Part I Solutions

1. (a) Unit step input, $X = 1/s$

$$Y=\frac{2}{s(s+2)(s^2+9)}=\frac{\alpha_1}{s}+\frac{\alpha_2}{s+2}+\frac{\alpha_3}{s+3j}+\frac{\alpha*_3}{s-3j}$$

with

$$\alpha_1=\left.\frac{2}{(s+2)(s^2+9)}\right|_{s=0}=\frac{2}{(2)(9)}=\frac{1}{9}$$

$$\alpha_2=\left.\frac{2}{s(s^2+9)}\right|_{s=-2}=\frac{2}{(-2)(13)}=\frac{-1}{13}$$

$$\alpha_3=\left.\frac{2}{s(s+2)(s-3j)}\right|_{s=-3j}=\frac{2}{(-3j)(2-3j)(-6j)}=\frac{-1}{9}\frac{1}{(2-3j)}=\frac{-(2+3j)}{(9)(13)}$$

Hence

$$y(t)=\frac{1}{9}-\frac{1}{13}e^{-2t}-\frac{1}{(9)(13)}\left[(2+3j)e^{-3jt}+(2-3j)e^{3jt}\right]$$

$$=\frac{1}{9}-\frac{1}{13}e^{-2t}-\frac{1}{(9)(13)}\left[(2+3j)(\cos 3t-j\sin 3t)+(2-3j)(\cos 3t+j\sin 3t)\right]$$

$$=\frac{1}{9}-\frac{1}{13}e^{-2t}-\frac{2}{(9)(13)}\left[2\cos 3t+3\sin 3t\right]$$

$$=\frac{1}{9}-\frac{1}{13}e^{-2t}-\frac{2\sqrt{13}}{(9)(13)}\sin(3t+\phi)\ ,\quad \phi=\tan^{-1}(\frac{2}{3})$$

With an impulse input, $X = 1$,

$$Y=\frac{2}{(s+2)(s^2+9)}=\frac{\alpha_1}{s+2}+\frac{\alpha_2}{s+3j}+\frac{\alpha*_2}{s-3j}$$

$$\alpha_1=\left.\frac{2}{(s^2+9)}\right|_{s=-2}=\frac{2}{13}$$

$$\alpha_2=\left.\frac{2}{(s+2)(s-3j)}\right|_{s=-3j}=\frac{2}{(2-3j)(-6j)}=\frac{1}{3}\frac{j}{(2-3j)}\frac{2+3j}{2+3j}=\frac{-3+2j}{(3)(13)}$$

And

$$y(t)=\frac{2}{13}e^{-2t}+\frac{1}{(3)(13)}\left[(-3+2j)e^{-3jt}+(-3-2j)e^{3jt}\right]$$

$$=\frac{2}{13}e^{-2t}+\frac{1}{39}\left[(-3+2j)(\cos 3t-j\sin 3t)+(-3-2j)(\cos 3t+j\sin 3t)\right]$$

$$=\frac{2}{13}e^{-2t}+\frac{2}{39}\left[-3\,cost\ 3t+2\sin 3t\right]$$

$$=\frac{2}{13}e^{-2t}+\frac{2}{3\sqrt{13}}\sin(3t+\phi)\ ,\quad \phi=\tan^{-1}(\frac{-3}{2})$$

(b) It is not obvious from $y(t)$ that all the terms cancel out at $t = 0$, but initial value theorem can be used to show that $y(0) = 0$. As for $y(t \to \infty)$, the pure sinusoidal term will not go away. There is no one final value and the final value theorem does not apply.

2.

$$y'' + y' + y = \sin \omega t$$

After Laplace transform,

$$Y = \frac{1}{s^2 + s + 1} \frac{\omega}{s^2 + \omega^2}$$

The root of $s^2 + s + 1 = 0$ are $-1/2 \pm j\sqrt{3/4}$.

We expect $y(t)$ to have a term of the form $\exp(-1/2\ t)\sin(\sqrt{3/4}\ t + \phi)$, which is an oscillation that decays away in time. Eventually, we are left with a pure sinusoidal term associated with $\omega/(s^2 + \omega^2)$. There is no final constant value and the final value theorem does not apply.

3.

$$y''+4y'+5y=f(t)$$

After Laplace transform,

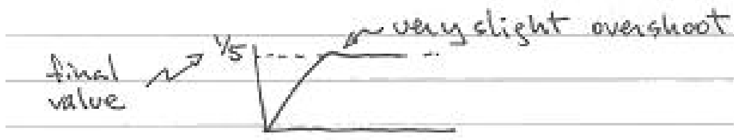
$$\frac{Y}{F}=\frac{1}{s^2+4s+5}=\frac{1/5}{(1/5)s^2+(4/5)s+1}$$

The roots of $s^2+4s+5=0$ are $s=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm j$

We expect the time dependent function $\exp(-2t)\sin(t+\phi)$, an oscillation that decays in time.

Indeed, $\tau=\frac{1}{\sqrt{5}}$, $2\zeta\tau=4/5$, $\zeta=(1/2)(4/5)\sqrt{5}=\frac{2}{\sqrt{5}}\sim0.89$, only very slightly underdamped.

If we use a unit step input and Eq. (3-24), we can find the overshoot to be only 0.2%.



4.

(a) $$Y=\frac{10}{(s+1)^2(s+3)}=\frac{a}{(s+1)}+\frac{b}{(s+1)^2}+\frac{c}{(s+3)}$$

$$b=\left.\frac{10}{(s+3)}\right|_{s=-1}=\frac{10}{2}=5$$

$$c=\left.\frac{10}{(s+1)^2}\right|_{s=-3}=\frac{10}{4}=\frac{5}{2}$$

$$\frac{10}{(s+3)}=a(s+1)+b+\{\text{c-terms with}\,(s+1)^2\}$$

Differentiate once,

$$\frac{-10}{(s+3)^2}=a+\{\text{c-terms with}\,(s+1)\}$$

Set $s=-1$, $a=-10/4=-5/2$,

$$y(t)=-\frac{5}{2}\mathrm{e}^{-t}+5t\,\mathrm{e}^{-t}+\frac{5}{2}\mathrm{e}^{-3\mathrm{t}}$$

$$=5\,\mathrm{e}^{-t}(-\frac{1}{2}+t)+\frac{5}{2}\mathrm{e}^{-3\mathrm{t}}$$

(b) $$Y=\frac{s+3}{s^2+2\mathrm{s}+5}$$

For

$$s^2+2\mathrm{s}+5=0\ ,\quad s=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm2\mathrm{j}$$

Or we can see that

$$(s^2+2s+5)=(s^2+2s+1)+4=(s+1)^2+2^2$$

But for now, we do the long, slow way,

$$Y=\frac{s+3}{s^2+2\mathrm{s}+5}=\frac{a}{s-(-1+2\mathrm{j})}+\frac{a^*}{s-(-1-2\mathrm{j})}$$

$$a=\left.\frac{s+3}{s-(-1-2\mathrm{j})}\right|_{s=-1+2\mathrm{j}}=\frac{-1+2\mathrm{j}+3}{(-1+2\mathrm{j})+1+2\mathrm{j}}=\frac{1+j}{2\mathrm{j}}=\frac{1-j}{2}$$

$$y(t)=\frac{1}{2}(1-j)\,\mathrm{e}^{(-1+2\mathrm{j})t}+\frac{1}{2}(1+j)\,\mathrm{e}^{(-1-2\mathrm{j})t}$$

$$=\frac{1}{2}\mathrm{e}^{-t}\left[(1-j)(\cos 2\mathrm{t}+j\sin 2\mathrm{t})+(1+j)(\cos 2\mathrm{t}-j\sin 2\mathrm{t})\right]$$

$$=\frac{1}{2}\mathrm{e}^{-t}\quad 2(\cos 2\mathrm{t}+\sin 2\mathrm{t})$$

$$=\sqrt{2}\,\mathrm{e}^{-t}\sin(2\mathrm{t}+\phi)\ ,\quad \phi=\tan^{-1}(1)=\pi/4$$

(c) $$Y = \frac{e^{-4s}}{s(2s^2+3s+2)} = \frac{e^{-4s}}{2s(s^2+3/2\,s+1)}$$

The roots of $s^2+3/2\,s+1=0$ are $s=-\frac{3}{4}\pm\frac{1}{2}\sqrt{\frac{9}{4}-4}=-\frac{3}{4}\pm j\frac{\sqrt{7}}{4}$

Or we can make use of

$$(s^2+\frac{3}{2}s+1)=(s^2+\frac{3}{2}s+\frac{9}{16})-\frac{9}{16}+1=(s+\frac{3}{4})^2+\frac{7}{16}$$

For now, we do it the long way. Consider first, without the time delay,

$$\frac{1}{s(s^2+\frac{3}{2}s+1)}=\frac{a}{s}+\frac{b}{s-(\frac{-3}{4}+j\frac{\sqrt{7}}{4})}+\frac{b^*}{s-(\frac{-3}{4}-j\frac{\sqrt{7}}{4})}$$

$$a=\left.\frac{1}{s^2+3/2\,s+1}\right|_{s=0}=1$$

$$b=\left.\frac{1}{s\left[s-(\frac{-3}{4}-j\frac{\sqrt{7}}{4})\right]}\right|_{s=\frac{-3}{4}+j\frac{\sqrt{7}}{4}}=\frac{1}{\left(\frac{-3}{4}-j\frac{\sqrt{7}}{4}\right)\left(j\frac{2\sqrt{7}}{4}\right)}=\frac{8}{(-3+j\sqrt{7})(j\sqrt{7})}$$

$$=\frac{8}{-7-3\sqrt{7}\,j}\;\frac{-7+j3\sqrt{7}}{-7+j3\sqrt{7}}=\frac{8(-7+j3\sqrt{7})}{49+(9)(7)}$$

$$=\frac{8}{112}(-7+j3\sqrt{7})=-\frac{1}{2}+j\frac{3\sqrt{7}}{14}$$

$$y(t)=\frac{1}{2}\left\{1+e^{\frac{-3}{4}t}\left[(\frac{-1}{2}+j\frac{3\sqrt{7}}{14})(\cos\frac{\sqrt{7}}{4}t+j\sin\frac{\sqrt{7}}{4}t)+\ldots\text{their conjugate terms}\right]\right\}$$

$$=\frac{1}{2}\left\{1+e^{\frac{-3}{4}t}(-\cos\frac{\sqrt{7}}{4}t-3\frac{\sqrt{7}}{7}\sin\frac{\sqrt{7}}{4}t)\right\}$$

We finally put the time delay back in,

$$y(t-4)=\frac{1}{2}\left\{1+e^{\frac{-3}{4}(t-4)}\left[-\cos\frac{\sqrt{7}}{4}(t-4)-3\frac{\sqrt{7}}{7}\sin\frac{\sqrt{7}}{4}(t-4)\right]\right\}u(t-4)$$

(d) This is like part (c). Consider first

$$\frac{1}{s(s^2+9)}=\frac{a}{s}+\frac{b}{3+3j}+\frac{b^*}{3-3j}$$

$$a=\left.\frac{1}{s^2+9}\right|_{s=0}=\frac{1}{9}$$

$$b=\left.\frac{1}{s(s-3j)}\right|_{s=-3j}=\frac{1}{(-3j)(-6j)}=\frac{-1}{18}=b^*$$

Without dead time,

$$y(t) = \frac{1}{9} - \frac{1}{18}\left(e^{3jt} + e^{-3jt}\right)$$
$$= \frac{1}{9}\left(1 - \cos 3\mathrm{t}\right)$$

Now with dead time,

$$y(t-2) = \left[\frac{1}{9}\left(1 - \cos 3(t-2)\right)\right]\mathrm{u}(t-2)$$

5.

$$G = \frac{s+1}{s^2(10\,s+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{10\,s+1}$$

$$c = \left.\frac{s+1}{s^2}\right|_{s=-1/10} = \frac{9/10}{1/100} = 90$$

$$b = \left.\frac{s+1}{10\,s+1}\right|_{s=0} = 1$$

$$\frac{s+1}{10\,s+1} = a\,s + b + \{\,c\text{ -terms with } s^2\,\}$$

Differentiate once,

$$\frac{1}{10\,s+1} - 10\,\frac{s+1}{(10\,s+1)^2} = a + \{\,c\text{ -terms with } s\,\}$$

Set $s = 0$, $a = 1 - 10 = -9$

$$G = \frac{-9}{s} + \frac{1}{s^2} + \frac{90}{10\,s+1}$$

Can also find

$$g(t) = -9 + t + 9\,\mathrm{e}^{-\frac{1}{10}t}$$

6. The two *s* in the original equation cancel out. So,

$$\frac{Y}{X} = \frac{3(s+2)(s-2)}{(5s^3+6s^2+2s+3)}$$

With $X = 1/s$,

$$y(t\rightarrow\infty) = \lim_{s\rightarrow 0}\left[s\frac{3(s+2)(s-2)}{5s^3+6s^2+2s+3}\frac{1}{s}\right] = \frac{(3)(2)(-2)}{3} = -4$$

This result is only valid if all three roots of $5s^3+6s^2+2s+3=0$ have negative real parts. We check with MATLAB using

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roots([5 6 2 3])
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and found –1.26, +0.03±0.69*j*. So the final value theorem does not apply and the value –4 obtained above is meaningless.

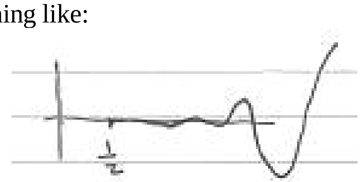
7. The roots of $s^2 - 2s - 5 = 0$ are $s = 1 \pm \frac{1}{2}\sqrt{4-20} = 1 \pm 2j$.

For

$$\frac{Y}{X} = \frac{s+1}{(s+2)(s^2-2s+5)} ,$$

we expect $y(t)$ to have time dependence e^{-2t} and $e^t \sin(2t+\phi)$. The e^{-2t} term decays away when approximately $\tau = 5$ (½) [time units]. The $e^t \sin(2t+\phi)$ term will grow exponentially with an oscillation.

With time delay due to $e^{-1/2s}$, it just means that any response to $x(t)$ will be shifted by ½ time units. Whether $X(s) = 1$ or $1/s$, we still expect something like:



8. The roots of $s^2+2s+2=0$ are $s=-1\pm\frac{1}{2}\sqrt{4-8}=-1+j$.
For

$$\frac{Y}{X}=\frac{10\,s\,(s+1)}{(s+2)(s^2+2s+2)}$$

we have zeros at 0 and –1, and poles at –2 and –1 ±*j*. The response has time dependent functions e^{-2t} and $e^{-t}\sin(t+\phi)$.

For

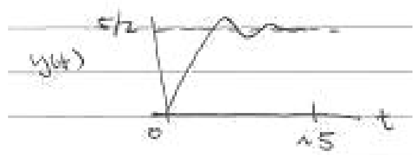
$$\frac{Y}{X}=\frac{10}{(s+2)(s^2+2s+2)}$$

we expect $y(t)$ to have the same time depdent functions as the first transfer function.

With $X = 1/s$,

$$y(t\to\infty)=\lim_{s\to 0}\left[s\frac{10}{(s+2)(s^2+2s+2)}\frac{1}{s}\right]=\frac{5}{2}$$

This 5/2 is the steady state gain of the transfer function. The response $y(t)$ will reach a final value of 5/2 in an oscillatory manner.



If we write $s^2+2s+2=2(\frac{1}{2}s^2+s+1)$, and equate

$$\frac{1}{s^2}+s+1=\tau^2s^2+2\zeta\tau s+1 ,$$

we find

$$\tau=\frac{1}{\sqrt{2}} \text{ , and } 2\zeta\tau=1 \text{ , or } \zeta=\frac{1}{2}\sqrt{2}=\frac{1}{\sqrt{2}} .$$

If

$$\frac{Y}{X}=\frac{10}{(s+2)(s^2+2)} ,$$

the poles are –2 and $\pm j\sqrt{2}$. We expect $y(t)$ to have the functional dependence e^{-2t} and $\sin(\sqrt{2}t)$. When $X = 1/s$, the e^{-2t} term will decay away and $y(t)$ will eventually become a pure sine wave that oscillates about the mean of 5/2 and with a frequency of $\sqrt{2}$. There is no final value.

Part I Solutions

9. The Laplace transform of $F(t)$ = 3 u(t) is $F(s)$ = 3/s, so

$$Y = \frac{18}{s^2+3s+9}\frac{3}{s}$$

(a) $y(t\to\infty) = \lim_{s\to 0}[sY] = \frac{(18)(3)}{9} = 6$

(b) $$\frac{Y}{X} = \frac{2}{\frac{1}{9}s^2+\frac{1}{3}s+1}$$

So

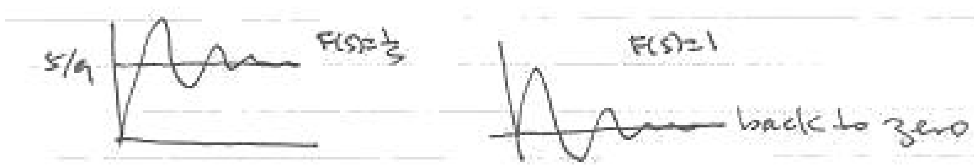
$$\tau = \frac{1}{3}\,,\quad 2\zeta\tau = \frac{1}{3}\,,\ \text{and}\ \zeta = \frac{1}{3}\frac{1}{2}3 = \frac{1}{2}$$

For ζ = ½, the overshoot is 0.16.

If $X = M/s$, the overshoot means $\frac{10-MK_p}{MK_p} = 0.16$, or MK_p = 8.62
From the transfer function, K_p = 2, so M = 8.62/2 = 4.31.

(c) $$\frac{Y}{F} = \frac{5/9}{\frac{1}{9}s^2+\frac{1}{9}s+1}\,,$$

$$\tau = \frac{1}{3}\,,\quad 2\zeta\tau = \frac{1}{9}\,,\ \text{and}\ \zeta = \frac{1}{9}\frac{1}{2}3 = \frac{1}{6} < 1$$



With both step and impulse response, we can calculate the period to be T = 2.12 [time units], and settling time to be approximately 4(τ/ζ) = (4)(1/3)(6) = 8 [time units].

For unit step input and ζ = 1/6, we can find that overshoot = 0.589, and decay ratio = 0.347. (The decay ratio applies to an impulse response too.)

10. The Laplace transform of

$$\tau_1 \frac{\mathrm{d}c_1}{\mathrm{d}t} = c_o - c_1 \ , \ \ c_1(0) = 0 \text{ , and } \quad \tau_2 \frac{\mathrm{d}c_2}{\mathrm{d}t} = c_1 - c_2 \ , \ \ c_2(0) = 0$$

would give

$$\frac{C_1(s)}{C_o(s)} = \frac{1}{\tau_1 s + 1} \ , \ \text{ and } \quad \frac{C_2(s)}{C_1(s)} = \frac{1}{\tau_2 s + 1}$$

$$\frac{C_2(s)}{C_o(s)} = \frac{C_2(s)}{C_1(s)} \frac{C_1(s)}{C_o(s)} = \frac{1}{(\tau_2 s + 1)} \frac{1}{(\tau_1 s + 1)}$$

(a) Now, $C_o(s) = 6$,

$$C_2(s) = \frac{6}{(\tau_2 s + 1)(\tau_1 s + 1)}$$

So, $c_2(t \to \infty) = 0$. We can confirm this with the final value theorem. And with the numerical values

$$\tau_1 = \frac{4}{0.02} = 200 \ \text{s} \ , \quad \tau_2 = \frac{3}{0.02} = 150 \ \text{s}$$

we can either do partial factions or simple table look up to find,

$$c_2(t) = \frac{6}{50} \left(\mathrm{e}^{-t/200} - \mathrm{e}^{-t/150} \right)$$

which indeed approaches zero as time increases.

(b) $\tau = 4/2 = 2$ min

$$\frac{C_1(s)}{C_o(s)} = \frac{1}{(\tau s + 1)} \ , \quad \frac{C_2(s)}{C_o(s)} = \frac{1}{(\tau s + 1)^2} \ , \ldots , \quad \frac{C_5(s)}{C_o(s)} = \frac{1}{(\tau s + 1)^5}$$

The plotting is an exercise in using MATLAB.

(c) If the poles are distinct, we identify the dominant pole (the largest time constant, τ) and choose the time of simulation to be at least 5τ (say 6τ), since $1 - \mathrm{e}^{-5} \sim 0.99$.

If there are multiple poles, as in part (b), we have $\mathrm{O}(t^4 \mathrm{e}^{-t/\tau})$ and it will take much longer than 5 or 6 τ to have the term decays away (i.e., reaching the new steady state). (To find the time t where $t^4 \mathrm{e}^{-t/\tau} = \varepsilon$, with $\varepsilon \ll 1$, we'll need a trial and error calculation for a chosen ε.)

11. The poles of a transfer function are not dependent on the input, so they stay at $-4.5 \pm 2.5j$ whether it is a step input or a rectangular pulse input.

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12. The process is likely nonlinear.

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13. The steady state gain is the sum of the individual steady state gains, i.e., $K_1 + K_2$. Or you can use the final value theorem and $X = 1/s$.

$$\lim_{s\to 0}[sY]=\left[s\left(\frac{K_1}{\tau_1 s+1}+\frac{K_2}{\tau_2 s+1}\right)\frac{1}{s}\right]=K_1+K_2$$

$$\frac{Y}{X}=\frac{K_1(\tau_2 s+1)+K_2(\tau_1 s+1)}{(\tau_1 s+1)(\tau_2 s+1)}=\frac{(K_1\tau_2+K_2\tau_1)s+(K_1+K_2)}{(\tau_1 s+1)(\tau_2 s+1)}$$

So the poles are at $-1/\tau_1$, and $-1/\tau_2$, and the zero is at $-(K_1+K_2)/(K_1\tau_2+K_2\tau_1)$.

14. $$Y = \frac{1}{s(s^2+2s+3)}$$

The roots of $s^2+2s+3=0$ are $s=\frac{-2\pm\sqrt{4-12}}{2}=-1\pm j\sqrt{2}$

$$Y=\frac{a}{s}+\frac{bs+c}{s^2+2s+3}$$

$$a=\left.\frac{1}{s^2+2s+3}\right|_{s=0}=\frac{1}{3}$$

$$\frac{1}{3}(s^2+2s+3)+bs^2+cs=1$$

$$(\frac{1}{3}+b)s^2+(\frac{2}{3}+c)s+1=1$$

So we have *b* = -1/3, and *c* = -2/3

$$(s^2+2s+3)=(s^2+2s+1)+2=(s+1)^2+(\sqrt{2})^2$$

$$-\frac{1}{3}s-\frac{2}{3}=-\frac{1}{3}(s+2)=-\frac{1}{3}(s+1+1)$$

$$Y=\frac{1/3}{s}-\frac{1}{3}\left[\frac{s+1}{(s+1)^2+2}+\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}\right]$$

After inverse transform,

$$y(t)=\frac{1}{3}-\frac{1}{3}e^{-t}\left[\cos\sqrt{2}t+\frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$$=\frac{1}{3}\left[1-e^{-t}\frac{1}{\sqrt{2}}(\sqrt{2}\cos\sqrt{2}t+\sin\sqrt{2}t)\right]$$

$$=\frac{1}{3}\left[1-e^{-t}\sqrt{\frac{3}{2}}\sin(\sqrt{2}t+\phi)\right],\ \ \phi=\tan^{-1}\sqrt{2}$$

FYI. We could have gone the slower route too:

$$\frac{1}{s(s^2+2s+3)}=\frac{1/3}{s}+\frac{\alpha}{s-(-1+\sqrt{2}j)}+\frac{\alpha^*}{s-(-1-\sqrt{2}j)}$$

$$\alpha=\left.\frac{1}{s(s+1+\sqrt{2}j)}\right|_{s=-1+\sqrt{2}j}=\frac{1}{(-1+\sqrt{2}j)(-1+\sqrt{2}j+1+\sqrt{2}j)}=\frac{1}{2\sqrt{2}j(-1+\sqrt{2}j)}$$

$$=\frac{-1}{2\sqrt{2}j(\sqrt{2}+j)}\frac{\sqrt{2}-j}{\sqrt{2}-j}=\frac{-(\sqrt{2}-j)}{2\sqrt{2}(2+1)}=\frac{-\sqrt{2}+j}{6\sqrt{2}}$$

Back in time domain:

$$y(t)=\frac{1}{3}-\frac{1}{6\sqrt{2}}\left[(\sqrt{2}-j)e^{(-1+\sqrt{2}j)t}+(\sqrt{2}+j)e^{(-1-\sqrt{2}j)t}\right]$$

$$=\frac{1}{3}\left\{1-\frac{e^{-t}}{2\sqrt{2}}\left[(\sqrt{2}-j)e^{\sqrt{2}jt}+(\sqrt{2}+j)e^{-\sqrt{2}jt}\right]\right\}$$

$$=\frac{1}{3}\left\{1-\frac{e^{-t}}{2\sqrt{2}}\left[(\sqrt{2}-j)(\cos\sqrt{2}+j\sin\sqrt{2}t)+(\sqrt{2}+j)(\cos\sqrt{2}-j\sin\sqrt{2}t)\right]\right\}$$

$$=\frac{1}{3}\left[1-\frac{e^{-t}}{2\sqrt{2}}2(\sqrt{2}\cos\sqrt{2}t+\sin\sqrt{2}t)\right]$$

$$=\frac{1}{3}\left[1-e^{-t}\sqrt{\frac{3}{2}}(\sin\sqrt{2}t+\phi)\right]\ ,\quad \phi=\tan^{-1}(\sqrt{2})$$

15. $\dfrac{Y}{F}=\dfrac{3}{s(s^2+2s+4)}$

The roots of $s^2+2s+4=0$ are $s=\dfrac{-2\pm\sqrt{4-16}}{2}=-1\pm j\sqrt{3}$

$$\frac{Y}{F}=\frac{3/4}{s(1/4\,s^2+1/2\,s+1)}$$

So

$$\tau=\frac{1}{2}\ ,\ \ 2\zeta\tau=\frac{1}{2}\ ,\ \ \zeta=\frac{1}{4}2=\frac{1}{2}$$

Indeed, $\zeta/\tau=1$.

