Signals and Systems

Principles and Applications

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Introduction to Signals

P 1.1 Find the recovered signal if a signal of frequency 50 Hz is sampled using a sampling frequency of 80 Hz. What is the phase value?

Solution

Let $x(t) = \sin(2\pi F t)$

Putting F = 50 Hz and $t = nT = n/F_S$, we get

$$x(t) = \sin\left(\frac{2\pi \times 50n}{80}\right) = \sin\left(\frac{5\pi n}{4}\right)$$

$$= \sin\left((2\pi - \frac{3\pi}{4})n\right) = \sin\left(-\frac{2\pi \times 3n}{8}\right)$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = -3/8 = -30/80$, so analog frequency recovered is 30 Hz. There is a negative sign so a phase shift of 180° is introduced.

P 1.2 An analog signal can be represented as

$$x(t) = \cos(150\pi t) + 2\sin(300\pi t) - 400\cos(600\pi t)$$

What is the Nyquist rate for this signal? If the signal is sampled with a sampling frequency of 300 Hz, what is the DT signal obtained after sampling? What is the recovered signal?

Solution

The signal is given by

$$x(t) = \cos(150\pi t) + 2\sin(300\pi t) - 400\cos(600\pi t)$$

The maximum frequency in the signal is 300 Hz. So the Nyquist frequency is 600 Hz. If the signal is sampled using a sampling frequency of 300 Hz, by putting $t = n/300$ in the equation, we get

$$x(t) = \cos\left(150\pi n/300\right) + 2\sin\left(300\pi n/300\right) - 400\cos\left(600\pi n/300\right)$$

$$= \cos\left(\pi n/2\right) + 2\cos(\pi n) - 400\cos(2\pi n)$$

$$= \cos\left(\pi n/2\right) + 2\cos(\pi n) - 400$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. $f = -1/4$, $1/2$, $0 = -75/300$, $150/300$ and 0, so the analog frequency recovered is 75 Hz and 150 Hz. Here the 300 Hz component will be aliased as the DC component.

P 1.3 Let an analog signal be represented as

$$x(t) = \sin(10\pi t) + 2\sin(20\pi t) - 2\cos(30\pi t)$$

What is the Nyquist rate for this signal? If the signal is sampled with a sampling frequency of 20 Hz, what is the DT signal obtained after sampling? What is the recovered signal?

Solution

The signal is given by

$$x(t) = \sin(10\pi t) + 2\sin(20\pi t) - 2\cos(30\pi t)$$

The signal has frequencies 5 Hz, 10 Hz and 15 Hz. The maximum frequency in the signal is 15 Hz, so the Nyquist frequency is 30 Hz.

If the signal is sampled using a sampling frequency of 20 Hz, by putting $t = n/20$ in the equation, we get

$$x(t) = \sin\left(10\pi n/20\right) + 2\sin\left(20\pi n/20\right) - 2\cos\left(30\pi n/20\right)$$

$$= \sin\left(\pi n/2\right) + 2\sin(\pi n) - 2\cos\left(3\pi n/2\right)$$

$$= \sin(\pi n/2) - \cos(2\pi n - \pi n/2)$$

$$= \sin(\pi n/2) - \cos(-\pi n/2)$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. $f = -1/4, 0, -1/4 = 5/20, 0, -5/20$ so the analog frequency recovered is 5 Hz, 0 Hz, –5 Hz. The 10 Hz component will be aliased as the DC component.

P 1.4 Find the recovered signal if a signal of frequency 150 Hz is sampled using sampling frequencies of 400 Hz and 200 Hz, respectively. What is the phase value in each case?

Solution

Let $x(t) = \sin(2\pi Ft)$. Putting $F = 150$ Hz and $t = nT = n/F_S$, $F_S = 400$ Hz, we get

$$x(t) = \sin\left(\frac{2\pi \times 150n}{400}\right) = \sin\left(\frac{2\pi \times 3}{8} \times n\right) = \sin\left(\frac{3\pi}{4} \times n\right) = \sin\left(2 \times \frac{3\pi n}{8}\right)$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = 3/8 = 150/400$, so the analog frequency recovered is 150 Hz. There is a positive sign so a phase shift of 0° is introduced.

Let $x(t) = \sin(2\pi Ft)$. Putting $F = 150$ Hz and $t = nT = n/F_S$, $F_S = 200$ Hz, we get

$$x(t) = \sin\left(\frac{2\pi \times 150n}{200}\right) = \sin\left(\frac{2\pi \times 3}{4} \times n\right) = \sin\left(2\pi - \frac{2\pi}{4} \times n\right) = \sin\left(-\frac{2\pi n}{4}\right)$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = -1/4 = -50/200$, so the analog frequency recovered is 50 Hz. There is a negative sign so a phase shift of 180° is introduced.

P 1.5 Design an anti-aliasing filter for a signal represented as

$$x(t) = \sin(80\pi t) + \sin(100\pi t) - 6\cos(150\pi t)$$

Solution

We have to design an anti-aliasing filter for a signal represented as

$$x(t) = \sin(80\pi t) + \sin(100\pi t) - 6\cos(150\pi t)$$

Compare the equation with $x(n) = \sin(2\pi Fn)$, where F is the analog frequency. We find that the signal contains frequencies 40 Hz, 50 Hz and 75 Hz. So the required sampling frequency is 150 Hz and the Nyquist frequency is 75 Hz. Anti-aliasing filter must have a cut-off frequency of 75 Hz.

P 1.6 Design an anti-aliasing filter for a signal represented as

$$x(t) = \cos(170\pi t) + \cos(190\pi t) - 3\cos(250\pi t)$$

Solution

We have to design an anti-aliasing filter for a signal represented as

$$x(t) = \cos(170\pi t) + \cos(190\pi t) - 3\cos(250\pi t)$$

Compare the equation with $x(n) = \sin(2\pi Fn)$, where F is the analog frequency. We find that the signal contains frequencies 85 Hz, 95 Hz and 125 Hz. So the required sampling frequency is 250 Hz. The Nyquist frequency is 125 Hz. Anti aliasing filter must have a cut-off frequency of 125 Hz.

P 1.7 Find the recovered signal, if a signal with frequencies 150 Hz and 250 Hz are sampled using a sampling frequency of 200 Hz. What is the phase value?

Solution

Let $x(t) = \sin(2\pi Ft)$. Putting F = 150 Hz and $t = nT = n/F_S$, F_S=200 Hz, we get

$$x[n] = \sin\left(2\pi \times \frac{150n}{200}\right) = \sin\left(2\pi \times \frac{3}{4} \times n\right)$$

$$= \sin\left(2\pi - \frac{-2\pi}{4} \times n\right) = \sin\left(\frac{-2\pi n}{4}\right)$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = -1/4 = -50/200$, so the analog frequency recovered is 50 Hz. There is a negative sign so a phase shift of 180° is introduced.

Let $x(t) = \sin(2\pi Ft)$. Putting F = 250 Hz and $t = nT = n/F_S$, F_S = 200 Hz, we get

$$x[n] = \sin\left(2\pi \times \frac{250n}{200}\right) = \sin\left(2\pi \times \frac{5}{4} \times n\right) = \sin\left[\left(2\pi + \frac{2\pi}{4}\right)n\right] = \sin\left(\frac{2\pi n}{4}\right)$$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = 1/4 = 50/200$, so the analog frequency recovered is 50 Hz. There is a positive sign so a phase shift of 0° is introduced.

P 1.8 Consider the analog sinusoidal signal

$$x(t) = 5\sin(500\pi t)$$

a. The signal is sampled with F_S = 1500 Hz. Find the frequency of the DT signal.

b. Find the frequency of the DT signal if F_S = 300 Hz.

Solution

a. Let $x(t) = 5\sin(2\pi Ft)$. Putting $F = 250$ Hz and $t = nT = n/F_S$, $F_S = 1500$ Hz, we get

$$x(t) = \sin\left(2\pi \times \frac{250n}{1500}\right) = \sin\left(2\pi \times \frac{1}{6} \times n\right) = \sin\left(\frac{2\pi}{3} \times n\right) = \sin\left(\frac{2\pi n}{3}\right)$$

make it $x(n)$ =

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = 1/6 = 250/1500$, so the analog frequency recovered is 250 Hz. There is a positive sign so a phase shift of 0° is introduced.

b. Let $x(t) = \sin(2\pi Ft)$. Putting F = 250 Hz and $t = nT = \mathrm{n}/F_S$, F_S = 300 Hz, we get

$$x(t) = \sin\left(2\pi \times \frac{250n}{300}\right) = \sin\left(2\pi \times \frac{5}{6} \times n\right) = \sin\left(2\pi - \frac{2\pi}{6} \times n\right) = \sin\left(-\frac{2\pi n}{6}\right)$$

make it $x(n)$ =

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = -1/6 = -50/300$, so the analog frequency recovered is 50 Hz. There is a negative sign so a phase shift of 180° is introduced.

P 1.9 An analog signal given by $x(t) = \sin(200\pi t) + 3\cos(250\pi t)$ is sampled at a rate 300 sample/s. Find the frequency of the DT signal.

Solution

Given that $x(t) = \sin(200\pi t) + 3\cos(250\pi t)$ and the frequencies in the signal are 100 Hz and 125 Hz. The sampling frequency F_S = 300 Hz.

Let us put F = 100 Hz and $t = nT = n/F_S$, F_S = 300 Hz. We get

$$x(t) = \sin\left(2\pi \times \frac{100n}{300}\right) = \sin\left(\frac{2\pi}{3} \times n\right) = \sin\left(\frac{2\pi}{3} \times n\right) = \sin\left(\frac{-2\pi n}{3}\right)$$

make it $x(n) =$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = 1/3 = 100/300$, so the analog frequency recovered is 100 Hz. There is a positive sign so a phase shift of 0° is introduced.

Let us put $F = 125$ Hz and $t = nT = n/F_S$, $F_S = 300$ Hz. We get

$$x(t) = \sin\left(2\pi \times \frac{125n}{300}\right) = \sin\left(2\pi \times \frac{5}{12} \times n\right)$$

$$= \sin\left(2\pi \times \frac{5}{12} \times n\right) = \sin\left(2\pi n \times \frac{5}{12}\right)$$

make it $x(n) =$

Compare the equation with $x(n) = \sin(2\pi fn)$, where f is the recovered digital frequency given by $f = F/F_S$. Here $f = 5/12 = 125/300$, so the analog frequency recovered is 125 Hz. There is a positive sign so a phase shift of 0° is introduced.

2

Signals and Operations on Signals

P 2.6 (Analog signal) Consider a signal given by $x(t) = \sin(2\pi ft)$, where f is the frequency of the signal equal to 100 Hz. Plot a signal.

Solution

The signal exists for all t. This is an analog signal. The signal is plotted in Fig. 2.1. The plot is shown only for some finite duration.

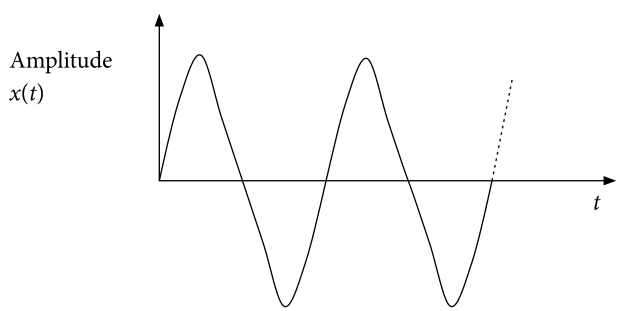


Fig. 2.1 A sine wave signal existing for all continuous values of t

Let us do it practically. We will use MATLAB to generate and plot this analog signal. A MATLAB program is given here. We will generate a vector of time values with a sampling interval of 0.001 seconds. Using a plot command, it joins all successive s values to get the appearance of a continuous signal.

```
clear all;
f=100;
w=2*pi*f;
t=0:0.0001:0.1;
s=sin(w*t);
plot(t,s);
```

```
title('plot of sine wave-approximation to analog sine
wave is plotted');
xlabel('time'); ylabel('amplitude');
```

The output of the MATLAB program is plotted in Fig. 2.2.

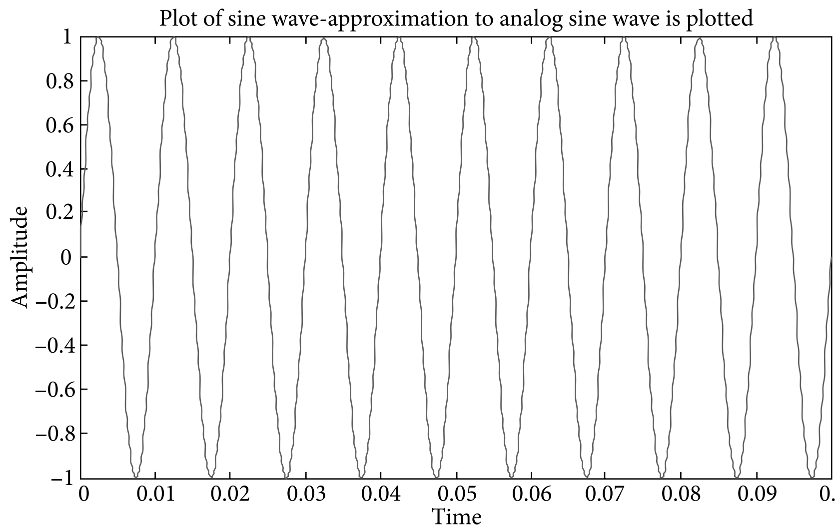


Fig. 2.2 Plot of a sine wave using MATLAB

P 2.7 Plot an analog signal given by

$x(t) = 0.1 \times t$ for $0 \le t \le 10$ seconds
$= 0$ otherwise

Solution

The signal is defined only for values of t between 0 to 5 seconds. The signal plot can be represented as shown in Fig. 2.3.

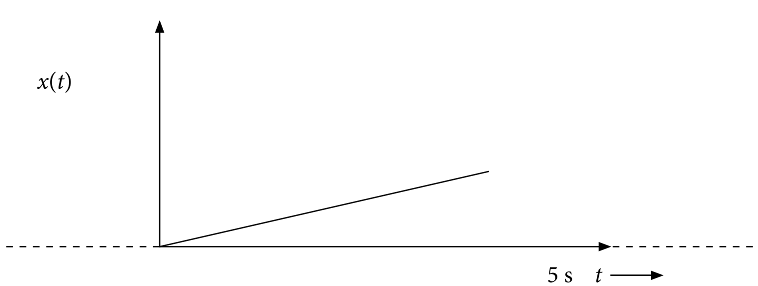


Fig. 2.3 Plot of signal $x(t)$ with slope = 0.1

Let us write the program to generate this signal. We will again generate a vector of time values with a sampling interval of 0.01 seconds. Using a plot command, it joins all successive x values to get the appearance of a continuous signal. The output of the program is plotted in Fig. 2.4.

