 0 \le r < \infty.$$

Thus, $R = \sqrt{X^2+Y^2}$ is a Rayleigh r.v. with a pdf given by $p_R(r) = (r/\sigma^2)\,e^{-r^2/(2\sigma^2)}$, $0 \le r < \infty$.

3.

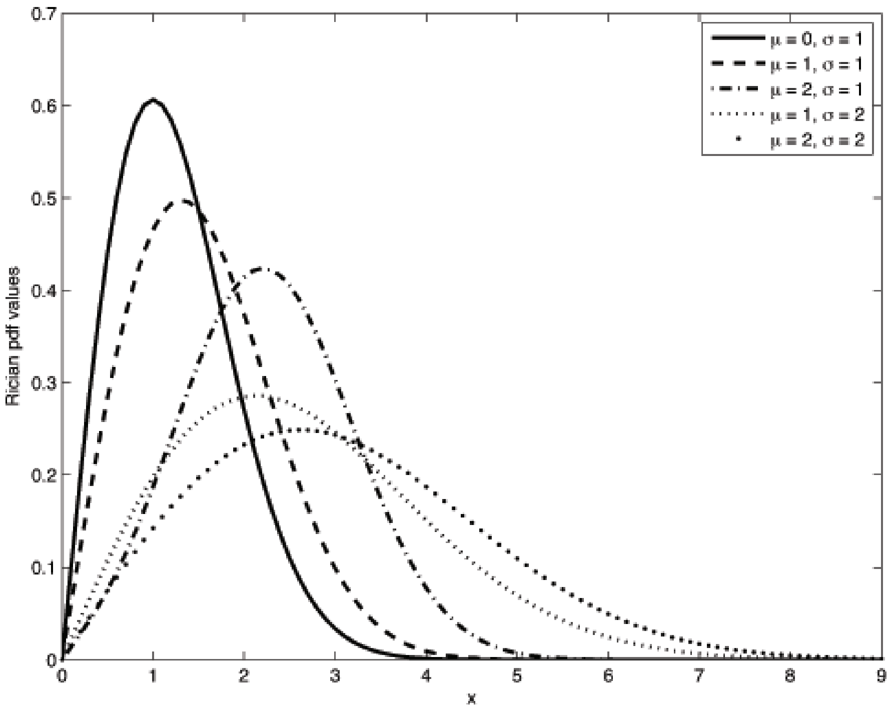


Fig. 1. Plots of Rician pdf's for five values of μ and σ.

5. **a.** Using the Matlab function *mean*, we obtained the estimated mean $\tilde{\mu}_R = mean(\mathbf{r}) = 1.2619$.

   b. Using the Matlab function *var*, we obtained the estimated variance $\tilde{\sigma}_R^2 = var(\mathbf{r}) = 0.43366$.

   c. Using $\tilde{\sigma}_1 = \tilde{\mu}_R \times \sqrt{2/\pi}$, then we obtained $\tilde{\sigma}_1 = 1.0069$.

   d. Using $\tilde{\sigma}_2 = \sqrt{\tilde{\sigma}_R^2/(2-\pi)}$, we obtained $\tilde{\sigma}_2 = 1.0052$.

   e. Thus, both $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are very close to the original $\sigma = 1$ of the Gaussian r.v.