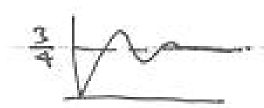
If F = 1/s,

$$Y=\frac{3}{s^2(S^2+2s+4)}=\frac{a}{s^2}+\frac{bs+c}{s^2+2s+4}$$

$$a=\left.\frac{3}{s^2+2s+4}\right|_{s=0}=\frac{3}{4}$$

So $y(t)$ will oscillates initially, but eventually it becomes a pure ramp function with slope 3/4.

If F = 1, $Y=\dfrac{3}{s(S^2+2s+4)}$, the response is oscillatory but will reach a final steady state of ¾.



16.
(a)

$$\tau \frac{\mathrm{d}c_1}{\mathrm{d}t} = c_o - c_1 - k\tau c_1 \; ,$$

$$\tau \frac{\mathrm{d}c_2}{\mathrm{d}t} = c_1 - c_2 - k\tau c_2$$

with $\tau = V/Q = 4$ min , and k = 1.5 min^{-1}. Take the first CSTR equation and rewrite as

$$\tau \frac{\mathrm{d}c_1}{\mathrm{d}t} + (1 + k\tau) c_1 = c_o$$

This becomes

$$\tau_p \frac{\mathrm{d}c_2}{\mathrm{d}t} + c_1 = K c_o \; , \text{ with } \; \tau_p = \frac{\tau}{1 + k\tau} \; \text{ and } \; K = \frac{1}{1 + k\tau}$$

This is a linear equation, so it takes the same form even in deviation variables, and after Laplace transform, we should get

$$\frac{C_1(s)}{C_o(s)} = \frac{K}{\tau_p s + 1}$$

And similarly for the second CSTR equation, we should get

$$\frac{C_2(s)}{C_1(s)} = \frac{K}{\tau_p s + 1}$$

since k and τ are constants. So

$$\frac{C_2(s)}{C_o(s)} = \left[\frac{K}{\tau_p s + 1} \right]^2$$

The steady state gain is K^2 . Response is critically damped. The time τ_p may appear as if it were the time constant because it is associated with e^{-t/τ_p} , but the actual time dependent term is really $t\,\mathrm{e}^{-t/\tau_p}$, which is much slower (See Part b.)

(b) The 63% response is only applicable to a first order function, not in this problem.
Now with k = 0, K = 1, and $\tau_p = \tau$, and the input, $C_o(s) = 1/s$,

$$C_2(s) = \frac{1}{(\tau s + 1)^2} \frac{1}{s} \; = \frac{a}{s} + \frac{b}{(\tau s + 1)} + \frac{c}{(\tau s + 1)^2}$$

$$a = \left. \frac{1}{(\tau s + 1)^2} \right|_{s=0} = 1$$

$$c = \left. \frac{1}{s} \right|_{s = -1/\tau} = -\tau$$

$$\frac{1}{s} = \{ a\text{-terms with } (\tau s + 1)^2 \} + b(\tau s + 1) + c$$

Differentiate once,

$$-\frac{1}{s^2} = \{ a\text{-terms with } (\tau s+1)\} + \tau b$$

Set s = - 1/τ

$$b = \frac{-\tau^2}{\tau} = -\tau$$

$$c_2(t) = 1 - \tau \frac{1}{\tau} \mathrm{e}^{-t/\tau} - \tau \frac{t}{\tau^2} \mathrm{e}^{-t/\tau} = 1 - e^{-t/\tau} - \frac{t}{\tau} e^{-t/\tau}$$

$$c_2(t) = 1 - (1 + \frac{t}{\tau}) \mathrm{e}^{-t/\tau}$$

At $t = \tau$,

$$c_2(\tau) = 1 - 2\,\mathrm{e}^{-1} = 0.264 \text{ , or } 26.4\%$$

This response is much slower than the first order 63.2% $(1-\mathrm{e}^{-t})$.

17.

$$\frac{Y}{F}=\frac{K_p}{\tau_p s+1}\ , \quad \text{with} \quad F=\frac{\alpha}{s^2}$$

$$Y=\frac{\alpha K_p}{s^2(\tau_p s+1)}=\frac{a}{s}+\frac{b}{s^2}+\frac{c}{\tau_p s+1}$$

$$b=\left.\frac{\alpha K_p}{\tau_p s+1}\right|_{s=0}=\alpha K_p$$

$$c=\left.\frac{\alpha K_p}{s^2}\right|_{s=-1/\tau_p}=\alpha K_p \tau_p^2$$

$$\frac{\alpha K_p}{\tau_p s+1}=a\,s+b+\{c\text{-terms with } s^2\}$$

Differentiate once,

$$-\frac{\alpha K_p \tau_p}{(\tau_p s+1)^2}=a+\{c\text{-terms with } s\}$$

So setting s = 0 gives

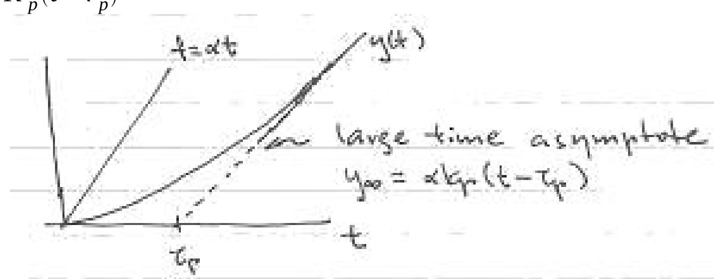
$$a=-\alpha K_p \tau_p$$

And the time domain response is

$$y(t)=\alpha K_p\left[-\tau_p+t+\tau_p \mathrm{e}^{-t/\tau_p}\right]$$

The large time asmptote has slope αK_p, and intercept τ_p:

$$y(t\rightarrow\infty)\sim\alpha K_p(t-\tau_p)$$



18.
(a)

$$y(x)=\frac{\alpha x}{1+(\alpha-1)x}$$

The first order approximation is

$$y(x)\sim y(x_s)+\left[\frac{\alpha}{1+(\alpha-1)x_s}-\frac{\alpha(\alpha-1)x_s}{(1+(\alpha-1)x_s)^2}\right](x-x_s)$$

$$y(x)\sim y(x_s)+\left[\frac{\alpha}{(1+(\alpha-1)x_s)^2}\right]x'\,,\ \text{with}\quad x'=x-x_s$$

(b)

$$P_i^s=P_c\exp\left(A_1-\frac{A_2}{T+A_3}\right)$$

The first order approximation is

$$P_i^s(T)=P_i^s(T_s)+P_c\left[\frac{A_2}{(T+A_3)^2}\exp\left(A_1-\frac{A_2}{T+A_3}\right)\right]_{T_s}(T-T_s)$$

$$=P_i^s(T_s)+\left[\frac{A_2}{(T_s+A_3)^2}P_i^s(T_s)\right]T'\,,\quad\text{with}\ \ T'=T-T_s$$

19.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -Dx + \mu(s)x \, , \tag{19.1}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = D(S_{in} - S) - \frac{1}{Y}\mu(s)x \, , \tag{19.2}$$

where $\mu(s) = \dfrac{\mu_m s}{K_m + s}$

Linearization of the nonlinear terms:

$$Dx \sim D_s x_s + D_s x' + x_s D'$$

$$Ds_{in} \sim D_s s_{in,s} + D_s s'_{in} + s_{in,s} D'$$

$$Ds \sim D_s s_s + D_s s' + s_s D'$$

where the deviation variables are:

$x' = x - x_s$, $D' = D - D_s$, $s' = s - s_s$, and $s'_{in} = s_{in} - s_{in,s}$

and also,

$$\mu(s)x \sim \mu(s_s)x_s + \left(\frac{\mu_m s}{K_m + s_s}\right)(x - x_s) + x_s\left(\frac{\mu_m}{K_m + s_s} - \frac{\mu_m s_s}{(K_m + s_s)^2}\right)(s - s_s)$$

$$= \mu(s_s)x_s + \mu(s_s)x' + \left[x_s \frac{\mu_m K_m}{(K_m + s_s)^2}\right]s'$$

Further define

$$\mu_s(s_s) = \frac{\mu_m K_m x_s}{(K_m + s_s)^2}$$

Eq. (19.1) can be written as

$$\frac{\mathrm{d}x'}{\mathrm{d}t} = -(D_s x' + x_s D') + \mu(s_s)x' + \mu_s(s_s)s'$$

$$\frac{\mathrm{d}x'}{\mathrm{d}t} + \left(D_s - \mu(s_s)\right)x' = -x_s D' + \mu_s(s_s)s'$$

And in terms of time constant and gains,

$$\tau_1 \frac{\mathrm{d}x'}{\mathrm{d}t} + x' = -K_1 D' + K_2 s' \tag{19.3}$$

where

$$\tau_1 = \frac{1}{D_s - \mu(s_s)} \ , \quad K_1 = \frac{x_s}{D_s - \mu(s_s)} \ , \ \text{and} \quad K_2 = \frac{\mu_s(s_s)}{D_s - \mu(s_s)}$$

and we have purposely chosen to have the negative sign to associate with the D' term because we know increase in D will lead to a decrease in x.

After Laplace transform,

$$X = \left[\frac{-K_1}{\tau_1 s+1}\right] D + \left[\frac{K_2}{\tau_1 s+1}\right] S \tag{19.4}$$

Repeating the same exercise with Eq. (19.2), we have

$$\frac{\mathrm{d}s'}{\mathrm{d}t} = -(D_s s'_{in} + s_{in,s} D') - (D_s s' + s_s D') - \frac{1}{Y}\left[\mu(s_s)x' + \mu_s(s_s)s'\right]$$

$$\frac{\mathrm{d}s'}{\mathrm{d}t} + \left(D_s + \frac{1}{Y}\mu_s(s_s)\right)s' = D_s s'_{in} + (s_{in,s} - s_s)D' - \frac{1}{Y}\mu(s_s)x'$$

And in terms of time constant and gains,

$$\tau_2 \frac{\mathrm{d}s'}{\mathrm{d}t} + s' = K_3 s'_{in} + K_4 D' - K_5 x' \tag{19.5}$$

where

$$\tau_2 = \frac{1}{D_s + \mu_s(s_s)/Y}\ ,\quad K_3 = \tau_2 D_s\ ,\quad K_4 = \tau_2(s_{in,s} - s_s)\ ,\ \text{and}\quad K_5 = \tau_2 \mu(s_s)/Y$$

$$S = \left[\frac{K_3}{\tau_2 s+1}\right] S_{in} + \left[\frac{K_4}{\tau_2 s+1}\right] D + \left[\frac{-K_5}{\tau_2 s+1}\right] X \tag{!9.6}$$

Now substitute Eq. (19.6) in (19.5):

$$X = \left[\frac{-K_1}{\tau_1 s+1}\right] D + \left[\frac{K_2}{\tau_1 s+1}\right]\left\{\left[\frac{K_3}{\tau_2 s+1}\right] S_{in} + \left[\frac{K_4}{\tau_2 s+1}\right] D + \left[\frac{-K_5}{\tau_2 s+1}\right] X\right\}$$

$$(\tau_1 s+1)(\tau_2 s+1)X = -K_1(\tau_2 s+1)D + K_2 K_3 S_{in} + K_2 K_4 D - K_2 K_5 X$$

Finally,

$$X = \left[\frac{K_2 K_4 - K_1(\tau_2 s+1)}{(\tau_1 s+1)(\tau_2 s+1) + K_2 K_5}\right] D + \left[\frac{K_2 K_3}{(\tau_1 s+1)(\tau_2 s+1) + K_2 K_5}\right] S_{in}$$

20. The nonlinear term is

$$r(C_A,\ C_B)=\frac{k\,C_A C_B}{1+K_A C_A+K_B C_B}$$

$$r(C_A,\ C_B)\sim r(C_A^s,\ C_B^s)+\left[\frac{k\,C_B}{1+K_A C_A+K_B C_B}-\frac{k\,K_A C_A C_B}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}(C_A-C_A^s)$$
$$+\left[\frac{k\,C_A}{1+K_A C_A+K_B C_B}-\frac{k\,K_B C_A C_B}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}(C_B-C_B^s)$$
$$=r(C_A^s,\ C_B^s)+\left[\frac{k\,C_B(1+K_A C_A+K_B C_B)-k\,K_A C_A C_B}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}C'_A$$
$$+\left[\frac{k\,C_A(1+K_A C_A+K_B C_B)-k\,K_B C_A C_B}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}C'_B$$
$$=r(C_A^s,\ C_B^s)+\left[\frac{k\,C_B(1+K_B C_B)}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}C'_A+\left[\frac{k\,C_A(1+K_A C_A)}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}C'_B$$

Define shorthand

$$\alpha=\left[\frac{k\,C_B(1+K_B C_B)}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}},\qquad \beta=\left[\frac{k\,C_A(1+K_A C_A)}{(1+K_A C_A+K_B C_B)^2}\right]_{\text{s.s.}}$$

So for the nonlinear equation,

$$\frac{\mathrm{d}C_A}{\mathrm{d}t}=\frac{1}{\tau}(C_{Ao}-C_A)-r(C_A,\ C_B)$$

At steady state, $0=\frac{1}{\tau}(C_{Ao}^s-C_A^s)-r(C_A^s,\ C_B^s)$

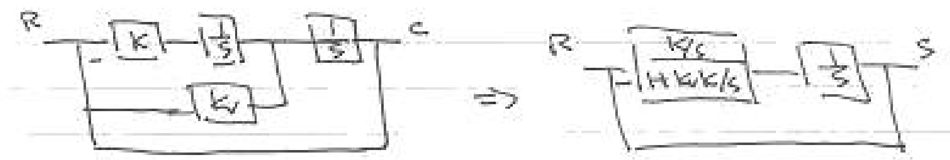
And after substituting for the nonlinear term and using deviation variables,

$$\frac{\mathrm{d}C'_A}{\mathrm{d}t}=\frac{1}{\tau}(C'_{Ao}-C'_A)-\alpha C'_A-\beta C'_B$$

$$\frac{\mathrm{d}C'_A}{\mathrm{d}t}+\left(\frac{1}{\tau}+\alpha\right)C'_A=\frac{1}{\tau}C'_{Ao}-\beta C'_B$$

where C'_A and C'_B are the deviation variables.

21.
First in closing the small internal loop,



$$\frac{K/S}{1+K_v K/s}=\frac{K}{s+K_v K}$$

$$\frac{C}{R}=\frac{\frac{K}{s+K_v K}}{1+\frac{K}{s+K_v K}}=\frac{K}{s^2+K\,K_v s+K}$$

$$\frac{C}{R}=\frac{1}{\frac{1}{K}s^2+K_v s+1}$$

If K = ¼, $\tau=1/\sqrt{K}$ = ½, and

$2\zeta\tau=K_v$, ζ = 0.7,

K_v = 2 (0.7) (1/2) = 0.7

22.

$$C_{p1} M_1 \frac{dT_m}{dt} = h_1 A_1 (T_i - T_m) \ , \quad \text{and} \quad C_{p2} M_2 \frac{dT_i}{dt} = h_2 A_2 (T_o - T_i)$$

can be written as

$$\tau_1 \frac{dT_m}{dt} = (T_i - T_m) \ , \quad \text{and} \quad \tau_2 \frac{dT_i}{dt} = (T_o - T_i)$$

where

$$\tau_1 = \frac{C_{p1} M_1}{h_1 A_1} \ , \quad \text{and} \quad \tau_2 = \frac{C_{p2} M_2}{h_2 A_2}$$

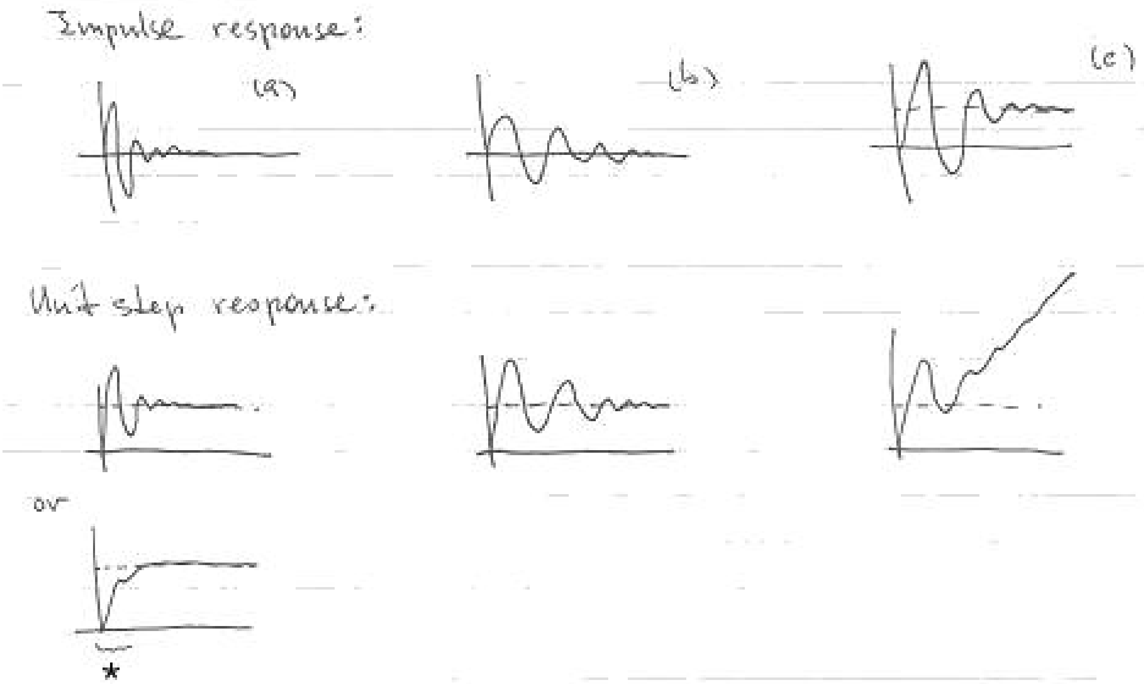
After putting the equation in deviation variables, the linear equations will retain the same form. Further with Laplace transform,

$$\frac{T_m}{T_i} = \frac{1}{\tau_1 s + 1} \ , \quad \text{and} \quad \frac{T_i}{T_o} = \frac{1}{\tau_2 s + 1}$$

$$\frac{T_m}{T_o} = \frac{1}{(\tau_1 s + 1)} \frac{1}{(\tau_2 s + 1)}$$

The response is 2nd order overdamped with time constants τ_1 and τ_2. The steady state gain is 1 and hence T_m will eventually be identical to T_o.

23. The key is that the dominant pole in (b) is complex and oscillations die out more slowly than in (a). In (c), the pole at $s = 0$ will lead to a constant from an impulse input and a ramp from a step input.



* A little "bump" will appear if the oscillations die out quickly.

24.

$$f_1 = y_1 - \alpha\, y_1\, y_2$$

$$f_1 \sim y_1 - \alpha[\, y_1^s\, y_2^s + y_2^s\, y'_1 + y_1^s\, y'_2] \,, \quad \text{with} \quad y'_1 = y_1 - y_1^s \text{ , and } \; y'_2 = y_2 - y_2^s$$

$$f_2 = -y_2 + \beta\, y_1\, y_2$$

$$f_2 \sim -y_2 + \beta[\, y_1^s\, y_2^s + y_2^s\, y'_1 + y_1^s\, y'_2]$$

At steady state,

$$0 = y_1^s - \alpha\, y_1^s\, y_2^s$$

$$0 = -y_2^s + \beta\, y_1^s\, y_2^s$$

The linearized equations are:

$$\frac{\mathrm{d}\, y'_1}{\mathrm{d}\, t} = y'_1 - \alpha\, y_2^s\, y'_1 - \alpha\, y_1^s\, y'_2$$

$$\frac{\mathrm{d}\, y'_2}{\mathrm{d}\, t} = -y'_2 + \beta\, y_2^s\, y'_1 + \beta\, y_1^s\, y'_2$$

In matrix form,

$$\frac{\mathrm{d}}{\mathrm{d}\, t}\begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix} = \begin{bmatrix} 1 - \alpha\, y_2^s & -\alpha\, y_1^s \\ \beta\, y_2^s & -1 + \beta\, y_1^s \end{bmatrix} \begin{bmatrix} y'_1 \\ y'_2 \end{bmatrix}$$

The characteristic polynomial is

$$(s - 1 + \alpha\, y_2^s)(s + 1 - \beta\, y_1^s) + \alpha\, \beta\, y_1^s\, y_2^s = 0$$

After expansion and cancellation of terms,

$$s^2 + (\alpha\, y_2^s - \beta\, y_1^s)\, s + (\alpha\, y_2^s + \beta\, y_1^s - 1) = 0$$

25. There are three poles. The transfer function is third order.

(a) $[s-(-2+j)]\,[s-(-2-j)]=[(s+2)-j]\,[(s+2)+j]=(s+2)^2+1$

$$G(s)=\frac{(s+1)\alpha}{(s+4)(s^2+4s+5)}=\frac{\alpha}{20}\frac{(s+1)}{(\frac{1}{4}s+1)(\frac{1}{5}s^2+\frac{4}{5}s+1)}$$

The steady state gain is given as 2, so 2 = α/20 or α = 40. Hence,

$$G(s)=\frac{2(s+1)}{(\frac{1}{4}s+1)(\frac{1}{5}s^2+\frac{4}{5}s+1)}$$

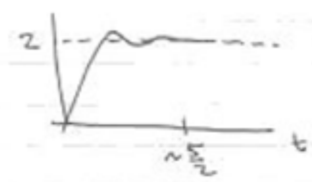
(b) The term (s + 4) gives rise to e^{-4t}, and (s^2+4s+5) gives $e^{-2t}\sin(t+\phi)$ in the time domain. The time constants are ¼ and ½.

We can double check with $(1/5\,s^2+4/5\,s+1)$,

$$\tau=1/\sqrt{5}\ ,\ \ 2\zeta\tau=4/5\ ,\ \ \zeta=(1/2)(4/5)(\sqrt{5})=2/\sqrt{5}$$

So, $\tau/\zeta=(1/\sqrt{5})\,(\sqrt{5}/2)=1/2$.

(c) With $\zeta=2/\sqrt{5}=0.89$, the response will only be very slightly underdamped.



A reasonable settling time can be either 3τ/ζ (for within 5%) or 4τ/ζ (within 2%), meaning 3/2 or 4/2. Note: in the rough hand sketch, we casually labeled "~5/2" only because the drawing is close to being at the steady state, and so we denoted that as roughly 5τ/ζ.

(d) Now

$$\frac{Y}{X}=G=\frac{40(s+1)}{s(s+4)(s^2+4s+5)}$$

With the additional pole at s = 0, a step input will lead to a ramp response.

26.
The poles and their time domain functional forms are:

$$\pm 2j \quad \Leftrightarrow \quad \sin 2t$$

$$-2 \quad \Leftrightarrow \quad e^{-2t}$$

$$-6 \pm 2j \quad \Leftrightarrow \quad e^{-6t} \sin(2t+\phi)$$

27.

$$Q_2(s) = \frac{1}{(6s+1)^2} \ , \quad \text{and the input} \quad Q_o(s) = 1$$

So $q_2(t)$ should have the form $(a_1 + a_2 t)\mathrm{e}^{-t/\tau}$, with τ = 6, and $q_2(t \to \infty) = 0$.

If $Q_2(s) = \dfrac{1}{s(6s+1)}$, then $q_2(t \to \infty) = 1$.

28.

$$\frac{1}{s(\tau_1 s+1)(\tau_2 s+1)}=\frac{a}{s}+\frac{b}{\tau_1 s+1}+\frac{c}{\tau_2 s+1}$$

with

$$a=\left.\frac{1}{(\tau_1 s+1)(\tau_2 s+1)}\right|_{s=0}=1$$

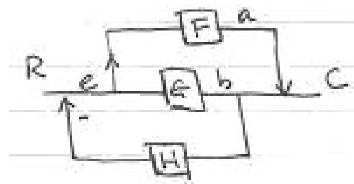
$$b=\left.\frac{1}{s(\tau_2 s+1)}\right|_{s=-1/\tau_1}=\frac{\tau_1}{(\tau_2/\tau_1-1)}=\frac{\tau_1^2}{(\tau_2-\tau_1)}$$

$$c=\left.\frac{1}{s(\tau_1 s+1)}\right|_{s=-1/\tau_2}=\frac{\tau_2}{(\tau_1/\tau_2-1)}=\frac{\tau_2^2}{(\tau_1-\tau_2)}=\frac{-\tau_2^2}{(\tau_2-\tau_1)}$$

$$y(t)=1+\frac{\tau_1^2}{\tau_2-\tau_1}\,\frac{1}{\tau_1}\mathrm{e}^{-t/\tau_1}-\frac{\tau_2^2}{\tau_2-\tau_1}\,\frac{1}{\tau_2}\mathrm{e}^{-t/\tau_2}$$

$$=1+\frac{1}{\tau_2-\tau_1}\left[\tau_1\mathrm{e}^{-t/\tau_1}-\tau_2\mathrm{e}^{-t/\tau_2}\right]$$

29.



$$e = R - Hb$$

$$C = a + b = Fe + Ge \ , \quad b = Ge$$

So

$$b = Ge = G(R - Hb)$$

$$b = \frac{GR}{1 + GH}$$

And

$$e = R\left[1 - \frac{HG}{1 + GH}\right] = R\frac{1}{1 + GH}$$

$$C = \left[\frac{F + G}{1 + GH}\right]R$$

30.

$$\frac{C}{R}=\frac{\frac{2K}{s(s+1)}}{1+\frac{2K}{s(s+1)}}=\frac{2K}{s(s+1)+2K}$$

$$\frac{C}{R}=\frac{1}{(\frac{1}{2K})s^2+(\frac{1}{2K})s+1}$$

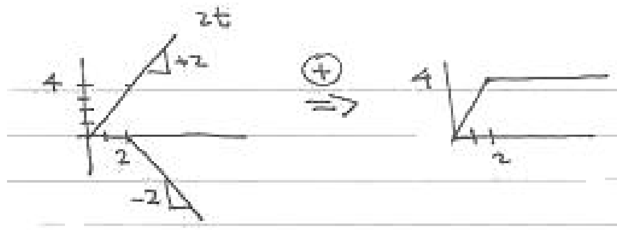
The transfer function has unity gain, so there is no steady state error.

