CN # Chapter 7

CT # Supplement: Sum-Frequency Mixing

In this chapter of the online supplement to "Compact Blue-Green Lasers," we elaborate on using sum-frequency mixing, generally of two infrared wavelengths, to produce blue-green light. In the usual configuration, a relatively weak signal at λ_1 is converted to an output at λ_3 by sum-frequency mixing with a strong pump field at λ_2. For example, one of the earliest interactions that was explored involved mixing the 809-nm signal from a diode laser with the 1064-nm signal from a diode-pumped solid-state laser to produce blue light at 459 nm. At the time this was first explored (mid-1980s), the power that could be produced by a 809-nm diode laser in a suitable spatial mode was relatively low, generally several tens of milliwatts, while powers of several watts could be obtained at 1064 nm.

In the first part of this supplement, we will derive equations for the plane-wave case that include the effect of depletion of the weak signal at λ_1—in the ideal case, all the power at λ_1 would be converted to λ_3, so that depletion cannot be ignored. However, in actual practice Gaussian beams rather than plane waves are used and complete conversion is usually not obtained. Hence, in the second part of the supplement, we will adapt the Boyd-Kleinman analysis presented in the previous supplement for second-harmonic generation to some special cases of sum-frequency mixing relevant to many of the practical implementations that have been explored for blue-green generation.

A

7.1 Plane Wave Treatment

The coupled equations relating the amplitude of the $1, 3$ fields are:

$$\frac{dA_3(x)}{dx} = \left(\frac{-4j}{n_3^2} d_{eff} \frac{A_2}{4} k_3 e^{j\Delta k x} e^{-j\Delta\phi}\right) A_1(x) \tag{7.1}$$

$$\frac{dA_1(x)}{dx} = \left(\frac{-4j}{n_1^2} d_{eff} \frac{A_2}{4} k_1 e^{-j\Delta k x} e^{j\Delta\phi}\right) A_3(x) \tag{7.2}$$

where we have assumed that the strong 2 pump field is not depleted.

We can introduce the constants:

$$\kappa_1 = \frac{k_1 d_{eff} A_2}{n_1^2} \tag{7.3}$$
$$\kappa_3 = \frac{k_3 d_{eff} A_2}{n_3^2}$$

and taking $\Delta\phi = 0$ we can then write the coupled equations as:

$$\frac{dA_3(x)}{dx} = -j\kappa_3 e^{j\Delta kx} A_1(x) \tag{7.4}$$
$$\frac{dA_1(x)}{dx} = -j\kappa_1 e^{-j\Delta kx} A_3(x) \tag{7.5}$$

Differentiating the first equation we obtain:

$$\frac{d^2 A_3(x)}{dx^2} = -j\kappa_3 \left(j\Delta k A_1(x) + \frac{dA_1(x)}{dx} \right) e^{j\Delta kx} \tag{7.6}$$

We can substitute for $A_1(x)$ using the first equation above and for $\frac{dA_1(x)}{dx}$ using the second to obtain:

$$\begin{aligned} \frac{d^2 A_3(x)}{dx^2} &= -j\kappa_3 \left(j\Delta k \cdot \frac{\frac{dA_3(x)}{dx}}{-j\kappa_3 e^{j\Delta kx}} + -j\kappa_1 e^{-j\Delta kx} A_3(x) \right) e^{j\Delta kx} \qquad (7.7) \\ &= j\Delta k \frac{dA_3(x)}{dx} - \kappa_1 \kappa_3 A_3(x) \end{aligned}$$

or

$$\frac{d^2A_3(x)}{dx^2} - j\Delta k \frac{dA_3(x)}{dx} + \kappa_1 \kappa_3 A_3(x) = 0 \tag{7.8}$$