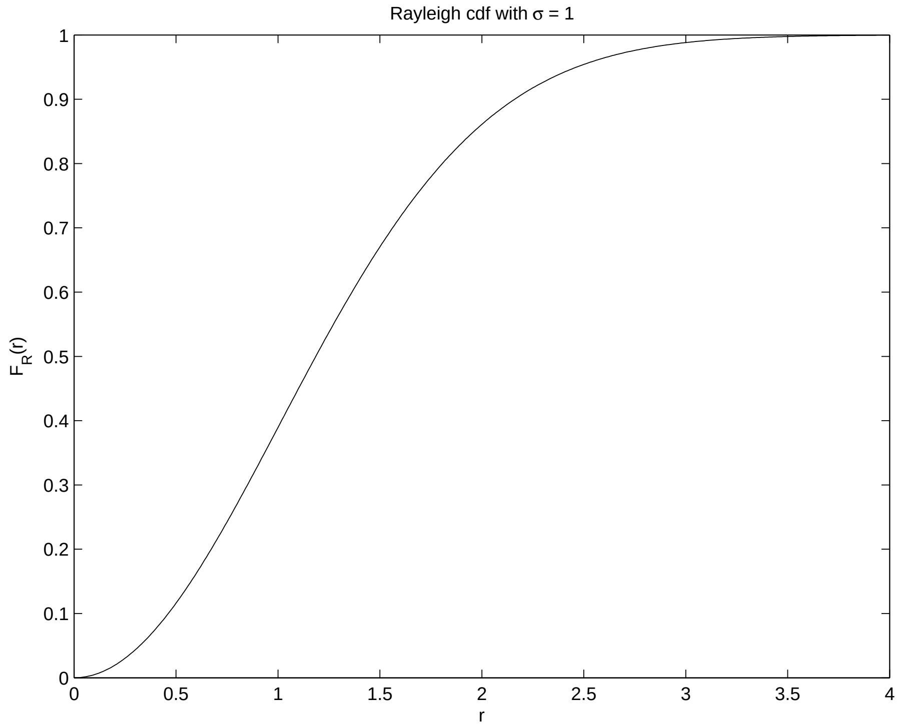


Fig. 2. Plot of Rayleigh cdf $F_R(r)$ vs. r.

f. Fig. 3 shows the the empirical cdf and the Rayleigh cdf match each other well.

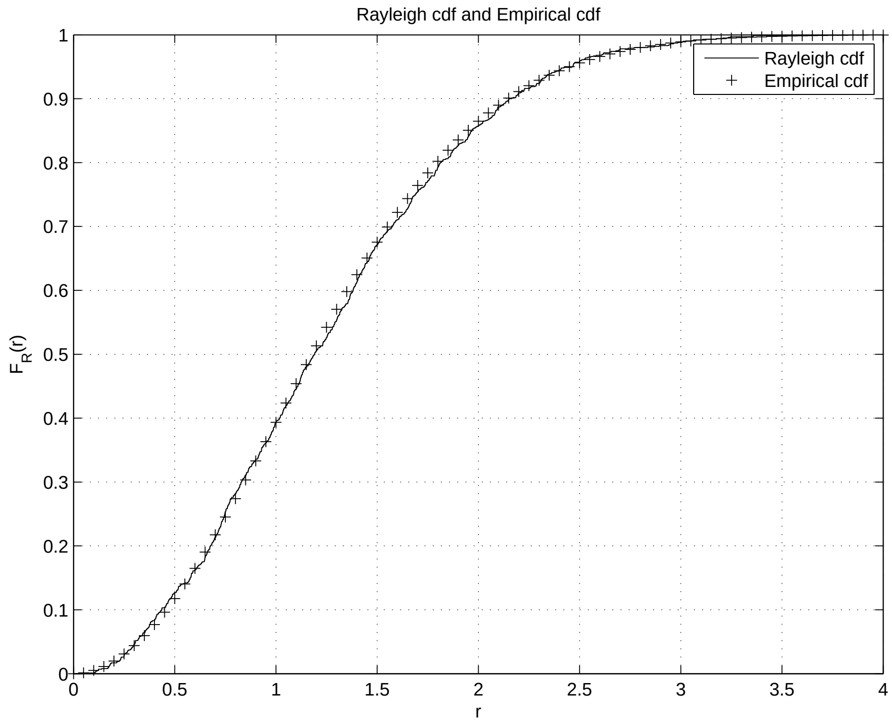


Fig. 3. Plots of empirical cdf and Rayleigh cdf $F_R(r)$ vs. r.

7. In Fig. 8, the Rician pdf is represented by the solid curve. The Gaussian pdf with mean of $\mu = 2$ and $\sigma = 0.5$ is represented by the dashed curve. We see this Gaussian approximation pdf appears to have an offset to the left of the Rician pdf. However, the Gaussian approximation pdf with a modified mean of $\sqrt{\mu^2 + \sigma^2}$ and $\sigma = 0.5$ is represented by the dotted curve and provide a much better approximation to the desired Rician pdf.