If the overshoot is 0.1, then ζ = 0.59,

$$\tau=\frac{1}{\sqrt{2K}}\ ,\ \ 2\zeta\tau=\frac{1}{2K}\ ,\ \ \zeta=\frac{1}{4K}\sqrt{2K}=\frac{1}{4}\sqrt{\frac{2}{K}}$$

$K = 0.35_7$

31.
(a)



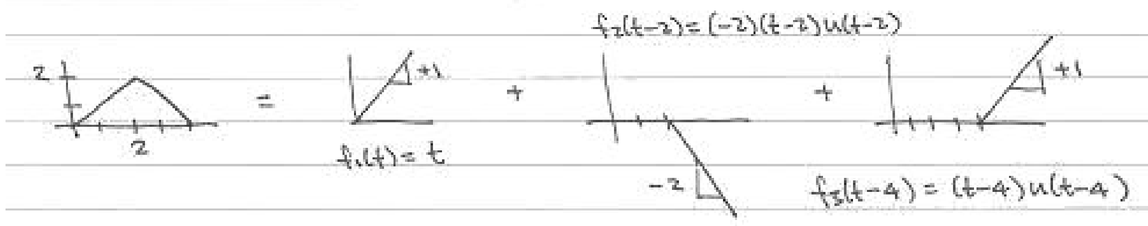
$$f(t)=2t-2(t-2)u(t-2)$$

The second term on the right really is the function $g(t-2)=g(t-2)u(t-2)$, where $g(t)=-2t$. You *cannot* write -2*t* + 4.

Here,

$$F(s)=\frac{2}{s^2}-\frac{2}{s^2}e^{-2s}=\frac{2}{s^2}(1-e^{-2s})$$

(b)

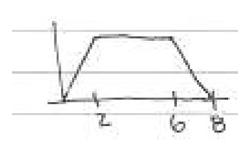


$$f(t)=t-2(t-2)u(t-2)+(t-4)u(t-4)$$

$$F(s)=\frac{1}{s^2}-\frac{2}{s^2}e^{-2s}+\frac{1}{s^2}e^{-4s}$$
$$=\frac{1}{s^2}(1-2e^{-2s}+e^{-4s})$$

(c) The answer to this part is based on parts (a) and (b), and we need to superimpose four functions:

$$f(t)=2t-2(t-2)u(t-2)-2(t-6)u(t-6)+2(t-8)u(t-8)$$
$$F(s)=\frac{2}{s^2}\left(1-e^{-2s}-e^{-6s}+e^{-8s}\right)$$



32. The algebraic relations based on Fig. PI.32:

$$E_2 = E_1 - \tilde{G}^* M$$

And skipping one step, we can still see from the diagram that

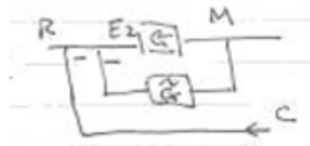
$$E_1 = R - C + e^{-\theta s} \tilde{C}_1 \ , \text{ with } \ \tilde{C}_1 = \tilde{G}^* M$$

After substituting for E_1 in the E_2 equation:

$$E_2 = (R - C) + e^{-\theta s} \tilde{G}^* M - \tilde{G}^* M$$

$$E_2 = (R - C) - \tilde{G} M \ , \text{ after defining } \ \tilde{G} = \tilde{G}^* (1 - e^{-\theta s})$$

From these reduced block diagram (see sketch on the right), the location where we usually have the controller function is:



$$G_c = \frac{G}{1 + G \tilde{G}} = \frac{G}{1 + G \tilde{G}^* (1 - e^{-\theta s})}$$

So finally,

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} = \frac{G G_p}{1 + G \tilde{G}^* (1 - e^{-\theta s}) + G G_p}$$

Alternate route: use Mason's gain formula.

There is only one forward path with the path gain:

$F_1 = GG_p$

There are three negative feedback loops: the big one plus two smaller ones within. The system determinant is

$$\Delta = 1 + G G_p + G \tilde{G}^* - G \tilde{G}^* e^{-\theta s}$$

or

$$\Delta = 1 + G G_p + G \tilde{G}^* (1 - e^{-\theta s})$$

Note that the last term is negative because as we go through this loop, we encounter two minus signs. Dividing these two quantities gives us the transfer function above.

33.

$$q = C_v R^{l-1}\left(\frac{\Delta P(h)}{\rho_s}\right)^{1/2} \quad , \quad \Delta P(h) = \frac{(P_o + \rho g h) - P_1}{g_c}$$

To linearize the nonlinear term:

$$q \sim q(l^s,\ h^s) + \left.\frac{\partial q}{\partial l}\right|_{\text{s.s.}} l' + \left.\frac{\partial q}{\partial h}\right|_{\text{s.s.}} h' \ , \quad l' = l - l^s \ , \ h' = h - h^s$$

$$\left.\frac{\partial q}{\partial l}\right|_{\text{s.s.}} = C_v\left(\frac{\Delta P(h^s)}{\rho_s}\right)^{1/2} (\ln R) R^{l^s - 1} = q(l^s,\ h^s) \ln R \equiv C_1$$

$$\left.\frac{\partial q}{\partial h}\right|_{\text{s.s.}} = C_v R^{l^s - 1} \frac{1}{2} \frac{\rho g}{g_c \rho_s}\left(\frac{\Delta P(h^s)}{\rho_s}\right)^{-1/2} \equiv C_2$$

where we have defined C_1 and C_2 as “shorthand” notations. The differential equation becomes:

$$A\frac{\mathrm{d}h'}{\mathrm{d}t} = q'_o - C_1 l' - C_2 h'$$

$$A\frac{\mathrm{d}h'}{\mathrm{d}t} + C_2 h' = q'_o - C_1 l'$$

$$\frac{A}{C_2}\frac{\mathrm{d}h'}{\mathrm{d}t} + h' = \frac{1}{C_2} q'_o - \frac{C_1}{C_2} l'$$

$$\tau\frac{\mathrm{d}h'}{\mathrm{d}t} + h' = K_1 q'_o - K_2 l' \ , \quad \tau = \frac{A}{C_2} \ , \quad K_1 = \frac{1}{C_2} \ , \quad K_2 = \frac{C_1}{C_2}$$

After Laplace transform:

$$H(s) = \left[\frac{K_1}{\tau s + 1}\right] Q_o(s) - \left[\frac{K_2}{\tau s + 1}\right] L(s)$$

34.
The mass balance of A is:

$$\tau \frac{\mathrm{d}C_A}{\mathrm{d}t} = C_{Ai} - C_A + \tau r_A \text{ , with } (-r_A) = 2k_o \mathrm{e}^{-E/RT} C_A^2 \tag{34-1}$$

which follows the chemical kinetics convention of $(\nu_A r)$. And the energy balance:

$$\tau \frac{\mathrm{d}T}{\mathrm{d}t} = T_i - T + \frac{(-\Delta H)\tau}{\rho C_p}(-r_A) - \frac{UA_c}{\rho C_p Q}(T - T_c) \tag{34-2}$$

Here, the linearized reaction rate is

$$(-r_A) \sim 2k_o \mathrm{e}^{-E/RT_s} C_{A,s}^2 + 4k_o \mathrm{e}^{-E/RT_s} C_{A,s}(C_A - C_{A,s}) + \frac{E}{RT_s^2} \, 2k_o \mathrm{e}^{-E/RT_s} C_{A,s}^2 (T - T_s)$$

Define

$$(-r_{A,s}) = 2k_o \mathrm{e}^{-E/RT_s} C_{A,s}^2 \text{ , } \quad k(T_s) = k_o \mathrm{e}^{-E/RT_s} \text{ ,}$$

$$C'_A = C_A - C_{A,s} \text{ , } \quad T' = T - T_s$$

and the linearized rate in a more compact form:

$$(-r_A) \sim (-r_{A,s}) + 4k(T_s) C_{A,s} C'_A + \frac{E}{RT_s^2}(-r_{A,s}) T' \tag{34-3}$$

After linearization, the reactant A mass balance becomes

$$\tau \frac{\mathrm{d}C'_A}{\mathrm{d}t} = -C'_A - \tau \left[4k(T_s) C_{A,s} C'_A + \frac{E}{RT_s^2}(-r_{A,s}) T' \right]$$

$$\tau \frac{\mathrm{d}C'_A}{\mathrm{d}t} + \left[1 + 4\tau k(T_s) C_{A,s} \right] C'_A = -\tau \frac{E}{RT_s^2}(-r_{A,s}) T'$$

$$\tau_A \frac{\mathrm{d}C'_A}{\mathrm{d}t} + C'_A = -K_A T' \tag{34-4}$$

where

$$\tau_A = \frac{\tau}{1 + 4\tau k(T_s) C_{A,s}} \text{ , and } \quad K_A = \frac{\tau \dfrac{E}{RT_s^2}(-r_{A,s})}{1 + 4\tau k(T_s) C_{A,s}}$$

After Laplace transform,

$$C_A(s) = -\left[\frac{K_A}{\tau_A s + 1} \right] T(s) = -G_A(s) T(s) \tag{34-5}$$

The linearized energy balance becomes

$$\tau\frac{\mathrm{d}T'}{\mathrm{d}t}=T'_i-T'+\frac{(-\Delta H)\tau}{\rho C_p}\left[4k(T_s)C_{A,s}C'_A+\frac{E}{RT_s^2}(-r_{A,s})T'\right]-\kappa(T'-T'_c)\ ,\quad \kappa=\frac{UA_c}{\rho C_p Q}$$

$$\tau\frac{\mathrm{d}T'}{\mathrm{d}t}+\left[1+\kappa-\frac{(-\Delta H)\tau}{\rho C_p}\left(\frac{E}{RT_s^2}\right)(-r_{A,s})\right]T'=T'_i+\left[\frac{(-\Delta H)\tau}{\rho C_p}4k(T_s)C_{A,s}\right]C'_A+\kappa T'_c$$

Define ”shorthand” notations,

$$\omega=1+\kappa-\frac{(-\Delta H)\tau}{\rho C_p}\left(\frac{E}{RT_s^2}\right)(-r_{A,s})\ ,\quad \beta=\frac{(-\Delta H)\tau}{\rho C_p}4k(T_s)C_{A,s}\ ,$$

then we can write

$$\tau\frac{\mathrm{d}T'}{\mathrm{d}t}+\omega T'=T'_i+\kappa T'_c+\beta C'_A$$

and finally,

$$\tau_T\frac{\mathrm{d}T'}{\mathrm{d}t}+T'=K_{T_1}T'_i+K_{T_2}T'_c+K_{T_3}C'_A \tag{34-6}$$

where

$$\tau_T=\frac{\tau}{\omega}\ ,\quad K_{T_1}=\frac{1}{\omega}\ ,\quad K_{T_2}=\frac{\kappa}{\omega}\ ,\quad K_{T_3}=\frac{\beta}{\omega}$$

After Laplace transform,

$$\begin{aligned}T(s)&=\left[\frac{K_{T_1}}{\tau_T s+1}\right]T_i(s)+\left[\frac{K_{T_2}}{\tau_T s+1}\right]T_c(s)+\left[\frac{K_{T_3}}{\tau_T s+1}\right]C_A(s)\\&=G_{T_1}(s)T_i(s)+G_{T_2}(s)T_c(s)+G_{T_3}(s)C_A(s)\end{aligned} \tag{34-7}$$

Sub Eq. (34-7) in (34-5):

$$C_A(s)=-G_A(s)\left[G_{T_1}(s)T_i(s)+G_{T_2}(s)T_c(s)+G_{T_3}(s)C_A(s)\right]$$

$$C_A(s)=\left[\frac{-G_A(s)G_{T_1}(s)}{1+G_A(s)G_{T_3}(s)}\right]T_i(s)+\left[\frac{-G_A(s)G_{T_2}(s)}{1+G_A(s)G_{T_3}(s)}\right]T_c(s) \tag{34-8}$$

The same procedure applies to the energy balance equation.

Part II Solutions

1. (a)

$$G_c = K_c \frac{\tau_D s + 1}{\alpha \tau_D s + 1} \ , \quad G_p = \frac{K_p}{\tau_p s + 1}$$

With simple unity feedback,

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} = \frac{K_c K_p (\tau_D s + 1)}{(\alpha \tau_D s + 1)(\tau_p s + 1) + K_c K_p (\tau_p s + 1)}$$

$$\frac{C}{R} = \frac{K_c K_p (\tau_D s + 1)}{\alpha \tau_D \tau_p s^2 + (\alpha \tau_D + \tau_p + K_c K_p \tau_D) s + (1 + K_c K_p)}$$

Rewriting it as

$$\frac{C}{R} = \frac{K}{\tau^2 s^2 + 2 \zeta \tau s + 1}$$

where we have the closed-loop steady state gain and natural time period defined as

$$K = \frac{K_c K_p}{1 + K_c K_p} \ \text{, and} \quad \tau = \left(\frac{\alpha \tau_D \tau_p}{1 + K_c K_p} \right)^{1/2}$$

To find the damping ratio,

$$\zeta = \frac{1}{2} \ \frac{\alpha \tau_D + \tau_p + K_c K_p \tau_D}{1 + K_c K_p} \left(\frac{1 + K_c K_p}{\alpha \tau_D \tau_p} \right)^{1/2}$$

$$\zeta = \frac{1}{2} \ \frac{\tau_D (\alpha + K_c K_p) + \tau_p}{\sqrt{\alpha \tau_D \tau_p (1 + K_c K_p)}}$$

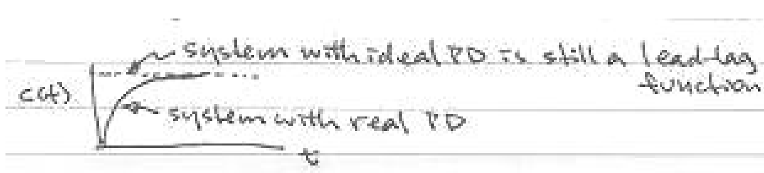
(b) From the natural time period equation, if α decreases, so does τ.

(c) Load change problem. We expect $G_L = K_d / (\tau_p s + 1)$, i.e., same process time constant τ_p, and

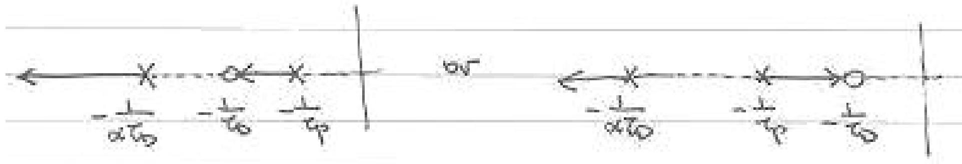
$$\frac{C}{L} = \frac{G_L}{1 + G_c G_p}$$

Offset due to load change = $0 - \dfrac{K_d}{1 + K_c K_d}$, where the zero represents no change in R.

(d)



(e) We can figure out with a simple root locus sketch, without doing much work, that the system with a real PID will not have complex closed-loop poles (or underdamped behavior).



Part II Solutions

1 (partial, without Part e).
(a)

$$G_c = K_c \frac{\tau_D s + 1}{\alpha \tau_D s + 1} \;,\quad G_p = \frac{K_p}{\tau_p s + 1}$$

With simple unity feedback,

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} = \frac{K_c K_p (\tau_D s + 1)}{(\alpha \tau_D s + 1)(\tau_p s + 1) + K_c K_p (\tau_p s + 1)}$$

$$\frac{C}{R} = \frac{K_c K_p (\tau_D s + 1)}{\alpha \tau_D \tau_p s^2 + (\alpha \tau_D + \tau_p + K_c K_p \tau_D) s + (1 + K_c K_p)}$$

Rewriting it as

$$\frac{C}{R} = \frac{K}{\tau^2 s^2 + 2 \zeta \tau s + 1}$$

where we have the closed-loop steady state gain and natural time period defined as

$$K = \frac{K_c K_p}{1 + K_c K_p} \;,\text{ and }\; \tau = \left(\frac{\alpha \tau_D \tau_p}{1 + K_c K_p} \right)^{1/2}$$

To find the damping ratio,

$$\zeta = \frac{1}{2} \, \frac{\alpha \tau_D + \tau_p + K_c K_p \tau_D}{1 + K_c K_p} \left(\frac{1 + K_c K_p}{\alpha \tau_D \tau_p} \right)^{1/2}$$

$$\zeta = \frac{1}{2} \, \frac{\tau_D (\alpha + K_c K_p) + \tau_p}{\sqrt{\alpha \tau_D \tau_p (1 + K_c K_p)}}$$

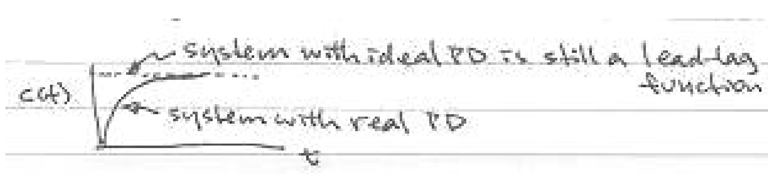
(b) From the natural time period equation, if α decreases, so does τ.

(c) Load change problem. We expect $G_L = K_d / (\tau_p s + 1)$, i.e., same process time constant τ_p, and

$$\frac{C}{L} = \frac{G_L}{1 + G_c G_p}$$

Offset due to load change = $0 - \dfrac{K_d}{1 + K_c K_d}$, where the zero represents no change in R.

(d)



2. The characteristic equation is

$$1+K_c\frac{-s+1}{(s+1)(s+2)}=0$$

(a) $$(s+1)(s+2)+K_c(-s+1)=0$$

$$s^2+(3-K_c)s+2+K_c=0$$

For a stable system, we need

$3-K_c>0$, and $2+K_c>0$

So

$$-2<K_c<3$$

(b) The term $(-s+1)/(s+1)$ is the first order Padé approximation of e^{-2s} . The presence of the open-loop positive zero ($s = 1$) makes the system unstable when $K_c > K_{cu}$, the ultimate gain. Otherwise, both "systems" with the respective characteristic equations,

$1+K_c\dfrac{1}{(s+1)(s+2)}=0$, and $1+K_c\dfrac{1}{(s+2)}=0$

are always stable.

FYI. To find the K_{cu} rigorously, we need to use

$$1+K_c\frac{e^{-2s}}{(s+2)}=0$$

and Bode plots as explained in Chapter 8 and MATLAB Session 7. We should find $K_{cu} = 2.38$.

(c) To explain stability using frequency response, we write

$$G^*=(-s+1)\,\frac{1}{(s+1)}\,\frac{1}{(s+2)}$$

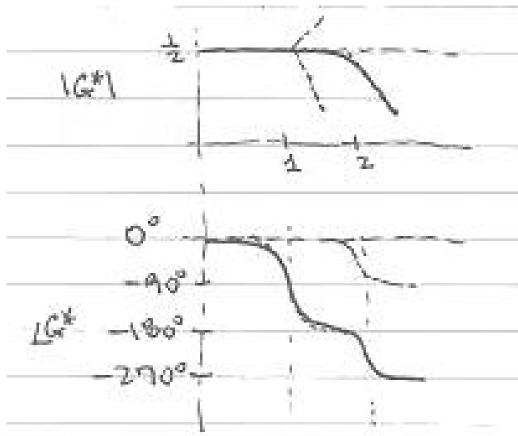
Before we do the Bode plot, we need to take a look at $G(s) = (-s + 1)$ and its Nyquist plot. For this function,

$G(j\omega) = 1 - j\omega$.

So its Nyquist plot is a vertical line that begins at +1 when $\omega = 0$, and goes downward to negative infinity. The format expressions for the magnitude and phase angle are

$|G(j\omega)| = \sqrt{(1 + \omega^2)}$, and $\angle G(j\omega) = \tan^{-1}(-\omega)$.

So the log-log magnitude plot of this function is like a first-order lead, with a slope of +1 past the break frequency, but its phase angle plot is like that of a first order lag.



The G^* magnitude plot is identical to that of $1/(s+2)$. The $(-s+1)$ and $(s+1)$ contributions cancel each other. In the phase angle plot, all three "terms" contribute a –90° lag each.

Note: Since we wrote the solution, MATLAB made some changes to their computational algorithm and the Bode plot of $G(s) = (-s + 1)$ begins at 360° instead of 0°, but its calculation of gain margin (or really critical gain) is still correct.

3. (a) The characteristic equation is

$$s(s+2)(s^2+1)+K_c\,2(s+1)=0$$

Expanding,

$$s^4+2s^3+s^2+2(1+K_c)s+2K_c=0$$

So the necessary conditions for stability is

$$2(1+K_c)>0\ ,\ \ K_c>-1 \quad \text{and} \quad 2K_c>0$$

That is, we need $K_c > 0$ for positive proportional gains.

Now with the Routh array

$$\begin{array}{ccc} 1 & 1 & 2K_c \\ 2 & 2(1+K_c) & 0 \\ b_1 & 2K_c & \\ c_1 & 0 & \\ 2K_c & & \end{array}$$

where

$$b_1=\frac{2-2(1+K_c)}{2}>0\ ,\ \text{meaning}\ \ -K_c>0\ ,\ \text{or}\ \ K_c<0$$

This requirement contradicts $K_c > 0$ from the coefficient test.
We cannot find a proportional controller if $K_c > 0$.

(b) With a PI controller,

$$1+K_c\frac{(\tau_I s+1)}{\tau_I s}\,\frac{2(s+1)}{s(s+2)(s^2+1)}=0$$

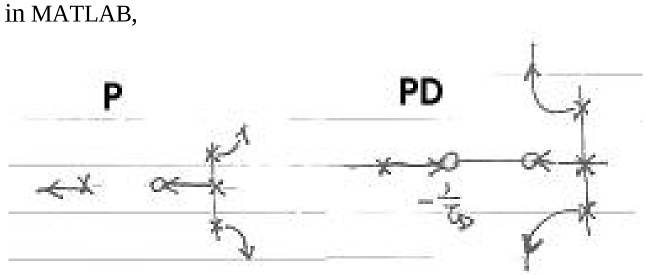
we now add one more open-loop pole at $s = 0$. It is unlikely to work.

With a PD controller,

$$1+K_c(\tau_D s+1)\,\frac{2(s+1)}{s(s+2)(s^2+1)}=0$$

By adding just one more open-loop zero, $-1/\tau_D$, this appears to be the more sensible thing to try.

(c) Use `rlocus( )` in MATLAB,



A system with PD control is stable if τ_D is large enough.

4. The closed-loop characteristic equation is

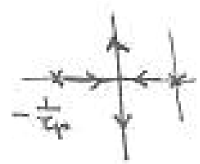
$$1+K_c\frac{K}{s(\tau_p s+1)}=0$$

The closed-loop transfer function is

$$\frac{C}{R}=\frac{G_cG_p}{1+G_cG_p}=\frac{K_cK}{s(\tau_p s+1)+K_cK}=\frac{1}{\left(\frac{\tau_p}{K_cK}\right)s^2+\left(\frac{1}{K_cK}\right)s+1}$$

(a) The system has unity steady state gain; there is no offset. We do not need integral control. We can use P or PD control.

(b)



(c)

$$\left(\frac{\tau_p}{K_cK}\right)s^2+\left(\frac{1}{K_cK}\right)s+1=\tau^2s^2+2\zeta\tau s+1$$

$$\tau=\left(\frac{\tau_p}{K_cK}\right)^{1/2} \quad , \quad \zeta=\frac{1}{2}\,\frac{1}{K_cK}\left(\frac{K_cK}{\tau_p}\right)^{1/2}=\frac{1}{2}\,\frac{1}{\sqrt{K_cK\tau_p}}$$

Now

$\zeta=1/\sqrt{2}$, K = 0.6, τ_p = 3,

$$K_c=\frac{1}{2}\,\frac{1}{(3)(0.6)}=0.28$$

The closed-loop poles are $-\zeta/\tau\pm j\sqrt{1-\zeta^2}/\tau$. With K_c = 0.28, and τ = 4.24, the poles are at –0.167± 0.167j.

(d) The time constant is τ/ζ = 4.24/0.707 = 6. Can double check with

$$\frac{\tau}{\zeta}=\left(\frac{\tau_p}{K_cK}\right)^{1/2}2(K_cK\tau_p)^{1/2}=2\tau_p=6$$

(e) $\tau/\zeta=2\tau_p$ is independent of ζ. Results in parts (c) to (e) are consistent with the root locus plot.