A MATLAB program can be written as

```
clear all;
t=0:0.1:10;
x=0.01*t;
plot(t,x);
title('plot of signal x');
xlabel('time'); ylabel('amplitude');
```

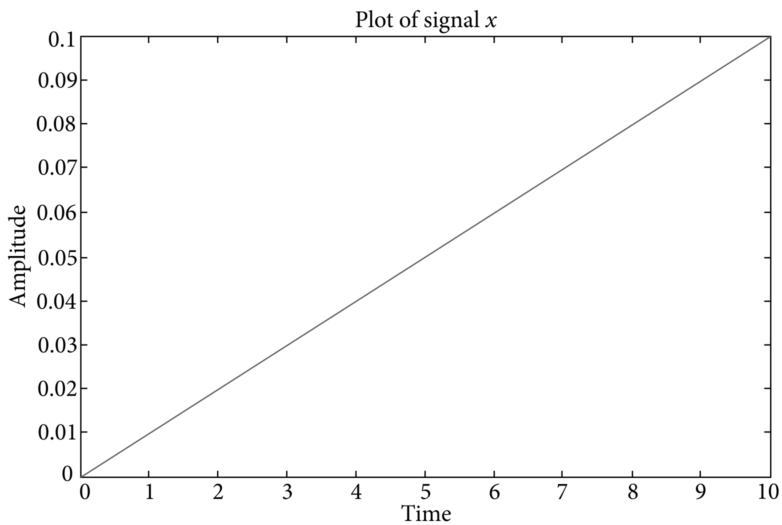


Fig. 2.4 Plot of signal x

P 2.8 Plot analog signal given by

$$x(t) = -1 \text{ for } 0 \le t \le 2 \text{ seconds}$$
$$= 1 \text{ for } 2 \le t \le 4 \text{ seconds}$$

Solution

The signal is defined only for values of t between 0 to 4 seconds. The signal plot can be represented as shown in Fig. 2.5.

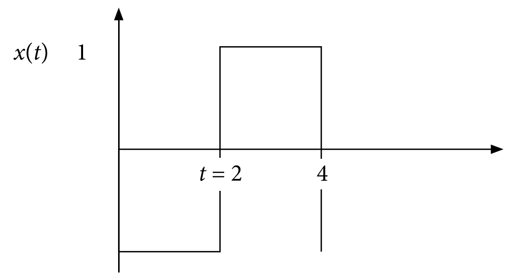


Fig. 2.5 Plot of signal $x(t)$ defined between $t = 0$ to $t = 4$ seconds

Let us write the program to generate this signal. We will again generate a vector of time values with a sampling interval of 0.01 seconds. Using a plot command, it joins all successive x values to get the appearance of a continuous signal. The output of the program is plotted in Fig. 2.6. Here, the signal values are discontinuous. This makes the program difficult to write in the discrete time domain. To generate an index as an integer in MATLAB, time values are multiplied by 10.

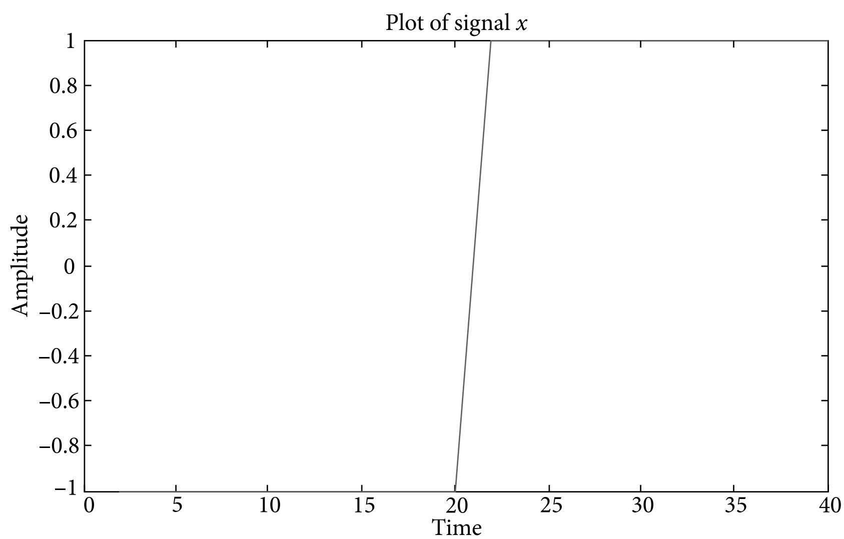


Fig. 2.6 Plot of signal x

A MATLAB program can be written as

```
clear all;
t=0:0.01:2;
i=1+t*10;
for i=1:20,
    x(i)= -1;
end
```

```
t=2:0.01:4;
i=11+(t-1)*10;
for i=21:40,
    x(i)= 1;
end
plot(x);
title('plot of signal x');
xlabel('time'); ylabel('amplitude');
```

P 2.9 (Discrete time signal) Consider a sampled signal $x(t) = \sin(20\pi t)$, where a sample is taken at $t = 0, T, 2T, 3T$ etc. T represents a sampling time given by

$$T = \frac{1}{f_s},\ f_s = \text{sampling frequency}.$$

The signal can be written as $x(n) = \sin(20\pi nT)$. Plot the signal.

Solution

The sampled signal represented as $x(n)$ exists for discrete time values, i.e., at $t = 0, 1 \times T, 2 \times t$, etc. The nth sample is represented as $x(n)$. Hence, it is termed as a discrete time signal or DT signal. It does not mean that the signal has a zero value at all other values of time.

Let us write the program to generate this signal. A MATLAB program to generate a signal is given as follows. The plot of the signal is shown in Fig. 2.7.

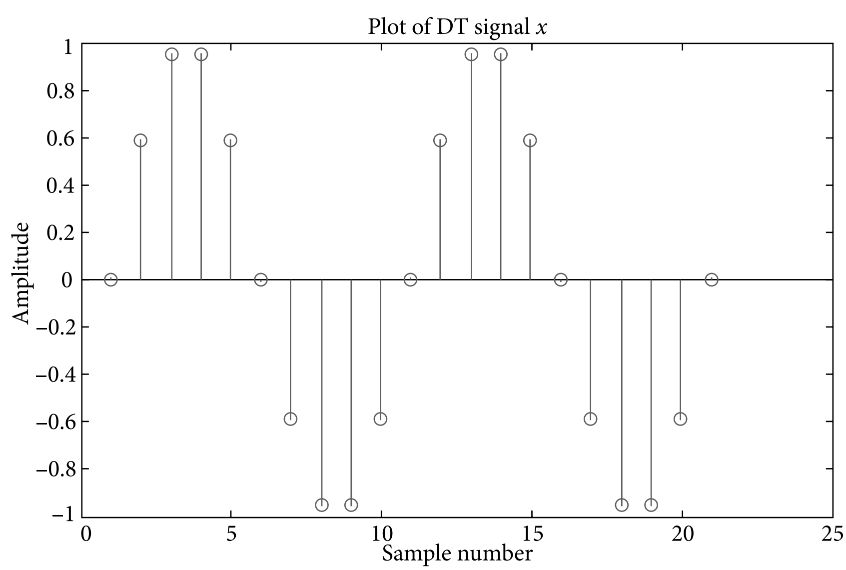


Fig. 2.7 Plot of signal x

```
clear all;
f=10;
T=0.01;
for n=1:21,
x(n)=sin(2*pi*f*(n-1)*T);
end
stem(x);title('plot of DT signal x');
xlabel('sample number');ylabel('amplitude');
```

P 2.10 (Discrete time signal) Plot a sampled signal

$$\begin{aligned} x(n) &= 1 \text{ for } 0 \le n \le 3 \\ &= 1 \text{ for } 4 \le n \le 6 \\ &= 0 \text{ otherwise} \end{aligned}$$

Solution

The signal is defined as 1 for values of n from 0 to 3 and equal to –1 for n = 4 to 6 only. It is equal to zero for all other values of n. The signal plot is shown in Fig. 2.8.

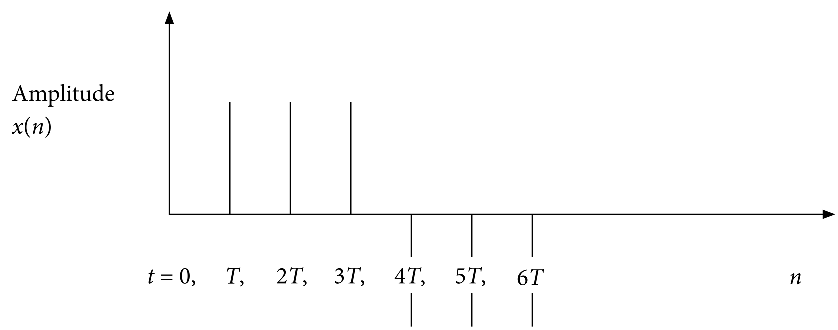


Fig. 2.8 Plot of signal $x(n)$ in P 2.10

Let us write the program to generate this signal. A MATLAB program to generate the signal is given as follows. The plot of the signal is shown in Fig. 2.9. MATLAB does not use the index as zero. Hence, we have to do a trick. Generate the s variable between 0 to 6 and plot the values of x against s.

```
clear all;
for n=1:4,
  x(n)=1;
end
for n=5:7,
  x(n)=-1;
end
s=0:1:6;
```

```
stem(s,x);title('plot of DT signal x');
xlabel('sample number');ylabel('amplitude');
```

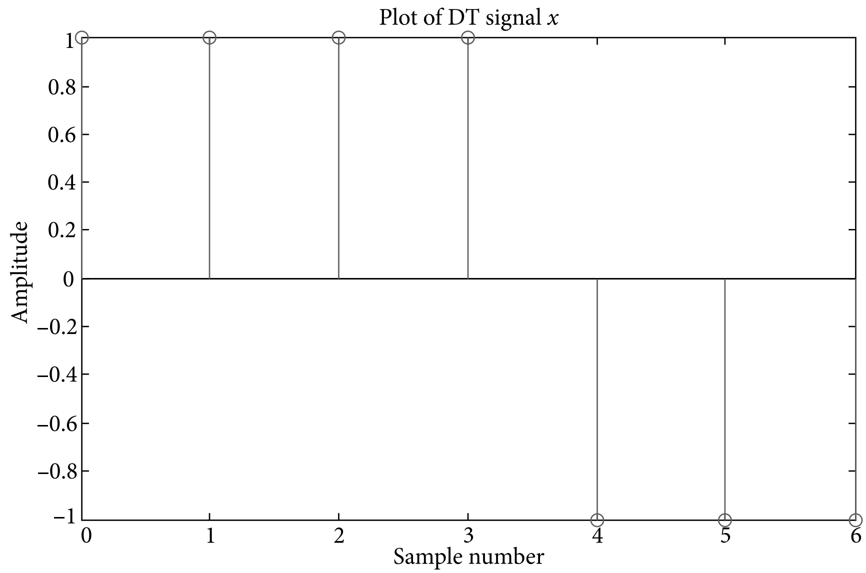


Fig. 2.9 Plot of signal x

P 2.11 (Discrete time signal) Plot a sampled signal

$$
\begin{aligned}
x(n) &= -2 \quad \text{for } 1 \le n \le 3 \\
&= 1 \quad \text{for } -2 \le n \le 0 \\
&= 0 \quad \text{otherwise}
\end{aligned}
$$

Solution

The signal is defined as 1 for values of n from −2 to 0 and equal to -1 from n from 1 to 3 only. It is equal to zero for all other values of n. The signal plot is shown in Fig. 2.10.

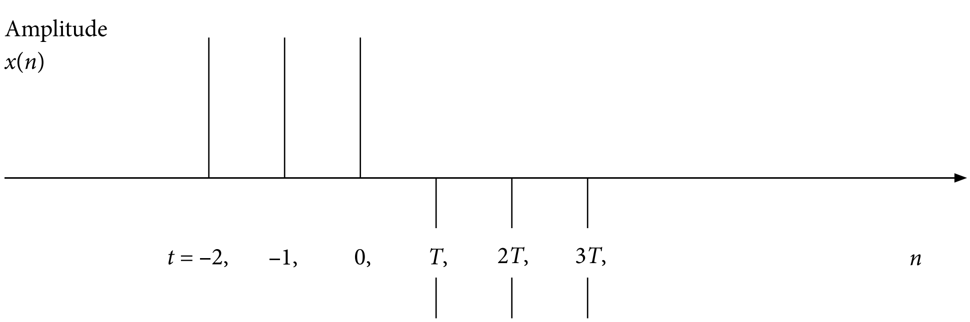


Fig. 2.10 Plot of signal $x(n)$ for P 2.11

Let us write the program to generate this signal. A MATLAB program to generate a signal is given as follows. The plot of the signal is shown in Fig. 2.11. MATLAB does not use the index as zero. Hence, we have to do a trick. Generate the *s* variable between –2 to 3 and plot the values of *x* against *s*.

```
clear all;
for n=1:2,
    x(n)=0;
end
for n=3:5,
  x(n)=1;
end
for n=6:8,
  x(n)=-2;
end
for n=9:10,
    x(n)=0;
end
s=-4:1:5;
stem(s,x);title('plot of DT signal x');
xlabel('sample number');ylabel('amplitude')
```

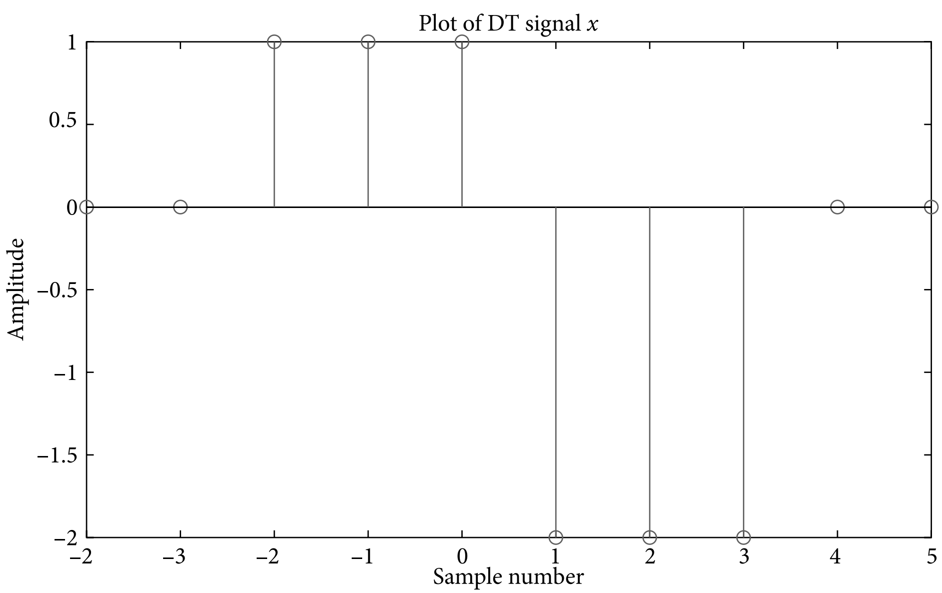


Fig. 2.11 Plot of signal $x(n)$ for P 2.11

P 2.12 Convert the samples in vector signal X to digital form.

