Chapter 4: Differentiation

Part A: Rules of Differentiation

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Definition of Derivative

A function $f\colon I \to \mathbb{R}$, where I is an open interval, has **derivative** m **at a point** $a \in I$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a) - m(x - a)| \leq \epsilon|x - a|$.

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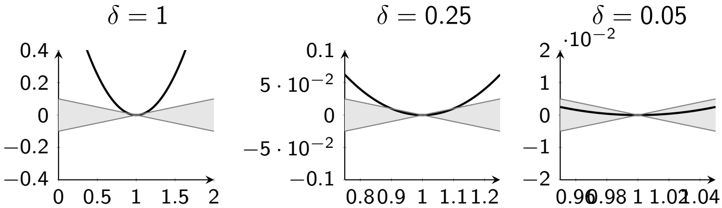
- If f has derivative m at a, the line $y = f(a) + m(x - a)$ is called the **tangent line** to the graph of f at $(a, f(a))$.
- If f has derivative m at a, we use the notation $f'(a)$ or $\dfrac{df}{dx}(a)$ or $\left.\dfrac{df}{dx}\right|_{x=a}$ for m.

Example

The sequence of graphs illustrates this definition for $y = x^2$, $a = 1$, $m = 2$ and $\epsilon = 0.1$. We see that $\delta = 0.05$ works for these values, and brings the curve inside the shaded zone.



Caratheodory's Characterization of Derivative

Theorem 1

A function f has derivative $f'(a)$ at a if and only if there is a function φ such that $f(x) - f(a) - f'(a)(x - a) = \varphi(x)(x - a)$ and $\lim_{x \to a} \varphi(x) = \varphi(a) = 0$.

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Proof. Suppose the derivative $f'(a)$ exists. Define

$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} - f'(a) & \text{if } x \neq a. \\ 0 & \text{if } x = a. \end{cases}$$

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Consider any $\epsilon > 0$. The definition of derivative gives $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a) - f'(a)(x-a)| \leq \epsilon |x-a|$.

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$$\varphi(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} - f'(a) & \text{if } x \neq a. \\ 0 & \text{if } x = a. \end{cases}$$

Consider any $\epsilon > 0$. The definition of derivative gives $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a) - f'(a)(x-a)| \leq \epsilon |x-a|$. Then $0 < |x-a| < \delta$ implies $|\varphi(x)| \leq \epsilon$, which corresponds to the desired $\lim_{x\to a} \varphi(x) = 0$.

The steps can be reversed to obtain the converse. $\square$

Differentiability implies Continuity

Theorem 2

If a function is differentiable at a point, then it is continuous at that point.

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There is a function φ such that
$f(x) - f(a) - f'(a)(x - a) = \varphi(x)(x - a)$ and $\lim_{x \to a} \varphi(x) = 0$.
Hence,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(f(a) + f'(a)(x - a) + \varphi(x)(x - a)\right) = f(a).$$

□

Higher Derivatives

Differentiating $f\colon D \to \mathbb{R}$ creates a new function $f'\colon D' \to \mathbb{R}$ where D' consists of all the points where f is differentiable.

Higher Derivatives

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We can further differentiate f' to get $f'' = (f')'$, called the **second derivative** of f, and so on.

Higher Derivatives

Differentiating $f : D \to \mathbb{R}$ creates a new function $f' : D' \to \mathbb{R}$ where D' consists of all the points where f is differentiable.
We can further differentiate f' to get $f'' = (f')'$, called the **second derivative** of f, and so on.
Other choices of notation are:

$$f^{(0)}(x) = f(x),$$
$$f^{(1)}(x) = f'(x) = \frac{df}{dx}(x),$$
$$\vdots$$
$$f^{(n)}(x) = \frac{d^n f}{dx^n}(x).$$

The function $f^{(n)}$, obtained by differentiating f successively n times, is called the ***n*th derivative** of f.