4 (partial, without part b).
The closed-loop characteristic equation is

$$1+K_c\frac{K}{s(\tau_p s+1)}=0$$

The closed-loop transfer function is

$$\frac{C}{R}=\frac{G_c G_p}{1+G_c G_p}=\frac{K_c K}{s(\tau_p s+1)+K_c K}=\frac{1}{\left(\frac{\tau_p}{K_c K}\right)s^2+\left(\frac{1}{K_c K}\right)s+1}$$

(a) The system has unity steady state gain; there is no offset. We do not need integral control. We can use P or PD control.

(c)

$$\left(\frac{\tau_p}{K_c K}\right)s^2+\left(\frac{1}{K_c K}\right)s+1=\tau^2 s^2+2\zeta\tau s+1$$

$$\tau=\left(\frac{\tau_p}{K_c K}\right)^{1/2} \quad , \quad \zeta=\frac{1}{2}\,\frac{1}{K_c K}\left(\frac{K_c K}{\tau_p}\right)^{1/2}=\frac{1}{2}\,\frac{1}{\sqrt{K_c K\tau_p}}$$

Now

$\zeta=1/\sqrt{2}$, K = 0.6, τ_p = 3,

$$K_c=\frac{1}{2}\,\frac{1}{(3)(0.6)}=0.28$$

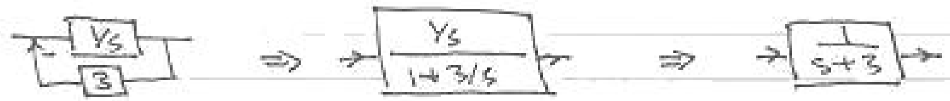
The closed-loop poles are $-\zeta/\tau\pm j\sqrt{1-\zeta^2}/\tau$. With K_c = 0.28, and τ = 4.24, the poles are at –0.167± 0.167j.

(d) The time constant is τ/ζ = 4.24/0.707 = 6. Can double check with

$$\frac{\tau}{\zeta}=\left(\frac{\tau_p}{K_c K}\right)^{1/2} 2(K_c K\tau_p)^{1/2}=2\tau_p=6$$

(e) $\tau/\zeta=2\tau_p$ is independent of ζ. Results in parts (c) to (e) are consistent with the root locus plot.

5. We first need to close the inner loop:



We also want to take note that the feedback function $1/(s + 10)$ has a steady state gain of 1/10. To have a system with consistent units, we need to add this 1/10 as K_m as in Fig. 5.5 (text), but which is omitted in this problem statement. You'd also find that you need this K_m if you need to show that the system, with a PI controller (Fig. PII. 5b) has no offset.

(a) For Fig. PII. 5A,

$$\frac{C}{R}=\frac{K_c(\tau_D s+1)/(s+3)}{1+\dfrac{K_c(\tau_D s+1)}{(s+3)(s+10)}}=\frac{K_c(\tau_D s+1)(s+10)}{(s+3)(s+10)+K_c(\tau_D s+1)}$$

Take $R = 1/s$,

$$c(t\to\infty)=\lim_{s\to 0}[sC]=\frac{10\,K_c}{30+K_c}$$

and

$$\text{offset} = 1-c(t\to\infty)=1-\frac{10\,K_c}{30+K_c}$$

We now repeat by adding K_m (Fig. 5.5 text) back, then

$$c(t\to\infty)=\frac{1}{10}\,\frac{10\,K_c}{30+K_c}=\frac{K_c}{30+K_c}\text{ , and offset = } 1-\frac{K_c}{30+K_c}=\frac{30}{30+K_c}$$

For Fig. PII 5b,

$$\frac{C}{R}=\frac{K_c(\tau_I s+1)(s+10)}{\tau_I s(s+3)(s+10)+K_c(\tau_I s+1)}$$

With $R = 1/s$, $c(t\to\infty)=10$. Indeed we need to introduce $K_m = 1/10$ so that $c(t\to\infty)=(1/10)10=1$. Now, there is no offset.

(b) The closed-loop characteristic equation is

$$1+K_c(\tau_D s+1)\,\frac{1}{(s+3)}\,\frac{1}{(s+10)}=0$$

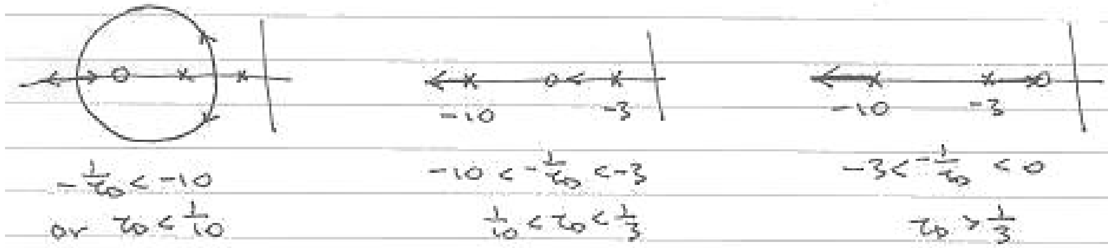
$$s^2+(13+K_c\tau_D)s+30+K_c=0$$

So we need

$$30+K_c>0\text{ , and } 13+K_c\tau_D>0$$

The system is always stable with $K_c > 0$ and $\tau_D > 0$.

(c) Three general possibilities:



(d) We have underdamped behavior only when $\tau_D < 1/10$. To find an expression for the damping ratio, we need the closed-loop characteristic equation in part (b) written as

$$\left(\frac{1}{30+K_c}\right)s^2+\left(\frac{13+K_c\tau_D}{30+K_c}\right)s+1=0$$

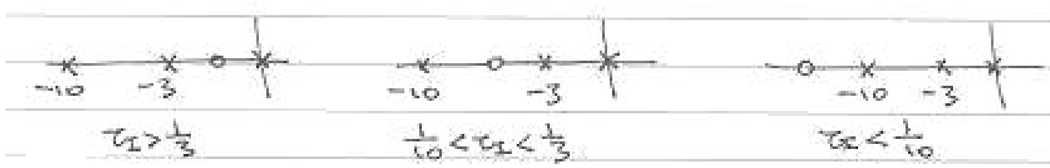
so

$$\tau=\left(\frac{1}{30+K_c}\right)^{-1/2}\text{, and } \zeta=\frac{1}{2}\left(\frac{13+K_c\tau_D}{30+K_c}\right)(30+K_c)^{1/2}=\frac{1}{2}\,\frac{13+K_c\tau_D}{(30+K_c)^{1/2}}$$

(But actual computation is much easier with root locus and using the $\zeta=\cos\theta$ line to find K_c.) The other two ranges of τ_D have no oscillations, but they are also slower—the closed-loop dominant poles are closer to the origin. So we prefer $\tau_D < 1/10$ as the basis of the controller.

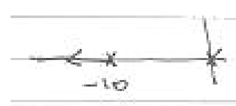
(e) $$1+K_c\frac{(\tau_I s+1)}{\tau_I s(s+3)(s+10)}=0$$

Possibilities of the range of τ_I:

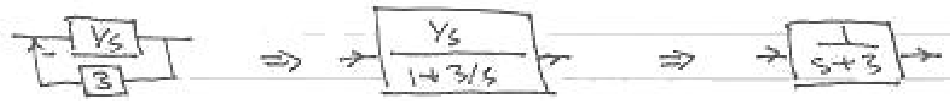


(f) Need to use MATLAB and root locus plots. When $\tau_I = 0.3$, cannot have $\zeta = 0.2$. For $\zeta = 0.9$, $K_c \sim 29$.

(g) $\tau_I = 1/3$ will lead to pole-zero cancellation. The root locus plot is that of a second order system with open-loop poles at 0 and -10.



5 (partial, Part a only).
We first need to close the inner loop:



We also want to take note that the feedback function $1/(s + 10)$ has a steady state gain of 1/10. To have a system with consistent units, we need to add this 1/10 as K_m as in Fig. 5.5 (text), but which is omitted in this problem statement. You'd also find that you need this K_m if you need to show that the system, with a PI controller (Fig. PII. 5b) has no offset.

(a) For Fig. PII. 5A,

$$\frac{C}{R}=\frac{K_c(\tau_D s+1)/(s+3)}{1+\dfrac{K_c(\tau_D s+1)}{(s+3)(s+10)}}=\frac{K_c(\tau_D s+1)(s+10)}{(s+3)(s+10)+K_c(\tau_D s+1)}$$

Take $R = 1/s$,

$$c(t\to\infty)=\lim_{s\to 0}[sC]=\frac{10\,K_c}{30+K_c}$$

and

$$\text{offset} = 1-c(t\to\infty)=1-\frac{10\,K_c}{30+K_c}$$

We now repeat by adding K_m (Fig. 5.5 text) back, then

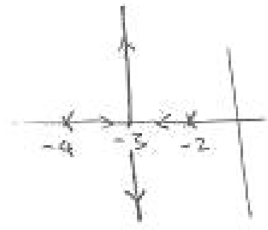
$$c(t\to\infty)=\frac{1}{10}\,\frac{10\,K_c}{30+K_c}=\frac{K_c}{30+K_c}\ \text{, and}\quad \text{offset} = 1-\frac{K_c}{30+K_c}=\frac{30}{30+K_c}$$

For Fig. PII 5b,

$$\frac{C}{R}=\frac{K_c(\tau_I s+1)(s+10)}{\tau_I s(s+3)(s+10)+K_c(\tau_I s+1)}$$

With $R = 1/s$, $c(t\to\infty)=10$. Indeed we need to introduce $K_m = 1/10$ so that $c(t\to\infty)=(1/10)10=1$. Now, there is no offset.

6. The characteristic equation is



$$1+\frac{K}{(s+2)(s+4)}=0$$

$$s^2+6s+8+K=0$$

The closed-loop poles are $s=-3\pm\frac{1}{2}\sqrt{36-4(8+K)}$.

When the system is critically damped, $36-4(8+K)=0$ or K = 1. Thus the system is overdamped when K < 1 and underdamped when K > 1.

$$\left(\frac{1}{8+K}\right)s^2+\left(\frac{6}{8+K}\right)s+1=0$$

$$\tau^2 s^2+2\zeta\tau s+1=0$$

$$\tau=\frac{1}{\sqrt{8+K}}\ ,\quad \zeta=\frac{1}{2}\,\frac{6}{8+K}\,\sqrt{8+K}=\frac{3}{\sqrt{8+K}}\ \text{, and thus}\quad 8+K=\left(\frac{3}{\zeta}\right)^2$$

When ζ = 0.707, K = 10.

The steady state gain should be $K/(8 + K)$. So when K = 10, the steady state error is $1 – K/(8 + K)$ = 0.44, quite large.

The time constant is $\tau/\zeta=(1/\sqrt{8+K})(\sqrt{8+K}/3)=1/3$, which is obvious from the closed-loop poles. So 95% settling time would be roughly $3\tau/\zeta=1$ [time units], and if we choose 98% settling time, the tmie constant would be roughly $4\tau/\zeta=4/3$.

If the overshoot is 0.1, then from $0.1=\exp(-\pi\zeta/\sqrt{1-\zeta^2})$, $\zeta=0.59$, and

$$8+K=(3/\zeta)^2 \text{ , or } K = 17.7$$

The load function should also have (s + 2) (s + 4) in its denominator.

6 (partial, Part b only).
The characteristic equation is

$$1+\frac{K}{(s+2)(s+4)}=0$$

$$s^2+6s+8+K=0$$

The closed-loop poles are $s=-3\pm\frac{1}{2}\sqrt{36-4(8+K)}$.

When the system is critically damped, $36-4(8+K)=0$ or $K = 1$. (Thus the system is overdamped when $K < 1$ and underdamped when $K > 1$.)

7. $1+\dfrac{K_c\,e^{-0.35\,s}}{(5.1\,s+1)(1.2\,s+1)}=0$

(a) Define (time constants are in minutes)

$$G(s)=\frac{K_c\,e^{-0.35\,s}}{(5.1\,s+1)(1.2\,s+1)}$$

With K_c = 7.5, ω = 0.8 rad/min,

$$|G(j\omega)| = 7.5\ \frac{1}{\sqrt{1+5.1^2 0.8^2}}\ \frac{1}{\sqrt{1+1.2^2 0.8^2}}=1.29$$

$$\begin{aligned}\measuredangle G(j\omega) &= \tan^{-1}(-5.1\times0.8)+\tan^{-1}(-1.2\times0.8)-(0.35)(0.8)(180/\pi)\\ &= -76.2^{o}-43.8^{o}-16^{o}=-136^{o}\end{aligned}$$

(b) This part follows Example 7.4A. Use MATLAB. First enter

```
p = conv([5.1 1], [1.2 1]);
g = tf(1,p);
tdead = 0.35;
```

Then follow the example, and after using `margin()`, should find K_{cu} = 19.1

When there is no dead time, it is a simple second order system and this is always stable.

(c) Use K_c = 19.1/1.7 = 11.2
(We can do a MATLAB `margin()` calculation to confirm that.)

(d) This follows Example 6.3D. After generating the results in part (b), put together

```
tmp = [freq; mag; phase]'
```

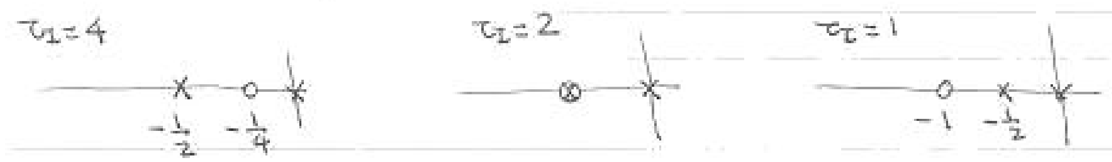
Should find where the frequency is approximately 1.02, that magnitude ~ 0.12, and phase angle ~ 150°.

So we use K_c = 1/0.12 = 8.3. (Again, we can do a MATLAB calculation to confirm that K_c = 8.3 will have a PM ~ 30°.)

8. The characteristic equation is

$$1+K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)\frac{2}{2s+1}=0$$

(a) Expected shape of the root locus plots:



Only when $\tau_I = 1$ will we have complex closed-loop poles. (Note that $\tau_I = 2$ leads to pole-zero cancellation, what direct synthesis tries to do.)

(b) Use MATLAB and $\zeta = 3/4$

```
Gp = tf(2, [2,1]);
taui = 1;
Gc = tf([taui 1], [taui 0]);
rlocus(Gc*Gp)
sgrid(3/4, 1)
rlocfind(Gc*Gp)
```

Should find $K_c \sim 0.26$ at the closed-loop poles $-0.38 \pm 0.34j$,
and $K_c \sim 1.01$ at the closed-loop poles $-0.76 \pm 0.66j$.

9. The characteristic equation is

$$1+\frac{2(1+\beta s)}{s(s+2)}=0$$

If β is a positive number, the roots will always have a negative real part and the system is always stable.

$$(1/2)s^2+(1+\beta)s+1=0$$

$\tau=1/\sqrt{2}$, $2\zeta\tau=(1+\beta)$, and thus $\zeta=(1/2)\sqrt{2}(1+\beta)=(1+\beta)/\sqrt{2}$

To be underdamped, we want

$\zeta=\frac{1+\beta}{\sqrt{2}}<1$, lead to $1+\beta<\sqrt{2}$ and $\beta<0.41$

10. The characteristic equation is

$$1+\frac{K(1+bs)}{s(\tau s+1)}=0$$

$$\tau s^2+(1+Kb)s+K=0$$

Now, K = 1, and τ = 1,

$$s^2+(1+b)s+1=0$$, leading to $2\zeta\tau=(1+b)$ or $\zeta=\frac{1}{2}(1+b)$

If ζ = 0.7, b = (2) (0.7) – 1 = 0.4.
If b increases, ζ increases and the system is less underdamped. It would be overdamped when b > 1.
If b = 0, the system would only have proportional control (as buried in this big K in this problem.)

For

$$\frac{c_1(s)}{c(s)}=1+bs$$, or $c_1(t)=c+b\frac{\mathrm{d}c}{\mathrm{d}t}$

the feedback information includes the rate of change of controlled variable c.

11.
Case (a): the characteristic equation is

$$1+\frac{K_c K_p(1+\tau_D s)}{\tau_p s+1}=0$$

Case (b): the characteristic equation is

$$1+\frac{K_c K_p(1+\tau_D s)}{\tau_p s+1}=0$$

They are identical. Their root locus plots are also identical. Can take on either one of the two possibilities:

But the two cases have different closed-loop functions.
Case (a):

$$\frac{C}{R}=\frac{K_c K_p(1+\tau_D s)}{(\tau_p s+1)+K_c K_p(1+\tau_D s)}=\frac{K_c K_p(1+\tau_D s)}{(\tau_p+K_c K_p \tau_D)s+(1+K_c K_p)}$$

$$\frac{C}{R}=\frac{K_1(1+\tau_D s)}{\tau_1 s+1}\ ;\ \text{with}\quad K_1=\frac{K_c K_p}{1+K_c K_p}\ ,\text{ and }\ \tau_1=\frac{\tau_p+K_c K_p \tau_D}{1+K_c K_p}$$

Cae (b):

$$\frac{C}{R}=\frac{K_c K_p}{(\tau_p s+1)+K_c K_p(1+\tau_D s)}=\frac{K_1}{\tau_1 s+1}$$

They have the same steady state gains K_1 and time constant τ_1, but case (a) has the response of a lead-lag element, while case (b) is just first order.

Both cases have offset = $1 - K_1 = 1/(1 + K_c K_p)$.

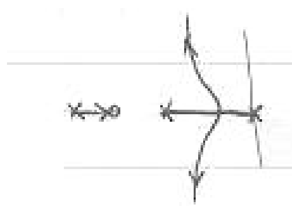
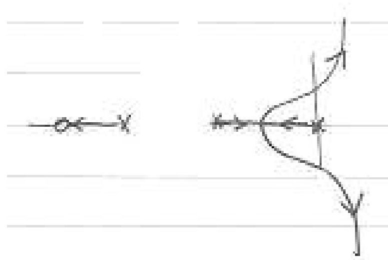
12. The characteristic equation is

$$1+K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)\frac{1}{(s+1)(s+2)}=0$$

The MATLAB statements are similar to those in Example 7.5A:

```
Gp = tf(1, conv([1 1], [2 1]));
taui = 2.5  %or 1.5 , 0.5
Gc = tf([taui 1], [taui 0]);
rlocus(Gc*Gp)
sgrid(0.7,1)
rlocfind(Gc*Gp)
```

(a) From the MATLAB results for $\zeta = 0.7$,

τ_I	Gain and closed-loop poles
$\tau_I = 2.5$ 	$K_c \sim 1.16$, poles at -0.33 and $-0.58 \pm 0.59j$
$\tau_I = 1.5$ 	$K_c \sim 0.57$, poles at $-0.33 \pm 0.34j$ and -0.83
$\tau_I = 0.5$ 	$K_c \sim 0.1$, poles at $-0.21 \pm 0.22j$ and -1.07

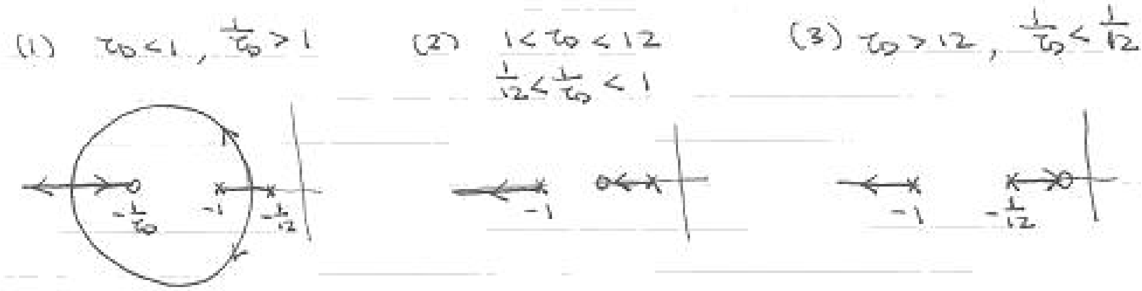
(b) The case with $\tau_I = 0.5$ is the least desirable. The system can become unstable with low K_c and for $\zeta = 0.7$, the dominant poles at $-2.1 \pm 0.22j$ are the slowest of all three cases. (Recall that the real part of the complex pole is $-\zeta/\tau$.)

For $\tau_I = 1.5$ and 2.5, the dominant poles are $-0.33 \pm 0.34j$ and -0.33, respectively. So in terms of "speed" and settling time, they behave the same. The difference is that with $\tau_I = 2.5$, the complex pole is "faster" and so oscillations will be damped out quicker than when we use $\tau_I = 1.5$, in which case the complex poles are the slower ones (c.f. real pole at -0.83.).

13. The closed-loop characteristic equation is

$$1+K_c\,\frac{(\tau_D s+1)}{(12s+1)(s+1)}=0$$

The possibilities (can be from doing MATLAB) are:

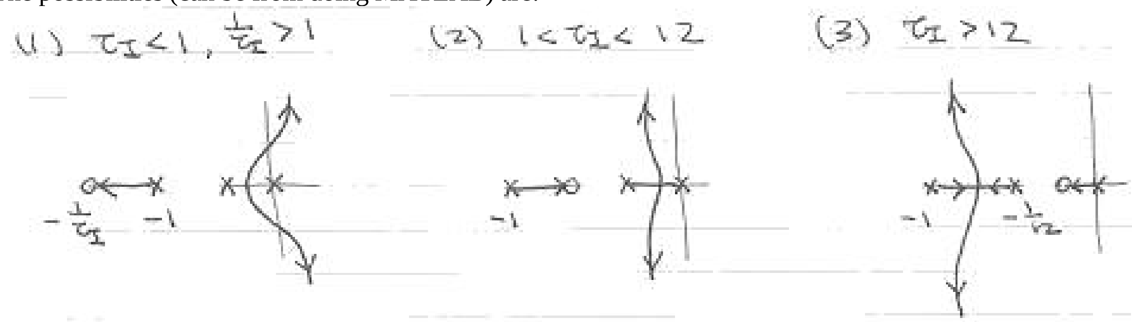


Only case (1) can have an underdamped system. In all cases, we have a second order system with no positive zero; the system is stable for all $K_c > 0$.

With a PI controller, we have

$$1+K_c\,\frac{(\tau_I s+1)}{\tau_I s(12s+1)(s+1)}=0$$

The possibilities (can be from doing MATLAB) are:



All choices of τ_I lead to an underdamped system. It can become unstable when $\tau_I < 1$.

Now given $\tau_I = 0.5$, the closed-loop characteristic equation is

$$1+K_c\,\frac{(\frac{1}{2}s+1)}{\frac{1}{2}s(12s+1)(s+1)}=0$$

$$\frac{1}{2}s(12s^2+13s+1)+K_c(\frac{1}{2}s+1)=0$$

$$12s^3+13s^2+(1+K_c)s+2K_c=0$$

So simply based on the coefficients, we must have

$1+K_c>0$, or $K_c>-1$ (Or simply $K_c > 0$ for positive proportional gains)

Now the Routh array:

$$\begin{array}{ll} 12 & 1+K_c \\ 13 & 2K_c \\ b_1 & \\ 2K_c & \end{array}$$

with

$$b_1=\frac{(13)(1+K_c)-24K_c}{13}>0$$

$$13(1+K_c)>24K_c \text{ , or } 1.18>K_c$$

So we need (for positive proportional gains)

$$0<K_c<1.18$$

Repeat with $s = j\omega$ substitution:

$$-12\omega^3 j-13\omega^2+(1+K_c)\omega j+2K_c=0$$

The real parts:

$$2K_c-13\omega^2=0 \text{ , } K_c=(13/2)\omega^2$$

The imaginary parts:

$$\omega(-12\omega^2+1+K_c)=0$$

Substituting for K_c:

$$-12\omega^2+1+\frac{13}{2}\omega^2=0$$

will lead to

$$\omega_n^2=0.18 \text{ , } K_{cu}=1.18$$

Finally, with PI control, there is no offset.

14.

(a) The closed-loop characteristic equation is

$$1+K_c\frac{3\,e^{-s}}{4s+1}=0$$

Using a first-order Padé approximation, it becomes

$$1+K_c\frac{3(-1/2s+1)}{(4s+1)(1/2s+1)}=0$$

$$(\frac{1}{2}s+1)(4s+1)+3K_c(-\frac{1}{2}s+1)=0$$

$$2s^2+\frac{3}{2}(3-K_c)s+(1+3K_c)=0$$

Stability requires

$3-K_c>0$, or $K_c<3$

$1+3K_c>0$, or $K_c>-1/3$

Together, for positive values of proportional gain,

$0<K_c<3$

(b) Substituting $s = j\omega$ in the original characteristic equation,

$$4\omega j+1+3K_c(\cos\omega-j\sin\omega)=0$$

The real parts:

$$1+3K_c\cos\omega=0$$

The imaginary parts:

$$4\omega-3K_c\sin\omega=0$$

Substituting for K_c:

$$4\omega+\tan\omega=0$$

Solving with MATLAB, should find $\omega_u=1.72$ and $K_{cu}=-1/(3\cos\omega)=2.3$.

(c) Need to add the dead time to the MATLAB `bode()` phase angle as in Example 8.6. We can also use our M-file "`ezbo.m`" which contains most of the statements that we need.
Note: the default frequency vector chosen by MATLAB in this problem is too low. Make sure you define your own as in

```
freq = logpsace(-2,1,100);
[mag, phase] = bode(G,freq);
```

From our `ezbo.m`, we found K_{cu} = 2.31 and ω_{cg} = 1.71, which are consistent with the results in part (b).