$X = \{0, 0.125, 0.5, 0.25, 0.125\}$

Solution

The digital signal X will be represented as

$X = \{1000, 1001, 1100, 1010, 1001\}$

The range of values between –1 to +1 is divided into 16 levels. The centre level 1000 represents a zero. Each level indicates a value of 0.125. Hence, the second level, i.e., if third bit is 1, i.e., 1010 will represent a value 0.25. Level 3, i.e., if the second bit is 1, i.e., 1100 will represent a value of 0.75.

P 2.13

i. Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-2t^2}\,\delta(t-1)dt.$$

Solution

Let us first define the delta function.

$$\delta(t-1) = \begin{cases} 1 & \text{for } \; t=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} e^{-2t^2}\,\delta(t-1)dt = [e^{-2t^2}] \rightarrow$$

$t = 1 = e^{-2\times 1} = e^{-2}$

ii. Evaluate the integral

$$\int_{-\infty}^{\infty} t^2\delta(t-6)dt\,.$$

Solution

Let us first define the delta function.

$$\delta(t-6) = \begin{cases} 1 & \text{for } \; t=6 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} t^2\delta(t-6)dt = [t^2] \rightarrow$$

$t = 6 = 36$

iii. Evaluate the integral

$$\int_{-\infty}^{\infty} \sin(\pi t)\delta(t-1)dt.$$

Solution

Let us first define the delta function.

$$\delta(t-1) = \begin{cases} 1 & \text{for } t=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \sin(\pi t)\delta(t-1)dt = \sin(\pi) \rightarrow$$

$t = 1 = 0$

iv. Evaluate the integral

$$\int_{-\infty}^{\infty} (t-1)^2\delta(t-1)dt.$$

Solution

Let us first define the delta function.

$$\delta(t-1) = \begin{cases} 1 & \text{for } t=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} (t-1)^2\delta(t-1)dt = 0$$

v. Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin(2t)\delta(t) + \sin(2t)\delta(t-2)]dt.$$

Solution

Let us first define the delta function.

$$\delta(t-2) = \begin{cases} 1 & \text{for } t=2 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(t) = \begin{cases} 1 & \text{for } t=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} [\sin(2t)\delta(t) + \sin(2t)\delta(t-2)]dt = \sin(4)$$

vi. Evaluate the integral

$$\int_{-\infty}^{\infty} e^{4j\omega t}\delta(t)dt.$$

Solution

Let us first define the delta function.

$$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} e^{4j\omega t}\delta(t)dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Note: $e^{4\omega \times 0} = 1$

P 2.14 Evaluate the summation

i. $\sum_{n=-\infty}^{\infty} e^{n}\delta(n).$

Solution

Let us define the DT delta function.

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.47}$$

$$\sum_{n=-\infty}^{\infty} e^{n}\delta(n) = 1$$

ii. $\sum_{n=-\infty}^{\infty} \cos(3n)\delta(n-2).$

Solution

Let us first define the delta function.

$$\delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} \cos(3n)\delta(n-2) = \cos(6)$$

iii. $\sum_{n=-\infty}^{\infty} e^{2n}\delta(n+1)$

Solution

Let us first define the delta function.

$$\delta(n+1) = \begin{cases} 1 & \text{for } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} e^{2n}\delta(n+1) = e^{-2}$$

P 2.15 Consider an analog signal given by

$$x(t) = A \text{ for } -3/2 \le t \le 3/2$$

Find if the signal is even.

Solution

We will plot the signal and find out if $x(-t) = x(t)$. The plot of the signal is shown in Fig. 2.12. We observe that $x(t) = x(-t)$ for all t. It is symmetrical with respect to the origin or the vertical axis, i.e., amplitude axis. Hence, the signal is even. The same signal can also be written as

$$\begin{aligned} x(t) = A \text{ rect}(t/3) &= 1 \text{ for } |t| \le 3/2 \\ &= 0 \text{ for } |t| > 3/2 \end{aligned} \tag{2.57}$$

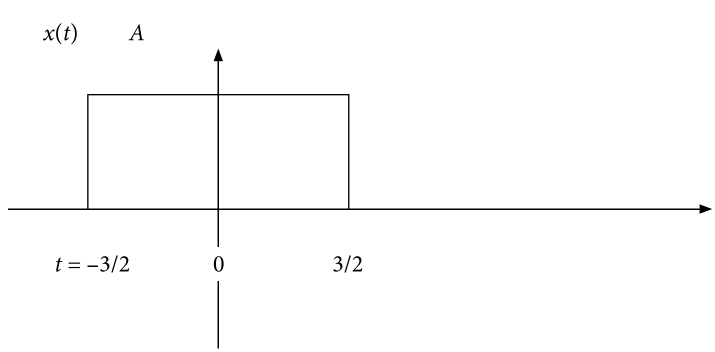


Fig. 2.12 Plot of signal $x(t)$ for P 2.15

At time instant t = + 3/2, the signal value is 1 and 0. This ambiguity can be clarified as at t = + $3/2^{+}$, $x(t)$ = 0 and at t = + 3/2, $x(t)$ = 1.

P 2.16 Consider an analog signal given by

$$x(t) = -4 \quad \text{for } -3/2 \le t \le 0 \text{ and}$$
$$x(t) = 4 \quad \text{for } 0 < t \le 3/2$$

Find if it is even or odd.

Solution

Let us plot the signal. The plot is shown in Fig. 2.13. We observe that $x(t) = -x(-t)$ for all t. The signal is anti-symmetrical with respect to the origin or the vertical axis, i.e., amplitude axis. Hence, the signal is an odd signal.

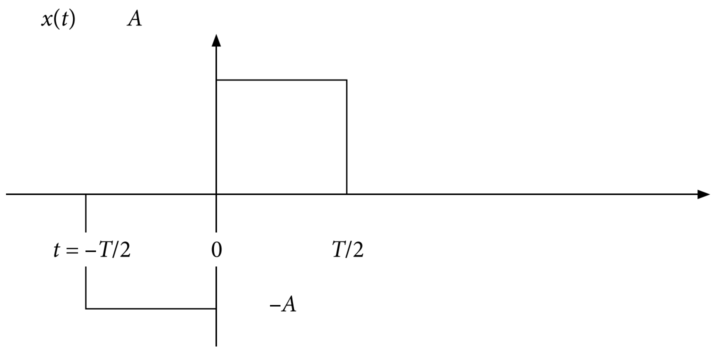


Fig. 2.13 Plot of signal $x(t)$ for P 2.16

P 2.17 Find the even and odd parts of the following signal.

$$x(t) = e^{j3\omega t}$$

Solution

We will write the signal as

$$x(t) = e^{j3\omega t} = \cos(3\omega t) + j\sin(3\omega t)$$

So,

$$x(-t) = e^{-3j\omega t} = \cos(-3\omega t) + j\sin(-3\omega t) = \cos(3\omega t) - j\sin(3\omega t)$$

$$x_e(t) = \left[x(t) + x(-t)\right]/2 = \cos(3\omega t)$$

$$x_o(t) = \left[x(t) - x(-t)\right]/2 = j\sin(3\omega t)$$

P 2.18 A signal is defined as $x(t) = e^{j2\omega t}$ for all $t \geq 0$. Find if the signal is even or odd.

Solution

The signal does not exist for negative values of *t*. The signal is neither even nor odd. Thus, it is simply undefined for all negative values of *t*.

P 2.19 Find the even and odd parts of the following signals.

i. $x(t) = \cos(2t) + \cos(3t) + \cos(t)\sin(2t)$.

Solution

$$x(t) = \cos(2t) + \cos(3t) + \cos(t)\sin(2t)$$

$$x(-t) = \cos(-2t) + \cos(-3t) + \cos(-t)\sin(-2t)$$

$$= \cos(2t) + \cos(3t) - \cos(t)\sin(2t)$$

$$x_e(t) = [x(t) + x(-t)]/2 = \cos(2t) + \cos(3t)$$

$$x_o(t) = [x(t) - x(-t)]/2 = \cos(t)\sin(2t)$$

ii. $x(t) = 1 + 3t + t^2 + t^3$.

Solution

$$x(t) = 1 + 3t + t^2 + t^3$$

$$x(-t) = 1 + (-3t) + (-t)^2 + (-t)^3 = 1 - 3t + t^2 - t^3$$

$$x_e(t) = [x(t) + x(-t)]/2 = 1 + t^2$$

$$x_o(t) = [x(t) - x(-t)]/2 = 3t + t^3$$

iii. $x(a) = \cos(2a) + \sin^3(a)$.

Solution

$$x(a) = \cos(2a) + \sin^3(a)$$

$$x(-a) = \cos(-2a) + \sin^3(-a) = \cos(2a) - \sin^3(a)$$

$$x_e(t) = [x(t) + x(-t)]/2 = \cos(2a)$$

$$x_o(t) = [x(t) - x(-t)]/2 = \sin^3(a)$$

iv. $x(a) = a^2\cos(2a) + a^3\sin(2a)$.

Solution

$$x(a) = a^2\cos(2a) + a^3\sin(2a)$$

$$x(-a) = (-a)^2\cos(-2a) + (-a)^3\sin(-2a) = a^2\cos(2a) + a^3\sin(2a)$$

$$x_e(t) = [x(t) + x(-t)]/2 = a^2\cos(2a) + a^3\sin(2a)$$

$$x_o(t) = [x(t) - x(-t)]/2 = 0$$

v. $x(t) = 1 + 2t\cos(t) + t^2\sin(3t) + t^3\sin(2t)\cos(5t)$

Solution

$$x(t) = 1 + 2t\cos(t) + t^2\sin(3t) + t^3\sin(2t)\cos(5t)$$

$$x(-t) = 1 + (-2t)\cos(-t) + (-t)^2\sin(-3t) + (-t)^3\sin(-2t)\cos(-5t)$$

$$= 1 - 2t\cos(t) - t^2\sin(3t) + t^3\sin(2t)\cos(5t)$$

$$x_e(t) = [x(t) + x(-t)]/2 = 1 + t^3\sin(2t)\cos(5t)$$

$$x_o(t) = [x(t) - x(-t)]/2 = 2t\cos(t) + t^2\sin(3t)$$

vi. $x(t) = (1 + t^2)\cos(4t)$

Solution

$$x(t) = (1 + t^2)\cos(4t) = \cos(4t) + t^2\cos(4t)$$

$$x(t) = (1 + (-t)^2)\cos(-4t) = \cos(-94t) + (-t)^2\cos(-4t) = \cos(4t) + t^2\cos(4t)$$

$$x_e(t) = [x(t) + x(-t)]/2 = \cos(4t) + t^2 \cos(4t)$$

$$x_o(t) = [x(t) - x(-t)]/2 = 0$$

vii. $x(t) = (t + t^3)\sin(5t)$

Solution

$$x(t) = (t + t^3)\sin(5t) = t\sin(5t) + t^3 \sin(5t)$$

$$x(-t) = (-t) + (-t)^3)\sin(-5t) = -t\sin(5t) - t^3 \sin(5t)$$

$$x_e(t) = [x(t) + x(-t)]/2 = t\sin(5t) + t^3 \sin(5t)$$

$$x_o(t) = [x(t) - x(-t)]/2 = 0$$

viii. $x(t) = (t + t^3)\tan(t)$

Solution

$$x(t) = (t + t^3)\tan(t) = t\tan(t) + t^3 \tan(t)$$

$$x(-t) = (-t) + (-t)^3)\tan(-t) = t\tan(t) + t^3 \tan(t)$$

$$x_e(t) = [x(t) + x(-t)]/2 = t\tan(t) + t^3 \tan(t)$$

$$x_o(t) = [x(t) - x(-t)]/2 = 0$$

ix. $x(t) = t \times (1 + t^2 + t^3)$

Solution

$$x(t) = t \times (1 + t^2 + t^3) = t + t^3 + t^4$$

$$x(-t) = (-t) \times (1 + (-t)^2 + (-t)^3) = -t - t^3 + t^4$$

$$x_e(t) = [x(t) + x(-t)]/2 = t^4$$

$$x_o(t) = [x(t) - x(-t)]/2 = t + t^3$$

P 2.20 Consider a discrete time signal given by

$$x(n) = 1 \text{ for } -3 \le n \le 3$$
$$= 0 \text{ otherwise}$$

Find if the signal is even or odd.

Solution

Let us plot the signal. A plot is shown in Fig. 2.14. The signal is symmetrical with respect to the origin and with respect to the amplitude axis. $x(-n) = x(n)$ for all n. Hence, the signal is even.

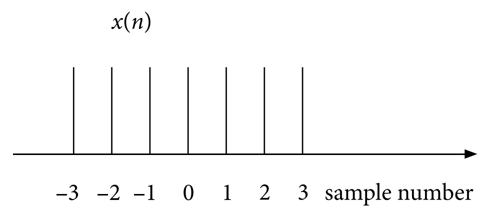


Fig. 2.14 Plot of signal for P 2.20

P 2.21 Consider a discrete time signal given by

$$x(n) = 1 \quad \text{for } 1 \le n \le 4$$
$$= -1 \quad \text{for } -1 \le n \le -4$$

Find if the signal is even or odd.

Solution

Let us plot the signal. The plot is shown in Fig. 2.15. The signal is anti-symmetrical with respect to the origin and with respect to the amplitude axis. $x(-n) = -x(n)$ for all n. Hence, the signal is odd.

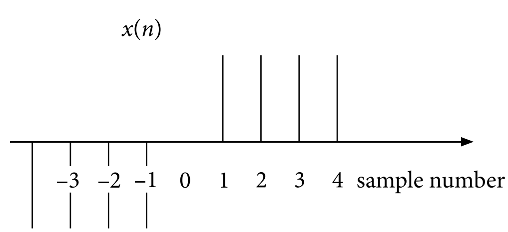


Fig. 2.15 Plot of signal for P 2.21

P 2.22 Consider a discrete time signal given by

$$x(n) = 1 \text{ for } 1 \le n \le 3$$
$$= 0 \text{ otherwise}$$

Find if the signal is even or odd.

Solution

Let us plot the signal. The plot is shown in Fig. 2.16. The signal is having a value zero for all negative values of n. It is neither symmetric nor anti-symmetrical with respect to the origin and with respect to the amplitude axis. Hence, the signal is neither even nor odd.