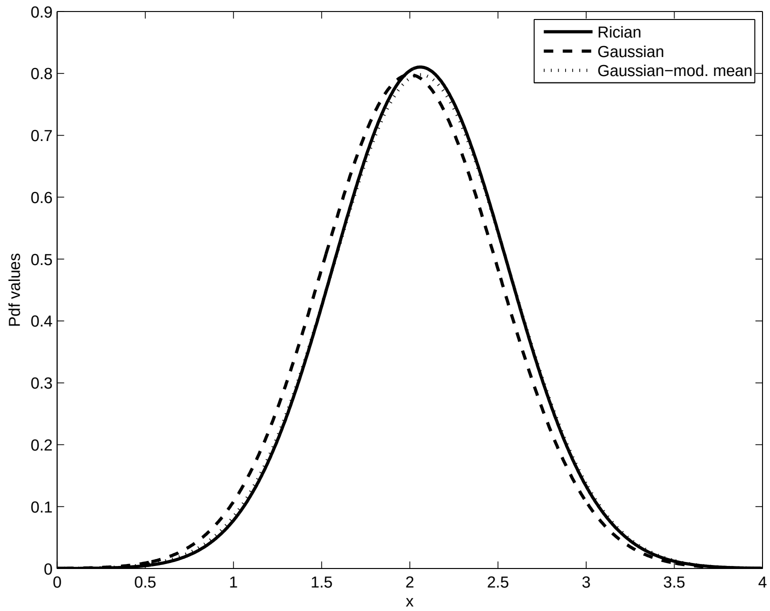


Fig. 8 Plots of Rician pdf, Gaussian pdf, and Gaussian pdf with a modified mean.

9. Fig. 9 compares the Rician pdf with different parameters of the Nakagame pdf's.

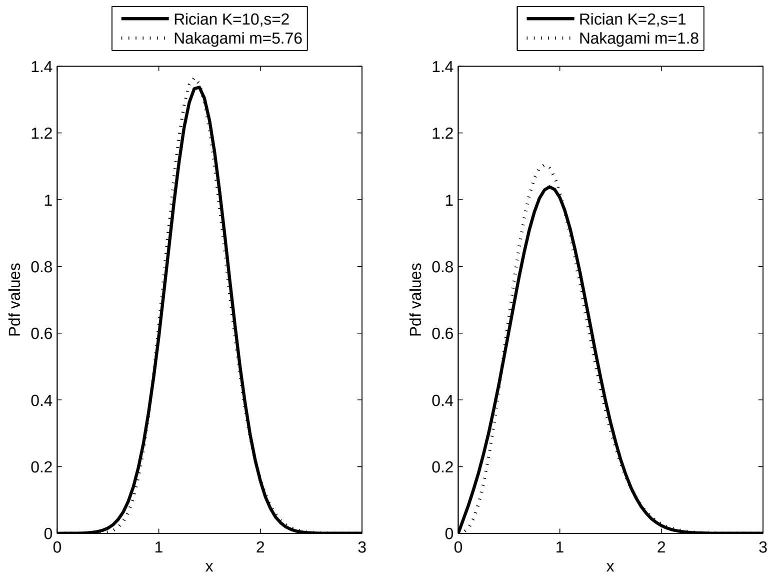


Fig. 9. Plots of Rician pdf and Nakagami-m pdf. (a). Rician (K=10, s=2), Nakagami (m = 5.76). (b). Rician (K=2, s=1), Nakagami (m = 1.80).

Homework #7 Answers

1. **a.**

$$\begin{aligned}
\epsilon^2 &= \mathrm{E}\{(s(t+\tau)-as(t))^2\} = R(0)+a^2R(0)-2aR(\tau) \\
\frac{\partial\epsilon^2}{\partial a} &= 2\hat{a}R(0)-2R(\tau)=0 \Longrightarrow \hat{a}=\frac{R(\tau)}{R(0)} \\
\epsilon^2_{Min} &= \mathrm{E}\{(s(t+\tau)-\hat{a}s(t))s(t+\tau)\} = R(0)-\hat{a}R(\tau) = R(0)-\frac{R^2(\tau)}{R(0)} \\
&= \frac{R^2(0)-R^2(\tau)}{R(0)}
\end{aligned}$$

b. For $R(t)=\delta(t), \quad -\infty<t<\infty$, denote

$$R_C(\tau) = \begin{cases} 1/c\,, & (-c/2) < \tau < (c/2)\,, \\ 0\,, & \text{elsewhere}\,, \end{cases}$$

then define $\delta(\tau) = \lim_{c\to 0} R_C(\tau)$. Thus,

$$\hat{a} = \begin{cases} lim_{c\to 0}R(\tau)/(1/c) = 0, & \tau\neq 0 \\ lim_{c\to 0}(1/c)/(1/c) = 1, & \tau = 0 \end{cases}, \qquad \epsilon^2_{Min} = \begin{cases} 0, & \tau = 0 \\ \infty, & \tau\neq 0. \end{cases}$$

c. Consider

$$\begin{aligned}
\mathrm{E}\{(s(t+\tau)-\hat{a}s(t))s(u)\} &= R(t+\tau-u)-\hat{a}R(t-u) \\
&= e^{-\alpha|t+\tau-u|}-e^{-\alpha|\tau|}e^{-\alpha|t-u|} \\
&= e^{-\alpha(t+\tau-u)}-e^{-\alpha\tau}e^{-\alpha(t-u)} \\
&= e^{-\alpha(t+\tau-u)}-e^{-\alpha(t+\tau-u)} \\
&= 0, \quad \tau>0, t\geq u.
\end{aligned}$$