15.

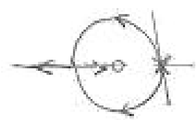
(a) With $G_p = K/s^2$, there are two poles at the origin. A step input ($1/s$) will give $1/s^3$, and the time response is to the order of t^2.

(b) With $G_c = K_c$,

$$\frac{C}{R} = \frac{K\,K_c}{s^2 + K\,K_c}$$, which is of the form $\omega \dfrac{\omega}{s^2+\omega^2}$

The closed-loop step response is a sinusoidal function with frequency $\sqrt{K\,K_c}$, so if we have picked K_c, we can compute K.

(c) With a PI controller, there will be three open-loop poles at the origin, which is unlikely to have stable system.
An ideal PD controller should provide a stable system:



(You can try both PI and PD controllers with MATLAB.)

15 (partial, without Part c).

(a) With $G_p = K/s^2$, there are two poles at the origin. A step input ($1/s$) will give $1/s^3$, and the time response is to the order of t^2.

(b) With $G_c = K_c$,

$$\frac{C}{R} = \frac{K K_c}{s^2 + K K_c} \text{, which is of the form } \omega \frac{\omega}{s^2 + \omega^2}$$

The closed-loop step response is a sinusoidal function with frequency $\sqrt{K K_c}$, so if we have picked K_c, we can compute K.

16.

(a) First, follow the text and set $C/R = e^{-\theta s}/(\tau_c s+1)$, which leads to

$$G_c = \frac{\tau_p s+1}{K_p e^{-t_d s}} \frac{e^{-\theta s}}{(\tau_c s+1) - e^{-\theta s}}$$

$$G_c = \frac{\tau_p s+1}{K_p} \frac{1}{(\tau_c s+1) - e^{-\theta s}}$$ if we choose to have $\theta = t_d$

Now substitute $e^{-\theta s} = (-\frac{\theta}{2}s+1)/(\frac{\theta}{2}s+1)$,

$$G_c = \frac{1}{K_p} \frac{(\tau_p s+1)(\frac{\theta}{2}s+1)}{(\tau_c s+1)(\frac{\theta}{2}s+1) - (\frac{-\theta}{2}s+1)}$$

$$= \frac{1}{K_p} \frac{(\tau_p s+1)(\frac{\theta}{2}s+1)}{(\tau_c+\theta)s\left(\frac{\tau_c\theta/2}{\tau_c+\theta}s+1\right)}$$

$$= \frac{\tau_p}{K_p(\tau_c+\theta)}\left[\left(1+\frac{1}{\tau_p}\frac{1}{s}\right)\frac{\frac{\theta}{2}s+1}{\tau^* s+1}\right], \text{ where } \tau^* = \frac{\tau_c\theta/2}{\tau_c+\theta}$$

So we have

$$K_c = \frac{\tau_p}{K_p(\tau_c+\theta)}$$, and $\tau_I = \tau_p$ (they are the same as direct synthesis)

Also $\tau_D = \frac{\theta}{2}$,

and for the real derivative part, $\tau^* = \frac{\tau_c}{\tau_c+\theta}\frac{\theta}{2} = \frac{\tau_c}{\tau_c+\theta}\tau_D$

The ratio $\tau_c/(\tau_c+\theta)$ is less than 1 but not that small. This is also the answer to part c.

(b) $\tilde{G}_p = \tilde{G}_{p+}\tilde{G}_{p-}$, where now

$$\tilde{G}_{p-} = \frac{K_p}{(\tau_p s+1)(\frac{\theta}{2}s+1)}$$, and $\tilde{G}_{p+} = -(\frac{\theta}{2}s+1)$

Now

$$G_c^* = \frac{1}{\tilde{G}_{p-}}\left(\frac{1}{\tau_c s+1}\right)^2 = \frac{(\tau_p s+1)(\frac{\theta}{2}s+1)}{K_p} \frac{1}{(\tau_c s+1)^2}$$

$$G_c = \frac{G_c^*}{1 - G_c^* \tilde{G}_p} = \frac{\dfrac{(\tau_p s+1)(\frac{\theta}{2}s+1)}{K_p} \dfrac{1}{(\tau_c s+1)^2}}{1 - \dfrac{1}{(\tau_c s+1)^2}(-\dfrac{\theta}{2}s+1)}$$

After simplification,

$$G_c = \frac{1}{K_p} \frac{1}{s} \frac{(\tau_p s+1)(\frac{\theta}{2}s+1)}{(\tau_c^2 s + 2\tau_c + \theta/2)}$$

This can be rearranged to a PID controller,

$$G_c = \frac{\tau_p}{K_p(2\tau_c+\theta/2)} (1 + \frac{1}{\tau_p} \frac{1}{s}) \left(\frac{\frac{\theta}{2}s+1}{\tau^* s+1} \right) \text{, with } \tau^* = \frac{\tau_c^2}{2\tau_c + \theta/2}$$

And we can identify

$$K_c = \frac{\tau_p}{K_p(2\tau_c+\theta/2)}\ , \quad \tau_I = \tau_p\ , \text{ and } \quad \tau_D = \theta/2$$

The value of K_c is slightly smaller than that of using direct synthesis, but the choice of τ_I is the same. Typically, $\tau_p > \theta$, so we have $\tau_I > \tau_D$ too. And for the (approximate) real derivative action, $\alpha\tau_D = \tau^*$, which is not that small a value.

17.

$$G_p = \frac{2}{20s+1}$$ has open-loop pole at –1/20 or –0.05

$$G_v = \frac{0.5}{s+1}$$ has open-loop pole at –1

(The problem statement has an open-loop pole at –5, but the original solution used –10. So here's the answer to the –5 in the text.)

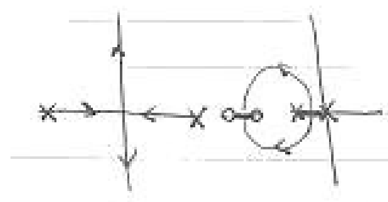
(a) Now with two additional open-loop poles at 0 and –5, and zeros at –0.1 and –0.5, the system must have a real PID controller.

Typically, $\tau_I > \tau_D$, so we should have $\tau_I = 10$ (1/0.1), and $\tau_D = 2$ (1/0.5).
And $-1/\alpha\tau_D = -5$, meaning $\alpha = 0.1$.

$$G_c = K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)\left(\frac{\tau_D s+1}{\alpha\tau_D s+1}\right) = K_c\left(\frac{10s+1}{10s}\right)\left(\frac{2s+1}{0.1s+1}\right)$$

(b) The order of the system with G_p, G_v, and the real PID controller is 4.

(c) You can tell what is on the real axis easily, but you need MATLAB to get the shape of the root locus plot.



18. The closed-loop characteristic equation is, after cancellation of (s + 5),

$$1 + K\frac{10}{s(s+20)} = 0$$

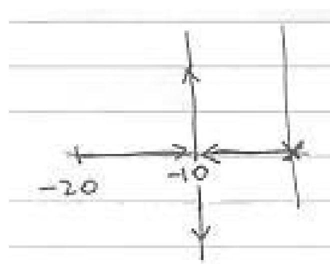
$$s^2 + 20s + 10K = 0$$

The poles are at

$$s = \frac{-20 \pm \sqrt{400 - 40K}}{2}$$

(a) The system is overdamped when $400 - 40K > 0$ or $K < 10$;
critically damped when $K = 10$, with the two repeated closed-loop poles at –10;
and underdamped when $400 - 40K < 0$ or $K > 10$;

(b)



(c) With integration in G_p, the system has no offset. We can confirm that by finding the closed-loop steady state gain.

$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p} = \frac{10K}{s(s+20) + 10K}$$

$$\frac{C}{R} = \frac{1}{\left(\frac{1}{10K}\right)s^2 + \frac{2}{K}s + 1}$$, which has unity gain

To find K such that the system has a damping ratio of 0.7, match

$$\left(\frac{1}{10K}\right)s^2 + \frac{2}{K}s + 1 = \tau^2 s^2 + 2\zeta\tau s + 1$$

$$\tau^2 = \frac{1}{10K} \ , \ \zeta = \frac{1}{2}\frac{2}{K}(10K)^{1/2} = \sqrt{\frac{10}{K}}$$

If $\zeta = 0.7$, $K = 10/0.49 = 20.4$

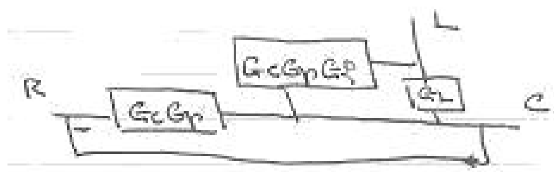
19.
Parts (a) and (b):
Fig. PII.19(a) can first be modified to

So

$$C=\left[\frac{G_c G_p}{1+G_c G_p}\right]R+\left[\frac{G_p G_f+G_L}{1+G_c G_p}\right]L$$

To elimintae the effect of disturbance L, we want $G_p G_f+G_L=0$, or $G_f=-G_L/G_p$.

Fig. PII.19(b) can be rearranged to



So

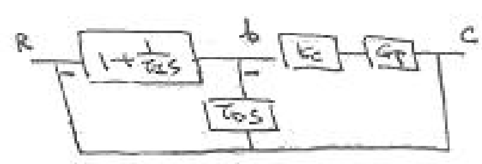
$$C=\left[\frac{G_c G_p}{1+G_c G_p}\right]R+\left[\frac{C_c G_p G_f+G_L}{1+G_c G_p}\right]L$$

To eliminate the effect of L, we want $G_c G_p G_f+G_L=0$, or $G_f=-G_L/G_c G_p$.

Part (c):

Fig. PII19(a) is better because we can compute G_f without having to worry about G_c. In Fig. PII.19(b), not only does G_f depends on G_c, but the function G_f can easily end up to have a higher order nominator polynomial.

Part (d): Fig. PII.19(c) is



Using the (dummy) variable b,

$$C=K_c G_p b$$

$$b=(1+\frac{1}{\tau_I s})(R-C)-\tau_D s C=(1+\frac{1}{\tau_I s})R-(1+\frac{1}{\tau_I s}+\tau_D s)C$$

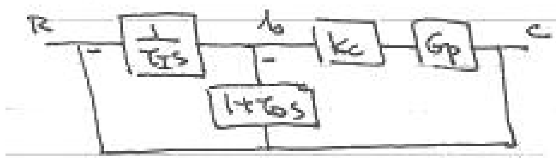
Substitute for b,

Part II Solutions

$$C = K_c G_p \left[(1+\frac{1}{\tau_I s}) R - (1+\frac{1}{\tau_I s} + \tau_D s) C \right]$$

$$\frac{C}{R} = \frac{K_c (1+\frac{1}{\tau_I s}) G_p}{1 + K_c (1+\frac{1}{\tau_I s} + \tau_D s) G_p}$$

With Fig. PII.19(d),

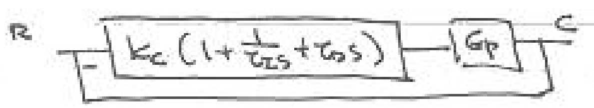


again with $C = K_c G_p b$

$$b = \frac{1}{\tau_I s}(R - C) - (1+\tau_D s) C = \frac{1}{\tau_I s} R - (1+\frac{1}{\tau_I s} + \tau_D s) C$$

$$\frac{C}{R} = \frac{K_c \frac{1}{\tau_I s} G_p}{1 + K_c (1+\frac{1}{\tau_I s} + \tau_D s) G_p}$$

Part (e): System with an ideal PID



$$\frac{C}{R} = \frac{K_c (1+\frac{1}{\tau_I s} + \tau_D s) G_p}{1 + K_c (1+\frac{1}{\tau_I s} + \tau_D s) G_p}$$

So the systems in parts (c) to (e) all have the same closed-loop characteristic polynomial.

20. With a simple unity feedback system, the characteristic equation is

$$1+K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)\left(\frac{0.5}{s-2}\right)=0$$

Now $\tau_I = 1$,

$$1+K_c\left(\frac{s+1}{s}\right)\left(\frac{0.5}{s-2}\right)=0$$

$$s^2-2s+K_c(0.5)(s+1)=0$$

$$s^2+(0.5K_c-2)s+0.5K_c=0$$

For stability, we need

$0.5K_c-2>0$, and $0.5K_c>0$

meaning,

$$K_c>4$$

So the ultimate gain is $K_{cu} = 4$, and we can confirm that with direct substitution.

$$-\omega^2-2\omega j+0.5K_c\omega j+0.5K_c=0$$

Imaginary parts: $-2\omega+0.5K_c\omega=0$, or $K_{cu}=4$

Real parts: $\omega^2=0.5K_c$, or $\omega_{n,u}=\sqrt{2}$

21.

(a) With PD control, the closed-loop characteristic equation $1 + G_c G_p = 0$ is

$$1+K_c(1+\tau_D s)\frac{K}{s-2}=0$$

$$(s-2)+K_c K(1+\tau_D s)=0$$

$$s=\frac{2-K_c K}{1+K_c K\tau_D}$$

For stability, we need

$1+K_c K\tau_D>0$, and $2-K_c K<0$

The first inequality simply means that all the gains and time constant should be positive, and the second leads to

$$K_c>2/K$$

Compared with proportional control, the characteristic equation and stability criterion are

$$(s-2)+K_c K=0$$

$$s=2-K_c K<0$$

Thus requiring $K_c>2/K$

In this problem, the stability criteria with P and PD controllers are the same.

(b) Now with a PI controller, the characteristic equation is

$$1+K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)\frac{K}{s-2}=0$$

$$\tau_I s(s-2)+K_c K(\tau_I s+1)=0$$

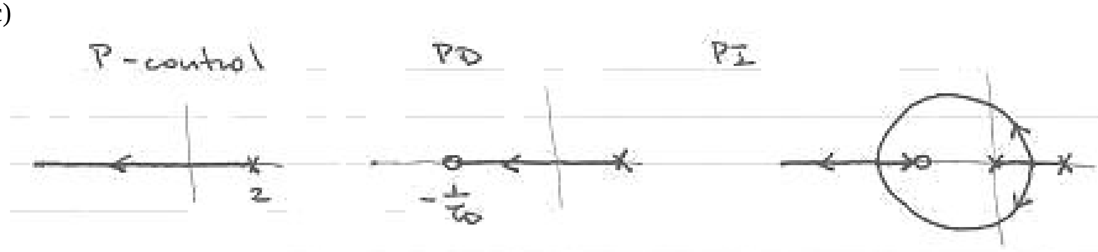
$$\tau_I s^2+\tau_I(K_c K-2)s+K_c K=0$$

For stability, we need $\tau_I > 0$, $K > 0$, $K_c > 0$, and

$K_c K - 2 > 0$, or $K_c > 2/K$.

The criterion stays the same.

(c)



(d) The ultimate gain in all these cases is $2/K$.
For P and PD control, the root locus sketch indicates that the ultimate frequency (on the real axis) is zero.

For PI control, we need to do a direct substitution, $s = j\omega$. The characteristic equation becomes

$$-\tau_I \omega^2 + \tau_I (K_c K - 2)\omega j + K_c K = 0$$

Re: $K_c K - \tau_I \omega^2 = 0$

Im: $\omega \tau_I (K_c K - 2) = 0$, meaning $K_{cu} = 2/K$

Substitute K_{cu} in the real part equation

$$K\left(\frac{2}{K}\right) - \tau_I \omega_u^2 = 0 \text{ , or } \omega_u = \sqrt{\frac{2}{\tau_I}}$$

The smaller the τ_I the larger is ω_u.

22. The load function is not in the closed-loop characteristic equation and has no effect on the stability.

(a) $1 + G_cG_p = 0$ leads to

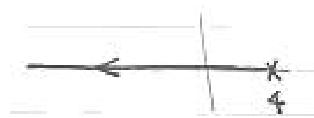
$$1+K_c\frac{2}{s-4}=0$$

$$(s-4)+2K_c=0$$

$$s=4-2K_c$$

For stability, we need $4 - 2K_c < 0$, or $K_c > 2$.

(b)



(c) The ultimate gain is $K_{cu} = 2$ (when $s = 0$). At this position, the system is stable with an impulse input but not a step input (the system response will be a ramp).

(d) With $G_c=\frac{1}{G_p}\frac{1}{\tau_c s}$

$$G_c=\frac{s-4}{2}\frac{1}{\tau_c s}=\frac{1}{2\tau_c}(1-\frac{4}{s})$$

The controller function has integrating action, but it could be done only if we could build a device with a positive zero.

23. The characteristic equation is

$$1+K_c\,\frac{1.2\,e^{-0.7\,s}}{(0.2\,s+1)(4\,s+1)}=0$$

(a) This problem follows Example 7.4A in Chapter 8. So with MATLAB, we can find
K_{cu} = 6.86 with ω_{cu} = 1.91 rad/min
(The time unit stays with that of the time constants in the transfer function.)

(b) Using the Ziegler-Nichols tuning relations and (K_{cu} ,ω_{cu}) from Part (a), we can calculate, for example,
K_c = 4, τ_I = 1.64, τ_D = 0.4
for a ¼-decay response.

The closed-loop characteristic equation is

$$1+K_c\left(1+\frac{1}{\tau_I\,s}+\tau_D\,s\right)\frac{1.2\,e^{-0.7\,s}}{(0.2\,s+1)(4\,s+1)}=0$$

Again using MATLAB and Bode plot, we can find GM = 1.9 for the Z-N ¼-decay settings.

(c) The basis of the calculation is the function

$$\left(1+\frac{1}{\tau_I\,s}+\tau_D\,s\right)\frac{1.2\,e^{-0.7\,s}}{(0.2\,s+1)(4\,s+1)}=0$$, with values of τ_I and τ_Δ from Part (b).

Again, using MATLAB, we find K_{cu} = 7.7

With GM = 1.7, K_c = 7.7/2.7 = 4.5

Repeating the calculation with K_c = 4.5, we find PM = 38°.

24. The system with a secondary loop is equivalent to

where

$$G_v^* = \frac{K_{c2} G_v}{1 + K_{c2} G_v}$$

Now

$$G_v + \frac{1}{5s+1} \text{ , and } \quad G_v^* = \frac{K_{c2}}{(5s+1) + K_{c2}} = \frac{\left(\frac{K_{c2}}{1+K_{c2}}\right)}{\left(\frac{5}{1+K_{c2}}\right)s + 1}$$

To speed up the valve, we make the time constant of G^*_v to be 1/10 of that with G_v. So

$$\frac{5}{1+K_{c2}} = (0.1)(5) \text{ , and } K_{c2} = 9, \; G_v^* = \frac{0.9}{0.5s+1}$$

The closed-loop characteristic equation is

$$1 + K_c \frac{0.9}{(0.5s+1)(0.1s+1)(s+1)} = 0$$

$$0.05s^3 + 0.65s^2 + 1.6s + 1 + 0.9K_c = 0$$

From the last constant coefficient, we need $1 + 0.9K_c > 0$, or $K_c > -1/0.9$

With the Routh array:

$$\begin{array}{ll} 0.05 & 1.6 \\ 0.65 & 1 + 0.9K_c \\ b_1 & \end{array}$$

where we need

$$b_1 = \frac{(0.65)(1.6) - (0.05)(1 + 0.9K_c)}{0.65} > 0$$

$$1.04 > 0.05 + 0.045K_c$$

$$K_c < 22$$

Hence, the ultimate gain is $K_{cu} = 22$.

If we want GM = 2, then we need $K_c = 22/2 = 11$.

The system equation without cascade control is

$$1 + K_c \frac{1}{(5s+1)(0.1s+1)(s+1)} = 0$$

On a Bode plot, the corner frequency of the valve function is 1/5. With cascade control, this corner frequency becomes 2 (time constant = 0.5), so the phase lag that it brings in is at a much higher frequency than when we have no cascade control. The consequence is that the cascade system has a wider bandwidth. Nonetheless, it still can become unstable as a third order system (containing 3 first-order lags).

25.

$$G(s)=\frac{18(2s+1)}{(s^2+3s+9)(s+4)}=\frac{\frac{1}{2}(2s+1)}{(\frac{1}{9}s^2+\frac{1}{3}s+1)(\frac{1}{4}s+1)}$$

(a) This part follows the properties of different functions in Example 8.2 to 8.4, and 8.9.

		High frequency asymptote	
Corner frequencies	Type	Magnitude slope	Phase lag
1/2	1st order lead	+1	+90°
1/3	2nd order lag	–2	–180°
4	1st order lag	–1	–90°

So on the magnitude plot, the slope of $|G(j\omega)|$ is –2 at high frequencies. At low frequency, the slope is, of course , zero and the value is ½, the steady state gain of $G(s)$.

On the phase angle plot, the total phase lag at high frequencies is –180°. It is of course 0° at very low frequencies. By first bringing in the first-order lead at lower frequencies, the total phase lag of $\measuredangle G(j\omega)$ never crosses over –180°.

(b) $\log|G(j\omega)|=\log(\frac{1}{2})+\log\sqrt{1+4\omega^2}-\log\sqrt{(1-\omega^2/9)^2+\omega^2/9}-\log\sqrt{1+\omega^2/16}$

$$\measuredangle G(j\omega)=\tan^{-1}(2\omega)+\tan^{-1}(\frac{-\omega/3}{1-\omega^2/9})+\tan^{-1}(\frac{-1}{4}\omega)$$

(c) The steady state gain of $G(s)$ is ½.
For the term

$$\frac{1}{9}s^2+\frac{1}{3}s+1=\tau^2s^2+2\zeta\tau s+1$$

$\tau=1/3$, $\zeta=\frac{1}{2}\,\frac{1}{3}\,3=\frac{1}{2}$, and $\frac{\tau}{\zeta}=\frac{2}{3}$

$$\frac{\sqrt{1-\zeta^2}}{\tau}=\frac{\sqrt{1-1/4}}{1/3}=\frac{3}{2}\sqrt{3}$$

So from the characteristic polynomial, the two time-domain functions are $e^{-\frac{3}{2}t}\sin(\frac{3\sqrt{3}}{2}t+\phi)$, and e^{-4t} . And $e^{-\frac{3}{2}t}$ decays much slower than e^{-4t} . We can consider the complex poles $-\frac{3}{2}\pm j\frac{3\sqrt{3}}{2}$ to be dominant.

(d) As noted in Part (a), the system is always stable. GM is not defined.

26.

(a) For the system $1 + G_c G_p = 0$, we have
open-loop poles at –0.5, $-3 \pm j\sqrt{3}$, $-6 \pm j\sqrt{5}$, –9
open-loop zero at –0.5

A good possibility is that we have a PD controller and more specifically, a real PD controller that also contributes the large open-loop pole at –9. Whether it is ideal or real PD, we probably have chosen $\tau_D = 2$ to cancel the open-loop pole from G_p.

(b) $(s+3-\sqrt{3}\,j)(s+3+\sqrt{3}\,j)=(s+3)^2+3=s^2+6s+12$

$(s+6-\sqrt{5}\,j)(s+6+\sqrt{5}\,j)=(s+6)^2+5=s^2+12s+41$

$$G_p=\frac{K}{(s^2+6s+12)(s^2+12s+41)(s+1/2)}$$

The steady state gain is 2, so

$2=\dfrac{K}{(12)(41)(1/2)}$, and $K = 492$

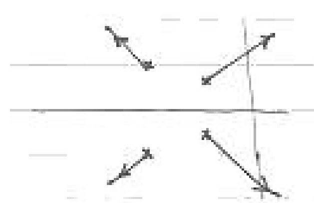
After pole-zero cancellation, the system characteristic equation is

$$1+K_c\frac{1}{(\frac{1}{9}s+1)}\,\frac{2}{(\frac{1}{12}s^2+\frac{1}{2}s+1)(\frac{1}{41}s^2+\frac{12}{41}s+1)}=0$$

Next, we need a root-locus plot to find K_c such that the decay ratio is 0.25 (ζ = 0.344 with Eq. 5.19). Using MATLAB, $K_c \approx 0.4$. The key is to use the more dominant poles (loci) that come off the $-3 \pm j\sqrt{3}$ open-loop poles.

(c) Simply repeat with DR = 0.1 (ζ = 0.215). $K_c \approx 0.64$.

(d) We should have done this first before we get the answers to Parts (b) and (c)! (Sketch from MATLAB:)



(e) The system characteristic equation stays the same. So we certainly can design a controller to handle disturbance.

27.

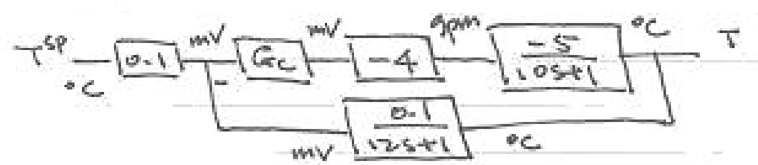
(a) We have to choose between Q_c and Q_j as the manipulated variable. There is no clear cut answer here; Q_j has a larger steady state gain but a much larger time constant. We shall select the cooling jacket flow Q_j anyway on the basis that we may compensate for its lack of speed by a proper system design (for example, with the use of cascade control).