We take a trial solution of the form

$$A_3(x) = \left(Ce^{j\gamma x} + De^{-j\gamma x}\right) e^{j\Delta k x/2} \tag{7.9}$$

from which we obtain:

$$\begin{aligned} \frac{dA_3(x)}{dx} &= \left(Ce^{j\gamma x} + De^{-j\gamma x}\right)\left(\frac{j\Delta k}{2} e^{j\Delta k x/2}\right) + j\gamma \left(Ce^{j\gamma x} - De^{-j\gamma x}\right) e^{j\Delta k x/2} \\ &= je^{j\Delta k x/2}\left[Ce^{j\gamma x}\left(\frac{\Delta k}{2} + \gamma\right) + De^{-j\gamma x}\left(\frac{\Delta k}{2} - \gamma\right)\right] \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} \frac{d^2A_3(x)}{dx^2} &= j\left(\frac{j\Delta k}{2} e^{j\Delta k x/2}\right)\left[Ce^{j\gamma x}\left(\frac{\Delta k}{2} + \gamma\right) + De^{-j\gamma x}\left(\frac{\Delta k}{2} - \gamma\right)\right] \\ &\quad + je^{j\Delta k x/2}\left[j\gamma Ce^{j\gamma x}\left(\frac{\Delta k}{2} + \gamma\right) - j\gamma De^{-j\gamma x}\left(\frac{\Delta k}{2} - \gamma\right)\right] \\ &= je^{j\Delta k x/2}\left\{Ce^{j\gamma x}\left[\left(\frac{j\Delta k}{2}\right)\left(\frac{\Delta k}{2} + \gamma\right) + j\gamma\left(\frac{\Delta k}{2} + \gamma\right)\right] + De^{-j\gamma x}\left[\left(\frac{j\Delta k}{2}\right)\left(\frac{\Delta k}{2} - \gamma\right) - j\gamma\left(\frac{\Delta k}{2} - \gamma\right)\right]\right\} \\ &= -e^{j\Delta k x/2}\left\{Ce^{j\gamma x}\left[\gamma^2 + \gamma\Delta k + \left(\frac{\Delta k}{2}\right)^2\right] + De^{-j\gamma x}\left[\gamma^2 - \gamma\Delta k + \left(\frac{\Delta k}{2}\right)^2\right]\right\} \end{aligned} \tag{7.11}$$

Inserting these expansions into the second-order differential equation above we obtain:

$$Ce^{j\gamma x}\left\{-\left[\gamma^2+\gamma\Delta k+\left(\frac{\Delta k}{2}\right)^2\right]+\Delta k\left[\frac{\Delta k}{2}+\gamma\right]+\kappa_1\kappa_3\right\}+De^{-j\gamma x}\left\{-\left[\gamma^2-\gamma\Delta k+\left(\frac{\Delta k}{2}\right)^2\right]-\Delta k\left[\gamma-\frac{\Delta k}{2}\right]+\kappa_1\kappa_3\right\}=0 \tag{7.12}$$

In order for the equality to be satisfied generally, the factor inside the braces must be zero, which reduces to:

$$-\gamma^2+\left(\frac{\Delta k}{2}\right)^2+\kappa_1\kappa_3=0 \tag{7.13}$$

$$\gamma^2=\left(\frac{\Delta k}{2}\right)^2+\kappa_1\kappa_3$$

Recall that the trial solution is $A_3(x) = \left(Ce^{j\gamma x}+De^{-j\gamma x}\right)e^{j\Delta kx/2}$.We expect that $A_3(0) = 0$, thus $C + D = 0$.Therefore $A_3(x) = C\left(e^{j\gamma x}-e^{-j\gamma x}\right)e^{j\Delta kx/2} = 2jC\sin(\gamma x)e^{j\Delta kx/2}$

Rearranging one of the original coupled equations, we find:

$$A_1(x)=\frac{dA_3(x)}{dx}\cdot\frac{1}{-j\kappa_3 e^{j\Delta kx}}=\frac{-2C\left[e^{j\gamma}\left(\frac{\Delta k}{2}+\gamma\right)-e^{-j\gamma}\left(\frac{\Delta k}{2}-\gamma\right)\right]e^{-j\Delta kx/2}}{\kappa_3} \tag{7.14}$$

