Appendix SA8.3 The Approximate Economy-wide Equivalence of Additive and Multiplicative SDA Effects[1]

A8.3.1 Two-factor SDA Setting

The main result for a two-factor additive SDA (ASDA) is expressed as in (8.7)

$$\Delta\mathbf{x} = \underbrace{(0.5)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change (ASDA)}} + \underbrace{(0.5)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change (ASDA)}} \tag{A8.3.1}$$

while that of the multiplicative SDA (MSDA) has the form shown in (8.49)

$$\frac{\mathbf{i}'\mathbf{x}^1}{\mathbf{i}'\mathbf{x}^0} = \underbrace{\left(\frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1} \times \frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\right)^{0.5}}_{\text{Technology change (MSDA)}} \times \underbrace{\left(\frac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0} \times \frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)^{0.5}}_{\text{Final-demand change (MSDA)}} \tag{A8.3.2}$$

or, in logarithmic form, (8.51),

$$\ln\left(\frac{\mathbf{i}'\mathbf{x}^1}{\mathbf{i}'\mathbf{x}^0}\right) = \underbrace{(0.5)\ln\left(\frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1} \times \frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\right)}_{\text{Technology change (MSDA)}} + \underbrace{(0.5)\ln\left(\frac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0} \times \frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)}_{\text{Final-demand change (MSDA)}} \tag{A8.3.3}$$

In what follows we will make use of the logarithmic mean function, $\mathcal{L}(\bullet,\bullet)$, defined as

$$\mathcal{L}(a,b) = \begin{cases} \dfrac{a-b}{\ln(a)-\ln(b)}, & \text{if } a \neq b \\ a, & \text{if } a = b \end{cases}$$

This function is symmetric, i.e., $\mathcal{L}(a,b) = \mathcal{L}(b,a)$, and this will be useful in what follows. The following theorem indicates conditions under which results from ASDA and MSDA decompositions will be approximately the same.

Theorem 1: *Consider the two-factor ASDA and MSDA frameworks given, respectively, in* (A8.3.1) *and* (A8.3.3). *If the condition* $\mathcal{L}(\mathbf{i}'\mathbf{x}^1,\ \mathbf{i}'\mathbf{x}^0) \approx \mathcal{L}(\mathbf{i}'\mathbf{L}^1\mathbf{f}^0,\ \mathbf{i}'\mathbf{L}^0\mathbf{f}^1)$ *holds, then the economy-wide relative contributions of the technology and final demand effects under these two decomposition formulations are approximately equal, i.e.,*

[1] This material is used, with the author's permission, from an unpublished paper by Umed Temursho, "Approximate Economy-wide Equivalence of the Additive and Multiplicative SDA Effects," dated March 14, 2019.

$$\underbrace{\frac{(0.5)\mathbf{i}'\Delta\mathbf{L}(\mathbf{f}^0+\mathbf{f}^1)}{\mathbf{i}'\Delta\mathbf{x}}}_{\substack{\text{Technology relative contribution}\\ \text{(ASDA)}}} \approx \underbrace{\frac{(0.5)\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\times\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1/\mathbf{i}'\mathbf{x}^0\right)}}_{\text{Technology relative contribution (MSDA)}} \tag{A8.3.4}$$

$$\underbrace{\frac{(0.5)\mathbf{i}'(\mathbf{L}^0+\mathbf{L}^1)\Delta\mathbf{f}}{\mathbf{i}'\Delta\mathbf{x}}}_{\substack{\text{Final-demand contribution}\\ \text{(ASDA)}}} \approx \underbrace{\frac{(0.5)\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\times\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1/\mathbf{i}'\mathbf{x}^0\right)}}_{\text{Final-demand contribution (MSDA)}} \tag{A8.3.5}$$

In words: if the logarithmic average of total outputs in periods 0 and 1, $\mathcal{L}(\mathbf{i}'\mathbf{x}^1,\ \mathbf{i}'\mathbf{x}^0)$, is close to the logarithmic average of total hypothetical outputs when the technology and final demand components of the two periods are interchanged, $\mathcal{L}(\mathbf{i}'\mathbf{L}^1\mathbf{f}^0,\ \mathbf{i}'\mathbf{L}^0\mathbf{f}^1)$, then the relative economy-wide technology and final demand effects found by the ASDA and MSDA formulations will be roughly the same.

