1 Chapter 1 Solution Set

Problems

1.1 Determine the relationship between the two Cauchy conditions for the one-dimensional wave equation such that (a) only a wave field propagating in the $+z$ direction is present and (b) only a wave propagating in the $-z$ direction is present. Are there any non-trivial Cauchy conditions that result in a zero field?

From Example 1.1 we require that the function $g(z)$ vanish for part (a) and $f(z)$ for part (b). From that problem we have that

$$\tilde{g}(K) = \frac{1}{2}[\tilde{u}_0(K) - \frac{i}{cK}\tilde{u}_0'(K)],$$

where

$$u(z,t)|_{t=0} = u_0(z) \quad \frac{\partial}{\partial t}u(z,t)|_{t=0} = u_0'(z).$$

The vanishing of $g(z)$ then requires that

$$\tilde{u}_0'(K) = -icK\tilde{u}_0(K).$$

On Fourier inversion of the above equation we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dK\,\tilde{u}_0'(K)e^{iKz} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dK\, -icK\tilde{u}_0(K)e^{iKz},$$

which yields

$$\frac{\partial}{\partial t}u(z,t)|_{t=0} = -c\frac{\partial}{\partial z}u(z,t)|_{t=0}.$$

Part b is done in a completely parallel manner yielding

$$\frac{\partial}{\partial t}u(z,t)|_{t=0} = +c\frac{\partial}{\partial z}u(z,t)|_{t=0}.$$

There are no non-trivial Cauchy conditions that generate a zero field since such a field would require that both $f(z)$ as well as $g(z)$ vanish which can only happen if both the field and its first time derivative both vanish.

1.2 **a** Compute the temporal Fourier transform of the "Rect" function

$$\text{Rect}(t) = \begin{cases} 1 & -T_0 \le t \le +T_0 \\ 0 & \text{else.} \end{cases}$$

We have that

$$\widetilde{\mathrm{Rect}}(\omega) = \int_{-T_0}^{T_0} dt\, e^{i\omega t} = \frac{e^{i\omega t}}{i\omega}|_{-T_0}^{T_0} = 2\frac{\sin\omega T_0}{\omega}.$$

We can also express the result in terms of the "Sinc" function as

$$\widetilde{\mathrm{Rect}}(\omega) = 2T_0\mathrm{Sinc}(\frac{\omega T_0}{\pi})$$

where

$$\mathrm{Sinc(x)} = \frac{\sin\pi x}{\pi x}.$$

b Use the Cauchy Riemann equations to prove that the transform that you computed is an entire analytic function of the frequency variable ω.

Put

$$\widetilde{\mathrm{Rect}}(\omega) = u(x,y) + iv(x,y)$$

where $\omega = x + iy$ with u, v, x, y all real. We have that

$$u(x,y) = \frac{\sin\omega T_0}{\omega} + \frac{\sin\omega^* T_0}{\omega^*}, v(x,y) = -i[\frac{\sin\omega T_0}{\omega} - \frac{\sin\omega^* T_0}{\omega^*}].$$

$$\begin{aligned}
\frac{\partial}{\partial x}u(x,y) &= [\frac{\partial}{\partial\omega}\frac{\sin\omega T_0}{\omega}]\frac{\partial\omega}{\partial x} + [\frac{\partial}{\partial\omega^*}\frac{\sin\omega^* T_0}{\omega^*}]\frac{\partial\omega^*}{\partial x} \\
&= \frac{\partial}{\partial\omega}\frac{\sin\omega T_0}{\omega} + \frac{\partial}{\partial\omega^*}\frac{\sin\omega^* T_0}{\omega^*} \\
\frac{\partial}{\partial y}v(x,y) &= -i\{[\frac{\partial}{\partial\omega}\frac{\sin\omega T_0}{\omega}]\frac{\partial\omega}{\partial y} - [\frac{\partial}{\partial\omega^*}\frac{\sin\omega^* T_0}{\omega^*}]\frac{\partial\omega^*}{\partial y}\} \\
&= -i[i\frac{\partial}{\partial\omega}\frac{\sin\omega T_0}{\omega} + i\frac{\partial}{\partial\omega^*}\frac{\sin\omega^* T_0}{\omega^*}] \\
&= \frac{\partial}{\partial\omega}\frac{\sin\omega T_0}{\omega} + \frac{\partial}{\partial\omega^*}\frac{\sin\omega^* T_0}{\omega^*}.
\end{aligned}$$

We thus find that

$$\frac{\partial}{\partial x}u(x,y) = \frac{\partial}{\partial y}v(x,y).$$

In an entirely parallel manner we find that

$$\frac{\partial}{\partial y}u(x,y) = -\frac{\partial}{\partial x}v(x,y)$$

thus establishing that $\widetilde{\mathrm{Rect}}(\omega)$ satisfies the Cauchy-Riemann equations.