¿From this

$$A_1(0) = \frac{-2C\gamma}{\kappa_3} \tag{7.15}$$

or

$$C = \frac{-\kappa_3 A_1(0)}{2} \tag{7.16}$$

Therefore,

$$A_3(x) = 2j\left(\frac{-\kappa_3 A_1(0)}{2}\right)\sin(\gamma x)e^{j\Delta k x/2} \tag{7.17}$$

$$I_3(x) = \frac{A_3\,(x)\,A_3^*\,(x)}{2\eta_3} = \frac{\eta_1}{\eta_3}\cdot\kappa_3^2\cdot I_1(0)\cdot\left(\frac{\sin\gamma x}{\gamma}\right)^2 \tag{7.18}$$

Under phasematching conditions ($\Delta k = 0$), $\gamma = \sqrt{\kappa_1\kappa_3}$ and for maximum conversion efficiency, we must have $\sqrt{\kappa_1\kappa_3}\ell_x = \pi/2$. When we write $\kappa_{1,3}$ in terms of intensities, we find that for a fixed crystal length ℓ_x, maximum conversion efficiency requires that the pump beam have a particular intensity $I_2 = \frac{\pi^2 n_1^2 n_3^2}{8\ell_x^2 k_1 k_3 \eta_2 d_{eff}^2}$. Thus, under these ideal conditions, $I_3(\ell_x) = \frac{\lambda_1}{\lambda_3}I_1(0)$. If we re-write this expression in terms of the average number of photons $N_{1,3}$ at the two wavelengths, we find that $N_3 = N_1$, meaning that under optimum conditions, all the photons at wavelength λ_1 that are introduced into the interaction are converted to photons at wavelength λ_3.

A

7.2 Focused Gaussian Beams

As mentioned in the text of “Compact Blue-Green Lasers”, the case of sum-frequency generation using focused Gaussian beams is somewhat more complicated than the case of second-harmonic generation, simply because there are now two input beams, each of which could be focused to a different degree, could be propagating in a different direction and could pass through the crystal at a different location. However, in most practical cases of blue-green generation, the goal is to maximize the power generated at the sum-frequency; hence, it is obviously desirable to overlap the beams well and to have them propagate in the same direction. However, the question of how each beam should be focused remains.

Recall that there were really two pieces to the solution of the second-harmonic generation problem: first, we had to determine the spatial variation of the nonlinear polarization induced by the electric field distribution of the input beam; second, we viewed this induced polarization as a source, and calculated the freely propagating electric field produced by it. Hence, if we can contrive a situation such that the two input beams in sum-frequency generation produce a polarization that has the same form as we obtained in second-harmonic generation, we may be apply some of the results from our earlier analysis of second-harmonic generation to the case of sum-frequency mixing.

We will use similar notation to that used in the supplement explaining the Boyd-Kleinman treatment of second-harmonic generation, with additional subscripts added where they are needed to distinguish between the two input beams required for sum-frequency mixing.

Suppose that the two input beams are coaxial Gaussian beams and that they have their waists at the same value of z. We can write:

$$\widetilde{E}_1\left(r,z\right)=\frac{E_{1o}}{2}\frac{w_{1o}}{w_1\left(z\right)}e^{-\frac{\alpha_1}{2}z}e^{-\frac{r^2}{w_1^2(z)}}e^{-jk_1z}e^{j\Psi_1(r,z)}e^{j\Phi_1(z)} \tag{7.19}$$

$$\widetilde{E}_2\left(r,z\right)=\frac{E_{2o}}{2}\frac{w_{2o}}{w_2\left(z\right)}e^{-\frac{\alpha_2}{2}z}e^{-\frac{r^2}{w_2^2(z)}}e^{-jk_2z}e^{j\Psi_2(r,z)}e^{j\Phi_2(z)} \tag{7.20}$$