Proof of Theorem 1 We start with (A8.3.4), ignoring the (0.5) factor on both sides.[2] The numerator on its left-hand side (lhs) can be written as

$$\mathbf{i}'\Delta\mathbf{L}(\mathbf{f}^0+\mathbf{f}^1)=\mathbf{i}'(\mathbf{L}^1-\mathbf{L}^0)(\mathbf{f}^0+\mathbf{f}^1)=\mathbf{i}'\mathbf{x}^1-\mathbf{i}'\mathbf{L}^0\mathbf{f}^1+\mathbf{i}'\mathbf{L}^1\mathbf{f}^0-\mathbf{i}'\mathbf{x}^0$$

So the lhs boils down to

$$\frac{\mathbf{i}'\Delta\mathbf{L}(\mathbf{f}^0+\mathbf{f}^1)}{\mathbf{i}'\Delta\mathbf{x}}=1+\frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0-\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{x}^1-\mathbf{i}'\mathbf{x}^0} \tag{A8.3.6}$$

Using the rule for the logarithm of a product, the right-hand side (rhs) of (A8.3.4), can be written as

$$\frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\times\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1/\mathbf{i}'\mathbf{x}^0\right)}=\frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\times\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1/\mathbf{i}'\mathbf{x}^0\right)}=\frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\right)+\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1/\mathbf{i}'\mathbf{x}^0\right)}=1+\frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1/\mathbf{i}'\mathbf{x}^0\right)} \tag{A8.3.7}$$

From (A8.3.6) and (A8.3.7), the approximate equivalence of the ASDA and MSDA technology effects in (A8.3.4) is equivalent to

[2] Since (0.5) appears on both sides of equations (A8.3.4) and (A8.3.5), it will be ignored in what follows in proving the approximate equivalences of the lhs and rhs in these equations.

$$\frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0 - \mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0} \approx \frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1 / \mathbf{i}'\mathbf{x}^0\right)}$$

or, rearranging,

$$\frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0 - \mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}\right)} \approx \frac{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0}{\ln\left(\mathbf{i}'\mathbf{x}^1 / \mathbf{i}'\mathbf{x}^0\right)}$$

Finally, using the logarithm of a quotient rule, this is equivalent to

$$\frac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0 - \mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\ln(\mathbf{i}'\mathbf{L}^1\mathbf{f}^0) - \ln(\mathbf{i}'\mathbf{L}^0\mathbf{f}^1)} \approx \frac{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0}{\ln(\mathbf{i}'\mathbf{x}^1) - \ln(\mathbf{i}'\mathbf{x}^0)} \qquad \text{(A8.3.8)}$$

From the definition of the logarithmic mean, the approximate equivalence of the ASDA and MSDA technology effects in (A8.3.8) is exactly $\mathcal{L}(\mathbf{i}'\mathbf{L}^1\mathbf{f}^0,\ \mathbf{i}'\mathbf{L}^0\mathbf{f}^1) \approx \mathcal{L}(\mathbf{i}'\mathbf{x}^1,\ \mathbf{i}'\mathbf{x}^0)$, as given in Theorem 1.

Finally, we show a similar connection for the second approximation, (A8.3.5) in Theorem 1. The lhs of (A8.3.5) can be written as

$$\frac{\mathbf{i}'(\mathbf{L}^0 + \mathbf{L}^1)\Delta\mathbf{f}}{\mathbf{i}'\Delta\mathbf{x}} = \frac{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0 + \mathbf{i}'\mathbf{L}^0\mathbf{f}^1 - \mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0} = 1 + \frac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1 - \mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0} \qquad \text{(A8.3.9)}$$

and the rhs of (A8.3.5) is equivalent to:

$$\frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0} \times \dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1 / \mathbf{i}'\mathbf{x}^0\right)} = \frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^1\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^0\mathbf{f}^0}\right) + \ln\left(\dfrac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1 / \mathbf{i}'\mathbf{x}^0\right)} = 1 + \frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1 / \mathbf{i}'\mathbf{x}^0\right)} \qquad \text{(A8.3.10)}$$

Thus, using (A8.3.9) and (A8.3.10), the approximate equivalence of the ASDA and MSDA final demand effects in (A8.3.5) is exactly equivalent to

$$\frac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1 - \mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0} \approx \frac{\ln\left(\dfrac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1}{\mathbf{i}'\mathbf{L}^1\mathbf{f}^0}\right)}{\ln\left(\mathbf{i}'\mathbf{x}^1 / \mathbf{i}'\mathbf{x}^0\right)}$$

As above, this can be rewritten as

$$\frac{\mathbf{i}'\mathbf{L}^0\mathbf{f}^1 - \mathbf{i}'\mathbf{L}^1\mathbf{f}^0}{\ln(\mathbf{i}'\mathbf{L}^0\mathbf{f}^1) - \ln(\mathbf{i}'\mathbf{L}^1\mathbf{f}^0)} \approx \frac{\mathbf{i}'\mathbf{x}^1 - \mathbf{i}'\mathbf{x}^0}{\ln(\mathbf{i}'\mathbf{x}^1) - \ln(\mathbf{i}'\mathbf{x}^0)} \qquad \text{(A8.3.11)}$$