1.3 Perform the steps leading from the Fourier integral representation of the causal Green function in Eq.(1.18) to its final form given in Eq.(1.20a).

We have already shown under Eq.(1.19) that

$$g_+(\mathbf{R},\tau) = -\frac{c}{(2\pi)^2 R}\int_{-\infty}^{\infty} dK\, \sin cK\tau \sin KR, \quad \tau > 0.$$

If we now make use of the Euler identity we can reduce the above equation to the form

$$g_+(\mathbf{R},\tau) = \frac{c}{16\pi^2 R}\int_{-\infty}^{\infty} dK\,[e^{iK(c\tau+R)} + e^{-iK(c\tau+R)} - e^{iK(c\tau-R)} - e^{-iK(c\tau-R)}]$$
$$= \frac{c}{4\pi R}[\delta(c\tau+R) - \delta(c\tau-R)] = -\frac{1}{4\pi R}\delta(\tau - R/c)$$

where the last equality follows from the fact that $c\tau + R > 0$ for $\tau > 0$ and that $\delta(ax) = \delta(x)/|a|$.

1.4 Prove using Cauchy's integral theorem that the difference between the causal (retarded) and a causal (advanced) Green functions satisfies the homogeneous wave equation and, hence, is not a Green function.

It follows from the definition of the causal and a causal Green functions and Eq.(1.17) that

$$g_+(\mathbf{R},\tau) - g_-(\mathbf{R},\tau) = \frac{1}{(2\pi)^4}\int_{C_0} d\omega \int d^3K\,\frac{e^{i(\mathbf{K}\cdot\mathbf{R}-\omega\tau)}}{-K^2+k^2}$$

where C_0 is a closed contour that extends from $-\infty + i\epsilon$ to $+\infty + i\epsilon$ in the u.h.p. and then from $+\infty - i\epsilon$ to $-\infty - i\epsilon$ in the l.h.p. with $\epsilon > 0$. If we now apply the D'Alembertion operator we find that

$$[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}][g_+(\mathbf{R},\tau) - g_-(\mathbf{R},\tau)] = \frac{1}{(2\pi)^4}\int_{C_0} d\omega \int d^3K\, e^{i(\mathbf{K}\cdot\mathbf{R}-\omega\tau)} = 0$$

using Cauchy's integral formula.

1.5 Compute the frequency domain outgoing and incoming wave Green functions $G_+(\mathbf{R},\omega)$ and $G_-(\mathbf{R},\omega)$ by performing spatial Fourier inversions of $\tilde{G}(\mathbf{K},\omega)$.

We showed in Section 1.2 that

$$\tilde{G}(\mathbf{K},\omega) = \frac{1}{k^2 - K^2},$$

which, on taking an inverse transform, yields

$$G(\mathbf{R},\omega) = \frac{1}{(2\pi)^3}\int d^3K\,\frac{e^{i\mathbf{K}\cdot\mathbf{R}}}{k^2-K^2}.$$

Since the transform $\tilde{G}(\mathbf{K},\omega)$ depends only on $K = |\mathbf{K}|$ it is convenient to transform the integral to spherical polar coordinates. Making this change of variables yields

$$G(\mathbf{R};\omega) = \frac{1}{(2\pi)^3}\int_0^{\infty} dK\,\frac{K^2}{k^2-K^2}\int_{4\pi} d\Omega\, e^{iK\mathbf{s}\cdot\mathbf{R}},$$

where $\mathbf{s}$ is the unit radial vector in $\mathbf{K}$ space and $d\Omega$ the differential solid angle in $\mathbf{K}$ space. By aligning the polar axis along the direction of $\mathbf{R}$ we have that $\mathbf{s}\cdot\mathbf{R} = R\cos\theta$ where θ is the polar angle in $\mathbf{K}$ space. The angular integral is then easily performed and we obtain after some minor algebra

$$G(\mathbf{R},\omega) = \frac{i}{(2\pi)^2 R}\int_{-\infty}^{\infty} KdK\,\frac{e^{iKR}}{K^2-k^2}. \tag{1.1}$$

The Fourier integral representation given in Eq.(1.1) is not uniquely defined until we define the contour of integration taken by the integration variable K. The outgoing wave Green function is obtained by selecting the contour to lie above the pole at $K = -k$ and below the pole at $K = +k$ while the incoming wave Green function results by selecting the contour to lie below the pole at $K = -k$ and above the pole at $K = +k$. Since either contour can be closed in the u.h.p. we then find using residue calculus that

$$G_{\pm}(\mathbf{R}, \omega) = -\frac{1}{4\pi}\frac{e^{\pm ikR}}{R},$$

with all other Green functions being linear combinations of $G_{\pm}$. It is clear that G_+ satisfies the outgoing wave radiation condition and G_- the incoming wave radiation condition.