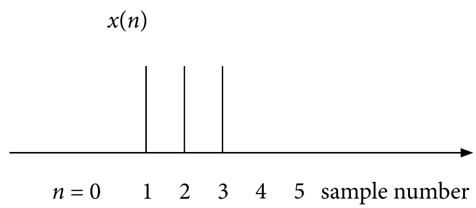


Fig. 2.16 Plot of signal for P 2.22

P 2.23

i. Consider a signal $x(t) = \sin(2t)$. Find if the signal is periodic and find the period.

Solution

$x(t+T) = \sin(2t+2T) = \sin(2t)$ if $2T = 2\pi$, i.e., $T = \pi$. So, the period of the signal is π seconds and frequency f is $1/\pi$. The value of the period is not rational, still the signal is periodic. Every analog sinusoidal signal is periodic.

ii. Consider the equation given by $x(t) = 2t + \cos(4\pi t)$. Is $x(t)$ a periodic signal?

Solution

Here, we will check if there is some T for which $x(t) = x(t+T)$. Put $t = t + T$ in $x(t) = 2t + \cos(4\pi t)$. We obtain

$$x(t+T) = 2t + T + \cos(4\pi t + 4\pi T).$$

The component with t increases as t increases so there is no T for which the signal will be periodic. Hence, the signal is aperiodic.

iii. Consider the equation given by $x(t) = (\sin(4\pi t))^2$. Is $x(t)$ a periodic signal?

Solution

Here, we will check if there is some T for which $x(t) = x(t+T)$. Put $t = t + T$ in $x(t) = (\sin(4\pi t))^2$. We obtain

$$x\left(t+T\right) = \left(\sin(4\pi\left(t+T\right)\right)^2 = \left(\sin\left(4\pi t\right)\right)^2 \text{ when } 4\pi T = \pi, \text{ i.e., } T = \frac{1}{4}.$$

The function is periodic with period equal to $\frac{1}{4}$.

iv. Consider the equation given by $x(t) = \left|\cos(4\pi t)\right|$. Is $x(t)$ a periodic signal?

Solution

Here, we will check if there is some T for which $x(t) = x(t+T)$. Put $t = t + T$ in $x(t) = \left|\cos(4\pi t)\right|$. We obtain

$$x(t+T) = \left|\cos(4\pi(t+T)\right| = \left|\cos(4\pi t)\right| \text{ when } 4\pi T = \pi, \text{ i.e., } T = \frac{1}{4}.$$

The function is periodic with period equal to $\frac{1}{4}$. Let us verify this by writing a MATLAB program. The plot of the function is shown in Fig. 2.17.

```
clear all;
f=10;
T=0.005;
for n=1:41,
y(n)=abs(cos(2*pi*f*(n-1)*T));
end
s=-20:1:20;
plot(s,y);title('plot of absolute value of
cosine function for positive and negative
angles');xlabel('angle pi divided in 20
points');ylabel('Amplitude');
```

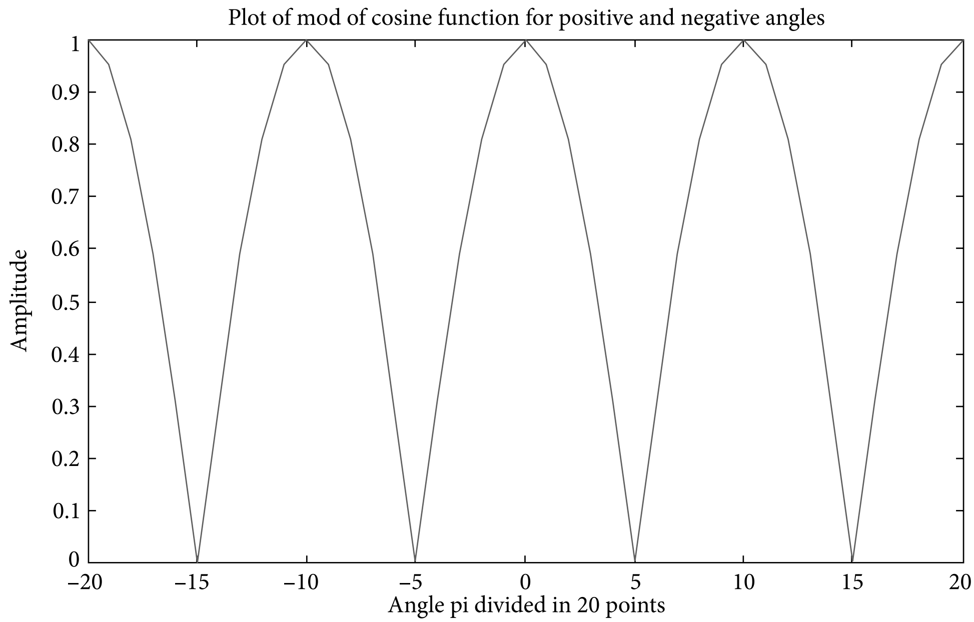


Fig. 2.17 A plot of a signal for P 2.23 iv

P 2.24 Consider the signal shown in Fig. 2.18. Is $x(t)$ a periodic signal?

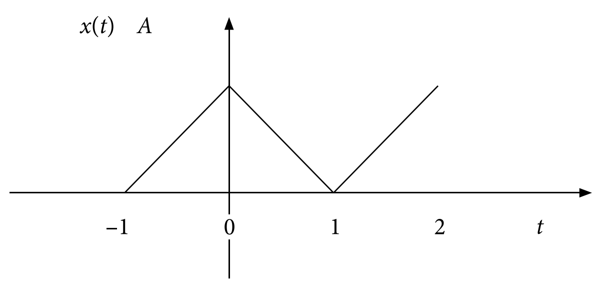


Fig. 2.18 A signal for P 2.24

Solution

The signal exists only over a small duration. It does not repeat. Hence, the signal is not periodic.

P 2.25 Consider the signal $x(t)$ shown in Fig. 2.19. If $y(t) = \sum_{k=-6}^{6} x(t-2k)$, is $y(t)$ a periodic signal?

Solution

Let us plot the signal to find if the signal is periodic. The plot of the signal is shown in Fig. 2.19.

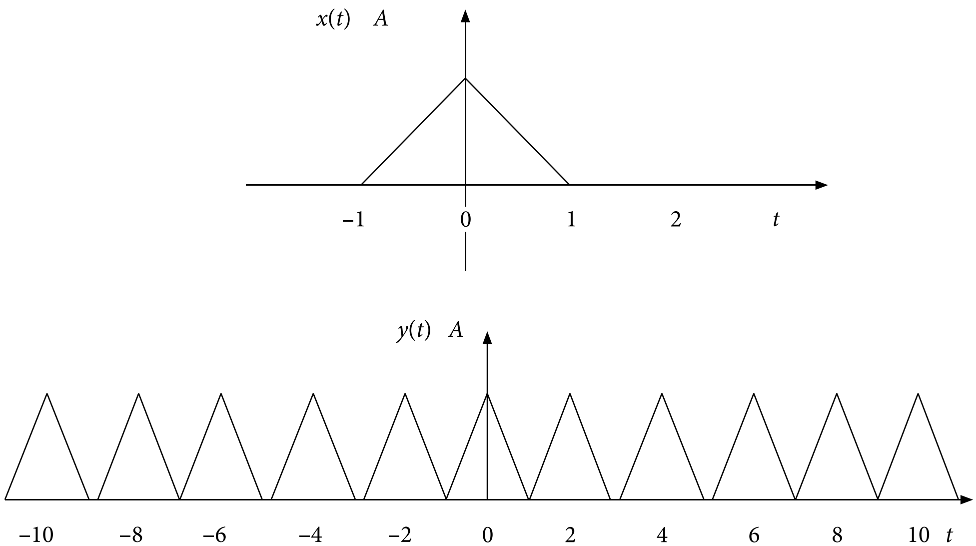


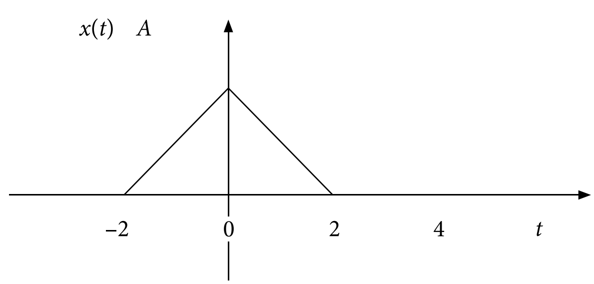
Fig. 2.19 A signal for P 2.25

Referring to Fig. 2.19 we can see that the signal repeats after $t = 2$. But, the signal does not exist before $t = -13$ and after $t = 13$. So, the signal can be considered as periodic over the period for which it is defined.

P 2.26 Consider the signal $x(t)$ shown in Fig. 2.20. If $y(t) = \sum_{k=-\infty}^{\infty} x(t-4k)$, is $y(t)$ a periodic signal?

Solution

Let us plot the signal to find if the signal is periodic. The plot of the signal is shown in Fig. 2.20.



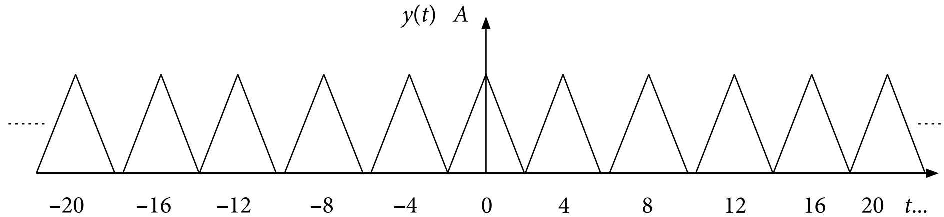


Fig. 2.20 A signal for P 2.26

Referring to Fig. 2.20, we can see that the signal repeats after $t = 4$. The signal is defined and it exists from $-\infty$ to $+\infty$. So, the signal can be considered as truly periodic.

P 2.27 Consider the equation given by $x(t) = e^{-5t}$. Is $x(t)$ a periodic signal?

Solution

Let us plot the signal to find if it is periodic. A MATLAB program is given here. Figure 2.21 shows a plot which indicates that the function is aperiodic.

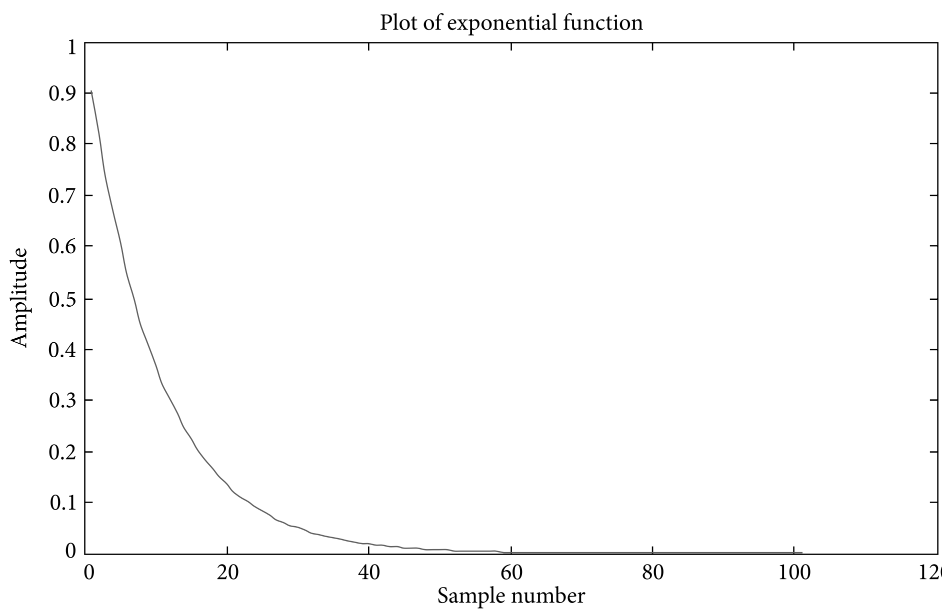


Fig. 2.21 Plot of signal for P 2.27

```
clear all;
t=0.05;
for n=1:101,
y(n)=exp(-5*t*n);
end
```

```
plot(y);title('plot of exponential function');xlabel
('samplenumber');ylabel('Amplitude');
```

P 2.28 Consider the signal $x(n)$ given by $x(n)=(-1/2)^n$. Is $x(n)$ a periodic signal?

Solution

Let us evaluate the values of the signal for different values of n and plot the signal. The value of the signal is positive for all even values of n and negative for all odd values of n.

$$x(0)=(-1/2)^0=1,\ x(1)=(-1/2)^1=-1/2,\ x(2)=(-1/2)^2=1/4 \text{ and so on}$$

$$x(-1)=(-1/2)^{-1}=\frac{1}{(-1/2)}=-2,\ x(-2)=(-1/2)^{-2}=\frac{1}{(-1/2)^2}=4 \text{ and so on.}$$

The signal plot is shown in Fig. 2.22. Hence, the signal is aperiodic.

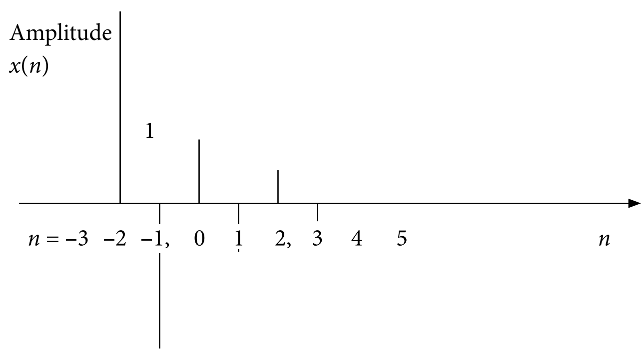


Fig. 2.22 Plot of signal for P 2.28

P 2.29 Consider the signal $x(n)$ given by $x(n)=(-1)^{n^3}$. Is $x(n)$ a periodic signal?

Solution

Let us evaluate the values of the signal for different values of n and plot the signal. The value of the signal is +1 for all even values of n and is equal to –1 for all odd values of n for positive as well as negative values of the exponent, i.e., n.

$$x(0)=(-1)^{0^3}=1,\ x(1)=(-1)^{1^3}=-1,\ x(2)=(-1)^{2^3}=1 \text{ and so on} \tag{2.96}$$

$$x(-1)=(-1)^{(-1)^3}=\frac{1}{(-1)^1}=-1,\ x(-2)=(-1)^{(-2)^3}=\frac{1}{(-1)^8}=1 \text{ and so on}$$

The signal plot is shown in Fig. 2.23. Hence, the signal is periodic with period equal to 2.

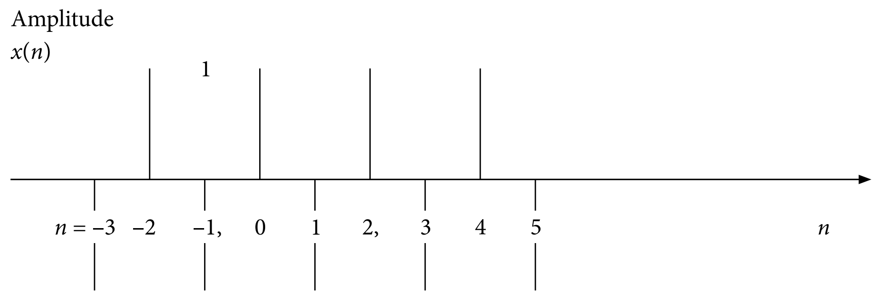


Fig. 2.23 Plot of the signal for P 2.29

P 2.30 Consider the signal shown in Fig. 2.24. Is $x(n)$ a periodic signal?