In terms of the logarithmic average, (A8.3.11) is exactly $\mathcal{L}(\mathbf{i}'\mathbf{L}^0\mathbf{f}^1,\ \mathbf{i}'\mathbf{L}^1\mathbf{f}^0) \approx \mathcal{L}(\mathbf{i}'\mathbf{x}^1,\ \mathbf{i}'\mathbf{x}^0)$. Because the logarithmic mean is symmetric, this is also $\mathcal{L}(\mathbf{i}'\mathbf{L}^1\mathbf{f}^0,\ \mathbf{i}'\mathbf{L}^0\mathbf{f}^1) \approx \mathcal{L}(\mathbf{i}'\mathbf{x}^1,\ \mathbf{i}'\mathbf{x}^0)$, and again this is exactly the condition given in Theorem 1. Q.E.D.

In the same paper Temursho uses world input-output tables (available from WIOD) for years from 1995 to 2011, encompassing 40 countries with 35 sectors each, for an empirical illustration. The base period (year 0) is fixed at 1995. Each successive year $(1996,\cdots,2011)$ becomes year 1 for which the technology and final demand effects were found using both ASDA and MSDA approaches. Percentage differences for each effect (technology and final demand) for each year are calculated as Diff% = $[$(ASDA effect)/(MSDA effect) $-1]\times 100$ and corresponding logarithmic mean differences are found as Diff $\mathcal{L}\% = [\mathcal{L}(\mathbf{i}'\mathbf{L}^t\mathbf{f}^0,\ \mathbf{i}'\mathbf{L}^0\mathbf{f}^t)/\mathcal{L}(\mathbf{i}'\mathbf{x}^t,\ \mathbf{i}'\mathbf{x}^0) - 1]\times 100$, for $0 = 1995$ and $t = 1996,\cdots,2011$. Table 8.3.1 shows the results of these experiments, where the relative economy-wide contributions of the technology and final effects are seen to be (approximately) equivalent as long as the logarithmic mean differences fall within the ±1 percent range.

Table 8.3.1 Relative Contributions of the Total Technology and Final Demand Effects

	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Technology Effect (Relative Economy-wide Contribution, %)																
ASDA	-2.02	21.64	23.51	5.44	11.11	11.10	2.73	2.76	3.17	5.05	5.73	5.97	6.45	5.10	5.28	4.74
MSDA	-2.03	21.64	23.50	5.42	11.10	11.11	2.76	2.74	3.13	4.93	5.55	5.68	6.12	4.77	4.91	4.32
Diff%	-0.20	0.00	0.00	0.30	0.10	-0.10	-0.80	0.70	1.40	2.30	3.20	5.00	5.40	6.90	7.60	9.80
Final Demand Effect (Relative Economy-wide Contribution, %)																
ASDA	102.02	78.36	76.49	94.56	88.89	88.90	97.27	97.24	96.83	94.95	94.27	94.03	93.55	94.90	94.72	95.26
MSDA	102.03	78.36	76.50	94.58	88.90	88.89	97.24	97.26	96.87	95.07	94.45	94.32	93.88	95.23	95.09	95.68
Diff%	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.10	-0.20	-0.30	-0.40	-0.30	-0.40	-0.40
Percentage Differences in Logarithmic Means of Total (Actual and Hypothetical) Outputs (%)																
Diff $\mathcal{L}$ %	-0.01	-0.01	-0.04	-0.03	-0.02	0.04	0.05	-0.04	-0.09	-0.26	-0.40	-0.64	-0.76	-0.73	-0.83	-0.92

A8.3.2 Three-factor SDA Setting

With three factors, labor inputs (employment, **h**), technology (**L**) and final demand (**f**), the corresponding main results are (where $\boldsymbol{\varepsilon} = \hat{\mathbf{h}}_c\mathbf{L}\mathbf{f}$ and $\varepsilon = \mathbf{i}'\boldsymbol{\varepsilon}$) :

Additive, as in (8.30),

$$\Delta\boldsymbol{\varepsilon} = (1/2)\underbrace{(\Delta\hat{\mathbf{h}}_c)(\mathbf{L}^0\mathbf{f}^0 + \mathbf{L}^1\mathbf{f}^1)}_{\text{Labor input coefficient change}} + (1/2)\underbrace{[\hat{\mathbf{h}}_c^0(\Delta\mathbf{L})\mathbf{f}^1 + \hat{\mathbf{h}}_c^1(\Delta\mathbf{L})\mathbf{f}^0]}_{\text{Technology change}} + (1/2)\underbrace{(\hat{\mathbf{h}}_c^0\mathbf{L}^0 + \hat{\mathbf{h}}_c^1\mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change}} \quad \text{(A8.3.12)}$$