Derivative via Limits

Theorem 3

Let $f : I \to \mathbb{R}$ where I is an open interval. Then f has derivative $f'(a)$ at $a \in I$ if and only if

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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Proof. We prove the first equality.

$$\begin{aligned} f'(a) = m \iff & \text{ there is } \varphi \text{ s.t. } f(x) - f(a) - m(x-a) = \varphi(x)(x-a) \\ & \text{ and } \lim_{x \to a} \varphi(x) = \varphi(a) = 0 \\ \iff & \lim_{x \to a} \frac{f(x) - f(a) - m(x-a)}{x - a} = 0 \\ \iff & \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m. \end{aligned}$$

Power Rule

Consider the function x^n, for a fixed $n \in \mathbb{N}$. Its derivative can be calculated as follows.

$$(x^n)' = \lim_{y \to x} \frac{y^n - x^n}{y - x} = \lim_{y \to x} \sum_{i=0}^{n-1} y^i x^{n-1-i}$$
$$= \sum_{i=0}^{n-1} x^i x^{n-1-i} = \sum_{i=0}^{n-1} x^{n-1} = n x^{n-1}.$$

The second equality uses the identity

$$y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}).$$

In particular, $x' = 1$, $(x^2)' = 2x$, etc.

One-Sided Derivative

- $f'_+(a) = \lim_{x \to a+} \dfrac{f(x) - f(a)}{x - a}$ is the **right derivative** of f at a.
- $f'_-(a) = \lim_{x \to a-} \dfrac{f(x) - f(a)}{x - a}$ is the **left derivative** of f at a.

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Task 1

Show that a function f is differentiable at $x = a$ if and only if the left and right derivatives of f at a exist and are equal.

Differentiability on Intervals

We say f is **differentiable on an interval** I if it is differentiable at every interior point of I, and has the appropriate one-sided derivative at any end-point which is included in I. We denote the one-sided derivative at an end-point c by $f'(c)$.

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Theorem 4

Suppose I is an interval and $f : I \to \mathbb{R}$ is a differentiable function. Then the following hold.

1. *If f is an increasing function then $f'(a) \geq 0$ for every $a \in I$.*
2. *If f is a decreasing function then $f'(a) \leq 0$ for every $a \in I$.*

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2. *If f is a decreasing function then $f'(a) \leq 0$ for every $a \in I$.*

Proof. Suppose f is an increasing function and a is not the right end-point of I:

$$x > a \implies \frac{f(x) - f(a)}{x - a} \geq 0 \implies f'_{+}(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a} \geq 0.$$

The other cases are proved in a similar fashion. □

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Algebra of Derivatives

Theorem 5

Let f and g be differentiable at p, and let $C \in \mathbb{R}$. Then their combinations satisfy the following rules.

1. *(Scaling)* $(Cf)'(p) = Cf'(p)$.
2. *(Sum Rule)* $(f+g)'(p) = f'(p) + g'(p)$.
3. *(Difference Rule)* $(f-g)'(p) = f'(p) - g'(p)$.
4. *(Product Rule)* $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$.
5. *(Reciprocal Rule)* $\left(\frac{1}{f}\right)'(p) = -\frac{f'(p)}{f(p)^2}$, *if* $f(p) \neq 0$.
6. *(Quotient Rule)* $\left(\frac{g}{f}\right)'(p) = \frac{g'(p)f(p) - g(p)f'(p)}{f(p)^2}$, *if* $f(p) \neq 0$.

Algebra of Derivatives - Proof

1. Scaling:

$$(Cf)'(p) = \lim_{x \to p} \frac{Cf(x) - Cf(p)}{x - p} = C \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = Cf'(p).$$

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2. Sum Rule:

$$\begin{aligned}(f+g)'(p) &= \lim_{x\to p} \frac{f(x) + g(x) - f(p) - g(p)}{x - p} \\ &= \lim_{x\to p} \frac{f(x) - f(p)}{x - p} + \lim_{x\to p} \frac{g(x) - g(p)}{x - p} \\ &= f'(p) + g'(p).\end{aligned}$$

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3. Difference Rule: Combine the sum rule with scaling by $C = -1$.

Algebra of Derivatives - Proof

4. $$(fg)'(p) = \lim_{x \to p} \frac{f(x)g(x) - f(p)g(p)}{x - p}$$
$$= \lim_{x \to p} \frac{f(x)g(x) - f(x)g(p) + f(x)g(p) - f(p)g(p)}{x - p}$$
$$= \lim_{x \to p} f(x)\frac{g(x) - g(p)}{x - p} + \lim_{x \to p} \frac{f(x) - f(p)}{x - p} g(p)$$
$$= f(p)g'(p) + f'(p)g(p).$$

(Since $f'(p)$ exists, f is continuous at p and $\lim_{x \to p} f(x) = f(p)$.)