1.6 Compute the one-dimensional incoming wave Green function $G_-(z, \omega)$ from Eq.(1.26) of example 1.3.

The incoming wave Green function results from taking $\Im k < 0$ and is obtained via contour integration as employed in obtaining the outgoing wave Green function in the example. We then find that

$$G_-(z, \omega) = +\frac{i}{2k}e^{-ik|z|},$$

a result that can also be obtained by taking the complex conjugate of $G_+(z, \omega)$ when the wavenumber k is real valued and via an analytic continuation of this result in the complex k plane for the general case of complex k.

1.7 Directly verify by differentiation that the one-dimensional causal Green function given in Eq.(1.29) of example 1.4 satisfies the defining equation Eq.(1.24) of example 1.3.

The Green function is given in Eq.(1.29) of Example 1.4:

$$g_+(z, \tau) = -\frac{c}{2}\Theta(c\tau - |z|)$$

where Θ is the step function. We then find that

$$\frac{\partial}{\partial z}g_+(z, \tau) = -\frac{c}{2}[\delta(c\tau - |z|)\frac{\partial}{\partial z}(c\tau - |z|)] = \frac{c}{2}\delta(c\tau - |z|)\mathrm{sgn}(z)$$

where $\mathrm{sgn}(z) = 2\Theta(z) - 1$ is the sign function. The second derivative is then

$$\frac{\partial^2}{\partial z^2}g_+(z, \tau) = \frac{c}{2}[\delta'(c\tau - |z|)(-\mathrm{sgn}^2(z)) + 2\delta(c\tau - |z|)\delta(z)] = \frac{c}{2}[-\delta'(c\tau - |z|) + 2\delta(c\tau)\delta(z)].$$

The τ derivatives are easily found to be

$$\frac{\partial}{\partial \tau}g_+(z, \tau) = -\frac{c^2}{2}\delta(c\tau - |z|), \quad \frac{\partial^2}{\partial \tau^2}g_+(z, \tau) = -\frac{c^3}{2}\delta'(c\tau - |z|).$$

Using the above results we then find that

$$[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}]g_+(z, \tau) = \frac{c}{2}[-\delta'(c\tau - |z|) + 2\delta(c\tau)\delta(z)] + \frac{c}{2}\delta'(c\tau - |z|) = \delta(\tau)\delta(z),$$

as required.

1.8 Verify by differentiation that the difference between the causal and acausal 1D Green function to the wave equation satisfies the homogeneous wave equation.

We have already verified in the preceding problem that the causal Green function to the 1D wave equation satisfies the defining Eq.(1.24) of Example 1.3. By following almost identical steps as employed in that problem it is easy to show that the acausal Green function also satisfies this equation. The difference between these two Green functions thus satisfies the homogeneous 1D wave equation.

1.9 Verify that the interior field representations given in Eqs.(1.37) remain valid with g_- replaced by g_+; i.e., show that the two new equations are also correct.

This follows immediately from the fact that these equations are derived only on the requirement that χ_2 vanish and this only depends on the field u_+ being causal and not on any requirement of the Green function.

1.10 Derive Eqs.(1.34a) and (1.34b) in Example 1.6.

Considered as a function of time t the function $\delta(\phi - v_\phi t)$ is periodic with period $T = 2\pi/v_\phi$ and hence can be expanded into the Fourier series

$$\delta(\phi - v_\phi t) = \sum_{n=-\infty}^{\infty} C_n(\phi) e^{-i\frac{2\pi}{T}nt} = \sum_{n=-\infty}^{\infty} C_n(\phi) e^{-inv_\phi t}$$

with the expansion coefficients given by

$$C_n(\phi) = \frac{1}{T}\int_0^T dt\, \delta(\phi - v_\phi t) e^{inv_\phi t} = \frac{1}{2\pi}\int_0^T dt\, \delta(t - \frac{\phi}{v_\phi}) e^{inv_\phi t} = \frac{e^{in\phi}}{2\pi}$$

which then yields the expansion

$$\delta(\phi - v_\phi t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\phi} e^{-inv_\phi t}.$$

which leads to the expansion Eq.(1.34a) for the source.

