10 Chapter 10 Solution Set

Problems

10.1 Prove that $G^*_{0_f} = -G_{0_f}$.

$$G_{0_f}(\mathbf{r},\mathbf{r}') = G_{0_+}(\mathbf{r},\mathbf{r}') - G_{0_-}(\mathbf{r},\mathbf{r}'),$$
$$G^*_{0_f}(\mathbf{r},\mathbf{r}') = \overbrace{G^*_{0_+}(\mathbf{r},\mathbf{r}')}^{G_{0_-}} - \overbrace{G^*_{0_-}(\mathbf{r},\mathbf{r}')}^{G_{0_+}} = -G_{0_f}(\mathbf{r},\mathbf{r}')$$

10.2 Derive Eqs.(10.2) and (10.3).

We first derive Eq.(10.2). The Porter-Bojarski integral equation was derived in Section 9.2.3 of the last chapter where it was found to be

$$\Phi(\mathbf{r}) = \int_{\tau_0} d^3r'\, G_{0_f}(\mathbf{r},\mathbf{r}')Q(\mathbf{r}')$$

where

$$\Phi(\mathbf{r}) = \int_{\partial\tau} dS' \,[U_{0_+}(\mathbf{r}')\frac{\partial}{\partial n'}G_{0_-}(\mathbf{r},\mathbf{r}') - G_{0_-}(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}U_{0_+}(\mathbf{r}')]$$

is the back propagated field into the source region from the boundary value data. In the case under consideration the boundary value data are $\psi(\mathbf{r}')$ and $\frac{\partial}{\partial n'}\psi(\mathbf{r}')$ and we find that

$$\Phi(\mathbf{r}) = \int_{\partial\tau} dS' \,[\psi(\mathbf{r}')\frac{\partial}{\partial n'}G_{0_-}(\mathbf{r},\mathbf{r}') - G_{0_-}(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}\psi(\mathbf{r}')]$$

which then yields

$$\Phi^*(\mathbf{r}) = \overbrace{\int_{\partial\tau} dS' \,[\psi^*(\mathbf{r}')\frac{\partial}{\partial n'}G_{0_+}(\mathbf{r},\mathbf{r}') - G_{0_+}(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}\psi^*(\mathbf{r}')]}^{U_2^{(in)}(\mathbf{r})} = -\int_{\tau_0} d^3r'\, G_{0_f}(\mathbf{r},\mathbf{r}')Q^*(\mathbf{r}')$$

where we have used the fact that for a lossless medium $G_{0_-} = G^*_{0_+}$ and that $G^*_{0_f}(\mathbf{r},\mathbf{r}') = -G_{0_f}(\mathbf{r},\mathbf{r}')$ (see previous problem). This establishes Eq.(10.2).

To establish Eq.(10.3) we use the simplified model for the scattering potential given in Eq.(10.1b) to find that

$$Q(\mathbf{r}) = V(\mathbf{r})U_1(\mathbf{r}) = \sum_{m=1}^{M} \mathcal{V}_m U_1(\mathbf{X}_m)\delta(\mathbf{r}-\mathbf{X}_m)$$

which when used in Eq.(10.2) yields the required result:

$$U_2^{(in)}(\mathbf{r}) = -\int_{\tau_0} d^3r'\, G_{0_f}(\mathbf{r},\mathbf{r}')Q^*(\mathbf{r}') = -\sum_{m=1}^{M} \mathcal{V}_m^* U_1^*(\mathbf{X}_m)G_{0_f}(\mathbf{r},\mathbf{X}_m).$$

10.3 Use the Lippmann Schwinger equations satisfied by the composite medium Green function to prove that the multistatic data matrix given in Eq.(10.7) is symmetric.

The composite medium Green function $G_+(\mathbf{r},\mathbf{r}')$ satisfies the two forms of the LS integral equations Eqs.(9.45) obtained in the last chapter which for the simplified model of the scattering potential given in Eq.(10.1b) assume the forms

$$G_+(\mathbf{r},\mathbf{r}') = G_{0_+}(\mathbf{r},\mathbf{r}') + \sum_{m=1}^{M} \mathcal{V}_m G_{0_+}(\mathbf{r},\mathbf{X}_m)G_+(\mathbf{X}_m,\mathbf{r}'),$$

$$G_+(\mathbf{r},\mathbf{r}') = G_{0_+}(\mathbf{r},\mathbf{r}') + \sum_{m=1}^{M} \mathcal{V}_m G_+(\mathbf{r},\mathbf{X}_m)G_{0_+}(\mathbf{X}_m,\mathbf{r}').$$