Multiplicative, as in (8.54)

$$\frac{\mathbf{i}'\boldsymbol{\varepsilon}^1}{\mathbf{i}'\boldsymbol{\varepsilon}^0} = \underbrace{\left(\frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^0}\right)^{0.5}}_{\text{Labor input coefficient change}} \times \underbrace{\left(\frac{(\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0}{(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0}\right)^{0.5}}_{\text{Technology change}} \times \underbrace{\left(\frac{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^0} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0}\right)^{0.5}}_{\text{Final-demand change}} \quad \text{(A8.3.13)}$$

And the logarithmic version of (A8.3.13) is

$$\begin{aligned} \ln\left(\frac{\mathbf{i}'\boldsymbol{\varepsilon}^1}{\mathbf{i}'\boldsymbol{\varepsilon}^0}\right) &= \underbrace{(0.5)\left(\frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^0}\right)}_{\text{Labor input coefficient change}} + \underbrace{(0.5)\ln\left(\frac{(\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0}{(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0}\right)}_{\text{Technology change}} \\ &\quad + \underbrace{(0.5)\ln\left(\frac{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^0} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0}\right)}_{\text{Final-demand change}} \end{aligned} \quad \text{(A8.3.14)}$$

The counterpart of Theorem 1 in the three-factor SDA case is:

Theorem 2: *Consider the three-factor ASDA and MSDA frameworks given in* (A8.3.12) *and* (A.8.3.14). *If the conditions* $\mathcal{L}(\mathbf{i}'\boldsymbol{\varepsilon}^1,\ \mathbf{i}'\boldsymbol{\varepsilon}^0) \approx \mathcal{L}[(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0, (\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1] \approx \mathcal{L}[(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0, (\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1]$ *hold, then the economy-wide relative contributions of the employment, technology and final demand effects are approximately equal under the two SDA formulations, i.e.*

$$\frac{(0.5)\Delta\mathbf{h}_c'(\mathbf{L}^0\mathbf{f}^0 + \mathbf{L}^1\mathbf{f}^1)}{\mathbf{i}'\Delta\boldsymbol{\varepsilon}} \approx \frac{(0.5)\ln\left(\frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^0}\right)}{\ln(\mathbf{i}'\boldsymbol{\varepsilon}^1 - \mathbf{i}'\boldsymbol{\varepsilon}^0)} \quad \text{(A8.3.15)}$$

$$\frac{(0.5)[(\mathbf{h}_c^0)'\Delta\mathbf{L}\mathbf{f}^1 + (\mathbf{h}_c^1)'\Delta\mathbf{L}\mathbf{f}^0]}{\mathbf{i}'\Delta\boldsymbol{\varepsilon}} \approx \frac{(0.5)\ln\left(\frac{(\mathbf{h}_c^0)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0}{(\mathbf{h}_c^1)'\mathbf{L}^0\mathbf{f}^0}\right)}{\ln(\mathbf{i}'\boldsymbol{\varepsilon}^1 - \mathbf{i}'\boldsymbol{\varepsilon}^0)} \quad \text{(A8.3.16)}$$

$$\frac{(0.5)[(\mathbf{h}_c^0)'\mathbf{L}^0 + (\mathbf{h}_c^1)'\mathbf{L}^1]\Delta\mathbf{f}}{\mathbf{i}'\Delta\boldsymbol{\varepsilon}} \approx \frac{(0.5)\ln\left(\frac{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^1}{(\mathbf{h}_c^0)'\mathbf{L}^0\mathbf{f}^0} \times \frac{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^1}{(\mathbf{h}_c^1)'\mathbf{L}^1\mathbf{f}^0}\right)}{\ln(\mathbf{i}'\boldsymbol{\varepsilon}^1 - \mathbf{i}'\boldsymbol{\varepsilon}^0)} \quad \text{(A8.3.17)}$$

The proof is similar to that for the two-factor case (although more complicated) and is omitted here. In an additional empirical illustration Temursho uses the same WIOD data, now including employment figures. In this particular case, the approximate equivalence of the ASDA and MSDA results holds only for the earliest two years (1996 and 1997, compared to 1995), where the percentage differences of the logarithmic means of the hypothetical and actual total employment figures are lower than ±1.5%. For other years the ASDA and MSDA results are often very different, increasing with increasing distance between t^0 and t^1. Hence, in general, for factors other than gross output the ASDA and MSDA approaches give different results.