We now define the parameters $\zeta_{1,2}=2z/b_{1,2}$ and write:

$$w_1(z)=w_{10}\left[1+\zeta_1^2\right] \tag{7.21}$$

$$e^{j\Psi_1(r,z)}=e^{j\frac{\zeta_1 r^2}{w_{10}^2\left(1+\zeta_1^2\right)}} \tag{7.22}$$

$$e^{j\Phi_1(z)}=\frac{1-j\zeta_1}{\sqrt{1+\zeta_1^2}} \tag{7.23}$$

with similar definitions for beam 2. Hence, we can write an expression for the first input beam as:

$$\widetilde{E}_1(r,z) = \frac{E_{1o}}{2}e^{-\frac{\alpha_1}{2}z}\frac{1}{\sqrt{1+\zeta_1^2}}e^{-\frac{r^2}{w_{1o}^2\left(1+\zeta_1^2\right)}}e^{-jk_1z}e^{j\frac{\zeta_1 r^2}{w_{1o}^2\left(1+\zeta_1^2\right)}}\frac{1-j\zeta_1}{\sqrt{1+\zeta_1^2}} \tag{7.24}$$

$$= \frac{E_{1o}}{2}e^{-\frac{\alpha_1}{2}z}\frac{1}{1+j\zeta_1}e^{-\frac{r^2}{w_{1o}^2(1+j\zeta_1)}}e^{-jk_1z}$$

Similarly, for the second input beam we write:

$$\widetilde{E}_2(r,z) = \frac{E_{2o}}{2}e^{-\frac{\alpha_2}{2}z}\frac{1}{\sqrt{1+\zeta_2^2}}e^{-\frac{r^2}{w_{2o}^2\left(1+\zeta_2^2\right)}}e^{-jk_2z}e^{j\frac{\zeta_{21} r^2}{w_{2o}^2\left(1+\zeta_2^2\right)}}\frac{1-j\zeta_2}{\sqrt{1+\zeta_2^2}} \tag{7.25}$$

$$= \frac{E_{2o}}{2}e^{-\frac{\alpha_2}{2}z}\frac{1}{1+j\zeta_2}e^{-\frac{r^2}{w_{2o}^2(1+j\zeta_2)}}e^{-jk_2z}$$

The Gaussian beams as we have written them her both have their waists at z=0, which is normally where we consider the input face of the crystal to be. We can make the expression more general by placing the waist at some position z=f and we can achieve this change by re-defining $\zeta_{1,2} = 2\,(z-f)\,/b_{1,2}$.

The nonlinear polarization induced by the interaction of these two beams through sum-frequency generation is given by:

$$\widetilde{P}^{(\omega_3)} = 4\epsilon_o d_{eff}\widetilde{E}^{(\omega_1)}\widetilde{E}^{(\omega 2)} \tag{7.26}$$

$$= 4\epsilon_o d_{eff}\left[\frac{E_{1o}}{2}e^{-\frac{\alpha_1}{2}z}\frac{1}{1+j\zeta_1}e^{-\frac{r^2}{w_{1o}^2(1+j\zeta_1)}}e^{-jk_1z}\right]\left[\frac{E_{2o}}{2}e^{-\frac{\alpha_2}{2}z}\frac{1}{1+j\zeta_2}e^{-\frac{r^2}{w_{2o}^2(1+j\zeta_2)}}e^{-jk_2z}\right]$$

$$= \epsilon_o d_{eff}E_{1o}E_{2o}e^{-\frac{1}{2}(\alpha_1+\alpha_2)z}\frac{1}{1+j\zeta_1}\frac{1}{1+j\zeta_2}e^{-\frac{r^2}{w_{1o}^2(1+j\zeta_1)}}e^{-\frac{r^2}{w_{2o}^2(1+j\zeta_2)}}e^{-j(k_1+k_2)z}$$