On setting $\mathbf{r} = \boldsymbol{\alpha}_j$ and $\mathbf{r}' = \boldsymbol{\alpha}_k$ we find from the second of the above two equations that

$$K_{j,k}(\omega) = G_+(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_k) - G_{0_+}(\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_k) = \sum_{m=1}^{M} \mathcal{V}_m G_+(\boldsymbol{\alpha}_j,\mathbf{X}_m)G_{0_+}(\mathbf{X}_m,\boldsymbol{\alpha}_k).$$

On the other-hand on making the same substitutions into the first of the two forms of the LS equations given above we obtain

$$K_{j,k}(\omega) = \sum_{m=1}^{M} \mathcal{V}_m G_{0_+}(\boldsymbol{\alpha}_j,\mathbf{X}_m)G_+(\mathbf{X}_m,\boldsymbol{\alpha}_k)$$

$$= \sum_{m=1}^{M} \mathcal{V}_m G_+(\boldsymbol{\alpha}_k,\mathbf{X}_m)G_{0_+}(\mathbf{X}_m,\boldsymbol{\alpha}_j) = K_{k,j}(\omega)$$

since $G_+(\mathbf{X}_m,\boldsymbol{\alpha}_k) = G_+(\boldsymbol{\alpha}_k,\mathbf{X}_m)$ and $G_{0_+}(\mathbf{X}_m,\boldsymbol{\alpha}_j) = G_{0_+}(\boldsymbol{\alpha}_j,\mathbf{X}_m)$. This then establishes the symmetry of the multistatic data matrix.

10.4 Determine the asymptotic form of the multistatic data matrix given in Eq.(10.7) in the limit when the transmit and receive locations lie on the surface on an infinite sphere.

The multistatic data matrix is given by

$$K_{j,k}(\omega) = \sum_{m=1}^{M} \mathcal{V}_m G_+(\boldsymbol{\alpha}_j,\mathbf{X}_m)G_{0_+}(\mathbf{X}_m,\boldsymbol{\alpha}_k).$$

If we then set $\boldsymbol{\alpha}_j = R_0\hat{\boldsymbol{\alpha}}_j$, $\boldsymbol{\alpha}_k = R_0\hat{\boldsymbol{\alpha}}_k$ where R_0 is the radius of an asymptotically large sphere and $\hat{\boldsymbol{\alpha}}_j$ and $\hat{\boldsymbol{\alpha}}_k$ are unit vectors we obtain

$$K_{j,k}(\omega) \sim \sum_{m=1}^{M} \mathcal{V}_m G_+(R_0\hat{\boldsymbol{\alpha}}_j, \mathbf{X}_m) G_{0_+}(\mathbf{X}_m, R_0\hat{\boldsymbol{\alpha}}_k).$$

The two Green functions are asymptotically defined in terms of the plane wave scattering states of the background medium and the composite medium via Eq.(9.9a):

$$G_{0_+}(\mathbf{X}_m, R_0\hat{\boldsymbol{\alpha}}_k) \sim -\frac{1}{4\pi}\frac{e^{ik_0R_0}}{R_0}\psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k),$$
$$G_+(R_0\hat{\boldsymbol{\alpha}}_j, \mathbf{X}_m) \sim -\frac{1}{4\pi}\frac{e^{ik_0R_0}}{R_0}\Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j),$$

where ψ_+ denotes the plane wave scattering state of the background medium and Ψ_+ that of the composite medium. Using the above expressions of the Green functions in the asymptotic expression for the multistatic data matrix then yields

$$K_{j,k}(\omega) \sim \frac{1}{(4\pi)^2}\frac{e^{2ik_0R_0}}{R_0^2}\sum_{m=1}^{M} \mathcal{V}_m \Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j)\psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k)$$

which due to symmetry of the multistatic data matrix can also be written in the form

$$K_{j,k}(\omega) \sim \frac{1}{(4\pi)^2}\frac{e^{2ik_0R_0}}{R_0^2}\sum_{m=1}^{M} \mathcal{V}_m \psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j)\Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k).$$

10.5 Prove that for the simple scattering potential model given in Eq.(10.1b) and co-incident point transmitter and receiver arrays that $K_{j,k}(\omega)$ is the scattered field component of the composite medium Green function $G_+(\alpha_j, \beta_k)$.