Algebra of Derivatives - Proof

4. $$(fg)'(p) = \lim_{x\to p} \frac{f(x)g(x) - f(p)g(p)}{x-p}$$
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$$= f(p)g'(p) + f'(p)g(p).$$
(Since $f'(p)$ exists, f is continuous at p and $\lim_{x\to p} f(x) = f(p)$.)

5. By continuity, $f(x) \neq 0$ for x near p. Hence,
$$\left(\frac{1}{f}\right)'(p) = \lim_{x\to p} \frac{1/f(x) - 1/f(p)}{x-p} = \lim_{x\to p} \frac{f(p)-f(x)}{f(x)f(p)(x-p)} = -\frac{f'(p)}{f(p)^2}.$$
(Since $f'(p)$ exists, we have $\lim_{x\to p} f(x) = f(p)$.)

Algebra of Derivatives - Proof

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4. $$\begin{aligned}(fg)'(p) &= \lim_{x\to p} \frac{f(x)g(x)-f(p)g(p)}{x-p} \\ &= \lim_{x\to p} \frac{f(x)g(x)-f(x)g(p)+f(x)g(p)-f(p)g(p)}{x-p} \\ &= \lim_{x\to p} f(x)\frac{g(x)-g(p)}{x-p} + \lim_{x\to p} \frac{f(x)-f(p)}{x-p}g(p) \\ &= f(p)g'(p)+f'(p)g(p).\end{aligned}$$

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   (Since $f'(p)$ exists, we have $\lim_{x\to p} f(x) = f(p)$.)

6. Quotient Rule: Combine the product rule and reciprocal rule.

Example

With these rules we can differentiate polynomials and rational functions. For example,

$$\begin{aligned}(x^{45} + 7x^4 + 99)' &= (x^{45})' + (7x^4)' + (99)' && \text{(sum rule)}\\ &= (x^{45})' + (7x^4)' && (C' = 0)\\ &= (x^{45})' + 7(x^4)' && \text{(scaling)}\\ &= 45x^{44} + 28x^3. && \text{(power rule)}\end{aligned}$$

Trigonometric Functions

Theorem 6

For every $x \in \mathbb{R}$, $\sin' x = \cos x$ *and* $\cos' x = -\sin x$.

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Proof. We differentiate the sine function, and leave the cosine for the reader.

$$\sin' x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \sin x + \frac{\sin h}{h} \cos x \right) = 0 \cdot \sin x + 1 \cdot \cos x = \cos x.$$

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□

Task 2

Use the reciprocal and quotient rules to show that

$$\sec' x = \sec x \tan x, \qquad \csc' x = -\csc x \cot x,$$

$$\tan' x = \sec^2 x, \qquad \cot' x = -\csc^2 x.$$

Logarithm

To differentiate the log function, we need the following inequalities:

Theorem 7

For $x > 0$, $1 - \dfrac{1}{x} \leq \log x \leq x - 1$.

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$$\int_1^x \frac{1}{x}\,dt \leq \int_1^x \frac{1}{t}\,dt \leq \int_1^x 1\,dt.$$

Substituting $1/x$ for x gives the inequalities for $0 < x \leq 1$. □

Logarithm

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Proof. We apply the limit definition of the derivative.

$$\log' x = \lim_{y \to x} \frac{\log y - \log x}{y - x} = \lim_{y \to x} \frac{\log(y/x)}{y - x}$$
$$= \lim_{h \to 1} \frac{\log(hx/x)}{hx - x} = \frac{1}{x} \lim_{h \to 1} \frac{\log h}{h - 1}.$$

Logarithm

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For $h > 1$, we have $\dfrac{1}{h} \le \dfrac{\log h}{h - 1} \le 1$ from Theorem 7. The Sandwich Theorem gives $\displaystyle\lim_{h \to 1+} \frac{\log h}{h - 1} = 1$.

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For $h > 1$, we have $\dfrac{1}{h} \leq \dfrac{\log h}{h - 1} \leq 1$ from Theorem 7. The Sandwich Theorem gives $\displaystyle\lim_{h \to 1+} \frac{\log h}{h - 1} = 1$.

If $h < 1$, the inequalities reverse and again give $\displaystyle\lim_{h \to 1-} \frac{\log h}{h - 1} = 1$. ☐

General Logarithm, Estimating e

Task 3

Let $a > 0$ *and* $a \neq 1$. *Show that* $\log_a' x = \dfrac{1}{x \log a}$.