Inserting this expression into the generating equation for the sum-frequency field, we obtain:

$$\begin{aligned} d\widetilde{E}_3(x) &= \frac{-j}{2\epsilon_o n_3^2} k_3 e^{jk_3 z} \tilde{P}^{(\omega_3)} dz \\ &= \frac{-jk_3}{2n_3^2} e^{j\Delta kz} d_{eff} E_{1o} E_{2o} e^{-\frac{1}{2}(\alpha_1+\alpha_2)z} \frac{1}{1+j\zeta_1} \frac{1}{1+j\zeta_2} e^{-\frac{r^2}{w_{1o}^2(1+j\zeta_1)}} e^{-\frac{r^2}{w_{2o}^2(1+j\zeta_2)}} dz \end{aligned} \tag{7.27}$$

If the two beams have the same confocal parameter $b_1 = b_2 = b$, then $\zeta_1 = \zeta_2 = \zeta$

then the expression above simplifies to:

$$d\widetilde{E}_3(x) = \frac{-jk_3}{2n_3^2} e^{j\Delta kz} d_{eff} E_{1o} E_{2o} e^{-\frac{1}{2}(\alpha_1+\alpha_2)z} \left[\frac{1}{1+j\zeta}\right]^2 e^{-\frac{r^2}{(1+j\zeta)}\left(\frac{1}{w_{1o}^2}+\frac{1}{w_{2o}^2}\right)} dz \tag{7.28}$$

If we define an effective beam waist

we can write

$$d\widetilde{E}_{3,SFG}(x) = \frac{-jk_3}{2n_3^2} e^{j\Delta kz} d_{eff} E_{1o} E_{2o} e^{-\frac{1}{2}(\alpha_1+\alpha_2)z} \frac{1}{1+j\zeta} \left\{ \frac{1}{1+j\zeta} e^{-\frac{r^2}{(1+j\zeta)}\left(\frac{1}{w_{1o}^2}+\frac{1}{w_{2o}^2}\right)} \right\} dz \tag{7.29}$$

This expression describes the contribution to the sum-frequency field from a slab of the nonlinear crystal with width dz, located at a position z inside the crystal

$(0 < z < l)$. Recall that in the case of second-harmonic generation, we had a similar expression:

$$d\widetilde{E}_{3,SHG}(x) = \frac{-jk_3}{2n_3^2} e^{j\Delta kz} d_{eff} \frac{E_o^2}{2} e^{-\alpha_1 z} \frac{1}{1+j\zeta} \left\{ \frac{1}{1+j\zeta} e^{-\frac{2r^2}{w_o^2(1+j\zeta)}} \right\} dz \qquad (7.30)$$

Note that if we define

$$\frac{2}{w_{eff}^2} = \frac{1}{w_{10}^2} + \frac{1}{w_{20}^2} \qquad (7.31)$$

$$E_{o,eff} = 2\sqrt{E_{1o}E_{2o}} \qquad (7.32)$$

$$\alpha_{eff} = \frac{1}{2}(\alpha_1 + \alpha_2) \qquad (7.33)$$

then we can write for this sum-frequency generation case the expression:

$$d\widetilde{E}_{3,SFG}(x) = \frac{-jk_3}{2n_3^2} e^{j\Delta kz} d_{eff} \frac{E_{o,eff}}{2} e^{-\alpha_{eff} z} \frac{1}{1+j\zeta} \left\{ \frac{1}{1+j\zeta} e^{-\frac{2r^2}{w_{eff}^2(1+j\zeta)}} \right\} d \qquad (7.34)$$

which has exactly the same form as the preceding one for second-harmonic generation.