See solution to Problem 10.3.

10.6 Prove that the antenna vectors $g(\mathbf{X}_1)$ and $g(\mathbf{X}_2)$ for the case of two co-located transmit and receive elements in a homogeneous background are linearly independent except for certain special scatterer and antenna locations. Determine these special situations where linear independence breaks down.

See Appendix in Devaney (2000) available on the CUP web site for this book.

10.7 Derive the representations of the K matrix given in Eq.(10.21) starting from the Lippmann Schwinger equations from the previous chapter.

See solution to Problem 10.3.

10.8 Formulate the SVD for the case of far field transmit and receive co-located antenna elements considered in Problem 10.4.

We found in Problem 10.4 that for the case of far field transmit and receive co-located antenna elements located on the surface of an asymptotically large sphere of radius R_0 that the multistatic data matrix assumes either of the two forms

$$K_{j,k}(\omega) \sim \frac{1}{(4\pi)^2} \frac{e^{2ik_0R_0}}{R_0^2} \sum_{m=1}^{M} \mathcal{V}_m \Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j)\psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k)$$
$$K_{j,k}(\omega) \sim \frac{1}{(4\pi)^2} \frac{e^{2ik_0R_0}}{R_0^2} \sum_{m=1}^{M} \mathcal{V}_m \psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j)\Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k),$$

where Ψ_+ are the plane wave scattering states of the composite medium and ψ_+ those of the background medium. In the formulation of the SVD for this case it is then natural to replace the data matrix $K_{j,k}$ by

$$K_{j,k}(\omega) \to \sum_{m=1}^{M} \mathcal{V}_m \Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j)\psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k)$$
$$= \sum_{m=1}^{M} \mathcal{V}_m \psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_j)\Psi_+(\mathbf{X}_m, -k_0\hat{\boldsymbol{\alpha}}_k).$$

The SVD is then formulated exactly as for the case of near field antennas but with the Green functions G_+ and G_{0_+} replaced by the plane wave scattering states Ψ_+ and ψ_+, respectively. The background and composite medium Green function vectors defined in Eqs.(10.9) are then replaced by the far field vectors

$$g_0(\mathbf{r}) \to [\psi_+(\mathbf{r}, -k_0\hat{\boldsymbol{\alpha}}_1), \psi_+(\mathbf{r}, -k_0\hat{\boldsymbol{\alpha}}_2), \cdots, \psi_+(\mathbf{r}, -k_0\hat{\boldsymbol{\alpha}}_N)]^T,$$
$$g(\mathbf{r}) \to [\Psi_+(\mathbf{r}, -k_0\hat{\boldsymbol{\alpha}}_1), \Psi_+(\mathbf{r}, -k_0\hat{\boldsymbol{\alpha}}_2), \cdots, \Psi_+(\mathbf{r}, -k_0\hat{\boldsymbol{\alpha}}_N)]^T,$$

where we have set $N_\alpha = N$. Under this replacement the remaining formulation is identical to that of the near field SVD.

10.9 Show that the singular values for the far field SVD considered in the previous problem are invariant under a finite translation of the scattering system.

By "scattering system" is meant the composite system comprised of the background with embedded scatterers. If we denote the plane wave scattering states for the background and composite medium of the system centered at an arbitrary point $\mathbf{x}$ by $\psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{x})$ and $\Psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{x})$ then we showed in Problem 9.3 that under a translation from say a reference point $\mathbf{x}_0'$ to a new reference point $\mathbf{x}_0$ then

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{x}_0) = e^{ik_0\mathbf{s}_0\cdot\delta\mathbf{x}_0}\psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{x}_0'),$$
$$\Psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{x}_0) = e^{ik_0\mathbf{s}_0\cdot\delta\mathbf{x}_0}\Psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{x}_0')$$

where $\delta\mathbf{x}_0 = \mathbf{x}_0 - \mathbf{x}_0'$ with $\mathbf{X}_0$ and $\mathbf{X}_0'$ being any two central locations of the scattering system. It then follows from the expression for the K matrix found in the previous problem that

$$K_{j,k}(\omega,\mathbf{x}_0)=\sum_{m=1}^{M}\mathcal{V}_m\Psi_+(\mathbf{X}_m,-k_0\hat{\boldsymbol{\alpha}}_j;\mathbf{x}_0)\psi_+(\mathbf{X}_m,-k_0\hat{\boldsymbol{\alpha}}_k;\mathbf{x}_0)$$