This means

$$\begin{aligned}
\epsilon_1^2 &= \mathrm{E}\Big\{(s(t+\tau)-\hat{a}s(t)-\sum_{i=1}^{N}b_i s(u_i))^2\Big\} \\
&= \mathrm{E}\{(s(t+\tau-\hat{a}s(t))^2\} + \mathrm{E}\Big\{\Big(\sum_{i=1}^{N}b_i s(u_i)\Big)^2\Big\} \\
&\geq \mathrm{E}\{(s(t+\tau)-\hat{a}s(t))^2\} = \epsilon^2_{Min}.
\end{aligned}$$

Thus, the minimum m.s. error is attained with all $b_i=0$. That is, the optimal linear estimate is $\hat{a}s(t)$, where $\hat{a}=R(\tau)/R(0)$, for $\tau>0$, and $t>u_i$, for all i.

3. **a.** From Orthogonal Principle, we obtain $E\{[S(n)-(\sum_{k=-\infty}^{\infty}h(k)X(n-k))]X(n-j)\} = 0\,, -\infty<j<\infty\,,$ or $R_{SX}(j) = \sum_{k=-\infty}^{\infty}h(k)X(j-k)\,, -\infty<j<\infty$. Without additional information, it is not possible to solve for the $h(k)$ directly.

b. By taking the Fourier transform of the last expression of part (a), we obtain $S_{SX}(f) = H(f)\,S_X(f)\,, -1/2\leq f<1/2$. Thus, $H(f) = S_{SX}(f)/S_X(f)\,, -1/2\leq f<1/2$.

c.

$R_{SX}(n) = E\{S(m)X(n+m)\} = E\{S(m)(S(n+m)+V(n+m))\}$
$= R_S(n) = a^{|n|}\sigma_S^2\,, -\infty<n<\infty\,,$
$R_X(n) = E\{X(m)X(n+m)\} = E\{(S(m)+V(m))(S(n+m)+V(n+m))\}$
$= R_S(n)+\delta(n)\,, -\infty<n<\infty\,,$
$S_{SX}(f) = S_S(f) = \sigma_S^2\sum_{k=-\infty}^{\infty}a^{|k|}\exp(-i2\pi kf) = \frac{\sigma_S^2(1-|a|^2)}{1-2a\cos(2\pi f)+|a|^2}\,, -\frac{1}{2}\leq f<\frac{1}{2}\,,$
$S_X(f) = R_S(n)+1 = \frac{\sigma_S^2(1-|a|^2)}{1-2a\cos(2\pi f)+|a|^2}+1 = \frac{1+|a|^2-2a\cos(2\pi f)+\sigma_S^2(1-|a|^2)}{1-2a\cos(2\pi f)+|a|^2}\,, -\frac{1}{2}\leq f<\frac{1}{2}\,,$
$H(f) = \frac{S_{SX}(f)}{S_X(f)} = \frac{\sigma_S^2(1-|a|^2)}{1+|a|^2-2a\cos(2\pi f)+\sigma_S^2(1-|a|^2)}\,, -\frac{1}{2}\leq f<\frac{1}{2}\,.$

d. From Orthogonal Principle, we have

$$\varepsilon^2_{Min} = E\{(S(n) - \hat{S}(n))^2\} = E\{(S(n) - \hat{S}(n))(S(n) - \hat{S}(n))\} = E\{(S(n) - \hat{S}(n))S(n)\}$$
$$= E\left\{\left(S(n) - \left(\sum_{k=-\infty}^{\infty} h(k)X(n-k)\right)\right) S(n)\right\} = R_S(0) - \sum_{k=-\infty}^{\infty} h(k)R_S(-k)\,.$$

But we also have

$$R_S(0) = \int_{-1/2}^{1/2} S_S(f)\,df\,, R_S(-k) = \int_{-1/2}^{1/2} \exp(i2\pi f(-k))\,S_S(f)\,df\,,\ H(f) = \sum_{k=-\infty}^{\infty} h(k)\,\exp(-i2\pi fk)\,.$$

Combining the above two sets of equations, we obtain

$$\varepsilon^2_{Min} = \int_{-1/2}^{1/2} [S_S(f) - H(f)\,S_S(f)]df = \int_{-1/2}^{1/2} [S_S(f) - \frac{(S_S(f))^2}{S_S(f)+S_V(f)}]df = \int_{-1/2}^{1/2} \frac{S_S(f)S_V(f)}{S_S(f)+S_V(f)}\,df.$$