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The limit calculation that we carried out in the last proof can also be expressed as

$$\lim_{h \to 0} \frac{\log(1+h)}{h} = 1 \quad \text{or} \quad \lim_{h \to 0} \log((1+h)^{1/h}) = 1.$$

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Applying the exponential function, and recalling that it is continuous, we get.

$$\lim_{h \to 0} (1+h)^{1/h} = e.$$

We can use this limit to get better estimates of e.

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Chain Rule

Theorem 9

Let g be differentiable at a and let f be differentiable at $b = g(a)$. Then the composition $f \circ g$ is differentiable at a and the derivative is given by

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

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1. $g(x) - g(a) = (n + \varphi(x))(x - a)$ and $\lim_{x \to a} \varphi(x) = \varphi(a) = 0$.
2. $f(y) - f(b) = (m + \psi(y))(y - b)$ and $\lim_{y \to b} \psi(y) = \psi(b) = 0$.

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2. $f(y) - f(b) = (m + \psi(y))(y - b)$ and $\lim_{y \to b} \psi(y) = \psi(b) = 0$.

Hence,
$$\begin{aligned} f(g(x)) - f(g(a)) &= \big(m + \psi(g(x))\big)(g(x) - b) \\ &= \big(m + \psi(g(x))\big)(n + \varphi(x))(x - a) \\ &= mn(x - a) + E(x)(x - a), \end{aligned}$$
and $\lim_{x \to a} E(x) = \lim_{x \to a} \big(m\varphi(x) + n\psi(g(x)) + \psi(g(x))\varphi(x)\big) = 0.$

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$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. Let $f'(g(a)) = m$ and $g'(a) = n$. By Theorem 1, we have functions φ and ψ such that:

1. $g(x) - g(a) = (n + \varphi(x))(x - a)$ and $\lim_{x \to a} \varphi(x) = \varphi(a) = 0$.
2. $f(y) - f(b) = (m + \psi(y))(y - b)$ and $\lim_{y \to b} \psi(y) = \psi(b) = 0$.

Hence,
$$\begin{aligned} f(g(x)) - f(g(a)) &= \big(m + \psi(g(x))\big)(g(x) - b) \\ &= \big(m + \psi(g(x))\big)\big(n + \varphi(x)\big)(x - a) \\ &= mn(x - a) + E(x)(x - a), \end{aligned}$$
and $\lim_{x \to a} E(x) = \lim_{x \to a} \big(m\varphi(x) + n\psi(g(x)) + \psi(g(x))\varphi(x)\big) = 0$.
This establishes that $(f \circ g)'(a) = mn = f'(g(a))g'(a)$. □

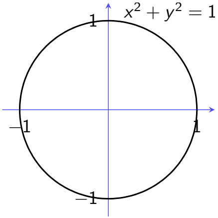
Chain Rule - Applications

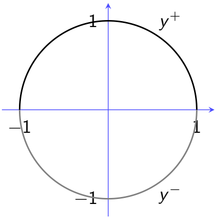
Differentiate the given functions:

1. $f(x) = (x^2 + 1)^{10}$.
2. $g(x) = |\cos x|$.
3. $h(x) = \cos |x|$.
4. $k(x) = \dfrac{\sin^2 x}{\sin x^2}$.

Implicit Differentiation

Consider the relation $x^2 + y^2 = 1$. For any $x \in [-1, 1]$ we can solve for corresponding $y = \pm\sqrt{1 - x^2}$. We say that $x^2 + y^2 = 1$ defines y **implicitly** in terms of x. In fact this implicit relation can be separated into two explicit functions $y^+ = \sqrt{1 - x^2}$ and $y^- = -\sqrt{1 - x^2}$.





Implicit Differentiation

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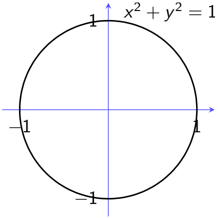

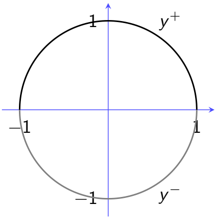

The Chain Rule allows us to calculate dy/dx without solving explicitly for y:

$$x^2 + y^2 = 1 \implies 2x + 2y\,y' = 0 \implies y' = -x/y \quad (\text{if } y \neq 0).$$

This works simultaneously for both cases of $y^\pm = \pm\sqrt{1 - x^2}$!