The derivation of Eq.(1.34b) follows directly from Fourier transformation of the source term given in Eq.(1.34a) we obtain

$$\begin{aligned} Q(\mathbf{r},\omega) &= \int_{-\infty}^{\infty} dt\, [\frac{\delta(\rho - a)}{2\pi a}\delta(z)\sum_{n=-\infty}^{\infty} e^{in\phi} e^{-inv_\phi t}] e^{i\omega t} \\ &= \frac{\delta(\rho - a)}{a}\delta(z)\sum_{n=-\infty}^{\infty} e^{in\phi}\delta(\omega - nv_\phi). \end{aligned}$$

1.11 Derive the expression for the radiated field given in Eq.(1.35a) of Example 1.6.

We use the primary field solution of the radiated field in the form

$$u_+(\mathbf{r},t) = \frac{-1}{4\pi}\int_{\tau_0} d^3r'\, \frac{q(\mathbf{r}', t - \frac{R}{c})}{R}, \tag{1.2}$$

with $R = |\mathbf{r} - \mathbf{r}'|$. On making use of the expansion Eq.(1.34a) for the source we find that

$$q(\mathbf{r}', t - \frac{R}{c}) = \frac{\delta(\rho' - a)}{2\pi a}\delta(z') \sum_{n=-\infty}^{\infty} e^{in\phi'} e^{-inv_\phi(t - \frac{R}{c})}$$

which, when substituted into Eq.(1.2), yields

$$\begin{aligned} u_+(\mathbf{r}, t) &= \frac{-1}{4\pi}\int_{\tau_0} d^3r' \, \frac{\frac{\delta(\rho'-a)}{2\pi a}\delta(z') \sum_{n=-\infty}^{\infty} e^{in\phi'} e^{-inv_\phi(t-\frac{R}{c})}}{R} \\ &= \frac{-1}{8\pi^2}\int_0^{2\pi} d\phi' \, \frac{\sum_{n=-\infty}^{\infty} e^{in\phi'} e^{-inv_\phi(t-\frac{R}{c})}}{R} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\{\frac{-1}{4\pi}\int_0^{2\pi} d\phi' \, e^{in\phi'} \frac{e^{i\frac{nv_\phi}{c}R}}{R}\} e^{-inv_\phi t} \end{aligned}$$

where now $R = \sqrt{(x - a\cos\phi')^2 + (y - a\sin\phi') + z^2}$. This reduces to Eq.(1.35a) on setting $\omega_n = nv_\phi$ and $k_n = \omega_n/c$.

1.12 Determine the Cauchy conditions satisfied by the free field propagator $g_f(\mathbf{R}, \tau)$ at $\tau = 0$ directly from its definition as the difference between the retarded and advanced Green functions to the wave equation.

From its defintion we have that

$$g_f(\mathbf{R}, \tau) = g_+(\mathbf{R}, \tau) - g_-(\mathbf{R}, \tau) = -\frac{\delta(\tau - R/c)}{4\pi R} + \frac{\delta(\tau + R/c)}{4\pi R}. \tag{1.3}$$

Setting $\tau = 0$ then yields

$$g_f(\mathbf{R}, \tau)|_{\tau=0} = -\frac{\delta(-R/c)}{4\pi R} + \frac{\delta(R/c)}{4\pi R} = 0,$$

since the delta function is an even function of its argument.

On taking the τ derivative of Eq.(1.3) we have that

$$\begin{aligned} \frac{\partial}{\partial\tau} g_f(\mathbf{R}, \tau) &= -\frac{\partial}{\partial\tau}\frac{\delta(\tau - R/c)}{4\pi R} + \frac{\partial}{\partial\tau}\frac{\delta(\tau + R/c)}{4\pi R} \\ = -\frac{\delta'(\tau - R/c)}{4\pi R} + \frac{\delta'(\tau + R/c)}{4\pi R} &\to -\frac{\delta'(-R/c)}{4\pi R} + \frac{\delta'(R/c)}{4\pi R} \end{aligned}$$

as $\tau \to 0$ and where $\delta'(x)$ denotes the derivative of the delta function. The derivative of the delta function is an odd function of its argument so we finally obtain

$$\frac{\partial}{\partial\tau} g_f(\mathbf{R}, \tau)|_{\tau=0} = \frac{\delta'(R/c)}{2\pi R}$$

which can also be expressed in the form

$$\frac{\partial}{\partial\tau} g_f(\mathbf{R}, \tau)|_{\tau=0} = c^2\frac{\delta'(R)}{2\pi R}.$$

1.13 Verify that the solution to the initial value problem given in Eq.(1.41) for the free field propagator reduces to the Cauchy conditions at $t = 0$.

On setting the field $u(\mathbf{r},t)$ in Eq.(1.41) equal to the free field propagator we find that

$$g(\mathbf{r},t) = \frac{1}{c^2}\int d^3r'\,[g_f(\mathbf{r}',t_0)|_{t_0=0}\frac{\partial}{\partial t_0}g_f(\mathbf{r}-\mathbf{r}',t) - g_f(\mathbf{r}-\mathbf{r}',t)\frac{\partial}{\partial t_0}g_f(\mathbf{r}',t_0)|_{t_0=0}]$$
$$= -\frac{1}{c^2}\int d^3r'\,g_f(\mathbf{r}-\mathbf{r}',t)\frac{\partial}{\partial t_0}g_f(\mathbf{r}',t_0)|_{t_0=0} = -\int d^3r'\,g_f(\mathbf{r}-\mathbf{r}',t)\frac{\delta'(r')}{2\pi r'} \tag{1.4}$$

where we have used the Cauchy conditions for $g_f(\mathbf{r},t)$ found in the previous problem.