Next, we need to make several assumptions to evaluate the Q_j regulating valve gain. Normal half-open operation provides 20 (gpm, flow units), so full range is 2 x 20. The valve may be driven by a 0-10 mV signal. We are guessing this on the basis of the sensor-transmitter output. The valve gain must be negative because the process gain is negative (and presuming K_c of the controller is positive). And without info on dynamics, we take G_v = K_v, and

$$K_v = \frac{-(2)(20)}{10} = -4\,\frac{\text{gpm}}{\text{mv}}$$

For the sensor, the measurement gain is $K_m = \dfrac{10-0}{200-100} = 0.1\,\dfrac{\text{mV}}{^\circ\text{C}}$

The system is



The closed-loop equation is

$$1 + K_c\,\frac{(0.1)(4)(5)}{(10s+1)(12s+1)} = 0$$

$$120s^2 + 22s + 1 + 2K_c = 0$$

$$\left(\frac{120}{1+2K_c}\right)s^2 + \left(\frac{22}{1+2K_c}\right)s + 1 = 0$$

So

$$\tau = \left(\frac{120}{1+2K_c}\right)^{1/2},\quad \zeta = \left(\frac{22}{1+2K_c}\right)\frac{1}{2}\left(\frac{1+2K_c}{120}\right)^{1/2} = \frac{11}{\sqrt{120}}\,\frac{1}{\sqrt{1+2K_c}}$$

Now ζ = 0.707, substitute in ζ expression; can find K_c = 0.51.

(b) The key is that we cannot use those empirical tuning relations based on first-order with dead time functions. Here, we may apply Example 6.2 (or Example 6.4 for a PI controller).

Say we choose τ_c = 3 [time unit] to be sufficiently faster than 10 and 12. Then with Example 6.2, we can calculate

$$K_c = \frac{10+12}{(2)(3)} = 3.7\ ,\quad \tau_I = 10+12 = 22\ ,\quad \tau_D = \frac{(10)(12)}{10+12} = 5.4_5$$

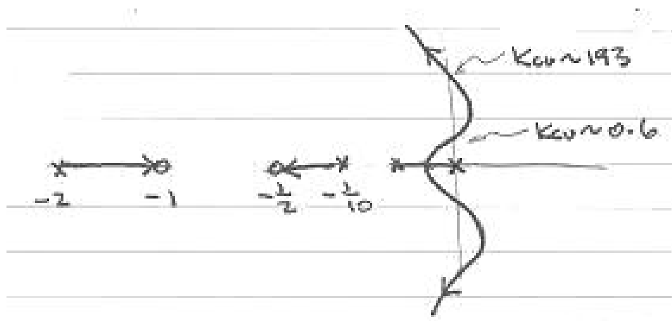
28. The closed-loop equation is

$$1+K_c \frac{1+\frac{1}{3s}+\frac{2}{3}s}{(10s+1)(20s+1)(0.5s+1)}=0$$

and $2s^2+3s+1=(2s+1)(s+1)$

So we have
open-loop zeros at –1/2, –1
open-loop poles at 0, –1/10, –1/20, –2

From MATLAB:



(b) $G_{PRC}=\frac{1}{(10s+1)(20s+1)(0.5s+1)} \approx \frac{e^{-10.5s}}{20s+1}$

(c) Only P control:

$$1+K_c \frac{1}{(10s+1)(20s+1)(0.5s+1)}=0$$

$$100s^3+215s^2+30.5s+1+K_c=0$$

Routh array:

$$\begin{array}{ll} 100 & 30.5 \\ 215 & 1+K_c \\ b_1 & \end{array}$$

So we need

$b_1=\frac{(215)(30.5)-100(1+K_c)}{215}>0$, leading to $K_c < 64.5$

We now have a "simple" third-order system and only one K_{cu} = 64.5.

(d) Apply the G_{PRC} approximation in Part (b) and Table 6.1. A matter of plug-and-chug or using the M-file "recipe.m."

(e) With the addition of derivative action, we can use a larger K_c.

29. Features of the plots:

- Phase lag goes from 0° to –250°. It appears that the phase lag would eventually reach –270° at very frequencies
- At large frequencies, the magnitude plot has slope approximately –3
- At frequency around ~ 4 rad/min, |G| is slightly larger than the value at very low frequencies

A logical guess is that

$$G(s) = \frac{K}{(\tau_1 s + 1)(\tau^2 s^2 + 2\tau\zeta s + 1)}$$

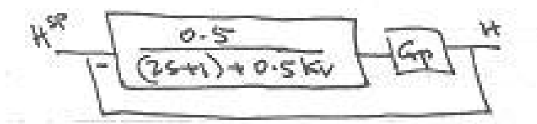
where

K ~ 1.1

ζ must be less than ½
$1/\tau$ is in between 4 and 10 rad/min
$1/\tau_1$ is between 1 and 10 rad/min; from the fact that the phase lag of G drops quite a bit "very soon" ($1/\tau_1$ is probably between 1 and 4)

For a closed-loop experiment, the magnitude and phase lag will be that of the closed-loop function C/R.

30. The block diagram in Fig. PII.30 is equivalent to



so the closed-loop characteristic equation is

$$1+\frac{0.5\,G_p}{2s+1+0.5\,K_v}=0$$

(a) For G_p, the model $A\frac{\mathrm{d}h'}{\mathrm{d}t}=q'_o$ should give us $G_p=\frac{1}{A}\,\frac{1}{s}=\frac{1}{2s}$ (A = 2 units)

So we have

$$1+\frac{0.5}{2s(2s+1+0.5\,K_v)}=0$$

$$4s^2+2(1+\frac{1}{2}K_v)s+0.5=0$$

$$8s^2+2(2+K_v)s+1=0$$

$$\tau=\sqrt{8}\ ,\ \text{and}\ \ \zeta=2(2+K_v)\,\frac{1}{2}\,\frac{1}{\sqrt{8}}$$

Now $\zeta = 1$ (critically damped), $2 + K_v = \sqrt{8}$, or $K_v = 0.82_8$

(b) $\zeta=(1/\sqrt{8})(2+K_v)$, so ζ is proportional to K_v, and the system is *less* underdamped if we increase K_v.

(c) Do not matter what K_v is. System has integrating action from G_p. There is no offset. Change in h will always match change in the set point.

31. The response at large times oscillates about ½ with an amplitude of approximately 0.34 and a phase lag of roughly 137°.

$$Y(s)=\frac{K}{s(\tau s+1)}\,\frac{A\omega}{s^2+\omega^2}=\frac{a}{s}+\frac{b}{\tau s+1}+\frac{cs+d}{s^2+\omega^2}$$

The first term on the far right (with coefficient a) contributes toward the constant, while the last term associates with the sinusoidal response. And the coefficient a is

$$a=\left.\frac{AK\omega}{(\tau s+1)(s^2+\omega^2)}\right|_{s=0}=\frac{AK\omega}{\omega^2}=\frac{AK}{\omega}$$

Given: ω = ½ rad/min [=1/2 (1/2π) cycle/s; can confirm this from plot]
Also from plot: A = 1 and a = ½, so

$$\frac{1}{2}=\frac{(1)(K)}{1/2}$$, or K = 1/4

With

$$G(s)=\frac{K}{s(\tau s+1)}\,,\quad \measuredangle G(j\omega)=-90^\circ+\tan^{-1}(-\tau\omega)$$

Now $\measuredangle G(j\omega)=-137^\circ$

$$-137^\circ=-90^\circ-\tan^{-1}(\tau\omega)$$, leading to $\tau\omega$ = 1/1.07

With ω = ½ rad/s, τ = 1.07/(1/2) = 2.14 s

Double check:

$$|G(j\omega)|=\frac{K}{\omega\sqrt{1+\tau^2\omega^2}}=\frac{0.25}{(1/2)\sqrt{1+(2.14)^2(1/2)^2}}=0.34$$, same as data

The very key is that the units of $\tau\omega$ are radians, and calculations are based on ω with the units of rad/time.

32. The characteristic equation is

$$1 + K_m G_c G_2 e^{-\frac{1}{2}s} G_5 = 0$$

where K_m = 0.15 V/°C, t_d = 0.5 min, $G_2 = \dfrac{4}{s+1}$ °C/V, and $G_5 = \dfrac{1}{10\,s+1}$

(a) Proportional control, $G_c = K_c$,

$$1 + K_c \frac{(0.15)(4)}{(s+1)(10\,s+1)} e^{-\frac{1}{2}s} = 0$$

To find the ultimate gain, we need to follow Example 7.4A in Chapter 8. From MATLAB, K_{cu} = 39.8 (with ω_{cg} = 1.39 rad/min).

Without transport lag, the system equation is

$$1 + K_c \frac{0.6}{(s+1)(10\,s+1)} = 0$$

It is a second order system that is always stable.

(b) $K_c = \dfrac{K_{cu}}{\text{GM}} = \dfrac{39.8}{2} = 19.9$
(Can check this with MATLAB; we should find also PM = 29.4°.)

(c) The basis of calculation is

$$G^* = \frac{0.6}{(s+1)(10\,s+1)} e^{-\frac{1}{2}s}$$

From MATLAB, at $\omega \approx$ 0.73 rad/min, $|G^*|$ = 0.066, $\angle G^*$ = –139° ≈ –140°

So we need K_c = 1/0.066 = 15.2 (Check this with MATLAB, the PM is 40.9°.)

(d) Plug and chug with Ziegler-Nichols tuning relation using K_{cu} = 39.8 and ω_{cg} = 1.39 rad/min

Should find K_c = 13.3, τ_I = 2.3 min, τ_D = 1.5 min
(Check with MATLAB using ideal PID and the equation

$$1 + K_c \left(1 + \frac{1}{\tau_I s} + \tau_D s\right) \frac{0.6}{(s+1)(10\,s+1)} e^{-\frac{1}{2}s} = 0 \text{ pg}$$

should find GM = 2.9, PM = 66°.)

(e) Approximate

$$G_{PRC} = \frac{0.6\, e^{-\frac{1}{2}s}}{(s+1)(10\,s+1)} \approx \frac{0.6\, e^{-\frac{3}{2}s}}{(10\,s+1)}$$

With the Ziegler-Nichols tuning relations, we found
K_c = 13.3, τ_I = 3 min, τ_D = 0.75 min
(Check this with MATLAB, GM = 5, PM = 45.5°.)

(f) Need to use MATLAB (or Simulink) for the simulations.

(g) Disturbance is Q_o,

$$G_{FF}=\frac{-G_3}{K_m G_2 G_5 e^{-\frac{1}{2}s}}\text{ , where }\quad G_3=\frac{-5\,e^{-\frac{1}{2}s}}{(10\,s+1)(s+1)}\quad \text{°C/gpm}$$

$$G_{FF}=\frac{5\,e^{-\frac{1}{2}s}}{(10\,s+1)(s+1)}\,\frac{(10\,s+1)(s+1)}{(0.15)(4)e^{-\frac{1}{2}s}}=\frac{5}{0.6}=8.3\quad \text{°C/gpm}$$

It is a steady-state compensator.

(h) Disturbance is V_2

$$G_{FF}=\frac{-G_4}{K_m G_2 G_5 e^{-\frac{1}{2}s}}\text{ , where }\quad G_4=\frac{2.5}{10\,s+1}\quad \text{°C/V}$$

Now $1/\exp(-\frac{1}{2}s)$ will become a time lead, $\exp(\frac{1}{2}s)$, and we need to redefine G_{FF} without it.

$$G_{FF}=\frac{-G_4}{K_m G_2 G_5}=\frac{-2.5}{10\,s+1}\,\frac{(10\,s+1)(s+1)}{(0.15)(4)}=-0.625\,(s+1)$$

In practice, we would either add a large pole as in Eq. (10.8) in text or simply omit the dynamic compensator and try first G_{FF} = –0.625 °C/V.

33. The process function

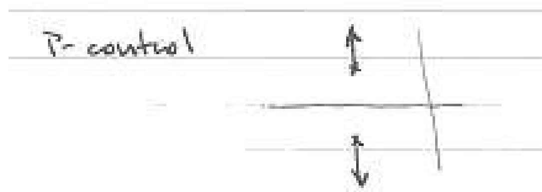
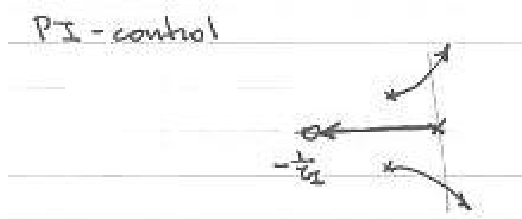
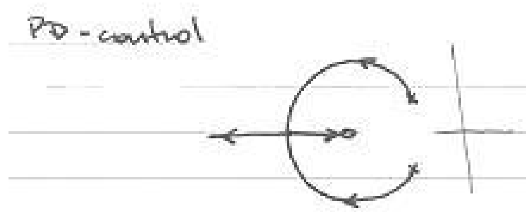
$$G_p = \frac{0.5}{0.01\,s^2 + 0.04\,s + 1}$$

has open-loop poles at $-2 \pm 9.8j$.

(a) With $\tau = 0.1$, $\zeta = (0.04)(1/2)(1/0.1) = 0.2$

The time constant is $\tau/\zeta = 0.1/0.2 = ½$, which is consistent with the pole

(b)

	System is stable but very underdamped and has offset; ζ of system < 0.2
	System can become unstable
	System is stable but has offset

Not allowing offset, we must use a PI controller. And we may want to select the larger τ_I = 1/4 s. (From the perspective of Chapter 8 analysis. τ_I has a corner frequency of 4 rad//s < 10 (1/0.1) of G_p and thus “eliminates” the integrator phase lag before the second order phase lag sets in.)

(c) The best is a PID controller. With $\tau_D < \tau_I$, we can choose τ_D = 1/8 s, with τ_I = 1/4 s. We do the root-locus analysis with the system equation

$$1 + K_c \left(1 + \frac{1}{\tau_I\, s} + \tau_D\, s\right) \frac{0.5}{0.01\,s^2 + 0.04\,s + 1} = 0$$

The dominant poles are the loci that come off the two complex open-loop poles.

34. The phase angle plot clearly indicates that presence of dead time. From the magnitude plot, the slope at high frequency is –1, so the function must be first order with dead time:

$$G(s)=\frac{K\,\mathrm{e}^{-\theta s}}{\tau s+1}$$

From the given corner frequency, $\tau = 1/2.5 = 0.4$ s
From the low frequency asymptote, $K = 5$

The phase angle equation is (in degrees)

$$\measuredangle G(j\omega)=-\tan^{-1}(\tau\omega)-\left(\frac{180}{\pi}\right)\theta\omega$$

Now $\tau = 0.4$ and given when $\omega = 1$ rad/s, and the phase lag = –33°

$$-33^{\circ}=-\tan^{-1}(0.4)-\left(\frac{180}{\pi}\right)\theta$$, so $\theta = 0.19$ s

and

$$G(s)=\frac{5\,\mathrm{e}^{-0.19 s}}{0.4 s+1}$$

This is one of several problems similar to Example 4.6. We do not really have to synthesize the state space model for each system as in the example. MATLAB can do that and much more, and there are more elegant approaches. However, these examples should help to take some of the mystery away and make us feel more comfortable in using canned packages or working with block diagrams.

II.35. The two equations that we can write are:

$$\frac{X_1}{X_2 + K_c(R - X_1)} = \frac{K_p}{\tau_p s + 1}\,, \quad \text{and} \quad \frac{X_2}{K_c(R - X_1)} = \frac{1}{\tau_I s}$$

They can be rearranged to appear as

$$\tau_p X_1\, s = -(1 + K_c K_p)\, X_1 + K_p X_2 + K_c K_p R\,, \text{ and } \tau_I X_2\, s = -K_c X_1 + K_c R$$

The resulting state space model is obvious from here on:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1 + K_p K_c}{\tau_p} & \dfrac{K_p}{\tau_p} \\ -\dfrac{K_c}{\tau_I} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + K_c \begin{bmatrix} \dfrac{K_p}{\tau_p} \\ \dfrac{1}{\tau_I} \end{bmatrix} r\,, \quad \text{and} \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The system matrix characteristic polynomial is

$$\det |sI - A| = \left(s + \frac{1 + K_c K_p}{\tau_p}\right) s + \frac{K_c K_p}{\tau_I \tau_p}$$

which is identical to what we'll get in Example 5.3.

This is one of several problems similar to Example 4.6. We do not really have to synthesize the state space model for each system as in the example. MATLAB can do that and much more, and there are more elegant approaches. However, these examples should help to take some of the mystery away and make us feel more comfortable in using canned packages or working with block diagrams.

II.36. The two equations that we can write are:

$$\frac{X_1}{X_2 + K_c(R - X_1)} = \frac{K}{s+a},$$

and

$$\frac{X_2}{K_c(R - X_1)} = \frac{z-p}{s+p}.$$

They can be rearranged to

$$sX_1 = -(a + K_c K)X_1 + KX_2 + K_c K R, \text{ and } sX_2 = -K_c(z-p)X_1 - pX_2 + K_c(z-p)R$$

The final state space model is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(a+K_c K) & K \\ -K_c(z-p) & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + K_c\begin{bmatrix} K \\ z-p \end{bmatrix} r, \text{ and } y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

You can further show that the system matrix characteristic equation is identical to

$$(s+p)(s+a) + K_c K(s+z) = 0$$

This problem follows the development in the two previous problems in II.35 and II.36.

II.37. With a serial PID, the derivative action is in the feedback path. The PI action in the forward path follows that of Problem II.35. Similarly, the derivative function, introduced in Eq. (5-7), is rearranged according to the lead-lag element in Problem II.36. With the locations of the state variables identified in the block diagram, we can write the four state equations:

$$\frac{X_1}{X_2} = \frac{K_p}{\tau_1 s + 1}, \text{ or } \tau_1 s X_1 = -X_1 + K_p X_2$$

$$\frac{X_2}{X_3 + K_c\left[R - \frac{1}{\alpha}(X_1 + X_4)\right]} = \frac{1}{\tau_2 s + 1}, \text{ or } \tau_2 s X_2 = -X_2 + X_3 - \frac{K_c}{\alpha}(X_1 + X_4) + K_c R$$

$$\frac{X_3}{K_c\left[R - \frac{1}{\alpha}(X_1 + X_4)\right]} = \frac{1}{\tau_I s}, \text{ or } \tau_I s X_3 = -\frac{K_c}{\alpha}(X_1 + X_4) + K_c R$$

and

$$\frac{X_4}{X_1} = \frac{1/\tau_D - 1/\alpha\tau_D}{s + 1/\alpha\tau_D}, \text{ or } sX_4 = -X_4/\alpha\tau_D + (1/\tau_D - 1/\alpha\tau_D)\, X_1$$

The resulting state space model is:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau_1} & \frac{K_p}{\tau_1} & 0 & 0 \\ -\frac{K_c/\alpha}{\tau_2} & -\frac{1}{\tau_2} & \frac{1}{\tau_2} & -\frac{K_c/\alpha}{\tau_2} \\ -\frac{K_c/\alpha}{\tau_I} & 0 & 0 & -\frac{K_c/\alpha}{\tau_I} \\ \frac{1}{\tau_D} - \frac{1}{\alpha\tau_D} & 0 & 0 & -\frac{1}{\alpha\tau_D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + K_c \begin{bmatrix} 0 \\ \frac{1}{\tau_2} \\ \frac{1}{\tau_I} \\ 0 \end{bmatrix} r \text{ , and } y = [1\ 0\ 0\ 0]\, \mathbf{x}$$

38.

(a) The closed-loop characteristic equation is

$$1+K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)\frac{2}{(4s+1)(5s+1)}=0$$

$$\tau_I s(20s^2+9s+1)+2K_c(\tau_I s+1)=0$$

$$20\tau_I s^3+9\tau_I s^2+\tau_I(1+2K_c)+2K_c=0$$

First, we need $2K_c > 0$, and $\tau_I(1 + 2K_c) > 0$,

meaning $K_c > 0$, and $\tau_I > 0$.

Next, with the Routh array,

$$\begin{array}{ll} 20\tau_I & \tau_I(1+2K_c) \\ 9\tau_I & 2K_c \\ b_1 & \\ 2K_c & \end{array}$$

So we need

$$b_1=\frac{9\tau_I^2(1+2K_c)-40\tau_I K_c}{9\tau_I}>0$$

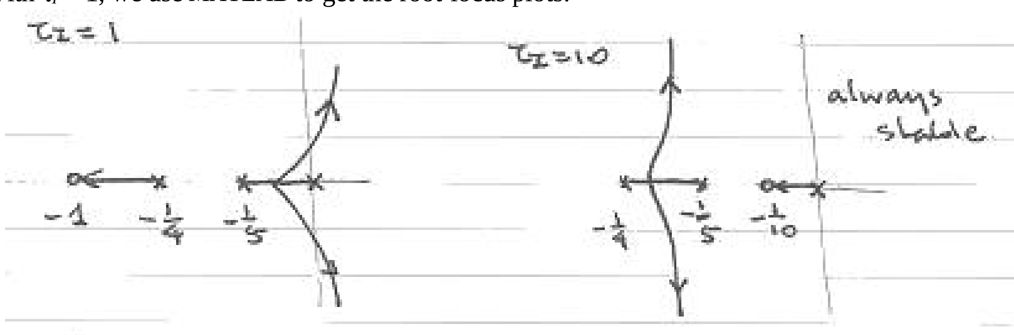
$$9\tau_I(1+2K_c)>40K_c$$

$$\tau_I>\frac{40K_c}{9(1+2K_c)}\text{, or }\frac{9\tau_I}{40-18\tau_I}>K_c$$

(b) When $\tau_I = 1$, we need $K_c<\dfrac{9}{40-18}=0.41$

And when $\tau_I = 10$, we need $\dfrac{9}{4}>\dfrac{K_c}{1+2K_c}$, which is always satisfied with $K_c > 0$; system is always stable.

(c) With $\tau_I = 1$, we use MATLAB to get the root-locus plots:



(d) When $\tau_I = 1$, from Part (b), $K_{cu} = 0.41$.

For GM = 2, we need $K_c = 0.41/2 = 0.205$.
To find PM, we need MATLAB. So from a Bode plot, we found PM = 14° (this is a bit low).

(e) With $\tau_I = 10$, the system is always stable.

(f) We need to do a root-locus plot. From MATLAB, for $\zeta = 1/\sqrt{2}$, $K_c \approx 0.033$.

The dominant poles are the two complex loci branching off from the real axis.

(g) With $\tau_I = 10$, again we need MATLAB and a root-locus plot. For a system with "embedded" response corresponding to $\zeta = 1/\sqrt{2}$, $K_c \approx 0.48$.

Here, the dominant pole is the loci on the negative real axis. The oscillations will damp out relatively quickly.

(h) We need MATLAB to do the Bode plots. With $\tau_I = 1$, the system equation is

$$1+K_c\left(\frac{s+1}{s}\right)\frac{2}{(4s+1)(5s+1)}=0$$

		Frequency asymptote limits	
Corner frequencies	Type	Log magnitude slope	Phase lag
–	integrator	–1	–90°
1/5	1st order lag	0 to –1	0 to –90°
1/4	1st order lag	0 to –1	0 to –90°
1	1st order lead	0 to +1	0 to +90°

The magnitude slope goes from –1 at low frequencies to –2 at high frequencies. In terms of the phase angle, the first-order lead comes in too late. The phase lag goes below –180° before the first-order lead brings the phase angle back to –180° at high frequencies.

With $\tau_I = 10$, the system equation is

$$1+K_c\left(\frac{10\,s+1}{10\,s}\right)\frac{2}{(4s+1)(5s+1)}=0$$

		Frequency asymptote limits	
Corner frequencies	Type	Log magnitude slope	Phase lag
–	integrator	–1	–90°
0.1	1st order lead	0 to +1	0 to +90°
1/5	1st order lag	0 to –1	0 to –90°
1/4	1st order lag	0 to –1	0 to –90°

Now, the first-order lead comes in and compensates for the integrator phase lag before the two first-order lags come in. The phase angle never crosses over –180°.

39.

(a) From Section 3.4.3 (text), $\tau_1 = A_1R_1$, $\tau_2 = A_2R_2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}=\begin{bmatrix}-1/\tau_1 & 1/\tau_1\\ (R_2/R_1)/\tau_2 & -(1+R_2/R_1)/\tau_2\end{bmatrix}\mathbf{x}+\begin{bmatrix}R_1/\tau_1\\ 0\end{bmatrix}u$$

where now $\mathbf{x} = [h_1\ \ h_2]^T$, and $u = q_o$.

The output is $y = [\ 0\ \ 1\ ]\ \mathbf{x}$.

With numerical values $A_1 = 5$ m^2, $A_2 = 2$ m^2, $R_1 = R_2 = 1$ min/m^2, $\tau_1 = 5$ min, and $\tau_2 = 2$ min,

$$\mathbf{A}=\begin{bmatrix}-1/5 & 1/5\\ 1/2 & -1\end{bmatrix}, \text{ and } \quad \mathbf{B}=\begin{bmatrix}1/5\\ 0\end{bmatrix}$$

(b) With det $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$(-\frac{1}{5}-\lambda)(-1-\lambda)-\frac{1}{10}=0$$

$$\lambda^2+\frac{6}{5}\lambda+\frac{1}{5}-\frac{1}{10}=0$$

$$10\,\lambda^2+12\,\lambda+1=0$$

The transfer function from Section 3.4.3 is

$$\frac{H_2(s)}{Q_o(s)}=\frac{1}{(5s+1)(2s+2)-1}=\frac{1}{10\,s^2+12\,s+1}$$

Since the characteristic equations are the same, the poles will too. (A quick quadratic root calculation will find them to be –0.90$_1$ and –1.1.)