$$=\sum_{m=1}^{M}\mathcal{V}_m e^{-ik_0\hat{\boldsymbol{\alpha}}_j\cdot\delta\mathbf{x}_0}\Psi_+(\mathbf{X}_m,-k_0\hat{\boldsymbol{\alpha}}_j;\mathbf{x}'_0)e^{-ik_0\hat{\boldsymbol{\alpha}}_k\cdot\delta\mathbf{x}_0}\psi_+(\mathbf{X}_m,-k_0\hat{\boldsymbol{\alpha}}_k;\mathbf{x}'_0)$$

$$=e^{-ik_0(\hat{\boldsymbol{\alpha}}_j+\hat{\boldsymbol{\alpha}}_k)\cdot\delta\mathbf{x}_0}\overbrace{\sum_{m=1}^{M}\mathcal{V}_m\Psi_+(\mathbf{X}_m,-k_0\hat{\boldsymbol{\alpha}}_j;\mathbf{x}'_0)\psi_+(\mathbf{X}_m,-k_0\hat{\boldsymbol{\alpha}}_k;\mathbf{x}'_0)}^{K_{j,k}(\omega,\mathbf{x}'_0)}.$$

Under a system translation we then find that the elements of the K matrix suffers a pure constant phase shift.

We now compute the normal equations satisfied by the $v_p(\mathbf{x}_0)$ and $v_p(\mathbf{x}'_0)$. We have that

$$[K^\dagger(\mathbf{x}_0)K(\mathbf{x}_0)]_{j,k}=\sum_{k'}K^*_{k',j}(\omega,\mathbf{x}_0)K_{k',k}(\omega,\mathbf{x}_0)$$

$$=\sum_{k'}e^{ik_0(\hat{\boldsymbol{\alpha}}_j+\hat{\boldsymbol{\alpha}}_{k'})\cdot\delta\mathbf{x}_0}K^*_{k',j}(\omega,\mathbf{x}'_0)e^{-ik_0(\hat{\boldsymbol{\alpha}}_k+\hat{\boldsymbol{\alpha}}_{k'})\cdot\delta\mathbf{x}_0}K_{k',k}(\omega,\mathbf{x}'_0)$$

$$=e^{ik_0(\hat{\boldsymbol{\alpha}}_j-\hat{\boldsymbol{\alpha}}_k)\cdot\delta\mathbf{x}_0}\sum_{k'}K^*_{k',j}(\omega,\mathbf{x}'_0)K_{k',k}(\omega,\mathbf{x}'_0)=e^{ik_0(\hat{\boldsymbol{\alpha}}_j-\hat{\boldsymbol{\alpha}}_k)\cdot\delta\mathbf{x}_0}[K^\dagger(\mathbf{x}'_0)K(\mathbf{x}'_0)]_{j,k}$$

We then conclude that

$$[K^\dagger(\mathbf{x}_0)K(\mathbf{x}_0)v_p(\mathbf{x}_0)]_j=\sum_k[K^\dagger(\mathbf{x}_0)K(\mathbf{x}_0)]_{j,k}v_p(k,\mathbf{x}_0)=\sigma_p^2(\mathbf{x}_0)v_p(j,\mathbf{x}_0).$$

Using the previous equation then yields

$$\sum_k e^{ik_0(\hat{\boldsymbol{\alpha}}_j-\hat{\boldsymbol{\alpha}}_k)\cdot\delta\mathbf{x}_0}[K^\dagger(\mathbf{x}'_0)K(\mathbf{x}'_0)]_{j,k}v_p(k,\mathbf{x}_0)=\sigma_p^2(\mathbf{x}_0)v_p(j,\mathbf{x}_0).$$

Which we can rewrite in the form

$$\sum_k[K^\dagger(\mathbf{x}'_0)K(\mathbf{x}'_0)]_{j,k}[e^{-ik_0\hat{\boldsymbol{\alpha}}_k\cdot\delta\mathbf{x}_0}v_p(k,\mathbf{x}_0)]=\sigma_p^2(\mathbf{x}_0)[e^{-ik_0\hat{\boldsymbol{\alpha}}_j\cdot\delta\mathbf{x}_0}v_p(j,\mathbf{x}_0)]$$

and from which we conclude that

$$\sigma_p(\mathbf{x}_0)=\sigma_p(\mathbf{x}'_0),\quad e^{-ik_0\hat{\boldsymbol{\alpha}}_k\cdot\delta\mathbf{x}_0}v_p(k,\mathbf{x}_0)=v_p(k,\mathbf{x}'_0).$$