5. From Orthogonal Principle, we obtain

$$E\left\{\left[S(t+a) - \int_0^{\infty} h(\tau)X(t-\tau)\,d\tau\right] X(t-s)\right\} = 0\,,\ -\infty < t < \infty\,,\ 0 \le s < \infty\,,$$

or

$$R_S(s+a) = \int_0^{\infty} h(\tau)R_X(s-\tau)\,d\tau\,, 0 \le s < \infty\,,$$

where

$$R_X(\tau) = R_S(\tau) + R_V(\tau)\,,\ -\infty < \tau < \infty\,.$$

7. By comparing the given autocorrelation function $R_S(k) = 4 \times 2^{-|k|}$, $-\infty < k < \infty$, to the autocorrelation function $R_S(k)$ of (2.3.2-6), we obtain

$$\frac{\sigma^2_W\,a^{-|k\}}}{1-a^2} = 4\,(1/2)^{-|k|}\,,\ -\infty < k < \infty\,,$$

or

$$\ln\left\{\frac{\sigma^2_W}{1-a^2}\right\} - |k|\,\ln\{a\} = \ln\{4\} - |k|\,\ln\left\{\frac{1}{2}\right\}\,,$$

which shows that $a = 1/2$ and $4\,\sigma^2_W/3 = 4$, or $\sigma^2_W = 3$. Thus, this random sequence $S(k)$ is a first-order autoregressive sequence modeled by (2.3.2 - 1). Then the recursive Kalman filtering algorithm of Sec. (2.3.2) is valid and the steady state version of the Kalman gain p of (2.3.2 -15) reduces to $a\,p^2 + b\,p + c = 0$ where $a = 1$ $b = 3 + SNR$, $c = 4\,SNR$, and $SNR = 3/\sigma^2_V$. A plot of p versus SNR is given here.

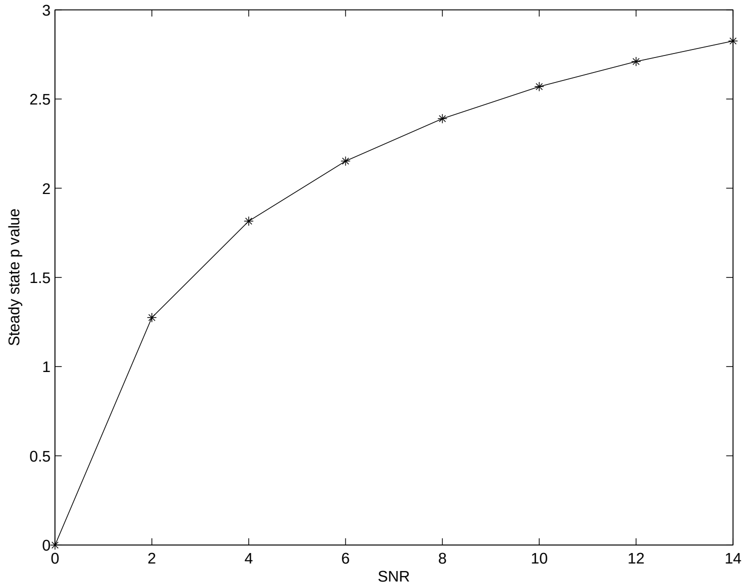
9. The density of X is given by $p_X(u;A) = p_W(u-A)$. Since $\frac{d}{dA}\ln p_X(u;A) = (\frac{d}{dA}p_X(u;A))/p_X(u;A) = -(\frac{d}{dA}p_W(u-A))/p_N(u-A)$, we have that

$$\left[E\left\{\left(\frac{d}{dA}p_X(u;A)\right)^2\right\}\right]^{-1} = \left[\int_{-\infty}^{\infty}\left(\frac{d}{dA}p_W(u-A)\right)^2 \frac{du}{p_W(u-A)}\right]^{-1}$$
$$= \left[\int_{-\infty}^{\infty}\left(\frac{d}{du}p_W(u)\right)^2 \frac{du}{p_W(u)}\right]^{-1}.$$

For the Laplacian pdf we have

$$\frac{dp_W(u)}{du} = \begin{cases} (-1/\sigma^2)e^{-\sqrt{2}u/\sigma}\,, & u \ge 0\,, \\ (1/\sigma^2)e^{\sqrt{2}u/\sigma}\,, & u < 0\,. \end{cases}$$

$(\frac{d}{du}p_W(u))^2/p_W(u) = \frac{\sqrt{2}\sigma}{\sigma^4}e^{-\sqrt{2}|u|/\sigma} = \frac{2}{\sigma^2}p_W(u)$. The CRLB is therefore equal to $\sigma^2/2$. For the Gaussian pdf, the CRLB is equal to σ^2.