(c) For proportional control, the characteristic equation is $1+K_c\dfrac{H_2}{Q_o}=0$, leading to

$$10\,s^2+12\,s+1+K_c=0$$

$$\frac{10}{1+K_c}s^2+\frac{12}{1+K_c}s+1=0$$

$$\tau=\left(\frac{10}{1+K_c}\right)^{1/2},\quad \zeta=\frac{12}{1+K_c}\,\frac{1}{2}\left(\frac{1+K_c}{10}\right)^{1/2}=\frac{6}{\sqrt{10}}\,\frac{1}{\sqrt{1+K_c}}$$

With $\zeta = 0.7$, $K_c = 6.35$ (with the poles at $-0.6\pm0.612j$)

(d) We can find the state feedback gain easily with the Ackermann's formula. (Details in the MATLAB M-file.)

(e) With PI control, the characteristic equation is

$$1+K_c\left(1+\frac{1}{\tau_I\,s}\right)\frac{1}{10\,s^2+12\,s+1}=0$$

with τ_I = 0.5 min, we can use root-locus plot to find that $K_c \approx 0.022$.

(f) and (g) – details also in MATLAB M-file.

But there is one important note regarding the time response simulation when we use the state feedback gain without integration. Here, x_2 (i.e., h_2) is the output, so we need to define $K_r = K_2$ such that Eq. (9.24) is

$$u(t) = -K_1 x_1 + K_2 (r - x_2)$$

and Eq. (9.25) becomes. in this problem,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B} K_2 r$$

40.

(a) Based on the given equation, the pairing is x_D-L and x_B-V, and the gain matrix is

$$\mathbf{K} = \begin{bmatrix} 0.6 & -0.5 \\ 0.3 & -0.4 \end{bmatrix}$$

The corresponding relative gain parameter is

$$\lambda_{x_D,L} = \frac{1}{1 - \frac{(-0.5)(0.3)}{(0.6)(-0.4)}} = 2.67 > 1$$

If we switch the pairing to x_D-V and x_B-L, the plant equation becomes

$$\begin{bmatrix} x_D \\ x_B \end{bmatrix} = \begin{bmatrix} \frac{-0.5\,e^{-\frac{1}{2}s}}{(7s+1)^2} & \frac{0.6}{(7s+1)^2} \\ \frac{-0.4}{(14s+1)(0.4s+1)} & \frac{0.3\,e^{-\frac{1}{2}s}}{(16s+1)(0.5s+1)} \end{bmatrix} \begin{bmatrix} V \\ L \end{bmatrix}$$

Now

$$\mathbf{K} = \begin{bmatrix} -0.5 & 0.6 \\ 0.4 & 0.3 \end{bmatrix}, \text{ and } \lambda_{x_D,V} = 1.67 > 1$$

If both cases, $\lambda > 1$, and from the values, it probably would not make that big of a difference in pairing. Nonetheless, we would choose the case with the slightly small λ (1.67) as the basis of the system design. The Simulink file is set up accordingly. It also includes the decouplers, which can be “turned off” by setting their gains to be zero.

(b) The first controller G_{c1} will be based on

$$G_{11} = \frac{-0.5\,e^{-\frac{1}{2}s}}{(7s+1)^2}$$

Using MATLAB and Bode plots, K_{cu} = 10.7 at ω_{cg} = 0.298 [rad/time unit]
If GM = 2, K_c = 10.7/2 = 5.35

The second controller G_{c2} is based on

$$\frac{0.3\,e^{-\frac{1}{2}s}}{(16s+1)(0.5s+1)}$$

From a Bode plot, K_{cu} = 126 at ω_{cg} = 1.76 [rad/time unit]
If GM = 2, K_c = 126/2 = 63

(c) From MATLAB, $|G_{11}|$ = 0.19 when $\angle G_{11} = -135°$
So we need K_c = 1/0.19 = 5.3

For G_{c2}, $|G_{22}|$ = 0.019_5 when $\angle G_{11} = -135°$
So we need K_c = 1/0.0195 = $51._2$

(d) Using Ziegler-Nichols tuning relation, and

K_{cu} = 10.7, ω_{cg} = 0.298, leads to K_c = 4.9, τ_I = 17.6 [time units]

K_{cu} = 126, ω_{cg} = 1.76, leads to K_c = 57, τ_I = 3 [time units]

(e) To detune the PI controller, reduce K_c and increase τ_I.

(f) For the two decoupling functions,

$$D_{21} = -\frac{G_{21}}{G_{22}} = \frac{0.4}{(14s+1)(0.4s+1)} \frac{(16s+1)(0.5s+1)}{0.3\,e^{-\frac{1}{2}s}}$$

We'll need to omit the exp(– ½ s) term because it will lead to advance in time. The time constants here are very similar and we may try to omit the dynamic terms. So we may begin with, simply,

$$D_{21} \approx \frac{0.4}{0.3} = 1.3$$

And with

$$D_{12} = \frac{-G_{12}}{G_{11}} = \frac{0.6}{(7s+1)^2} \frac{(7s+1)^2}{0.5\,e^{-\frac{1}{2}s}}$$

The $(7s + 1)^2$ term of course cancels out, and again we need to omit exp(– ½ s), so

$$D_{12} \approx \frac{0.6}{0.5} = 1.2$$

Those values are used to set up the Simulink file. In this problem, they help a bit, and if they are not set right, the system actually becomes unstable.

1. First, we need to get the transfer functions from the information given.

$$\frac{CO_2(s)}{Q(s)} = G_p = \frac{K_p}{\tau_p s + 1}\ ,\quad K_p = 0.23\ \frac{\text{ppm}}{\text{ml/min}}\ ,\quad \tau_p = 5\text{ min}$$

$$G_m = \frac{K_m}{\tau_m s + 1}\ ,\quad K_m = 5/400 = 0.0125\text{ V/ppm} = 12.5\text{ mV/ppm}\ ,\ \tau_m = 0.1\text{ min}$$

$$G_v = \frac{K_v}{\tau_v s + 1}\ ,\quad K_v = 0.2\ \frac{\text{ml/min}}{\text{mV}}\ ,\ \tau_m = 0.02\text{ min}$$

$$K_{amp} = 10\ \frac{\text{mV}}{\text{mV}}$$

We also have transport lag, $t_d = 0.75$ min in the feedback path.
The closed-loop characteristic equation is

$$1 + G_c K_{amp} G_v G_p G_m e^{-t_d s} = 0$$

With $K_{amp}K_vK_pK_m = 5.75$ [mV/mV], we have

$$1 + G_c\ \frac{5.75\, e^{-0.75\, s}}{(0.02\, s + 1)(5s + 1)(0.1\, s + 1)} = 0$$

(a) $G_c = K_c$. We define

$$G^* = \frac{5.75\, e^{-0.75\, s}}{(0.02\, s + 1)(5s + 1)(0.1\, s + 1)} = 0$$

From a plot of $|G^*|$ versus ω and $\angle G^*$ versus ω (i.e., Bode plot of G^*), the ultimate gain is $K_{cu} = 1.7$ at $\omega_{cg} = 1.9$ rad/min. Hence for system with GM = 1.7,

$K_c = K_{cu}/1.7 = 1$

(b) See the MATLAB file. Briefly, we use the Ziegler-Nichols ultimate gain tuning relations for a slight overshoot response. To further tune the controller, we cannot use root locus because of the dead time. And if we do not use techniques such as closed-loop log modulus, we just have to do a trial and error search to find $\zeta \approx 0.45$ (20% overshoot). But it is not as bad as it sounds because the Ziegler-Nichols tuning relations have settings that give us the K_c that would not have overshoot.

III.2.

(a) Key features from Fig. PIII.2a:
- High frequency asymptote of the magnitude plot has a slope approximately –2
- Phase lag varies from 0° to –180°
- Low frequency asymptote of the magnitude plot is 10 (= K_p)
- The magnitude curve rises above 10 before approaching the high frequency asymptote

All these observations are consistent with G_p being an underdamped second-order function.

From the magnitude plot, the corner frequency is approximately 0.3 rad/s. So

$1/\tau \approx 0.3$, or $\tau \approx 3.3$ s, and $\tau^2 \approx 11$

Also given that the function has a 25% overshoot in a unit step response experiment,
OS = 0.25 (or $\zeta \approx 0.4$). So

$2\zeta\tau \approx (2)(0.4)(3.3) = 2.7$

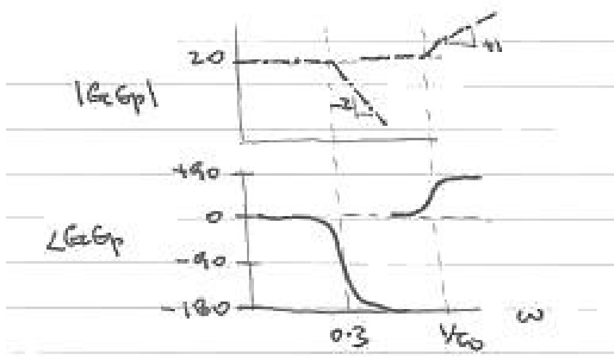
$$G_p = \frac{10}{11\,s^2 + 2.7\,s + 1}$$

(b) The plots now are for $|G_cG_p|$ and $\angle G_cG_p$.
- The phase angle now varies from 0° to –90° at very high frequencies. No phase change at low frequency, but G_c brings in phase lead at high frequencies; G_c must be a PD controller.
- The system is always stable.
- Compare the phase plots in (a) and (b), the corner frequency due to the PD controller ($1/\tau_D$) is likely to be higher than $1/\tau = 0.3$
- From the magnitude plot at low frequencies, $|G_cG_p| \approx 20$, so $K_c \approx 2$
- The slope of the high frequency asymptote in the magnitude plot is approximately –1 (no longer –2). The PD controller must be ideal, not real.

The key features can be summarized as:

	Corner	Frequency asymptotes	
Function	frequencies	Log magnitude slope	Phase lag
$1/(11s^2 + 2.7s + 1)$	≈ 0.3	0 to –2	0 to –180°
$(\tau_D s + 1)$	$1/\tau_D$	0 to +1	0 to +90°
Net value at very high frequencies		–1	–90°

The high and low frequency asymptotes of G_cG_p are shown next:



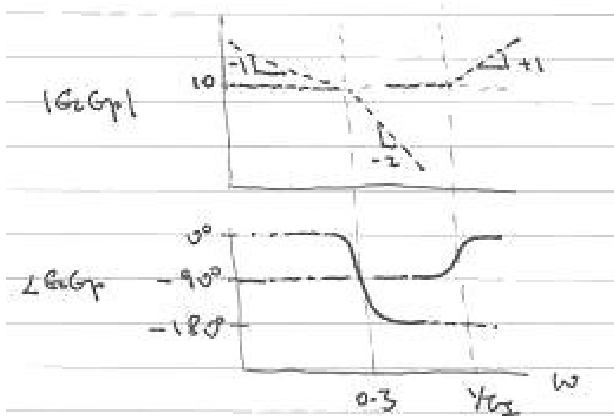
(c)

- The phase angle now begins at –90° at very low frequencies, then goes below –180° before approaching –180° at very high frequency. So the system can become unstable. We can jump to the conclusion that G_c is a PI controller. (If nothing else, the integrator contributes a constant –90°.)
- From the magnitude plot, the slope is about –1 at low frequencies, suggesting again the presence of an integrator. The slope at high frequency is about –2, consistent with a PI controller together with a second-order function.
- Comparing the phase plots in (a) and (c), it is likely that the corner frequency of the ($\tau_I s + 1$) term, $1/\tau_I$, is higher than $1/\tau = 0.3$.

The key features can be summarized as:

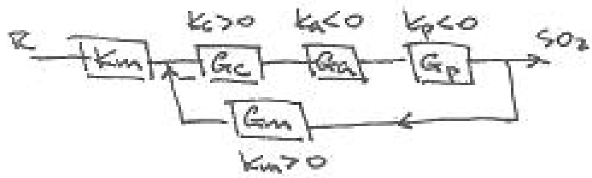
Function	Corner frequencies	Frequency asymptotes: Log magnitude slope	Phase lag
$1/s$	–	–1	–90°
$1/(11s^2 + 2.7s + 1)$	≈ 0.3	0 to –2	0 to –180°
$(\tau_I s + 1)$	$1/\tau_I$	0 to +1	0 to +90°
Net value at very high frequencies		–2	–180°

Asymptotes of G_c and G_p that may explain Fig. PIII.2c are sketched below:



3.

(a) The closed-loop equation is $1 + G_c G_a G_p G_m = 0$
Now, $K_m = +1$, $K_c > 0$, and $K_p = -0.05$ [ppm/gpm], so
K_a must be negative, or be -1 [gpm/mV], to have a negative feedback system.



If [SO_2] exceeds the set point, the controller output will decrease. But with a negative K_a, the action will increase the water flow to remove more SO_2 and bring its concentration back down.

(b) With the characteristic equation

$$1+K_c\left(\frac{\tau_I s+1}{\tau_I s}\right)(-1)\left(\frac{-0.05}{2s+1}\right)=0$$

we have a second-order system with no positive open-loop zeros. It is always stable for $K_c > 0$ (and $\tau_I > 0$).

(c) Same as Part (b). The system with a PD controller is always stable with $K_c > 0$, and $\tau_D > 0$. (You should confirm with a coefficient test or a root locus sketch.)

(d) With PD control, whether it is ideal or real (see Problem II.1), the system is always overdamped. (It is first-order if the PD is ideal.) To get underdamped behavior, we need a PI controller with $\tau_I < \tau_p$ (i.e., $\tau_I < 2$).

Now τ_I is given as 0.5 min in Part (b), so the closed-loop equation is

$$1+K_c\left(\frac{0.5\,s+1}{0.5\,s}\right)\left(\frac{0.05}{2s+1}\right)=0$$

To find K_c such that the system has a damping ratio of 0.7, the quick way is to use MATLAB to do a root-locus plot. With that, we found either
$K_c \approx 3.5$ with closed-loop poles at $-0.3 \pm 0.31j$, or
$K_c \approx 113$ with closed-loop poles at $-1.67 \pm 1.7j$
In actual application, we may saturate the system with $K_c = 113$. If so, we need to choose $K_c = 3.5$.

The slow way is to actually solve for K_c analytically. A couple of steps and the closed-loop characteristic equation should become

$$\left(\frac{1}{0.05\,K_c}\right)s^2+\left(\frac{0.05+0.025\,K_c}{0.05\,K_c}\right)s+1=0$$

from which we can find

$$\tau = \left(\frac{1}{0.05\,K_c} \right)^{1/2}, \text{ and } \zeta = \frac{1}{2}\,\frac{0.05 + 0.025\,K_c}{(0.05\,K_c)^{1/2}}$$

Substitute $\zeta = 0.7$, and after a couple of algebraic steps, we should find

$$(6.25 \text{ x } 10^{-4})\,K_c^2 - 0.073\,K_c + 0.25 = 0$$

The two solutions are $K_c = 3.53$ and 113.3.

(e) The problem statement implies that we now include

$$G_m = 1.2\,e^{-0.2\,s} \;\; \frac{\text{mV}}{\text{gpm}} \;;\;\; K_m = 1.2 \;\; \frac{\text{mV}}{\text{gpm}}$$

$$\text{and} \quad G_a = \frac{-0.9}{0.3\,s + 1} \;\; \frac{\text{mV}}{\text{gpm}}$$

And if we are to use the result from Part (d), we certainly want to use the much more conservative $K_c = 3.5$ because of the dead time in the system that was not accounted for before.

To do a simulation with $R = -10/s$, using Simulink is easy. To use `feedback()` and `step()`, we need to approximate the dead time with the Padé function and multiply the result with –10.

4.

(a) Both open-loop and closed-loop results point to the presence of an integrator in the process function. The system can be underdamped with only a proportional controller. Hence the process function must also have at least a first-order term. The simplest possible function that can explain the result is of the form:

$$G = \frac{K}{s(\tau s + 1)}$$

So even with just a proportional controller, we have a second-order system that has no offset.

(b) We write the process function as $Y/X = K/s(\tau s + 1)$ to explain the constant term in the sinusoidal response. With a given sinusoidal input, $X = \omega A/(s^2 + \omega^2)$, the response is

$$Y = \frac{K}{s(\tau s+1)} \frac{\omega A}{s^2+\omega^2} = \frac{\alpha_1}{s} + \frac{\alpha_2}{\tau s+1} + \frac{\alpha_3 s + \alpha_4}{s^2+\omega^2}$$

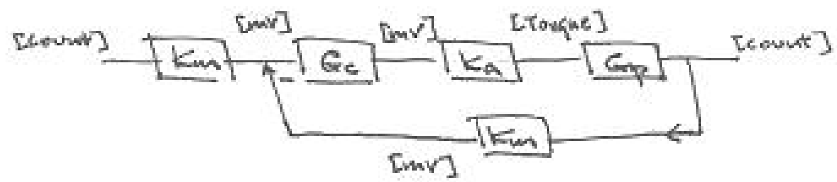
The last term on the right leads to sustained oscillations, while the second (the middle) term decays in time. The first term gives rise to the constant value

$$\alpha_1 = \left. \frac{K \omega A}{(\tau s+1)(s^2+\omega^2)} \right|_{s=0} = \frac{K A}{\omega}$$

This is the mean of the sinusoidal response in Fig. PIII.4. Since we know A and ω from the input, we can calculate K.

Also with the function $K/s(\tau s + 1)$, the integrator contributes a –90° lag and the first-order lag contributes another –90° at high frequencies to give a –180° total. The experimental results are consistent with these features. (The experimental procedures must use an actuator (G_a) and a sensor (G_m). That's why the functional form $K/s(\tau s + 1)$ is a lumped function that hides these details.)

(c) With a fast (relative to G_p) actuator and sensor, we take $G_a = K_a$, and $G_m = K_m$. Also $\tau = \tau_p$.



(d) Here,

$$G = \frac{K}{s(\tau_p s+1)} \;;\; K = K_m K_a K_p$$

(e) From the small Fig. PIII. 4(b), the phase lag is approximately 153°. Based on $G = K/s(\tau_p s + 1)$,

$$\measuredangle G = -153^\circ = -90^\circ - \tan^{-1}(\tau_p \omega)$$

The experiment used $\omega = 2(2\pi)$ rad/s, so $\tau_p = 0.16$ s

(f) With

$$|G| = K \frac{1}{\omega} \frac{1}{\sqrt{1+\tau_p^2 \omega^2}} \,,$$

we could have measured the amplitude of the normalized response in Fig. PIII.4(b), which is $|G|$, and with τ_p from Part (e), and ω chosen in doing the experiment, we can calculate K.

(g) To analyze the time response curve, we first need to go back to the closed-loop equation

$$1+K_c\,\frac{K}{s(\tau_p s+1)}=0\,,\ \text{with}\ K=K_m K_a K_p$$

$$\left(\frac{\tau_p}{K_c K}\right)s^2+\left(\frac{1}{K_c K}\right)s+1=0$$

In the form $\tau^2 s^2+2\zeta\tau s+1=0$

$$\tau=\left(\frac{\tau_p}{K_c K}\right)^{1/2}\,,\quad \zeta=\frac{1}{2}\left(\frac{1}{K_c K\,\tau_p}\right)^{1/2}$$

From the small Fig. PIII.4(a), the overshoot is roughly 0.5. So with OS = 0.5, we can find $\zeta = 0.215$.

With $\tau_p = 0.16$ s from Part (e) and $K_c = 0.02$ from the problem statement,
we can calculate $K = 17_{33} \approx 1700$.

(h) Take $K = 1733$, $\tau_p = 0.16$, $K_c = 0.02$,

$$\tau=\left(\frac{\tau_p}{K_c K}\right)^{1/2}=0.067$$

Estimation of settling time $T_s\approx\frac{4\tau}{\zeta}=\frac{(4)(0.067)}{0.215}\approx 1.3$

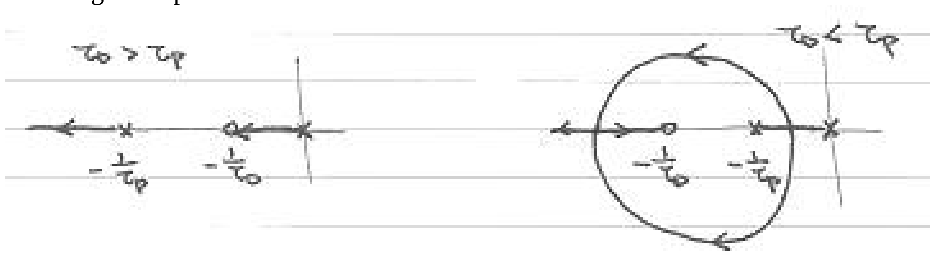
And time to peak $T_p=\frac{\pi\tau}{\sqrt{1-\zeta^2}}\approx 0.22$

These values are consistent (to one significant figure) with what we can estimate from Fig. PIII.4(a). We need the overshoot to calculate ζ. So of course, we get the overshoot back if we begin with ζ.

(i) With a PD controller, the system characteristic equation is

$$1+K_c(\tau_D s+1)\,\frac{K}{s(\tau_p s+1)}=0$$

There are two general possibilities:



Now $\tau_p \approx 0.16$ s from Part (e). This explains why the system does not oscillate when $\tau_D \approx 0.2$ (> 0.16), but underdamped behavior is observed when $\tau_D \approx 0.1$ (< 0.16).

(j) From Parts (d) and (g), we write the closed-loop characteristic function in the form

$$\frac{C}{R}=\frac{1}{\tau^2 s^2+2\zeta\tau s+1}$$

The magnitude is $\left|\frac{C}{R}\right|=\frac{1}{\sqrt{(1-\tau^2\omega^2)^2+(2\zeta\tau\omega)^2}}$

Now with $\tau \approx 0.067$, $\omega = 2(2\pi)$ rad/s, and $\zeta = 0.215$,

$$\left|\frac{C}{R}\right|=2.15\approx 2$$

Indeed the magnitude can be doubled at certain frequencies.

(k) The phase lag is $\measuredangle C/R=\tan^{-1}\left(\frac{-2\zeta\tau\omega}{1-\tau^2\omega^2}\right)$

If we choose ω to do an experiment and measure $|C/R|$ and $\angle C/R$, we theoretically have 2 equations (also magnitude equation in Part j) with 2 unknowns τ and ζ. With two highly nonlinear equations. However, the exercise of solving them would not be that much fun.

(l) We certainly want to use PD control to improve the system response. If we use direct synthesis,

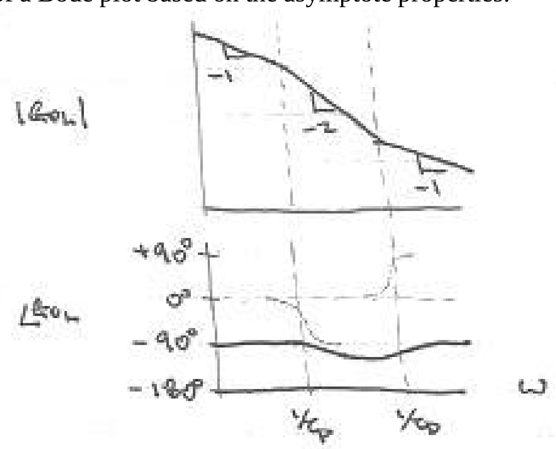
$$G_c=\frac{s(\tau_p s+1)}{K}\,\frac{1}{\tau_c s}=\frac{1}{K\tau_c}(\tau_p s+1)$$

which also is a PD controller.

(m) With a PD controller, the system is always stable (see Part (i) root locus) and we do not need to consider stability criteria. The Bode plot of the system will be based on

$$G_{OL}=\frac{K}{s}\,(\tau_D s+1)\,\frac{1}{(\tau_p s+1)}$$

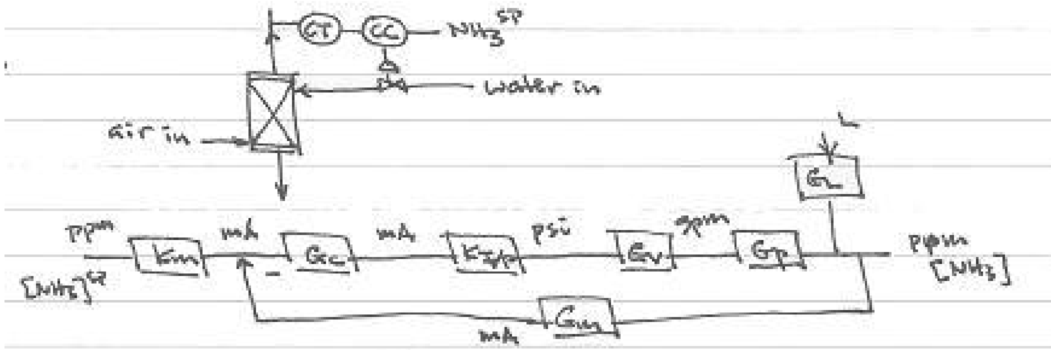
It has corner frequencies $1/\tau_D$ and $1/\tau_p$. And we want $\tau_D < \tau_p$ such that we have an underdamped system, $1/\tau_D > 1/\tau_p$. The sketch of a Bode plot based on the asymptote properties:



(n) We have a second-order system and we could find the answer analytically. Here, after so many parts, we'll take the easy route and use a root locus plot. So from MATLAB, we found $K_c \approx 0.003$ for a system with $\zeta = 0.7$.

(o) To make the system less underdamped, the easiest is to reduce K_c. Otherwise, we need to choose a larger τ_D and repeat the calculations.