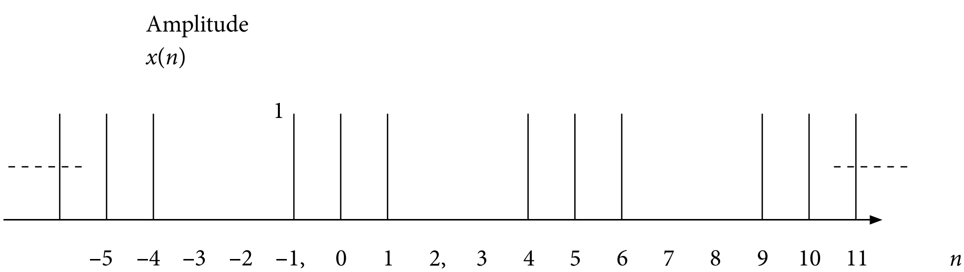


Fig. 2.24 Plot of signal for P 2.30

Solution

We can refer to the plot of the signal to see that the signal repeats itself after every 5 samples (i.e., from $n = -1$ to $n = 3$). So, the period is 5 samples.

P 2.31 Consider the signal shown in Fig. 2.25. Is $x(t)$ a periodic signal?

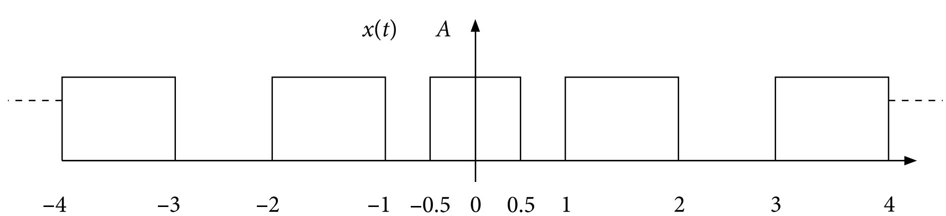


Fig. 2.25 Signal plot for P 2.31

Solution

The signal repeats after every two time unit period after time unit 1. But, it has a spacing of only ½ between the first and the second square wave. Therefore, the signal is aperiodic.

P 2.32 Find if the following DT signals are periodic.

i. Consider the signal $x(n) = \cos(0.03n\pi)$. Is $x(n)$ a periodic signal?

   We have to check if $x(n) = x(n + N)$ for some integer N

   That is,

$$x(n) = \cos(0.03n\pi) = \cos(0.03\pi(n+N))$$

$$\text{i.e.,}\quad 2\pi\frac{3}{200}N = 2\pi k \Rightarrow \frac{3}{200} = \frac{k}{N} \Rightarrow \text{ period is } N = 200$$

   As k and N are relatively prime, the fundamental period of the sinusoid is $N = 200$ samples.

ii. Consider the signal

$$x(n) = \cos\left(\frac{10n}{105}\pi\right).$$

   Is $x(n)$ a periodic signal?

   We have to check if

   $x(n) = x(n + N)$ for some integer N

   That is,

$$x(n) = \cos\left(\frac{10}{105}n\pi\right) = \cos\left(\frac{10}{105}\pi\left(n+N\right)\right)$$

$$\text{i.e.,}\quad 2\pi\frac{5}{105}N = 2\pi k \Rightarrow \frac{1}{21} = \frac{k}{N} \Rightarrow \text{ period is } N = 21$$

   As k and N are relatively prime, the fundamental period of the sinusoid is $N = 7$.

iii. Consider the signal $x(n) = \cos(5\pi n)$. Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N

That is,

$$x(n) = \cos(5n\pi) = \cos(4\pi n + \pi n) = \cos(\pi n) = \cos(\pi(n + N))$$

i.e., $2\pi \frac{1}{2} N = 2\pi k \Rightarrow \frac{1}{2} = \frac{k}{N} \Rightarrow$ period is $N = 2$

As k and N are relatively prime, the fundamental period of the sinusoid is $N = 2$.

iv. Consider the signal $x(n) = \sin(2n)$. Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N

That is,

$$x(n) = \sin(2n) = \sin(2(n + N))$$

i.e., $2\pi \frac{1}{\pi} N = 2\pi k \Rightarrow \frac{1}{\pi} = \frac{k}{N}$

As k and N are not relatively prime, the signal is aperiodic.

v. Consider the signal

$$x(n) = \sin\left(\frac{82n}{10}\pi\right).$$

Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N

That is,

$$x(n) = \sin\left(\frac{82n}{10}\pi\right) = \sin\left(8n\pi + \frac{2}{10}n\pi\right)$$

$$= \sin\left(\frac{2}{10}n\pi\right) = \sin\left(\frac{2}{10}\pi(n + N)\right)$$

i.e., $2\pi \frac{2}{20} N = 2\pi k \Rightarrow \frac{2}{20} = \frac{k}{N} = \frac{1}{10}$

As k and N are relatively prime, the period of the signal is $N = 10$.

vi. Consider the signal

$$x(n) = 5\cos\left(3n + \frac{\pi}{4}\right).$$

Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N

That is,

$$x(n) = 5\cos\left(3n + \frac{\pi}{4}\right) = 5\cos\left(3(n+N) + \frac{\pi}{6}\right)$$

i.e., $2\pi\frac{3}{2\pi}N = 2\pi k \Rightarrow \frac{3}{2\pi} = \frac{k}{N}$

As k and N are not relatively prime, the signal is aperiodic.

vii. Consider the signal

$$x(n) = 2\ \exp\left(j\left(\frac{n}{4} - \frac{\pi}{3}\right)\right).$$

Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N

That is,

$$x(n) = 2\ \exp\left(j\left(\frac{n}{4} - \frac{\pi}{2}\right)\right) = 2\exp\left(j\frac{(n+N)}{4} - \frac{\pi}{2}\right)$$

i.e., $2\pi\frac{1}{8\pi}N = 2\pi k \Rightarrow \frac{1}{8\pi} = \frac{k}{N}$

As k and N are not relatively prime, the signal is aperiodic.

viii. Consider the signal

$$x(n) = \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{n\pi}{8}\right).$$

Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N

That is,

$$x(n)=\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{n\pi}{8}\right)=\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{(n+N)\pi}{8}\right)$$

i.e., $2\pi\frac{1}{16}N=2\pi k\Rightarrow\frac{1}{16}=\frac{k}{N}$

As k and N are relatively prime, the signal is periodic with period $N = 16$.

ix. Consider the signal

$$x(n)=\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{8}\right)+3\cos\left(\frac{n\pi}{4}+\frac{\pi}{3}\right).$$

Is $x(n)$ a periodic signal?

We have to check if $x(n) = x(n + N)$ for some integer N for all 3 terms.

That is,

$$x(n)=\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{8}\right)+3\cos\left(\frac{n\pi}{4}+\frac{\pi}{3}\right)$$

$$=\cos\left(\frac{(n+N)\pi}{2}\right)-\sin\left(\frac{(n+N)\pi}{8}\right)+3\cos\left(\frac{(n+N)\pi}{4}+\frac{\pi}{3}\right)$$

i.e., for the first term, $2\pi\frac{1}{4}N=2\pi k\Rightarrow\frac{1}{4}=\frac{k}{N}$

As k and N are relatively prime, hence the signal is periodic with period $N = 4$ for the first term.

For the second term, $2\pi\frac{1}{16}N=2\pi k\Rightarrow\frac{1}{16}=\frac{k}{N}$

As k and N are relatively prime, the signal is periodic with period $N = 16$ for the second term.

For the third term, $2\pi\frac{1}{8}N=2\pi k\Rightarrow\frac{1}{8}=\frac{k}{N}$

As k and N are relatively prime, the signal is periodic with period $N = 8$ for the second term.

Considering the periods for all three terms, the period for the signal is the highest common divisor, i.e., LCM value of N for all terms which is equal to 16.

P 2.33 Consider a linear combination of two analog sinusoidal functions. $x(t) = 3\sin(6\pi t) + \cos(4\pi t)$. Find if the signal is periodic.

Solution

We will check if $x(t+T) = 3\sin(6\pi(t+T)) + \cos(4\pi(t+T)) = x(t)$ for some T.

$$x(t+T) = 3\sin\left(6\pi\left(t+T\right)\right) + \cos\left(4\pi\left(t+T\right)\right)$$

$$= 3\sin\left(6\pi t + 6\pi T\right) + \cos\left(4\pi t + 4\pi T\right)$$

$$3\sin\left(6\pi t + 6\pi T\right) = 3\sin\left(6\pi t\right)\ x\left(n\right) \text{ if } 6\pi T = 2\pi \Rightarrow T = 1/3$$

$$\cos\left(4\pi t + 4\pi T\right) = \cos\left(4\pi t\right) \text{ if } 4\pi T = 2\pi \Rightarrow T = 1/2$$

Common period T can be found using $\dfrac{M}{N} = \dfrac{2/3}{2\pi/3} = \dfrac{1}{\pi}$

where M and N stand for the period of the first term and the second term respectively.

Hence, the period for the combination signal can be found as follows.

The period for the linear combination of two terms is

$$2 \times M = 3 \times N = 2 \times \frac{1}{2} = 3 \times \frac{1}{3} = 1 \text{ seconds}$$

P 2.34 Consider a linear combination of two analog sinusoidal functions. $x(t) = 2\cos(3\pi t) + \sin(3t)$. Find if the signal is periodic.

Solution

We will check if $x(t+T) = 3\cos(4\pi(t+T) + \sin(5(t+T)$ for some T.

$$x\left(t+T\right) = 2\cos\left(3\pi\left(t+T\right)\right) + \sin\left(3\left(t+T\right)\right)$$

$$= 2\cos\left(3\pi t + 3\pi T\right) + \sin\left(3t + 3T\right)$$

$$2\cos\left(3\pi t + 3\pi T\right) = 2\cos\left(3\pi t\right) \text{ if } 3\pi T = 2\pi \Rightarrow T = 2/3$$

$$\sin\left(3t + 3T\right) = \sin\left(3T\right) \text{ if } 3T = 2\pi \Rightarrow T = 2\pi/3$$

Common period T can be found using $\frac{M}{N}=\frac{2/3}{2\pi/3}=\frac{1}{\pi}$

where M and N stand for the period of the first term and the second term respectively.

Hence, the period for the combination signal can be found as follows.

The period for the linear combination of two terms is

$$\pi\times M=1\times N=\pi\times\frac{2}{3}=1\times\frac{2\pi}{3}=2\pi/3\text{ seconds.}$$

The period is not a rational number. But, because it is an analog sinusoid, the combination signal is periodic.

P 2.35 Consider a sequence $x(n) = u(n) - u(n - 8)$. Find if it is causal.

Solution

The sequence is defined as

$$u(n)=\begin{cases}1 & \text{for all } n\geq 0\\ 0 & \text{otherwise}\end{cases}\qquad u(n-8)=\begin{cases}1 & \text{for } n\geq 8\\ 0 & \text{otherwise}\end{cases}$$

$$x(n)=u(n)-u(n-8)=\begin{cases}1 & \text{for } 0\leq n\leq 7\\ 0 & \text{otherwise}\end{cases}$$

This is a right-handed sequence and is causal.

P 2.36 Consider the following sequence $x(n)=u(-n-1)-u(-n-5)$. Find if the signal is causal.

Solution

$$u(-n-1)=\begin{cases}1 & \text{for } -n-1\geq 0 \text{ or } n\leq -1\\ 0 & \text{otherwise}\end{cases}$$

$$u(-n-5)=\begin{cases}1 & \text{for } -n-5\geq 0 \text{ or } n\leq -5\\ 0 & \text{otherwise}\end{cases}$$

$$x(n)=u(-n-1)-u(-n-5)=\begin{cases}1 & \text{for } -1\leq n\leq -5\\ 0 & \text{otherwise}\end{cases}$$

The signal exists for negative values of n. This is a left-handed and non-causal sequence.

P 2.37 Consider a CT signal given by $x(t) = e^{5t}u(t-1)$. Find if the signal is causal.

Solution

Let us first write the definition of $u(t - 1)$.

$$u(t-1) = \begin{cases} 1 & \text{for } t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

As the signal is appended by $u(t - 1)$, it exists for positive values of $t \geq 1$, and is zero for all $t < 0$. Hence, it is causal.

P 2.38 Consider a signal given by $x(t) = 2\sin c(7t)$. Find if the signal is causal.

Solution

The sinc function exists from minus infinity to infinity. Hence, the signal exists for negative values of t and is anti-causal.

P 2.39 Consider a CT signal $x(t) = e^{t}[u(t+4) - u(t-3)]$. Find if the signal is causal.

Solution

Let us write the definitions of the u functions.

$$u(t-3) = \begin{cases} 1 & \text{for } t \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$u(t+4) = \begin{cases} 1 & \text{for } t \geq -4 \\ 0 & \text{otherwise} \end{cases}$$

The signal is both sided. It exists for negative values of t as well and hence, it is anti-causal.

P 2.40 Consider a DT signal $x(t) = \left(\frac{1}{2}\right)^{n} u(n+5) - \left(\frac{1}{3}\right)^{n} u(n-4)$. Find if the signal is causal.

Solution

Let us write the definitions of the unit step functions.

$$u(n-4)=\begin{cases}1 & \text{for } n\ge 4\\ 0 & \text{otherwise}\end{cases}$$

$$u(n+5)=\begin{cases}1 & \text{for } n\ge -5\\ 0 & \text{otherwise}\end{cases}$$

The sequence is both sided. But, the function exists for negative values of n and is anti-causal.

P 2.41 Find if the following are deterministic signals.

i. $x(t)=2t+\sin(3\pi ft)+\cos(4\pi ft)$

ii. $x(n)=\cos\left(\frac{n\pi}{4}\right)+3\sin\left(\frac{n\pi}{3}+\frac{\pi}{5}\right)$

Solution

The equation for the signal is provided. We can use this equation to find the value of the signal at any time t. Hence, the signal is deterministic.

P 2.42 Consider a sinusoid of frequency 2 kHz. Is it a power signal?

Solution

The signal is a sinusoid which exists from $-\infty$ to ∞. Hence, the energy of a signal is infinite. Let us calculate the average power. The period of a signal is 1/2 kHz, i.e., 0.5 milli seconds.