We now note that

$$\delta(\mathbf{r}') = \frac{1}{(2\pi)^3}\int d^3K\,e^{i\mathbf{K}\cdot\mathbf{r}'} = \frac{1}{(2\pi)^2}\int_0^\infty K^2 dK \int_{-\pi}^{\pi} d\theta\,\sin\theta e^{iKr'\cos\theta}$$
$$= -\frac{i}{(2\pi)^2 r'}\int_{-\infty}^{\infty} dK\,Ke^{iKr'} = -\frac{\delta'(r')}{2\pi r'}$$

which when used in Eq.(1.5) yields

$$g(\mathbf{r},t) = -\int d^3r'\,g_f(\mathbf{r}-\mathbf{r}',t)\frac{\delta'(r')}{2\pi r'} = \int d^3r'\,g_f(\mathbf{r}-\mathbf{r}',t)\delta(\mathbf{r}') = g_f(\mathbf{r},t).$$

1.14 Derive the time-domain Porter-Bojarski integral equation from the interior field solution Eq.(1.37a):

$$\int_0^{T_0} dt'\int_{\tau_0} d^3r'\,g_f(\mathbf{r}-\mathbf{r}',t-t')q(\mathbf{r}',t') = \phi(\mathbf{r},t), \quad \mathbf{r}\in\tau,$$

where

$$g_f(\mathbf{R},\tau) = g_+(\mathbf{R},\tau) - g_-(\mathbf{R},\tau)$$

is the free field propagator and

$$\phi(\mathbf{r},t) = \int_{-\infty}^{\infty} dt'\int_{\partial\tau} dS'\,[u_+\frac{\partial}{\partial n'}g_- - g_-\frac{\partial}{\partial n'}u_+].$$

The Porter-Bojarski integral equation results from substituting the primary field solution Eq.(1.33) into the l.h.s. of Eq.(1.37a) and re-arranging the resulting equation.

1.15 Compute the frequency domain solution and radiation pattern for the one-dimensional wave equation from the time-domain solution found in Example 1.5.

The frequency domain field can be obtained directly the time domain field found in Example 1.5 by expressing $q(z',t')$ in a Fourier integral and simplifying the resulting expression for $u_+(z,t)$:

$$u_+(z,t) = -\frac{c}{2}\int_{-a_0}^{a_0} dz' \int_0^{t-\frac{|z-z'|}{c}} dt' \overbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, Q(z',\omega)e^{-i\omega t'}}^{q(z',t')}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\{-\frac{c}{2}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)\int_0^{t-\frac{|z-z'|}{c}} dt'\, e^{-i\omega t'}\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\{-\frac{c}{2}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)\frac{e^{-i\omega(t-\frac{|z-z'|}{c})}-1}{-i\omega}\}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\{-\frac{c}{2}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)\frac{e^{i\omega\frac{|z-z'|}{c}}}{-i\omega}\}e^{-i\omega t}.$$

where we have made use of the fact that

$$\int_{-\infty}^{\infty} d\omega\,\{-\frac{c}{2}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)\frac{-1}{-i\omega}\} = 0$$

due to the requirement that $q(z',t)$ vanishes for negative time so that $Q(z',\omega)$ goes to zero exponentially fast in the u.h.p. The ω integral lies above the real axis and can then be closed in the u.h.p. where there are no poles thus yielding zero. It then follows from the above final expression for $u(z,t)$ that

$$U(z,\omega) = -\frac{i}{2k}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)e^{ik|z-z'|}$$

where $k=\omega/c$.

The radiation pattern is found by taking z outside the source strip $[-a_0,a_0]$:

$$U(z,\omega) \sim \{-\frac{i}{2k}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)e^{\mp ikz'}\}e^{\pm ikz}$$

yielding

$$f_\pm = -\frac{i}{2k}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)e^{\mp ikz'} = -\frac{i}{2k}\tilde{Q}(\pm k,\omega)$$

where $f_\pm$ is the radiation pattern in the right (+) and left (-) half-lines and $\tilde{Q}(K,\omega)$ is the spatial Fourier transform of $Q(z,\omega)$.