Folium of Descartes

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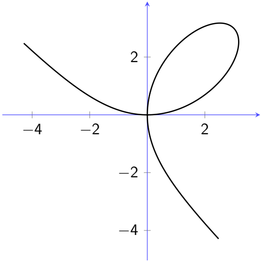
Consider the relation $x^3 + y^3 = 6xy$. Its solutions plot as follows.



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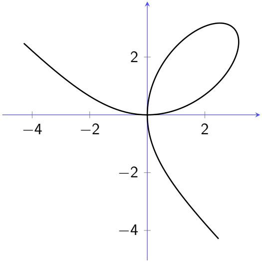

It is hard to separate this into explicit functions, but easy to differentiate implicitly:

$$x^3 + y^3 = 6xy \implies 3x^2 + 3y^2 y' = 6y + 6xy'$$

$$\implies (y^2 - 2x)y' = 2y - x^2 \implies y' = \frac{2y - x^2}{y^2 - 2x}.$$

Tangent to Ellipse

The equation of an ellipse in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

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If (x_0, y_0) is a point on the ellipse, the slope m of the tangent line there is given by

$$\frac{2x_0}{a^2} + \frac{2y_0}{b^2} m = 0 \quad \text{or} \quad m = -\frac{x_0}{y_0} \frac{b^2}{a^2}.$$

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Hence the equation of the tangent line at (x_0, y_0) is

$$y = y_0 - \frac{x_0}{y_0}\frac{b^2}{a^2}(x - x_0) \quad \text{or} \quad \frac{yy_0 - y_0^2}{b^2} + \frac{xx_0 - x_0^2}{a^2} = 0 \quad \text{or} \quad \frac{yy_0}{b^2} + \frac{xx_0}{a^2} = 1.$$

Derivative of Inverse Function

Theorem 10

Let f be a continuous and monotonic bijection between two intervals. Let $f'(a)$ exist and be non-zero. Then f^{-1} is differentiable at $b = f(a)$ and the derivative is given by

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

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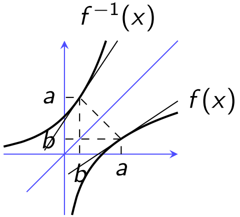
Proof. If a line with slope $m \neq 0$ is reflected in the $y = x$ line, the resulting line has slope $1/m$. The following picture now represents a proof.



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$$g(y) = \begin{cases} \dfrac{f^{-1}(y) - f^{-1}(b)}{y - b} & \text{if } y \neq b, \\ 1/f'(a) & \text{if } y = b. \end{cases}$$

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Substituting $y = f(x)$ and $b = f(a)$ gives

$$g(f(x)) = \begin{cases} \dfrac{x - a}{f(x) - f(a)} & \text{if } x \neq a, \\ 1/f'(a) & \text{if } x = a. \end{cases}$$

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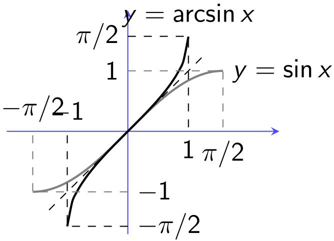
So $g \circ f$ is continuous at a. Therefore $g = g \circ f \circ f^{-1}$ is continuous at b. This gives the result. $\square$

Inverse Trigonometric Functions

The restriction $\sin\colon [-\pi/2, \pi/2] \to [-1, 1]$ is a bijection, hence has an inverse function that is called arcsine and is denoted by $\sin^{-1} x$ or $\arcsin x$.



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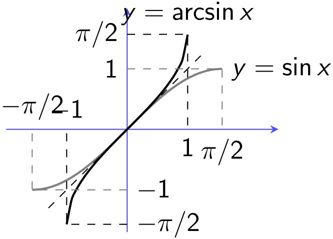

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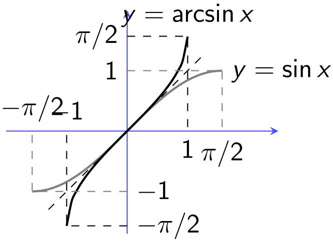

Similarly, $\cos\colon [0, \pi] \to [-1, 1]$ has an inverse function called arccosine, and denoted by $\cos^{-1} x$ or $\arccos x$.
Finally $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ has an inverse function called arctan, and denoted by $\tan^{-1} x$ or $\arctan x$.