Solution

$$P=\frac{2000}{2}\int_{-0.0005}^{0.0005}\left|\sin(2\pi 2000t)\right|^2 dt=\frac{1000}{2}\int_{-0.0005}^{0.0005}\left|1-\cos(4\pi 2000t)\right|dt$$

$$P=\frac{500}{1}\Big[t-4\pi 2000t\sin\big(4\pi 2000t\big)\Big]_{-0.0005}^{0.0005}=0.5$$

The average power is finite. The signal is a power signal.

P 2.43 If $x(t)=\sin(2\pi 50t)$ for $0\le t\le 1/2$, is $x(t)$ an energy signal?

Solution

The signal is a sinusoid existing only over a finite period between t = 0 to ½. Hence, the energy of a signal is finite and the signal is an energy signal. The period of the signal is $T=\frac{1}{f}=\frac{1}{50}=0.02$ seconds. Let us find the total energy of the signal.

$$E = \int_{t=0}^{0.5} \sin^2\left(2\pi 50t\right) dt$$

$$= \frac{1}{2}\int_0^{0.5} \left(1 - \cos\left(2\pi 100t\right)\right) dt$$

$$= \frac{1}{2}\left[t \downarrow_0^{0.5} - \frac{\sin\left(2\pi 100t\right)}{200\pi} \downarrow_0^{0.5} \right]$$

$$= \frac{1}{2}\left[0.5 - 0\right] = \frac{1}{4}$$

P 2.44 Consider a signal defined as

$$x(t) = \begin{cases} 2t & \text{for } 0 \le t \le 1 \\ 4 - 2t & \text{for } 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the energy and power of the signal and classify it as an energy or a power signal.

Solution

Let us find the total energy of the signal.

$$E = \int_{t=0}^{1} 2t dt + \int_{t=1}^{2} (4 - 2t) dt$$

$$E = \frac{2t^2}{2} \downarrow_{t=0}^{t=1} + 4t \downarrow_{t=1}^{t=2} - \frac{2t^2}{2} \downarrow_{t=1}^{t=2}$$

$= 1 + [8 - 4] - [4 - 1]$

$= 2$

The total energy is finite. So, the signal is an energy signal.

P 2.45 Consider a signal defined as

$$x(t) = \begin{cases} 2\cos(\pi t / 2) & \text{for } -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the energy and power of the signal and classify it as an energy or a power signal.

Solution

Let us find the total energy of the signal.

$$E = \int_{t=-1}^{1} 2\cos(\pi t/2)dt$$

$$E = 2\sin(\pi t/2)\big/(\pi/2) \downarrow_{t=-1}^{t=1}$$

$$= 2(\sin(\pi/2)\big/(\pi/2)) - 2(\sin(-\pi/2)\big/(\pi/2))$$

$$= 4$$

The total energy is finite. So, the signal is an energy signal.

P 2.46 Consider a signal defined as

$$x(t) = \begin{cases} 2 & \text{for } -3 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the energy and power of the signal and classify it as an energy or a power signal.

Solution

Let us first find the instantaneous power of the signal.

Instantaneous power is given by

$$\begin{aligned} P_X(t) &= |x(t)|^2 = 4 \text{ for } -3 \le t \le 3 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

The total energy of the signal is given by

$$E = \int_{-3}^{3} 4dt = 4 \times t \downarrow_{-3}^{3} = 4 \times 6 = 24$$

The average power is given by

$$P_X = \lim_{T\to\infty} \frac{E_X}{T} = 0$$

P 2.47 Consider a signal given by

$$x(t) = \{3\cos(\pi t) + 2\cos(\pi t) \ for -\infty \le t \le \infty\}$$

Find the energy and power of the signal and classify it as an energy or a power signal.

Solution

The total energy of the signal is given by

$$E = \int_{-\infty}^{\infty} \left[9\cos^2(\pi t) + 4\cos^2(\pi t)\right] dt$$

$$= \int_{-\infty}^{\infty} \left[4.5(1 - \cos(2\pi t)) + 2(1 - \cos(2\pi t)\right] dt$$

$$= \int_{-\infty}^{\infty} \left[4.5(1 - \cos(2\pi t)) + 2(1 - \cos(2\pi t)\right] dt$$

$$= \infty$$

The period of the signal is $T = \frac{1}{f}, 2\pi ft = \pi t, f = \frac{1}{2}, T = 2.$

The average power is given by

$$P_X = \frac{E_X}{2} = (9+4)/2 = 6.5$$

$$P_X = \frac{1}{2}\int_0^2 \left[9\cos^2(\pi t) + 4\cos^2(\pi t)\right] dt$$

$$= \frac{1}{2}\int_0^2 \left[4.5(1 - \cos(2\pi t)) + \left[2(1 - \cos(2\pi t)\right] dt$$

$$= \frac{1}{2}\left[4.5(t \downarrow_0^2 - \cos(2\pi t)/2\pi\right] \downarrow_0^2) + \left[2\left(t \downarrow_0^2 - \frac{\cos(2\pi t)}{2\pi} \downarrow_0^2\right)\right]$$

$$= \frac{1}{2}[9+4] = 6.5$$

The signal has infinite energy and finite average power. Hence, the signal is a power signal.

P 2.48 Consider a periodic DT sinusoid given by

$$g(k) = 5\cos\left(\frac{\pi k}{20}\right)$$

Find if the signal is an energy signal or a power signal.

Solution

Let us first find the period of the signal.

We have to check if $x(n) = x(n + N)$ for some integer N, i.e.,

$$g(k+N) = 5\cos\left(\frac{\pi k}{20} + \frac{\pi N}{20}\right)$$

i.e., $2\pi \frac{1}{40} N = 2\pi k' \Rightarrow \frac{1}{40} = \frac{k'}{N}$

As k' and N are relatively prime, the signal is periodic with period $N = 40$.

The signal exists for all k. Hence, the total energy of the signal is infinite. Let us find the average power for one cycle.

$$P = \frac{1}{40}\sum_{k=0}^{39} 25\cos^2\left(\frac{\pi k}{20}\right) = \frac{25}{40}\sum_{k=0}^{39}\frac{1}{2}\left[1+\cos\left(\frac{2\pi k}{20}\right)\right] = \frac{25}{40}\frac{40}{2} = 12.5 = (5)^2/2$$

5 – peak amplitude of the signal

We find that the average power is finite. Hence, the signal is a power signal.

P 2.49 Consider the analog periodic signal sketched in Fig. 2.26. Find if the signal is an energy signal or a power signal?

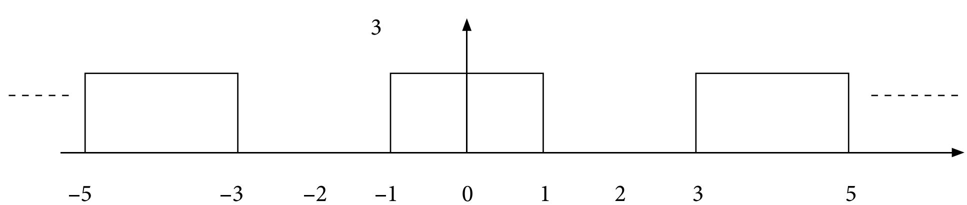


Fig. 2.26 Plot of signal for P 2.49

Solution

Figure 2.26 shows that the signal is a periodic signal with period from –4 to +4, i.e., period is 8. The signal varies from minus infinity to plus infinity. So, the total energy of the signal will be infinity. Let us find the average power of the signal. The average power is given by

$$P_X = \frac{1}{4}\int_{-1}^{+1} 9dx = \frac{1}{4}\times 9\times x\downarrow_{-1}^{1} = \frac{9}{4}\times 2 = \frac{9}{2} = 4.5.$$

Average power is finite. So, the signal is a power signal.

P 2.50 Consider an analog periodic signal sketched in Fig. 2.27. Find if the signal is an energy signal or a power signal?

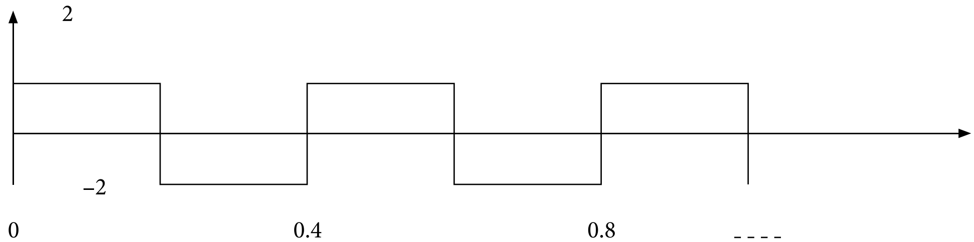


Fig. 2.27 Plot of signal for P 2.50

Solution

Figure 2.27 shows that the signal is a periodic signal with period from 0 to 0.4, i.e., period is 0.4 seconds. The signal varies from minus infinity to plus infinity. So, the total energy of the signal will be infinity. Let us find the average power of the signal. The average power is given by

$$P_X = \frac{1}{0.4}\left[\int_{0}^{0.2} (2)^2 \times dx + \int_{0.2}^{0.4} (-2)^2 \times dx\right] = \frac{1}{0.4}\times[0.8+0.8] = 4.$$

Average power is finite. So, the signal is a power signal.

P 2.51 Consider an analog periodic signal, a triangular wave sketched in Fig. 2.28. Find if the signal is an energy signal or a power signal?

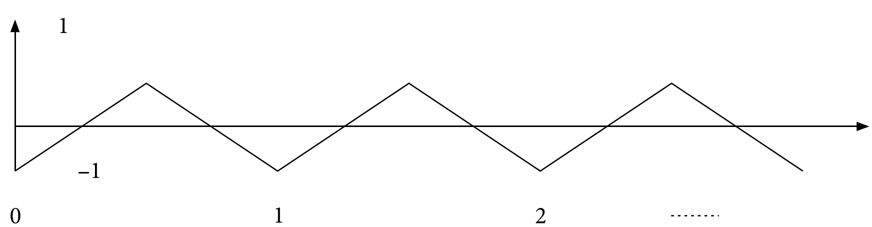


Fig. 2.28 Plot of signal for P 2.51

Solution

Figure 2.28 shows that the signal is a periodic signal with period from 0 to 0.1, i.e., period is 0.1 seconds. The signal varies from minus infinity to plus infinity. So, the total energy of the signal will be infinity. Let us find the average power of the signal. The average power is given by (the slope of the straight line between 0 to 0.05 is 40 and that between 0.05 to 0.1 is –40)

$$P_X = \frac{1}{1}\left[\int_0^{0.5}[4x-1]^2\,dx + \int_{0.5}^{1}[3-4x]^2\right]$$

$$= \frac{1}{1}\left[\int_0^{0.5}(16x^2-8x+1)dx + \int_{0.5}^{1}(9-24x+16x^2)dx\right]$$

$$= \frac{1}{1}\left[16\frac{x^3}{3}\downarrow_0^{0.5} - 8\frac{x^2}{2}\downarrow_0^{0.5} + x\downarrow_0^{0.5} + 9x\downarrow_{0.5}^{1} - 24\frac{x^2}{2}\downarrow_{0.5}^{1} + 16\frac{x^3}{3}\downarrow_{0.5}^{1}\right]$$

$$= \frac{1}{1}\left[\frac{2}{3} - 2 + 0.5 + 4.5 - 9 + \frac{14}{3}\right]$$

$$= \frac{25}{3}$$

Average power is finite. So, the signal is a power signal.

P 2.52 Consider a DT periodic signal as shown in Fig. 2.29. Find if the signal is an energy signal or a power signal? Find the average power.

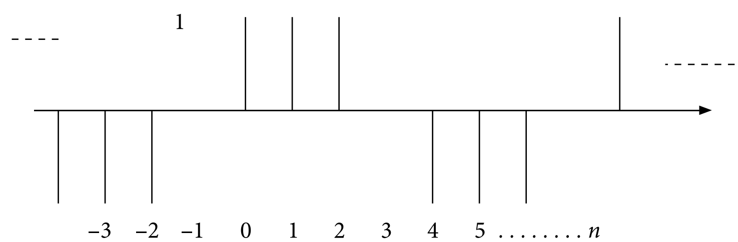


Fig. 2.29 Plot of signal for P 2.52

Solution

The signal is periodic with period equal to 8 samples. The signal extends over infinite duration. Hence, it has infinite energy. The average power is the power for one period.

$$P = \frac{1}{8}\left[\sum_{n=0}^{n=2} 1 + \sum_{4}^{6} 1\right] = 3/4.$$

P 2.53 Consider a DT periodic signal as shown in Fig. 2.30. Find if the signal is an energy signal or a power signal?

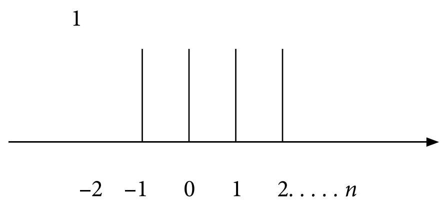


Fig. 2.30 Plot of signal for P 2.53

Solution

The signal exists only for a finite duration. Hence, the signal is an energy signal. Let us find the total energy of the signal.

$$E = \sum_{n=-1}^{n=2} 1 = 4.$$

The period of the signal is infinity. The average power is given by

$$P = \frac{1}{T}\sum_{n=-1}^{n=2} (1)^2 = \frac{4}{\infty} = 0.$$

P 2.54 Consider a signal defined as

$$x(n) = \begin{cases} n & \text{for } 0 \le n \le 4 \\ 9-n & \text{for } 5 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Find the energy and power of the signal and classify it as an energy or a power signal.

Solution

Let us plot the signal first. The plot of the signal is shown in Fig. 2.31.

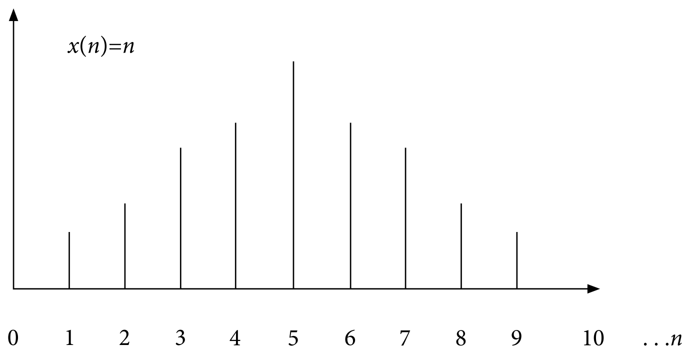


Fig. 2.31 Plot of signal for P 2.54

The total energy of the signal is given by

$$E = \sum_{0}^{4} n^2 + \sum_{5}^{9} (9-n)^2 = 1 + 4 + 9 + 16 + 16 + 9 + 4 + 1 = 60$$

Energy is finite. Hence, the signal is an energy signal.

The power of the signal is zero as the period is infinite.

P 2.55 Consider a signal defined as

$$x(n) = \begin{cases} \sin(\pi n/2) & \text{for } -4 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the energy and power of the signal and classify it as an energy or a power signal.