Hence, from this point on, the development is the same as what was presented in the supplemental chapter on second-harmonic generation. We can therefore skip to the end and modify the result obtained for the generated SHG power to derive an expression for the generated SFG power. Using the result from the supplement on focused SHG, and making the substitutions described above, we expect that

$$P_3 = \frac{1}{32\eta_3} \frac{k_3^2}{n_3^4} \left(4E_{1o}^2 E_{2o}^2\right) d_{eff}^2 b^2 \pi w_{eff}^2 e^{-\alpha' l} \xi h \tag{7.35}$$

where

$$\alpha' = \alpha_{eff} + \tfrac{1}{2}\alpha_3 = \frac{1}{2}\left(\alpha_1 + \alpha_2 + \alpha_3\right) \tag{7.36}$$

We note that since $b_1 = b_2 = b$

$$\frac{2}{w_{eff}^2} = \frac{1}{w_{10}^2} + \frac{1}{w_{20}^2} = \frac{2\pi n_1}{\lambda_1 b} + \frac{2\pi n_2}{\lambda_2 b} = \frac{2\pi}{b}\left(\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_2}\right) = \frac{2\pi n_3}{b\lambda_3} \tag{7.37}$$

or

$$w_{eff}^2 = \frac{b\lambda_3}{\pi n_3}$$

where the last equality results from the use of momentum conservation and the assumption of perfect phasematching ($\Delta k = 0$). In addition, we recall that the power of the input beams is given by

$$
\begin{aligned}
P_1 &= \frac{E_{1o}^2}{2\eta_1}\frac{\pi w_{1o}^2}{2} \\
P_2 &= \frac{E_{2o}^2}{2\eta_2}\frac{\pi w_{2o}^2}{2}
\end{aligned}
\tag{7.38}
$$

Using these relations, we can write:

$$
\begin{aligned}
P_3 &= \frac{1}{8\eta_3}\frac{k_3^2}{n_3^4}\left(\frac{4\eta_1 P_1}{\pi w_{1o}^2}\right)\left(\frac{4\eta_2 P_2}{\pi w_{2o}^2}\right) d_{eff}^2 b^2 \pi \left(\frac{b\lambda_3}{\pi n_3}\right) e^{-\alpha' l}\frac{l}{b}h \\
&= \frac{2\eta_1\eta_2}{\pi^2\eta_3}\frac{1}{n_3^5}\left(\frac{2\pi n_3}{\lambda_3}\right)^2\left(\frac{2\pi n_1}{b\lambda_1}\right)\left(\frac{2\pi n_2}{b\lambda_2}\right)\lambda_3 P_1 P_2 d_{eff}^2 b^2 e^{-\alpha' l} l h \\
&= \frac{32\pi^2 d_{eff}^2}{\epsilon_o c n_3^2 \lambda_1\lambda_2\lambda_3} P_1 P_2 e^{-\alpha' l} l h
\end{aligned}
\tag{7.39}
$$

B

7.2.1 Special Cases

Since the analysis follows that of the SHG case, we might expect that with no walk-off, optimum SFG will occur when both of the input beams are focused such that $l/b = 2.84$. Under these conditions, $h \approx 1$, so that with negligible loss,

$$
P_{3,opt} \approx \frac{32\pi^2 d_{eff}^2}{\epsilon_o c n_3^2 \lambda_1\lambda_2\lambda_3} P_1 P_2 l
\tag{7.40}
$$

In the case of loose focusing, i.e., $l/b \ll 1, h \to l/b$,so that

$$P_{3,loose} \approx \frac{32\pi^2 d_{eff}^2}{\epsilon_o c n_3^2 \lambda_1 \lambda_2 \lambda_3} P_1 P_2 \frac{l^2}{b} \tag{7.41}$$

For cases other than these, the reader is referred to the paper by S. Guha and J. Falk, "The effects of focusing in the three-frequency parametric upconverter," *J. Appl. Phys.,* **51**(1), 50–60 (1980), in which the Boyd-Kleinman analysis described here is carried out for more general cases. Their analysis leads to the following conclusions:

- When there is no walk-off, SFG efficiency is maximized when both beams have the same confocal parameter
- When there is walk-off, but when both beams have the same propagation constant, SFG efficiency is maximized when both beams also have the same confocal parameter.
- When there is walk-off and when the two beams have different propagation constants, the optimum values for the confocal parameters of the two beams must be found numerically. Their paper contains numerous plots from which the values can be estimated.