11.

$$p(\mathbf{x};a) = \frac{1}{\sqrt{(2\pi)^N \det(\mathbf{C})}} e^{-\frac{1}{2}(\mathbf{x}-a\mathbf{1})^\mathsf{T}\mathbf{C}^{-1}(\mathbf{x}-a\mathbf{1})},$$

where $\mathbf{1} = [1,\ldots,1]^\mathsf{T}$. Let $\mathbf{C}^{-1} = (\bar{C})_{ij}$ and, by symmetry, $\bar{C}_{ij} = \bar{C}_{ji}$.

$$\begin{aligned} \frac{d}{da}\ln p(\mathbf{x};a) &= -\frac{1}{2}\frac{d}{da}\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}(x_i-a)(x_j-a) = \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N\left[\bar{C}_{ij}(x_j-a)+\bar{C}_{ij}(x_j-a)\right] \\ &= \sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}x_i - a\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij} = \left(\frac{\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}x_i}{\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}} - a\right)\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}. \\ \text{CRLB} &= \left[-\mathrm{E}\left\{\frac{d^2}{da^2}p(\mathbf{X};a)\right\}\right]^{-1} = \frac{1}{\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}}. \end{aligned}$$

The estimator $\hat{a} = \sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}x_i / \sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}$ is unbiased ($\mathrm{E}\{X_i\} = a$, $\mathrm{E}\{\hat{A}\} = a$). Therefore, it is efficient, with variance equal to $\left(\sum_{i=1}^N\sum_{j=1}^N \bar{C}_{ij}\right)^{-1}$.

13.

$$\begin{aligned} p(\mathbf{x};n_0) &= \prod_{n=0}^{n_0-1}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-x^2(n)/(2\sigma^2)}\prod_{n=n_0}^{n_0+M-1}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x(n)-s(n-n_0))^2/(2\sigma^2)}\prod_{n=n_0+M}^{N-1}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-x^2(n)/(2\sigma^2)} \\ &= \frac{1}{\sqrt{(2\pi\sigma^2)^N}}e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2(n)}\ e^{-\frac{1}{2\sigma^2}\sum_{n=n_0}^{n_0+M-1}[-2x(n)s(n-n_0)+s^2(n-m_0)]}. \end{aligned}$$

Therefore, the maximum likelihood estimator is found by minimizing

$$\sum_{n=n_0}^{n_0+M-1}[s^2(n-n_0)-2x(n)s(n-n_0)].$$

Since $\sum_{n=n_0}^{n_0+M-1}s^2(n-n_0) = \sum_{n=0}^{M-1}s^2(n)$ does not depend on n_0, the maximum likelihood estimator for n_0 is found by maximizing $\sum_{n=n_0}^{n_0+M-1}x(n)s(n-n_0)$.

Chapter 8
Odd Numbered Homework Solutions

1.

```
%%
rand('state',49);
u=rand(1,501);
un=u(1:500);
un1=u(2:501);
plot(un,un1,'k*')
xlabel('u_n')
ylabel('u_{n+1}')
%%
Yields
```

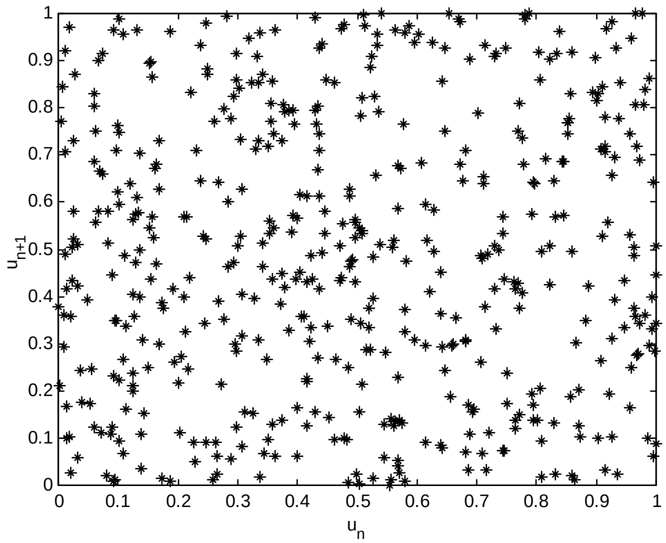


Fig. 1. Plot of U_{n+1} vs. U_n , n = 1, …, 500, for the rand generator.
The 500 points in this plot seems to be fairly randomly distributed with no obvious similar patters among adjacent values.