Derivatives of Arcsin and Arccos

Theorem 11

$$\arcsin' x = \frac{1}{\sqrt{1-x^2}} \qquad \textit{for } x \in (-1,1),$$
$$\arccos' x = \frac{-1}{\sqrt{1-x^2}} \qquad \textit{for } x \in (-1,1).$$

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The calculation for arccosine is similar and is left to the reader. □

Derivative of Arctan

Theorem 12

$\arctan' x = \dfrac{1}{1+x^2}$ *for* $x \in \mathbb{R}$.

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Proof.

$$\arctan' x = \frac{1}{\tan'(\arctan x)} = \frac{1}{\sec^2(\arctan x)}$$
$$= \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}.$$

□

Exponential Function

Theorem 13

The derivative of the exponential function is itself:

$$(e^x)' = e^x.$$

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Proof. Consider $f(x) = \log x$. Its inverse function is $f^{-1}(x) = e^x$. Applying the formula for differentiating an inverse function, we get

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Task 4

Let $a > 0$. *Show that* $(a^x)' = a^x \log a$.

Task 5

Prove that $\cosh' x = \sinh x$ *and* $\sinh' x = \cosh x$.

Power Rule

Theorem 14 (Power Rule)

If $r \in \mathbb{R}$ *then* $(x^r)' = r\,x^{r-1}$ *for* $x > 0$.

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Proof. $(x^r)' = (e^{r\log x})' = \frac{r}{x}e^{r\log x} = \frac{r}{x}x^r = rx^{r-1}$. □

Example 15

We'll differentiate the function $y = x^x$, with $x > 0$. We use the same technique as in the proof of the Power Rule.

$$(x^x)' = (e^{x\log x})' = e^{x\log x}(x\log x)' = x^x(1 + \log x).$$

Inverse Hyperbolic Functions

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Task 6

Show that $\sinh\colon \mathbb{R} \to \mathbb{R}$ *is a strictly increasing bijection.*

So $\sinh x$ has an inverse which is strictly increasing as well as continuous. We denote it by $\sinh^{-1}$ or $\operatorname{arsinh} x$.

Inverse Hyperbolic Functions

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The $\cosh x$ function is even and hence not one-one. So we restrict its domain to $[0, \infty)$.

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Show that $\cosh\colon [0, \infty) \to [1, \infty)$ *is a strictly increasing bijection.*

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Table of Contents

First Fundamental Theorem

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Theorem 16 (First Fundamental Theorem)

Let I be an interval and $f : I \to \mathbb{R}$ be integrable on each subinterval $[a, b] \subseteq I$. Fix $a \in I$ and consider the indefinite integral $F : I \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F'(c) = f(c)$ if f is continuous at c. (If c is an end-point, use the appropriate one-sided notion of continuity and differentiability.)

First Fundamental Theorem – Proof

For $h \neq 0$ we have

$$F(c+h) - F(c) = \int_a^{c+h} f(t)\,dt - \int_a^c f(t)\,dt = \int_c^{c+h} f(t)\,dt.$$

First Fundamental Theorem – Proof

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Hence, $$F(c+h) - F(c) - hf(c) = \int_c^{c+h} f(t)\,dt - \int_c^{c+h} f(c)\,dt$$
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Define $\varphi(h) = \frac{1}{h}\int_c^{c+h}(f(t) - f(c))\,dt$. Consider $\epsilon > 0$. If f is continuous at c, there is a $\delta > 0$ such that $|t - c| < \delta$ implies $|f(t) - f(c)| < \epsilon$.

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$$|\varphi(h)| = \frac{1}{|h|}\Big|\int_c^{c+h}(f(t) - f(c))\,dt\Big| \leq \frac{1}{|h|}|h|\epsilon = \epsilon.$$

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Therefore, $\varphi(h) \to 0$ as $h \to 0$, and so $F'(c) = f(c)$.

Examples

Example 17

Suppose we have to differentiate $F(x) = \int_0^x \sin\sqrt{t}\,dt$. By the First Fundamental Theorem we know immediately that $F'(x) = \sin\sqrt{x}$.

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Hence, by the Chain Rule,

$$G'(x) = F'(x^2)2x - F'(x) = 2x\sin\sqrt{x^2} - \sin\sqrt{x} = 2x\sin|x| - \sin\sqrt{x}.$$