1.16 Compute the frequency-domain radiation pattern for the time periodic source considered in Example 1.6.

We have from that example that

$$u(\mathbf{r},t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} U_n(\mathbf{r},\omega_n)e^{-i\omega_n t},$$

from which we conclude that

$$U(\mathbf{r},\omega) = \sum_{n=-\infty}^{\infty} U_n(\mathbf{r},\omega_n)\delta(\omega-\omega_n)$$

where

$$U_n(\mathbf{r},\omega_n) = -\frac{1}{4\pi}\int_0^{2\pi} d\phi'\, e^{in\phi'}\frac{e^{ik_nR}}{R}$$

where $R = \sqrt{(x-a\cos\phi')^2 + (y-a\sin\phi') + z^2}$ and $k_n = \omega_n/c$ with c being the velocity of the background medium and $\omega_n = nv_\phi$ with v_ϕ the radial velocity of the point source on the plane $z = 0$. We then find that

$$U_n(r\mathbf{s},\omega_n) \sim \{-\frac{1}{4\pi}\int_0^{2\pi} d\phi'\, e^{in\phi'}e^{-ik_n\mathbf{s}\cdot\mathbf{r}'}\}\frac{e^{ik_nr}}{r}$$

leading to the following expression for the radiation pattern

$$f(\mathbf{s},\omega) = \sum_{n=-\infty}^{\infty} f_n(\mathbf{s},\omega_n)\delta(\omega-\omega_n)$$

where

$$f_n(\mathbf{s},\omega_n) = -\frac{1}{4\pi}\int_0^{2\pi} d\phi'\, e^{in\phi'}e^{-ik_n\mathbf{s}\cdot\mathbf{r}'}. \tag{1.5}$$

To proceed we represent the unit vector $\mathbf{s} = \mathbf{r}/r$ in spherical polar coordinates with direction cosines $\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha$ so that

$$\mathbf{s}\cdot\mathbf{r}' = a\sin\alpha\cos\beta\cos\phi' + a\sin\alpha\sin\beta\sin\phi'$$

since $z' = 0$. Eq.(1.5) then becomes

$$\begin{aligned} f_n(\mathbf{s},\omega_n) &= -\frac{1}{4\pi}\int_0^{2\pi} d\phi'\, e^{in\phi'}e^{-ik_na\sin\alpha(\cos\beta\cos\phi'+\sin\beta\sin\phi')} \\ &= -\frac{1}{4\pi}\int_0^{2\pi} d\phi'\, e^{in\phi'}e^{-ik_na\sin\alpha\cos(\phi'-\beta)} \\ &= -\frac{1}{4\pi}e^{in\beta}\int_0^{2\pi} d\phi'\, e^{in\phi'}e^{-ik_na\sin\alpha\cos\phi'} = -\frac{(-i)^n}{4\pi}e^{in\beta}J_n(k_na\sin\alpha) \end{aligned}$$

where $J_n(\cdot)$ is the Bessel function of the first kind of order n. The radiation pattern then becomes

$$f(\mathbf{s},\omega) = f(\alpha,\beta,\omega) = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}(-i)^n e^{in\beta}J_n(k_na\sin\alpha)\delta(\omega-\omega_n).$$

1.17 Transform the SRC as defined in Eq.(1.48) into the time-domain and interpret the result.

Under inverse Fourier transform we have that

$$\frac{\partial U_+(\mathbf{r},\omega)}{\partial r} \to \frac{\partial u_+(\mathbf{r},t)}{\partial r}, \quad -ikU_+(\mathbf{r},\omega) \to \frac{1}{c}\frac{\partial}{\partial t}u_+(\mathbf{r},t)$$

so that the SRC becomes

$$\lim_{r\to\infty} r[\frac{\partial u_+(\mathbf{r},t)}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t}u_+(\mathbf{r},t)] \to 0.$$

We can interpret the above "time-domain SRC" as stating that at large distances from the source, the rates of change of the field amplitude w.r.t. time and distance r balance each other so that a positive increase (decrease) in one is balanced by a negative increase (decrease) in the other.

1.18 Prove that the far field approximation given in Eq.(1.49) is causal.

We have from Eq.(1.49)

$$F(\mathbf{s}, t - \frac{r}{c}) = -\frac{1}{4\pi}\int_{\tau_0} d^3r'\, q(\mathbf{r}', t + \frac{\mathbf{s}\cdot\mathbf{r}'}{c} - \frac{r}{c}).$$

For any given observation direction $\mathbf{s}$ and $\forall \mathbf{r}' \in \tau_0$ we have that

$$\mathbf{s}\cdot\mathbf{r}' \leq a_0$$

where a_0 is the distance from the origin (assumed inside τ_0) to the boundary $\partial\tau_0$ along the direction of the unit vector $\mathbf{s}$. It then follows that the first contribution to F occurs when

$$t + \frac{\mathbf{s}\cdot\mathbf{r}'}{c} - \frac{r}{c} \leq t + \frac{a_0}{c} - \frac{r}{c} = 0$$

or at the time

$$t = \frac{r}{c} - \frac{a_0}{c}$$

which is the retarded time for a pulse emitted at $t = 0$ to reach the field point at $\mathbf{r} = r\mathbf{s}$ from the boundary point in the direction of $\mathbf{s}$ thus verifying that the far field approximation is causal.