%%

3.

```
s0=49;
xx= [ ];
m = 2^31 -1;

for n=1:502,
    s = mod((7^5)*s0, m);
    x= s/m;
    xx=[xx,x];
    s0=s;
end
subplot(1,2,1)
plot(xx(1:500),xx(2:501),'k*')
xlabel('x_n')
```

```
ylabel('x_{n+1}')
axis([0 1 0 1])
%%
subplot(1,2,2)
plot(xx(1:500),xx(3:502),'k*')
xlabel('x_n')
ylabel('x_{n+2}')
axis([0 1 0 1])
```

yields

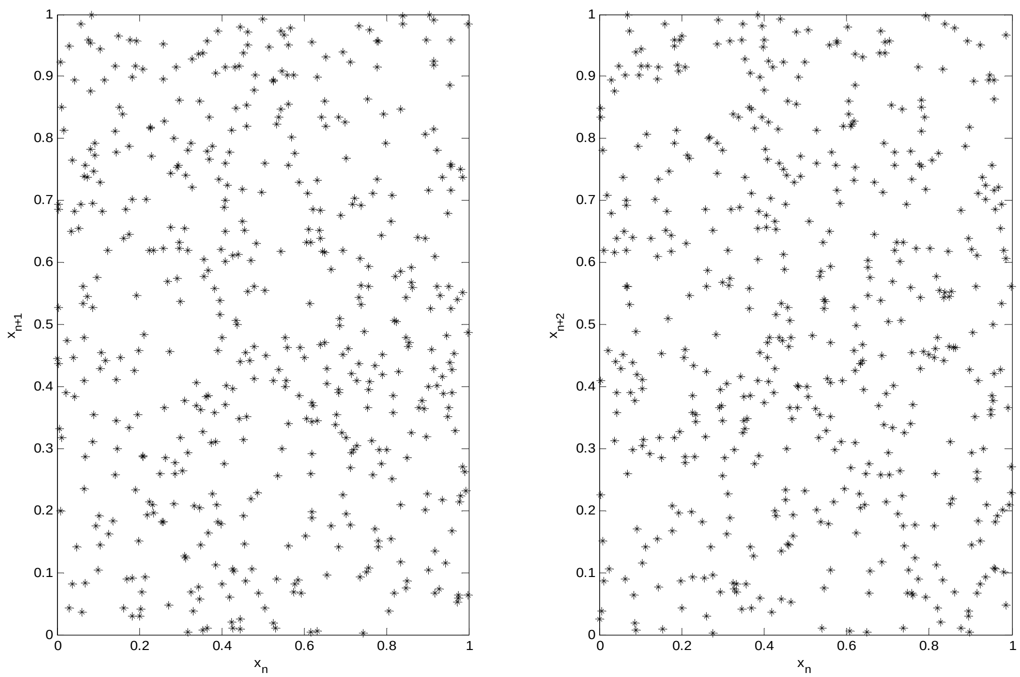


Fig. 3a. Plot of x_{n+1} vs. x_n , n = 1, …, 500. Fig. 3b. Plot of x_{n+2} vs. x_n , n = 1, …, 500.
From these two plots, there does not appear to have any regular patterns. Thus, from these simple visual inspections, there does not appear to be much correlations among elements of this sequence.