Finally, using the singular equation $Kv_p=\sigma_p u_p$ we find that

$$e^{-ik_0\hat{\boldsymbol{\alpha}}_k\cdot\delta\mathbf{x}_0}u_p(k,\mathbf{x}_0)=u_p(k,\mathbf{x}'_0).$$

10.10 Show that the singular vectors for the far field SVD considered in the previous two problems only suffer a phase shift under a finite translation of the scattering system and determine what that phase shift is.

See solution to preceding problem.

10.11 Show that the Green function vectors $g(\mathbf{r})$ and $g_-(\mathbf{r})$ are related via the equation

$$g(\mathbf{r}) = g_0(\mathbf{r}) + \sum_m \mathcal{V}_m G_{0_+}(\mathbf{r}, \mathbf{X}_m) g(\mathbf{X}_m).$$

This results from using the LS equation

$$G_+(\mathbf{r}, \mathbf{r}') = G_{0_+}(\mathbf{r}, \mathbf{r}') + \int d^3r'' G_{0_+}(\mathbf{r}, \mathbf{r}'') V(\mathbf{r}'') G_+(\mathbf{r}'', \mathbf{r}')$$

with the model given in Eq.(10.1b) for the scattering potential. The Green function vectors $g(\mathbf{r}), g_0(\mathbf{r})$ are then formed from column vectors of these Green functions to give the desired result.

10.12 Show that in place of Eqs.(10.25) we can also represent the matrices $K^\dagger K$ and $KK^\dagger$ in the alternative form

$$K^\dagger K = \sum_{m,m'} \Lambda_{0_\beta}(\mathbf{X}_m, \mathbf{X}_{m'}) g^*_\alpha(\mathbf{X}_m) g^T_\alpha(\mathbf{X}_{m'}), \tag{10.1a}$$

$$KK^\dagger = \sum_{m,m'} \Lambda^*_{0_\alpha}(\mathbf{X}_m, \mathbf{X}_{m'}) g_\beta(\mathbf{X}_m) g^\dagger_\beta(\mathbf{X}_{m'}), \tag{10.1b}$$

where

$$\Lambda_{0_\beta}(\mathbf{X}_m, \mathbf{X}_{m'}) = \mathcal{V}^*_m \mathcal{V}_{m'} H_{0_\beta}(\mathbf{X}_m, \mathbf{X}_{m'}), \quad \Lambda_{0_\alpha}(\mathbf{X}_m, \mathbf{X}_{m'}) = \mathcal{V}^*_m \mathcal{V}_{m'} H_{0_\alpha}(\mathbf{X}_m, \mathbf{X}_{m'}).$$

The first of the two above representations results from using the following representation of the K matrix

$$K(\omega) = \sum_{m=1}^{M} \mathcal{V}_m g_{0_\beta}(\mathbf{X}_m) g^T_\alpha(\mathbf{X}_m),$$

while the second results from using

$$K(\omega) = \sum_{m=1}^{M} \mathcal{V}_m g_\beta(\mathbf{X}_m) g^T_{0_\alpha}(\mathbf{X}_m).$$

10.13 Discuss the implications of using the representations Eqs.(10.1) rather than those in Eqs.(10.25) in the definition of a scatterer being well resolved and also in the actual generation of the DORT image of a well resolved scatterer.

Eqs.(10.1) employ the CPSF's of the background medium $H_{0_\beta}(\mathbf{X}_m, \mathbf{X}_{m'})$ and $H_{0_\alpha}(\mathbf{X}_m, \mathbf{X}_{m'})$ rather than those of the composite medium. This then requires that a scatterer be well resolved relative to the background rather than in the composite medium. Also, the DORT images have to be generated from the singular vectors computed using these representations with the composite medium Green functions which are unknown in general.

10.14 Discuss the implications of using the representations Eqs.(10.1) rather than those in Eqs.(10.25) in constructing the SVD of the K matrix and in time reversal MUSIC.

The same disadvantages apply here as for the generation of the DORT images discussed in the preceding problem. In particular, the steering vector would have to be constructed from the composite medium Green function vectors which are not known.

10.15 Express the filtered DORT and MUSIC algorithms using multiple frequencies.

The reader should have no difficulty doing this problem.