1.19 Express the energy spectrum $E_Q(\omega)$ directly in terms of the source $Q(\mathbf{r}, \omega)$.

We have from Eq.(1.55)

$$E_Q(\omega) = \frac{\kappa\omega^2}{8\pi^2 c}\int_{4\pi} d\Omega_s\, |\tilde{Q}(k\mathbf{s}, \omega)|^2.$$

On substituting the definition of $\tilde{Q}$ we obtain after some re-arrangement

$$E_Q(\omega) = \frac{\kappa\omega^2}{8\pi^2 c}\int_{\tau_0} d^3r \int_{\tau_0} d^3r'\, Q^*(\mathbf{r}, \omega)Q(\mathbf{r}', \omega)\int_{4\pi} d\Omega_s\, e^{ik\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')}.$$

If we now make use of the expansion (cf., Example 3.4 of Chapter 3)

$$j_0(kR) = \frac{1}{4\pi}\int_{4\pi} d\Omega_s\, e^{ik\mathbf{s}\cdot\mathbf{R}}$$

where $j_0(\cdot)$ is the zero order spherical Bessel function we obtain the required result

$$E_Q(\omega) = \frac{\kappa\omega^2}{2\pi c}\int_{\tau_0} d^3r \int_{\tau_0} d^3r'\, Q^*(\mathbf{r}, \omega)Q(\mathbf{r}', \omega)j_0(k|\mathbf{r} - \mathbf{r}'|).$$

1.20 Derive the general expression for a non-radiating source for the one-dimensional wave equation.

Following the same analysis as used for the 3D NR source to the wave equation we conclude that

$$q_{nr}(z,t) = [\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial z^2}]\pi(z,t)$$

where $\pi(z,t)$ has continuous first partials in some compact space-time region $\mathcal{S}_0 | z \in \tau_0, t \in [0, T_0]$ but is other-wise arbitrary.

1.21 Show that the solution to the one-dimensional radiation problem can be expressed entirely in terms of $\tilde{q}(\pm k, \omega)$ everywhere outside the source region. Using this solution show that the field everywhere outside the source region is uniquely determined by the value of the field at any two points $z_1 < -a_0$ and $z_2 > a_0$ where $[-a_0, a_0]$ is the space support for the source. Give an expression for the field in terms of the field amplitude at these two points.

We have shown in Problem 1.15 that the frequency domain solution to the one-dimensional radiation problem is given by

$$U(z,\omega) = -\frac{i}{2k}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)e^{ik|z-z'|}.$$

If z lies outside the source strip $[-a_0, a_0]$ the above reduces to

$$\begin{aligned} U(z,\omega) &= -\frac{i}{2k}\int_{-a_0}^{a_0} dz'\, Q(z',\omega)e^{\mp ikz'}e^{\pm ikz} \\ &= -\frac{i}{2k}\tilde{q}(\pm k,\omega)e^{\pm ikz} \end{aligned}$$

where the plus (+) sign is used in the r.h.s $z > a_0$ and the minus sign (-) in the l.h.s. $z < -a_0$. This then establishes that the solution to the one-dimensional radiation problem can be expressed entirely in terms of $\tilde{q}(\pm k, \omega)$ everywhere outside the source region.

We conclude from the above result that

$$-\frac{i}{2k}\tilde{q}(\pm k,\omega) = U(z_0,\omega)e^{\mp ikz_0}$$

where $|z_0| > a_0$ thus establishing that the field everywhere outside the source region is uniquely determined by the value of the field at any two points $z_1 < -a_0$ and $z_2 > a_0$ lying outside the source strip. On making use of the above we find that

$$U(z,\omega) = U(z_0,\omega)e^{\pm ik(z-z_0)}$$

where the plus sign is used if both z and z_0 lie in the r.h.s. and the minus sign if they both lie in the l.h.s.

1.22 Derive the equation satisfied by a frequency domain NR source Eq.(1.57) directly from the equation satisfied by the time-domain NR source Eq.(1.56).