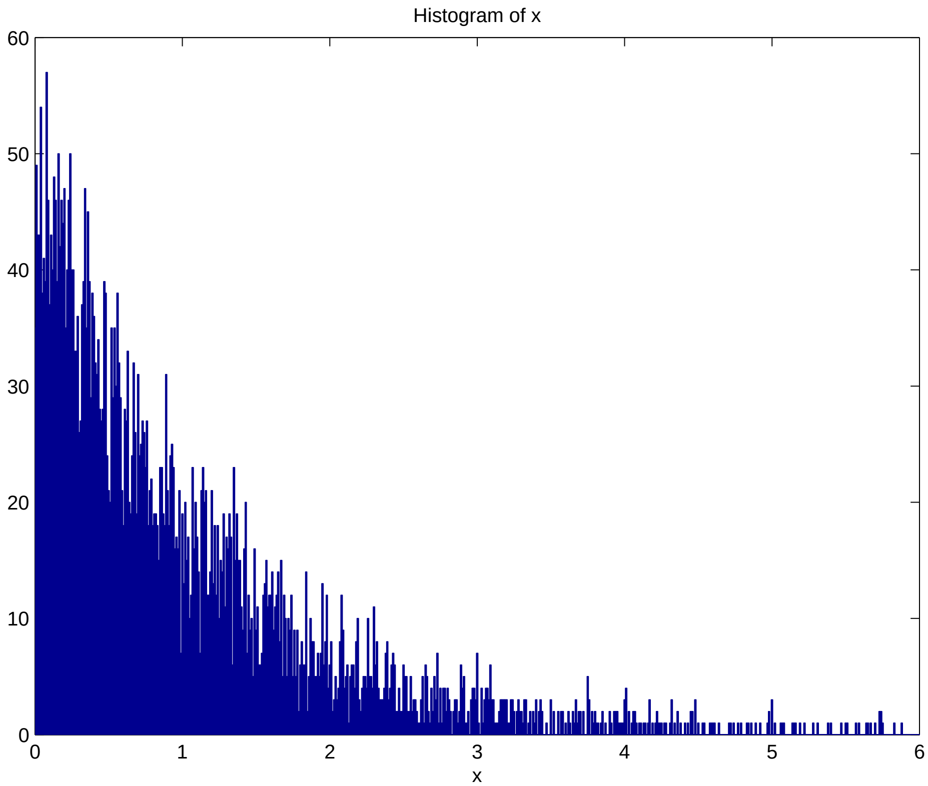
5. For these two $\mathbf{x}_1$ and $\mathbf{x}_2$ sequences, the Matlab function kstest($\mathbf{x}_1$, $\mathbf{x}_2$) yields 0. This means we can not reject the null hypothesis that these two sequences have the same distribution with a 5 % significance statistical level test. Loosely speaking, while the KS (and some other) statistical tests can not confirm these two sequences have the same distribution, it can only state with fairly high confidence that one can not conclude the two sequences came from two different distributions.

7. See Table 8.1 of Ex. 8.2 of Chapter 8 on p. 275.

9. See Table 8.4 of Ex. 8.4 of Chapter 8 on p. 278.

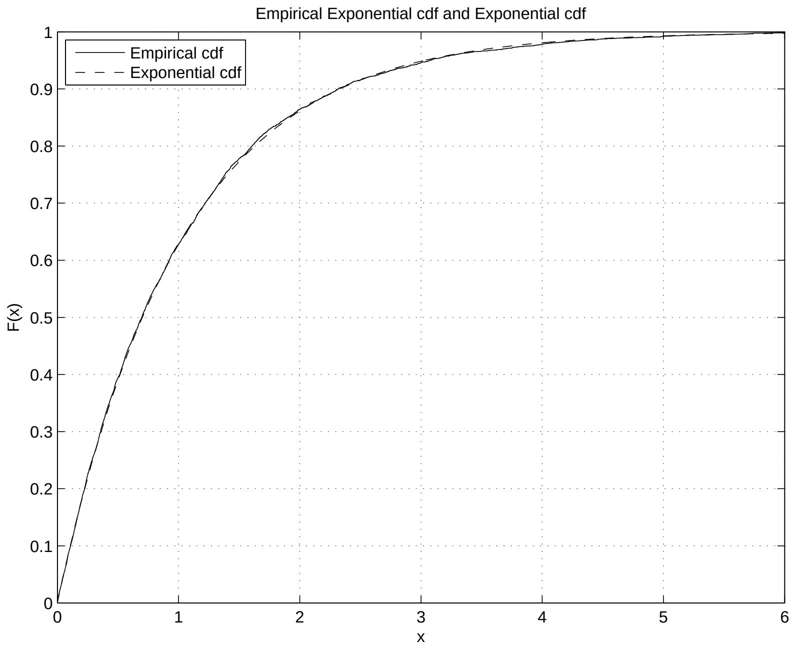
11.

```
randn('state',19);
r = raylrnd(1,[1,5000]);
randn('state',47);
x = r.*v;
mean(x) = 1.0122;
xx=0:.01:6;
hist(x,xx), axis[[0 6 0 60])
```



The histogram of X appears to have an exponential pdf with a mean of 1.0122.
Compare the empirical cdf of X with an exponential cdf with a men of 1.0122.

```
F = cdfplot(x), axis([0 6 0 1]);
ecdf=expcdf(xx,1.0122);
hold on
G = plot(xx,expcdf(xx,1.0122),'r--');
title('Empirical Exponential cdf and Exponential cdf')
legend([F,G], 'Empirical cdf', 'Exponential cdf','Location','NW')
```



The empirical exponential cdf and the true exponential cdf seem to fit each other very well.