5.
(a)



(b)

Controlled variable = measured variable = $[NH_3]$ in air outlet stream (ppm)
Manipulated variable = water inlet flow rate (gpm)
Disturbance possibilities are air inlet flow, and NH_3 inlet concentration

$$G_m = K_m = \frac{20-4}{200} = 0.08 \quad \frac{\text{mA}}{\text{ppm}}$$

$$K_{I/P} = \frac{15-3}{20-4} = 0.75 \quad \frac{\text{psi}}{\text{mA}}$$

$$K_v = \frac{-500}{10} = -50 \quad \frac{\text{gpm}}{\text{psi}}$$

(We need K_v negative as we'll see that K_p is negative.) Further, we need a fail-open or air-to-close valve for safety. So

$$G_v = \frac{-50}{5s+1} \quad \frac{\text{gpm}}{\text{psi}}$$

(c) From the units given the data are for the process function G_p only. The NH_3 concentration is the actual measurement, so the deviation is $[NH_3] - [NH_3]_{s.s.}$, where $[NH_3]_{s.s} = 50$ ppm.

For the plot (see MATLAB statements in the M-file), we approximate the process itself as a first-order with dead time function. The dead time is approximately 25 s, and the time constant is approximately 55 s.

The steady state gain is (51.77 – 50)/(–50) = –0.0354 (ppm/gpm). So

$$G_p = \frac{-0.0354\, e^{-25s}}{55s+1} \quad \frac{\text{ppm}}{\text{gpm}}$$

(d) So the process gain K_p is negative and if the controller gain K_c is also positive, then the actuator (valve) gain K_v must be negative, as we did in Part (b).

(e) One possibility is to use empirical tuning. Here, $G_{PRC} = K_{I/P} K_m G_v G_p$. So

$$G_{PRC} = \frac{K_{I/P} K_m K_v K_p}{(5s+1)(55s+1)} e^{-25s} \approx \frac{(K_{I/P} K_m K_v K_p) e^{-30s}}{55s+1}$$

With this first-order with dead time process reaction curve function, we can easily find the controller settings (see MATLAB statements in the M-file).

(f) and (g). See MATLAB Statements.
For example, if we use the ITAE settings, we have a 44% overshoot and we'll have to detune the controller by, for example, reducing K_c.

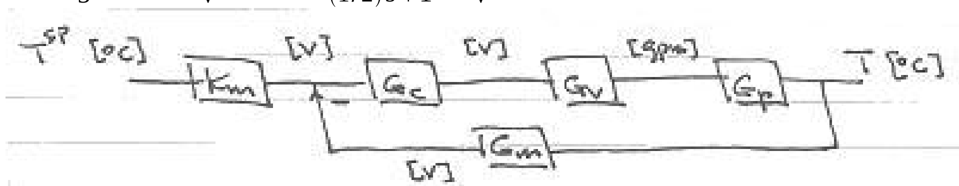
6.

(a) Use the cooling jacket, which has a much larger steady state gain. It takes about 10 min to reach steady state. So the time constant is approximately 2 min, consistent with other information given. So we have

$$G_p = \frac{-5}{2s+1} \quad \frac{^\circ\mathrm{C}}{\mathrm{gpm}}$$

$$K_m = \frac{5-0}{120-70} = 0.1 \quad \frac{\mathrm{V}}{^\circ\mathrm{C}}\ ; \quad G_m = \frac{0.1}{(1/4)s+1} \quad \frac{\mathrm{V}}{^\circ\mathrm{C}}$$

$$K_v = \frac{(2)(10)}{5} = 4 \quad \frac{\mathrm{gpm}}{\mathrm{V}}\ ; \quad G_v = \frac{4}{(1/2)s+1} \quad \frac{\mathrm{gpm}}{\mathrm{V}}$$



(b) We want a fail-open valve to ensure the temperature in the reactor may stay low. If the controller gain K_c is positive, a rise in temperature above the set point will lead to lower controller output. So we need an air-to-close valve to increase coolant flow rate. Hence we really need K_v to be negative and G_v should be

$$G_v = \frac{-4}{0.5s+1} \quad \frac{\mathrm{gpm}}{\mathrm{V}}$$

(c) Possible disturbances are reactant flow, reactant concentration, inlet temperature, and water flow in the condenser. Transfer functions are given in Parts (a) and (b) above.

(d) The closed-loop characteristic equation is

$$1 + K_c\left(\frac{-4}{0.5s+1}\right)\left(\frac{-5}{2s+1}\right)\left(\frac{0.1}{0.25s+1}\right) = 0$$

Expanding,

$$s^3 + \frac{13}{2}s^2 + 11s + 4(1+2K_c) = 0$$

We must have $(1 + 2K_c) > 0$, or $K_c > -1/2$

Routh array:

$$\begin{array}{cc} 1 & 11 \\ 13/2 & 4(1+2K_c) \\ b_1 & \\ 4(1+2K_c) & \end{array}$$

So we also must have

$$b_1 = \frac{(13/2)(11) - 4(1+2K_c)}{(13/2)} > 0$$

$$\frac{(13)(11)}{8} > 1 + 2K_c$$, or $K_c < 8.44$

So for positive K_c, the stability criterion is $0 < K_c < 8.44$

(e) From a root locus plot using MATLAB, we found $K_{cu} \approx 8.5$ at $\pm 3.3j$.
Overshoot of 5% means $\zeta = 0.69$.
Again from a root locus plot, we found $K_c \approx 0.66$.

(f) Here $G_{PRC} = G_v G_p G_m \approx \dfrac{2\,e^{-0.5\,s}}{2\,s+1}$

With this process reaction curve function, we can use empirical tuning relations to find the controller settings.

(g) We can use IMC too because we can apply the method to a first-order with dead time function (Example 6.5).

(h) With a Bode plot of $G^* = G_v G_p G_m$, we found $K_{cu} = 8.5$ at $\omega_{cg} = 3.3$ rad/min.

(i) Here, we use K_{cu} and ω_{cg} with the Ziegler-Nichols ultimate gain tuning relations.
For slight overshoot, $K_c = 2.8$, $\tau_I = 0.95$ min, $\tau_D = 0.63$ min.

(j) The closed-loop characteristic equation is now

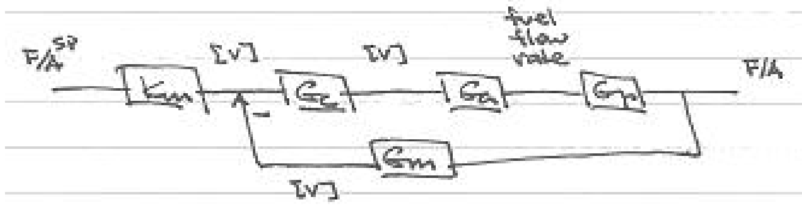
$$1+K_c\left(1+\frac{1}{\tau_I\,s}+\tau_D\,s\right)\left(\frac{-4}{0.5\,s+1}\right)\left(\frac{-5}{2\,s+1}\right)\left(\frac{0.1}{0.25\,s+1}\right)=0$$

With $\tau_I = 0.95$ min, $\tau_D = 0.63$ min, the PID controller contributes a +180° lead at low enough frequencies. The system has a net –90° lag at high frequencies. It is always stable.

(k) See MATLAB statements in the M-file for the simulation.

7.

(a)



(b) With a sensor that is sensitive to the fuel-air ratio and its fluctuations (noise measurement), we should omit derivative action and use only a PI controller.

(c) In this system, the measurement gain K_m (slope in Fig. PIII.7b) varies from extremely large about the steady state value to almost zero away from it. So if the fuel-air ratio deviates just a bit from steady state, we “lose” the feedback signal. The system operates almost on an on-off basis.

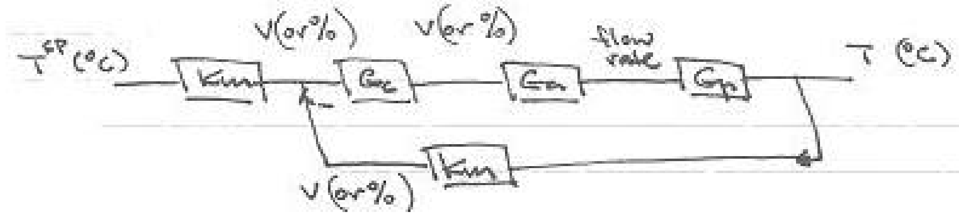
In this system design, we'll not estimate K_m from Fig. PIII.7b. Instead, we'll lump K_m together with K_c, on the presumption that some day, we may learn to design an adaptive controller that can adjust K_c according to the instantaneous value of K_m. So the closed-loop characteristic equation is

$$1+(K_m K_c)\left(1+\frac{1}{\tau_I s}\right) 0.5\left(\frac{1}{0.02\,s+1}+\frac{1}{s+1}\right)\frac{e^{-0.2\,s}}{0.1\,s+1}=0$$

where the sensor transfer function is split into two parts: the gain is lumped with K_c, separated from the dynamic part. See MATLAB statements in the M-file for details.

8.

(a)



The odd situation that we face is that the controller output change is given only as 5%. We could assume that its full range is 0–1 V as is the transmitter, such that

$$K_m = \frac{1-0\ \text{V}}{500-300\ ^\circ\text{C}} = 0.005 \quad \frac{\text{V}}{^\circ\text{C}} = 5\ \frac{\text{mV}}{^\circ\text{C}}$$

Another approach is simply to use “%” as a unit for both the transmitter and the controller. So

$$K_m = \frac{100\ \%}{500-300\ ^\circ\text{C}} = \frac{1}{2} \quad \frac{\%}{^\circ\text{C}}$$

But we do not really need to use K_m explicitly; its value is embedded in the PRC data. From Fig. PIII.8, we can estimate

$$G_{PRC} = G_a G_p G_m \approx \frac{K\,\text{e}^{-t_d s}}{\tau s + 1}$$

where $K = K_a K_p K_m$. From the figure,

$$K = \frac{445-425\ ^\circ\text{C}}{5\ \%} = 4 \quad \frac{^\circ\text{C}}{\%}\,,\quad t_d \approx 2 \text{ min, and } \tau = 6 \text{ min}$$

(b) and (d).

With the first-order with dead time function, we can calculate controller settings based on IMC or empirical tuning relations. (See MATLAB statements.)

(c) The closed-loop equation is

$$1 + K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right)\frac{4\,\text{e}^{-2s}}{6s+1} = 0$$

and with τ_I = 7 min, τ_D = 0.86 min chosen with IMC in Part (b), and using Bode plots, we need K_c = 1.16 to have PM = 30° (See MATLAB statements).

(e) See MATLAB statements in the M-file for the simulation.

9.

(a) With T_i, F, and C_A taken as constants, T_c is the only input to the process equation, and the temperature dependence is the only nonlinear term. And its linearized approximation is

$$e^{-E/RT} \approx e^{-E/RT_o} + \left(\frac{E}{RT_o^2}\right) e^{-E/RT_o}(T - T_o)$$

In terms of deviation variable $T' = T - T_o$, the linearized energy balance is

$$\frac{dT'}{dt} = \frac{-F}{V} T' + \frac{(-\Delta H)C_A}{\rho C_p} k_o e^{-E/RT_o}\left(\frac{E}{RT_o^2}\right) T' - \frac{U A_t}{\rho C_p V}(T' - T'_c)$$

$$\frac{dT'}{dt} + \left(\frac{F}{V} + \frac{U A_t}{\rho C_p V} - \frac{(-\Delta H)C_A}{\rho C_p} k_o e^{-E/RT_o} \frac{E}{RT_o^2}\right) T' = \frac{U A_t}{\rho C_p V} T'_c$$

leading to

$$\frac{dT'}{dt} + a T' = K T'_c$$

where

$$a = \left(\frac{F}{V} + \frac{U A_t}{\rho C_p V} - \frac{(-\Delta H)C_A}{\rho C_p} k_o e^{-E/RT_o} \frac{E}{RT_o^2}\right), \text{ and } \quad K = \frac{U A_t}{\rho C_p V}$$

After Laplace transform of the equation in deviation variables,

$$\frac{T(s)}{T_c(s)} = \frac{K}{s+a} = G_p(s)$$

(b) For the process to be stable, we need $a > 0$,

$$\frac{F}{V} + \frac{U A_t}{\rho C_p V} > \frac{(-\Delta H)C_A}{\rho C_p} k_o e^{-E/RT_o} \frac{E}{RT_o^2}$$

which can loosely be interpreted that the heat removal rate must be larger than the heat generation rate by the chemical reaction.

(c) With the data given, $a = -0.123 < 0$. So the chemical reactor is unstable. (Calculation details are in the MATLAB statements.)

(d) With the data given, we also find $K = 0.0134$. So

$$G_p = \frac{0.0134}{s - 0.123}$$

The closed-loop characteristic equation with proportional control is

$$1 + K_c \frac{K}{s+a} = 0 \text{ , or } \; s = -(KK_c + a)$$

For stability, we need $(KK_c + a) > 0$, or $K_c > a/K$
Substitute the numerical values, we need $K_c > 9.18$

(e) and (f)

We need integral control to eliminate offset. Thus we need a PI controller (and a PID if we need a bit more flexibility). See MATLAB statements for details.

10. We first need to linearize the nonlinear terms:

$$D(S_i - S) \approx D^s(S_i^s - S^s) + (S_i^s - S^s)D' + D^s S_i' - D^s S'$$

$$\frac{S}{K_m + S} \approx \frac{S^s}{K_m + S^s} + \frac{K_m}{(K_m + S^s)^2} S'$$

So the linearized equation in deviation variable is

$$\frac{\mathrm{d}S'}{\mathrm{d}t} = (S_i^s - S^s)D' + D^s S_i' - D^s S' - \frac{\mu_m C}{Y} \frac{K_m}{(K_m + S^s)^2} S'$$

$$\frac{\mathrm{d}S'}{\mathrm{d}t} + \left(D^s + \frac{\mu_m C}{Y} \frac{K_m}{(K_m + S^s)^2} \right) S' = (S_i^s - S^s)D' + D^s S_i'$$

$$\tau_p \frac{\mathrm{d}S'}{\mathrm{d}t} + S' = K_p D' + K_L S_i'$$

where

$$\tau_p = \frac{1}{a}, \quad K_p = \frac{S_i^s - S^s}{a}, \quad K_L = \frac{D^s}{a}, \text{ and } \quad a = D^s + \frac{\mu_m C}{Y} \frac{K_m}{(K_m + S^s)^2}$$

Choosing D' as the manipulated variable, and after Laplace transform, the process function is

$$\frac{S(s)}{D(s)} = \frac{K_p}{\tau_p s + 1} = G_p(s)$$

Next, with the data and equations given, we find the steady state values

$S^s = (1 - 0.95)\, S_i = 0.5$ g/L

$D^s = \mu_m S^s/(K_m + S^s) = 0.364$ h^{-1}

$C = Y(S_i^s - S^s) = 3.8$ g/L

and finally, $K_p = 9.6$ g·h/L and $\tau_p = 1$ h.

The closed-loop characteristic equation of the system is

$$1 + G_c \left(\frac{3}{0.06\,s + 1} \right) \left(\frac{9.6}{s + 1} \right) \left(\frac{e^{-0.15\,s}}{0.24\,s + 1} \right) = 0$$

With dead time in the system, we should use frequency response analysis to find K_{cu} and ω_{cg}. After that, we can use Ziegler-Nichols tuning relation to find the PID settings and tune the response with simulation. (See MATLAB statement for details.)

11.

(a) From the units of Fig. PIII.11, the plot gives us G_aG_p (the K_m has to be factored out). Let's consider

$$G_a G_p = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\ , \quad \text{where } K = K_a K_p$$

with $G_m = K_m$, and a proportional controller, $G_c = K_c$, the closed-loop transfer function is

$$\frac{C}{R} = \frac{K_c K_m K}{\tau^2 s^2 + 2\zeta\tau s + 1 + K_c K_m K}$$

So the system steady state gain is $(K_c K_m K)/(1 + K_c K_m K)$,

and the offset is $1 - \{(K_c K_m K)/(1 + K_c K_m K)\} = 1/(1 + K_c K_m K)$

Next, we find the numerical values. From Fig. PIII.11, we approximate

Overshoot = (2.5 – 1.5)/1.5 = 0.67
Oscillation period $T \approx 0.32$ s

From these values, we find $\zeta \approx 0.13$, and $\tau \approx 0.05$.
And from the fact that offset is 0.4 and when $K_c = 1$,

$$0.4 = \frac{1}{1 + K_m K}\ , \quad \text{leading to } K_m K = 1.5$$

(From Fig. PIII.11, $K = K_aK_p$ = (1.5 cm)/(0.2 V) = 7.5 cm/V. Hence, K_m = 1.5/7.5 = 0.2 V/cm)

In this problem, all we need is $K_mK = K_mK_aK_p = 1.5$, and

$$G_a G_p G_m = \frac{1.5}{(0.05)^2 s^2 + 2(0.13)(0.05)s + 1}$$

(b) The function in Part (a) is only approximate. The system can only become unstable if it is third order in this problem. The closed-loop characteristic equation is

$$1 + K_c\, \frac{1.5}{(\tau_a s + 1)(\tau^2 s^2 + 2\zeta\tau s + 1)} = 0$$

When GM = 1, $K_c = K_{cu}$, and using the magnitude of the "open-loop" function, we can write

$$1 = \frac{K_{cu}(1.5)}{\sqrt{(1 + \tau_a^2\omega^2)}\ \sqrt{(1 - \tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}}$$

Substitute K_{cu} = 1.65, $\omega = \omega_{cg}$ = 36.6 rad/s, τ = 0.05, ζ = 0.13, we can find τ_a = 0.007 s

(c) The time constant of the second-order function is

τ/ζ = 0.38 s >> τ_a = 0.007 s

So indeed the influence of the $1/(\tau_a s + 1)$ term is masked in the open-loop response experiment (see MATLAB statements for the plotting).

(d) To find a set of PID controller settings, we cannot use those relations that depend on a first-order with dead time process reaction curve function. But we can use the Ziegler-Nichols ultimate gain relations with K_{cu} and ω_{cg}. (See MATLAB statements for details.)

And if we use an ideal PID function, it is possible to have complex open-loop zeros. In fact, such a design may allow us to have a faster and less oscillatory system response.

12.

(a) For the first CSTR,

$$V\,\frac{\mathrm{d}C_1}{\mathrm{d}t}=Q(C_o-C_1)-V\,K\,C_1^2$$

We need to linearize the nonlinear terms first:

$$Q\,C_o\approx Q^s C_o^s+Q^s C_o{}'+C_o^s Q'$$

$$Q\,C_1\approx Q^s C_1^s+Q^s C_1{}'+C_1^s Q'$$

$$C_1^2\approx(C_1^s)^2+2\,C_1^s C_1{}'$$

Substitute these expansions and cancel out the steady state terms:

$$V\,\frac{\mathrm{d}C_1{}'}{\mathrm{d}t}=(C_o^s-C_1^s)Q'+Q^s C_o{}'-Q^s C_1{}'-(2\,V\,k\,C_1^s)C_1{}'$$

Define $\tau = V/Q^s$,

$$\tau\,\frac{\mathrm{d}C_1{}'}{\mathrm{d}t}+(1+2\tau\,k\,C_1^s)C_1{}'=C_o{}'+\left(\frac{C_o^s-C_1^s}{Q^s}\right)Q'$$

$$\tau_{p1}\,\frac{\mathrm{d}C_1{}'}{\mathrm{d}t}+C_1{}'=K_{11}C_o{}'+K_{21}Q'$$

where the time constant and steady-state gains are

$$\tau_{p1}=\frac{\tau}{1+2\tau\,k\,C_1^s}\;,\quad K_{11}=\frac{1}{1+2\tau\,k\,C_1^s}\;,\;\text{ and }\quad K_{21}=\frac{(C_o^s-C_1^s)/Q^s}{1+2\tau\,k\,C_1^s}$$

After Laplace transform,

$$C_1(s)=\left(\frac{K_{11}}{\tau_{p1}s+1}\right)C_o(s)+\left(\frac{K_{21}}{\tau_{p1}s+1}\right)Q(s)$$

(b) The equations for the other two CSTRs are

$$V\,\frac{\mathrm{d}C_2}{\mathrm{d}t}=Q(C_1-C_2)-V\,K\,C_2^2$$

$$V\,\frac{\mathrm{d}C_3}{\mathrm{d}t}=Q(C_2-C_3)-V\,K\,C_3^2$$

as V and k (constant temperature) are the same in all CSTRs. Following the derivation in Part (a), and by induction, we should arrive at

$$C_2(s)=\left(\frac{K_{12}}{\tau_{p2}s+1}\right)C_1(s)+\left(\frac{K_{22}}{\tau_{p2}s+1}\right)Q(s)$$

$$C_3(s)=\left(\frac{K_{13}}{\tau_{p3}s+1}\right)C_2(s)+\left(\frac{K_{23}}{\tau_{p3}s+1}\right)Q(s)$$

where

$$\tau_{p2} = \frac{\tau}{1 + 2\tau k C_2^s}\ , \quad \tau_{p3} = \frac{\tau}{1 + 2\tau k C_3^s}$$

$$K_{12} = \frac{1}{1 + 2\tau k C_2^s}\ , \quad K_{22} = \frac{(C_1^s - C_2^s)/Q^s}{1 + 2\tau k C_2^s}$$

$$K_{13} = \frac{1}{1 + 2\tau k C_3^s}\ , \quad K_{23} = \frac{(C_2^s - C_3^s)/Q^s}{1 + 2\tau k C_3^s}$$

The numerical values are handled in the MATLAB statements. From the steady state gain values, the inlet concentration C_o will be a more effective manipulated variable than Q. So the process function can be written as

$$G_p = \frac{C_3(s)}{C_1(s)} = \left(\frac{K_{11}}{\tau_{p1}s + 1}\right)\left(\frac{K_{12}}{\tau_{p2}s + 1}\right)\left(\frac{K_{13}}{\tau_{p3}s + 1}\right)$$

From the MATLAB statements, it appears numerically as

$$G_p = \left(\frac{0.28}{0.55\,s + 1}\right)\left(\frac{0.4}{0.8\,s + 1}\right)\left(\frac{0.5}{s + 1}\right)$$

(c) So now C_o is the manipulated variable, Q is the disturbance, and C_3 is the controlled variable. The process function is given at the end of Part (b).

(d) The closed-loop characteristic equation is $1 + K_cG_p = 0$, or

$$1 + K_c\,\frac{0.056}{(0.55\,s + 1)(0.8\,s + 1)(s + 1)} = 0$$

$$0.446\,s^3 + 1.81\,s^2 + 2.36\,s + (1 + 0.056\,K_c) = 0$$

We must have $1 + 0.056\,K_c > 0$, or $K_c > -17.9$ (For positive K_c, that just means $K_c > 0$)
And with the Routh array,

$$\begin{array}{cc} 0.446 & 2.36 \\ 1.81 & 1 + 0.056\,K_c \\ b_1 & \\ 1 + 0.056\,K_c & \end{array}$$

We need

$$b_1 = \frac{(1.8)(2.36) - 0.45\,(1 + 0.056\,K_c)}{1.82} > 0$$

So for stability, we need $0 < K_c < 153$ (for positive K_c), and $K_{cu} = 153$.
For GM = 2, we need $K_c = 153/2 = 76.6$

(e) The closed-loop characteristic equation is now

$$1 + K_c\left(1 + \frac{1}{\tau_I\,s}\right)\frac{0.056}{(0.55\,s + 1)(0.8\,s + 1)(s + 1)} = 0$$

Intuitively, we should choose the larger τ_I to have a more stable system. Indeed, we can see from root locus plots (see MATLAB statements) that with $\tau_I = 2$ min, the system is always stable.

(f) So we choose τ_I = 2 min to continue. With the characteristic equation in Part (e) and frequency response analysis, we find K_{cu} = 97.7. So for GM = 2, we need K_c = 97.7/2 = 48.9.

(g) Now we need to use a root locus plot. From MATLAB, to have a system ζ = 0.7, we need K_c = 9.75.

(h) See MATLAB statements for the comparative plot.

(i) This is just a matter of plug and chug with the empirical tuning relations. See the MATLAB statements for the calculation and time response simulation.

(j) If we go all the way back to Part (d) and apply K_{cu} and ω_{cg} to the Ziegler-Nichols tuning relations, we'd find that the tuning relations would recommend τ_D = 0.3 or 0.9 in this problem. If we repeat the exercise in the MATLAB statements of this problem, we'd see that the point of this Part is to see when a controller brings in phase lead.

With τ_D = 0.3 min, the derivative action does not bring in the phase lead soon enough and so only τ_I = 2 min (lower corner frequency ½) can stabilize the system. If we had chosen τ_D = 0.9 min, its corner frequency is low enough that its phase lead can stabilize the system even when τ_I = 0.5 min.

(k), (l), and (m)
Work that really needs MATLAB. See the MATLAB statements for details.

(n) Need the differential equations back in Parts (a) and (b). With only C_o as the input, we can omit all the terms associated with Q. For

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u \;; \quad y = \mathbf{C}\,\mathbf{x}$$

We have now

$$\mathbf{x} = \begin{bmatrix} C_1' \\ C_2' \\ C_3' \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1/\tau_{p1} & 0 & 0 \\ K_{12}/\tau_{p2} & -1/\tau_{p2} & 0 \\ 0 & K_{13}/\tau_{p3} & -1/\tau_{p3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} K_{11}/\tau_{p1} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

The numerical values are computed in the MATLAB statements.

(o) This is in the MATLAB statements too.

(p) We can compute that $\mathbf{C_o}$ = [$\mathbf{B}$ $\mathbf{AB}$ $\mathbf{A}^2\mathbf{B}$] and $\mathbf{O_b}$ = [$\mathbf{C}$ $\mathbf{CA}$ $\mathbf{CA}^2$]^T are both of rank 3.

(q) This is a mater of applying the Ackermann's formula. See the MATLAB statements for details.

(r) and (s). They are also in the MATLAB statements.