Solution

Let us plot the signal first. The period of the signal is

$$2\pi \frac{1}{4} N = 2\pi k \Rightarrow \frac{k}{N} = \frac{1}{4}, \text{ period is 4 samples.}$$

The total energy of the signal is given by

$$E = \sum_{n=-4}^{4} \sin^2(\pi n/2) = 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 = 4.$$

The energy is finite and the signal is an energy signal.

P 2.56 Consider a trapezoidal signal given by

$$X(t)=\begin{cases} t+5 & -5\le t\le -3 \\ 2 & -3\le t\le 3 \\ 5-t & 3\le t\le 5 \\ 0 & \text{otherwise} \end{cases}$$

Find the total energy of the signal.

Solution

The total energy is given by

$$E=\int_{-5}^{-3}(25+10t+t^2)dt+\int_{-3}^{3}2\times dt+\int_{3}^{5}(25-10t+t^2)dt$$

$$=50+10\frac{t^2}{2}\downarrow_{-5}^{-3}+\frac{t^3}{3}\downarrow_{-5}^{-3}+12+50-10\frac{t^2}{2}\downarrow_{3}^{5}+\frac{t^3}{3}\downarrow_{3}^{5}$$

$$=112-80+98/3-80+98/3$$

$$=52/3$$

P 2.57 Find if the signal $x(n)=\left(\frac{1}{3}\right)^n u(n)$ is an energy signal or a power signal?

Solution

Energy of the signal $E=\lim_{N\to\infty}\sum_{m=-N}^{N}|x(m)|^2$

$$E=\lim_{N\to\infty}\sum_{m=-N}^{N}\left|\left(\frac{1}{3}\right)^m u(m)\right|^2$$

$$E=\lim_{N\to\infty}\sum_{m=0}^{N}\left|\left(\frac{1}{9}\right)^m\right|$$

$$E=\sum_{m=0}^{\infty}\left|\left(\frac{1}{9}\right)^m\right|=\frac{1}{1-\frac{1}{9}}=\frac{9}{8}\text{ joules}$$

Power of the signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} |x(m)|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} \left(\frac{1}{9}\right)^m$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=0}^{N} \left(\frac{1}{9}\right)^m$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\left(\frac{1}{9}\right)}\right] \to 0$$

The signal has finite energy and zero power. So, the signal is an energy signal.

P 2.58 Find if the signal $x(n) = 3u(n)$ is an energy signal or a power signal?

Solution

$$u(n) = 1 \quad \text{for all } \ n \geq 0$$

The signal has infinite samples. The energy of the signal is

$$E = \sum_{n=0}^{\infty} 3 = \infty$$

The average power of the signal is

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 3 = 3 \times \frac{N+1}{2N+1} \to \frac{3}{2} \ \text{ is finite}$$

So, the signal is a power signal.

P 2.59 Find if the signal $x(n) = u(n) - u(n-7)$ is an energy signal or a power signal.

Solution

$$u(n) = 1 \quad \text{for all } \ n \geq 0 \ \text{ and } \ u(n-7) = 1 \quad \text{for all } \ n \geq 7.$$

$x(n) = u(n) - u(n-7) = 1$ for $0 \le n \le 6$.

The total energy of the signal is $E = \sum_{n=0}^{6} 1 = 7$ is finite. So, the signal is an energy signal.

P 2.60 Find the power of the signal given by $x(t) = e^{j5t}\cos(3t)$

Solution

$$x(t) = (\cos(5t) + j\sin(5t))\cos(3t)$$

$$= \frac{1}{2}[\cos(8t) + \cos(2t)] + j\frac{1}{2}[\sin(8t) - \sin(2t)]$$

Power of the signal is given by

$$P = 4\left(\frac{1}{2}\right)^2 \Big/ 2 = \frac{1}{2}$$

P 2.61 Determine if the signal is an energy signal or a power signal. Find the energy or power of the signal given by $x(t) = \sin^2(3t)$

Solution

The signal extends over infinite duration and is periodic. Let us find the power of the signal.

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sin^2(3t)\,dt$$

$$\sin^2(3t) = (\sin^2(3t)) = (1 - \cos^2(3t))$$

$$= \left(1 - \frac{1}{2}(1 + \cos(6t))\right) = \left(1 - \frac{1}{2} - \frac{1}{2}\cos(6t)\right)$$

$$= \left(\frac{1}{2} - \frac{1}{2}\cos(6t)\right)$$

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(\frac{1}{2}-\frac{1}{2}\cos(6t)\right)dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\frac{1}{2}(2T) = \frac{1}{2}$$

Power is finite and the signal is a power signal.

P 2.62 Determine if the signal is an energy signal or a power signal. Find the energy or power of the signal given by $x(t) = \sin(2t)[u(t-1)-u(t-5)]$.

Solution

The signal exists only over finite duration from $t = 1$ to $t = 5$. Hence, the signal is an energy signal. Let us find the total energy of the signal.

$$E = \int_1^5 |\sin(2t)|^2 dt = \int_1^5 \left(1-\cos^2(2t)\right)dt = \int_1^5 \left[1-\frac{1}{2}(1-\cos(4t)\right]dt$$

$$= \frac{1}{2}\int_1^5 dt = 2 \text{ joules}$$

P 2.63 Find the energy and average power of the signal given by $x(n) = e^{j[(\pi/2)n+\pi/6]}$

Solution

Energy of the signal is

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N}\left|e^{j[(\pi/2)n+\pi/6]}\right|^2 = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} 1 = \frac{2N+1}{2N+1} = 1$$

$$E = \lim_{N\to\infty} \sum_{-N}^{N}\left|e^{j[(\pi/3)n+\pi/2]}\right|^2 = \lim_{N\to\infty}(2N+1) = \infty$$

Note that $\left|e^{j[(\pi/3)n+\pi/2]}\right| = 1$

P 2.64 Find if the following signal is an energy or a power signal.

$$x(n) = \begin{cases} n & \text{for } 0 \le n \le 4 \\ 10-n & \text{for } 6 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Solution

The signal exists only for a small duration. Hence, it is an energy signal. Let us find the energy.

$$E = \sum_{n=0}^{4} |n|^2 + \sum_{n=6}^{9} (10-n)^2 = (1+4+9+16)+(16+9+4+1)$$

$$= 60$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\left[\sum_{n=0}^{4} |n|^2 + \sum_{n=6}^{9} (10-n)^2\right.$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}[(1+4+9+16)+(16+9+4+1)]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}[60] = \lim_{N\to\infty} \frac{1}{2N+1}[60] = 0$$

The signal is an energy signal.

P 2.65 Find energy of the signal shown in Fig. 2.32.

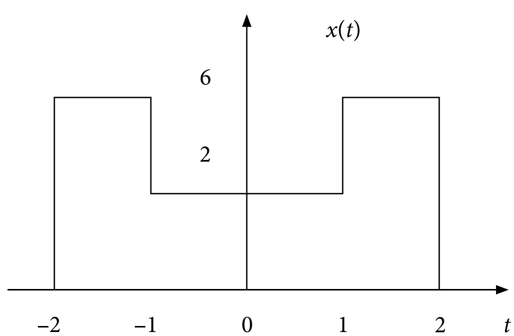


Fig. 2.32 Plot of signal for P 2.65

Solution

The energy of the signal is given by

$$E = \int_{-2}^{-1} 6^2\,dt + \int_{-1}^{1} 2^2\,dt + \int_{1}^{2} 6^2\,dt$$

$$= 36 + 8 + 36 = 80$$

P 2.66 Consider a rectangular pulse given by

$$x(t) = \text{rect}\left(\frac{t}{4}\right) = \begin{cases} 1 & \text{for } |t| \le 2 \\ 0 & \text{for } |t| > 2 \end{cases}$$

Draw the following functions derived from the rectangular pulse.

$x(3t)$, $x(3t + 4)$, $x(-2t - 2)$, $x(2(t + 2))$, $x(2(t - 2))$, $x(3t) + x(3t+4)$.

Solution

Let us first draw $x(t)$. It is shown in Fig. 2.33. We will now time scale it by a factor of 3 to get $x(3t)$. It is a compressed signal as shown in Fig. 2.34.

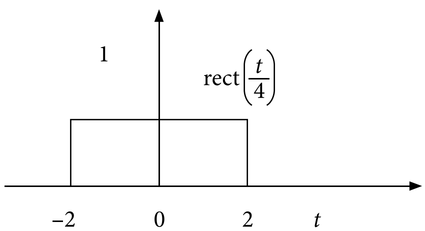


Fig. 2.33 Plot of $x(t)$

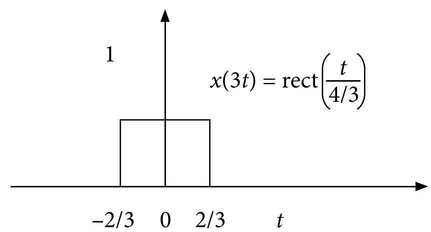


Fig. 2.34 Plot of $x(3t)$

$x(3t + 4) = x(3(t + 4/3))$ is the $x(3t)$ signal shifted left by 4/3 time units as shown in Fig. 2.35.

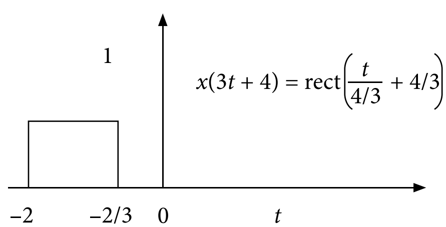


Fig. 2.35 Plot of $x(3t + 4)$

Let us plot $x(2t)$ and invert it to get $x(-2t)$. We will find that $x(-2t)$ is the same as $x(2t)$, as the signal is symmetrical about y-axis. It is shown in Fig. 2.36. $x(-2t - 2) = x(-2(t + 1))$ is the signal $x(-2t)$ shifted towards the left by 1 time unit. Let us draw $x(-2t - 2)$. It is shown in Fig. 2.37.

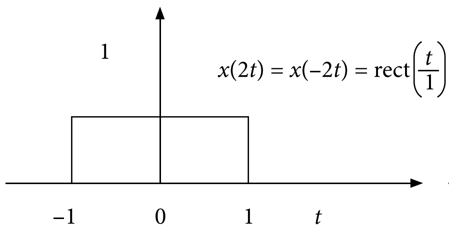


Fig. 2.36 Plot of $x(-2t)$

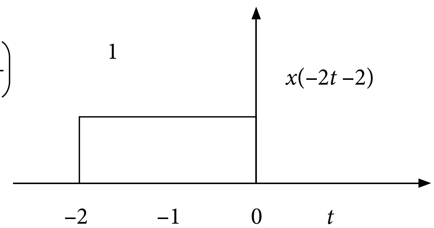


Fig. 2.37 Plot of $x(-2t - 2)$

We have already plotted $x(2t)$. So, let us shift it to the left by 2 time units to get $x(2(t + 2))$. It is shown in Fig. 2.38. $x(2(t - 2))$ is $x(2t)$ shifted towards the right by 2 time units as shown in Fig. 2.39.

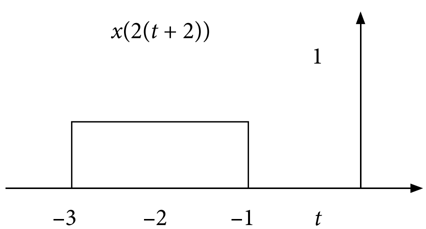


Fig. 2.38 Plot of $x(2(t + 2))$

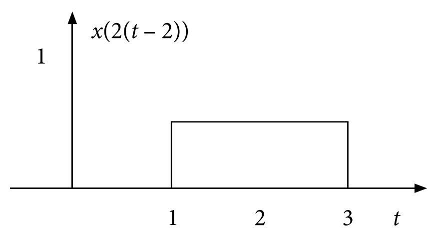


Fig. 2.39 Plot of signal $x(2(t - 2))$

We have already plotted $x(3t)$. We will now plot $x(3(t + 4/3))$ to get $x(3t + 4)$ which is signal $x(3t)$ shifted towards the left by 4/3 time units. It is plotted in Fig. 2.40. Let us add $x(3t)$ to $x(3t + 2)$ to get $x(3t) + x(3t + 2)$. It is shown in Fig. 2.41.

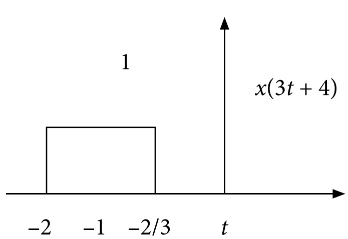


Fig. 2.40 Plot of $x(3t + 2)$

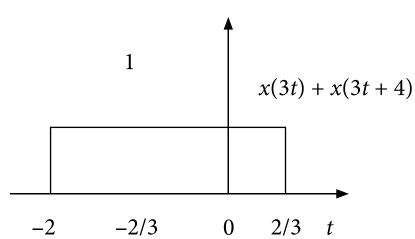


Fig. 2.41 Plot of $x(3t) + x(3t + 2)$

P 2.67 Let us solve the same problem using precedence rule.

P 2.68 Consider signal $x(n)$ shown in Fig. 2.42. Plot $x(2n)$ and $x(1/2n)$.

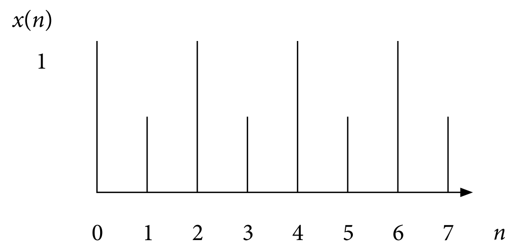


Fig. 2.42 Plot of signal $x(n)$

Solution

$x(2n)$ will compress the signal. Here, in the DT domain, compression by a factor of 2 will actually decimate the signal by 2, i.e., we have to collect alternate samples only. The signal $x(2n)$ is shown plotted in Fig. 2.43. We can observe that the samples with value equal to 1 are lost when we collect alternate samples.

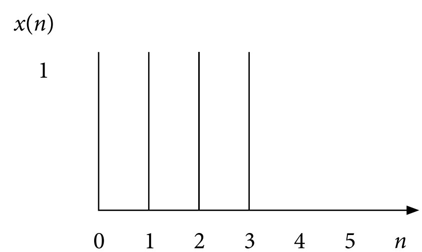


Fig. 2.43 Plot of signal $x(2n)$

The signal $x(n/2)$ is shown plotted in Fig. 2.44.