This result follows immediately by making use of the transform relationships

$$\nabla^2\pi(\mathbf{r},t) \to \nabla^2\Pi(\mathbf{r},\omega), \quad \frac{\partial^2}{\partial t^2}\pi(\mathbf{r},t) \to -\omega^2\Pi(\mathbf{r},\omega)$$

from which it follows that

$$q_{nr}(\mathbf{r},t) = [\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\pi(\mathbf{r},t) \to Q_{nr}(\mathbf{r},\omega) = [\nabla^2 + K^2]\Pi(\mathbf{r},\omega).$$

1.23 Construct a non-radiating source using the classical testing function of distribution theory

$$\Pi(\mathbf{r}) = \begin{cases} 0 & r \geq a_0 \\ \exp\frac{1}{r^2-a_0^2} & r < a_0. \end{cases}$$

Defining

$$\phi(r) = (r^2 - a_0^2)^{-1}, \quad r < a_0$$

we have that $\Pi(r) = \exp\phi(r)$, $r < a_0$. The NR source is then given by

$$Q_{nr}(\mathbf{r}) = [\nabla^2 + k_0^2]e^{\phi(r)} = [\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k_0^2]e^{\phi(r)},$$

since ϕ is spherically symmetric. On evaluating the above expression we find that

$$Q_{nr}(\mathbf{r}) = [-6\phi^2(r) + 8r^2\phi^3(r) + 4r^2\phi^4(r) + k_0^2]e^{\phi(r)}, \quad r < a_0.$$

1.24 Determine whether the rotating point source considered in Example 1.6 can ever be NR at one or more temporal frequencies.

To check this possibility we have to compute the radiation pattern and see if it vanishes at one or more temporal frequencies. We computed the radiation pattern for this source in Problem 1.16 where it was found to be

$$f(\mathbf{s},\omega) = f(\alpha,\beta,\omega) = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}(-i)^n e^{in\beta}J_n(k_n a\sin\alpha)\delta(\omega-\omega_n).$$

where $J_n(\cdot)$ are the Bessel functions of the first kind of order n and $k_n = \omega_n/c$ with $\omega_n = nv_\phi$ and v_ϕ the radial velocity of the point source. For the radiation pattern to vanish at any frequency then requires that

$$J_n(k_n a\sin\alpha) = 0,$$

for all polar angles $\alpha \in [0,\pi]$ which is clearly impossible. This source then cannot be NR at any single or group of frequencies.

1.25 Determine whether the rotating point source considered in Example 1.6 can ever be *essentially* NR at one or more temporal frequencies.

As in the previous problem it is necessary to check to see if the radiation pattern will be essentially vanishing at any frequencies. The radiation pattern is given by

$$f(\mathbf{s},\omega) = f(\alpha,\beta,\omega) = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}(-i)^n e^{in\beta}J_n(k_n a\sin\alpha)\delta(\omega-\omega_n).$$

where $J_n(\cdot)$ are the Bessel functions of the first kind of order n and $k_n = \omega_n/c$ with $\omega_n = nv_\phi$ and v_ϕ the radial velocity of the point source. For large index

$n >> 1$ the Bessel functions $J_n(x)$ decay exponentially fast for $x < n$ so that the source will be essentially NR at frequencies ω_n, $n >> 1$ such that

$$k_n a \sin\alpha \leq k_n a < n \to v_\phi < \frac{c}{a}.$$

1.26 Use the second Helmholtz identity to verify that the cloaking field within the interior τ_0 generated from the surface source given in Eq.(1.73) is not modified when the incident field is replaced by the total field (incident plus scattered). Discuss why this modification (using total rather than incident field measurements) requires that both the field and its normal derivative be separately measured.

It follows from the first Helmholtz identity written for a source located outside τ_0 that the surface source in Eq.(1.73) will cancel the field radiated into τ_0 by the exterior source but will generate zero field outside of τ_0. This is then a cloaking field since any scatterer located with τ_0 will experience a zero incident field and, hence, radiated a null field while the field exterior to τ_0 is not modified. If now we replace the field incident from outside τ_0 by the total field (this field plus an assumed scattered field from a scatterer within τ_0) in the construction of the cloaking field via Eq.(1.73) we find that the total field thus generated will be equal to the sum of the original cloaking field plus a contribution from the surface source generated from the interior scattered field[1]. However, since this scattered field is outgoing from τ_0 it follows from the second Helmholtz identity that this contribution will be null within τ_0 and thus not change the cloaking field within τ_0. The net field within τ_0 will thus be null and so no scattered field component will actually be generated.

This second scheme for cloaking requires that both the field as well as its normal derivative be known (measured) over $\partial\tau_0$. The reason is that the total field contains both incoming (from the incident field outside τ_0) as well as outgoing (from the scattered field generated from a scatterer within τ_0) and, hence, does not satisfy a second Helmholtz identity. The field and its normal derivative over τ_0 are thus not independent and have to be both independently measured to generate the cloaking surface source.

[1] In constructing an actual cloaking field using, for example, sensors and antennas, it is important to make sure the system is stable and insensitive to measurement errors. Thus, the cloaking field will never be exactly null within τ_0 so that a scattered field component will always be present and must be accounted for.