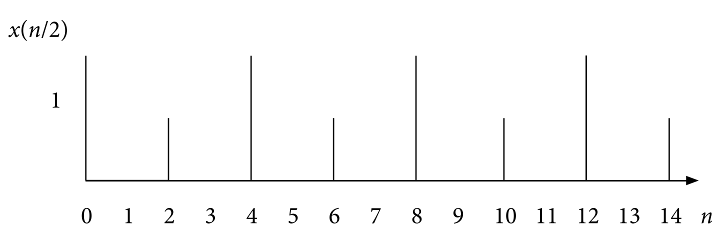


Fig. 2.44 Plot of $x(n/2)$

P 2.69 Consider two CT signals $x(t)$ and $y(t)$ as shown in Fig. 2.45. Find $x(t) + y(t)$, $x(t) - y(t)$ and $x(t)y(t)$, $x(t)y(t-1)$, $x(t+1)y(t-2)$, $x(t-1)y(-t)$, $x(t)$ $y(-t-1)$, $x(2t)y(-t+2)$, $x(2t) + y(2t)$.

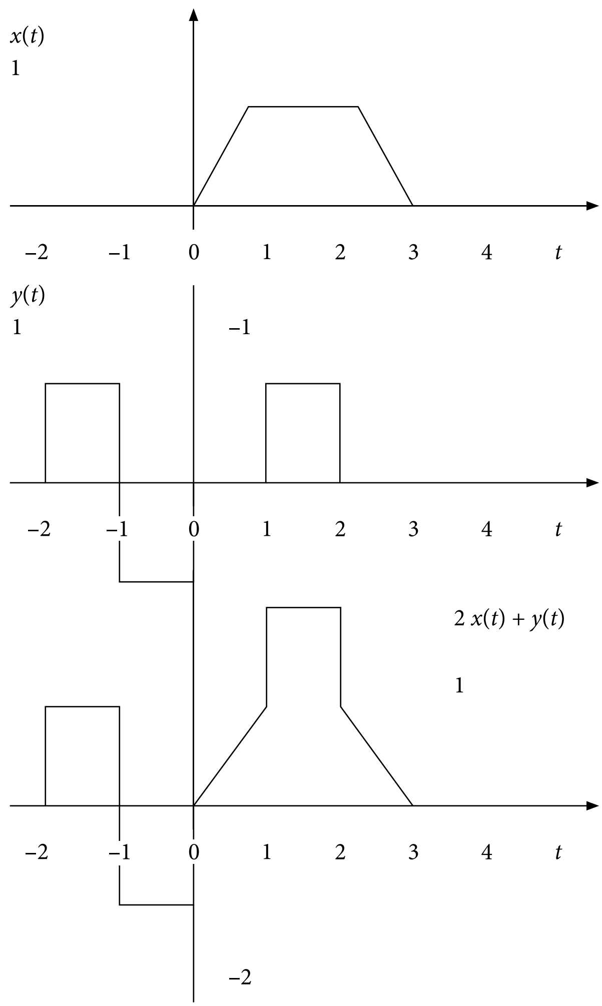


Fig. 2.45 Plot of $x(t)+y(t)$

Solution

Plot of $x(t)$, $y(t)$ and $x(t) + y(t)$ is all shown in Fig. 2.45. Plot of $x(t)$, $y(t)$ and $x(t) - y(t)$ is shown in Fig. 2.46. Let us now plot $x(t)\ y(t)$ in Fig. 2.47. $y(t - 1)$ is a signal $y(t)$ shifted towards the right by 1 time unit. We have plotted $x(t)$, $y(t - 1)$ and $x(t)y(t - 1)$ in Fig. 2.48. $x(t + 1)$ is signal $x(t)$ shifted left by 1 time unit and $y(t - 2)$ is signal $y(t)$ shifted towards the right by 2 time units. We have plotted $x(t + 1)$, $y(t - 2)$ and the product $x(t + 1)y(t - 2)$ in Fig. 2.49. $x(t - 1)$ is signal $x(t)$ shifted towards the right by 1 time unit and $y(-t)$ is the time reversed signal $y(t)$. The product $x(t - 1)y(-t)$ is shown in Fig. 2.50. $y(-t - 1)$ is signal $y(-t)$ shifted left by 1 time unit. $x(t)$ $y(-t - 1)$ is plotted in Fig. 2.51. $x(2t)$ and $y(2t)$ both are compressed signals by a factor of 2. The plot of $x(2t) + y(2t)$ is shown in Fig. 2.52.

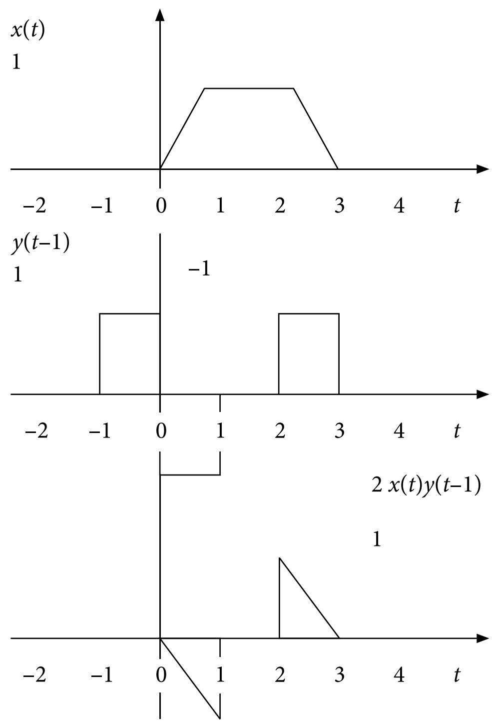


Fig. 2.46 Plot of $x(t) - y(t)$

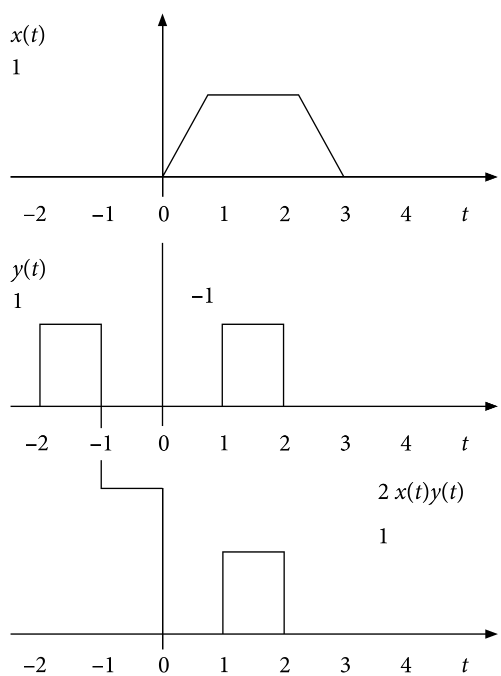


Fig. 2.47 Plot of $x(t)$, $y(t)$ and $x(t)y(t)$

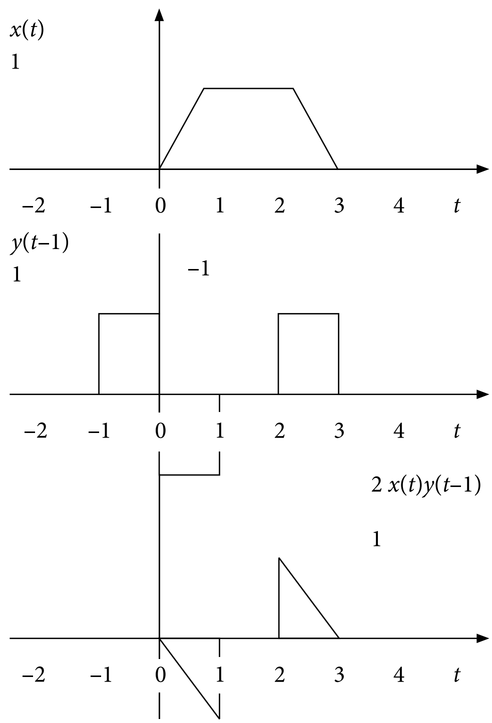


Fig. 2.48 Plot of $x(t)$, $y(t-1)$, $x(t)\ y(t-1)$

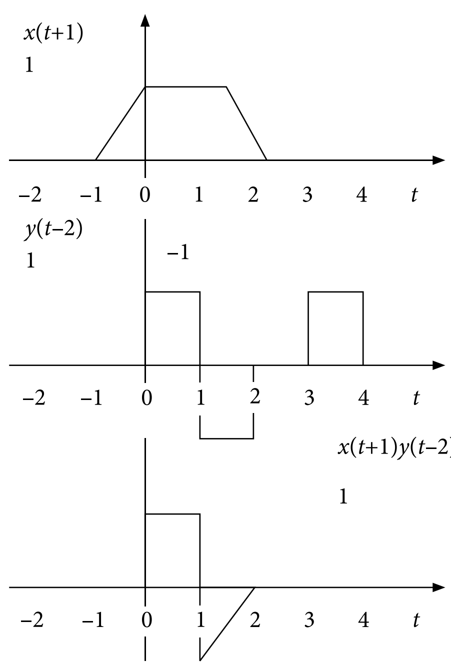


Fig. 2.49 Plot of $x(t+1)y(t-2)$

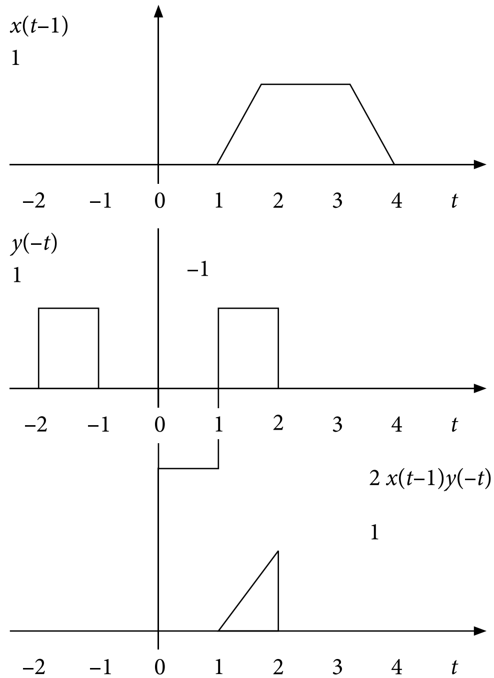


Fig. 2.50 Plot of $x(t-1)y(-t)$

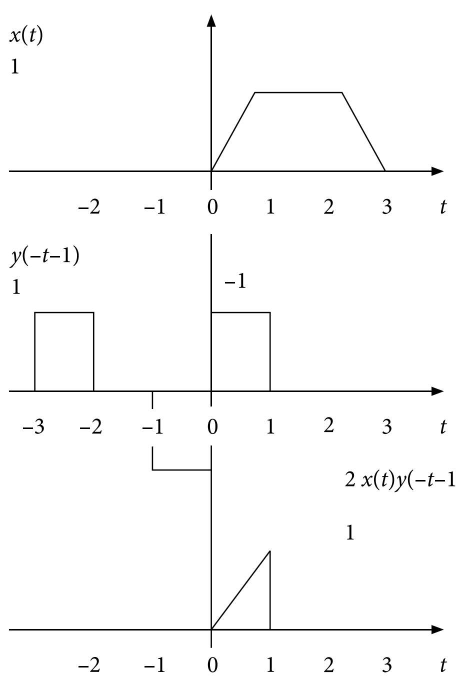


Fig. 2.51 Plot of $x(2t)y(-t+2)$

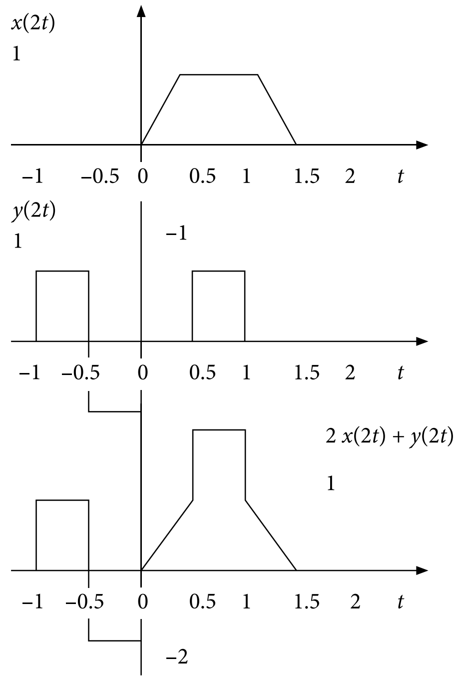
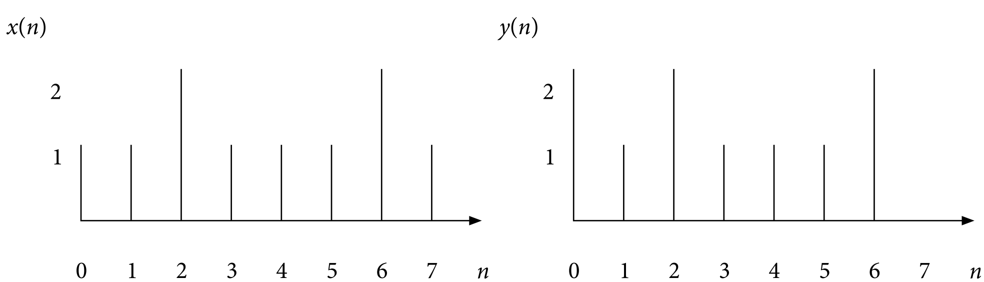


Fig. 2.52 Plot of $x(2t)y(3t)$

(Note: scale on the x-axis)

P 2.70 Consider DT signals $x(n)$ and $y(n)$ as shown in Fig. 2.53. Plot $x(n) + y(n)$, $x(n)y(n)$, $x(2n)y(n)$, $x(n-1)y(n+2)$.

$$x(n) = \{1,1,2,1,1,1,2,1\},\ y(n) = \{2,1,2,1,1,1,2,0\}$$



Plot of $x(n)$ Plot of $y(n)$

Fig. 2.53 Plot of signal $x[n]$ and $y[n]$

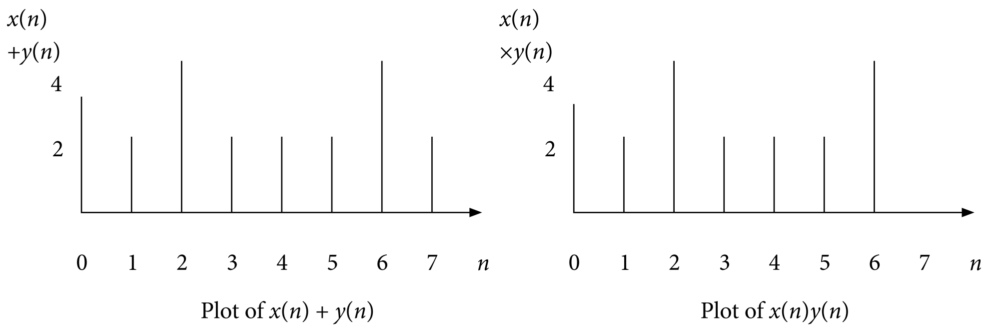


Fig. 2.54 a. Plot of $x[n] + y[n]$ and $x[n]y[n]$

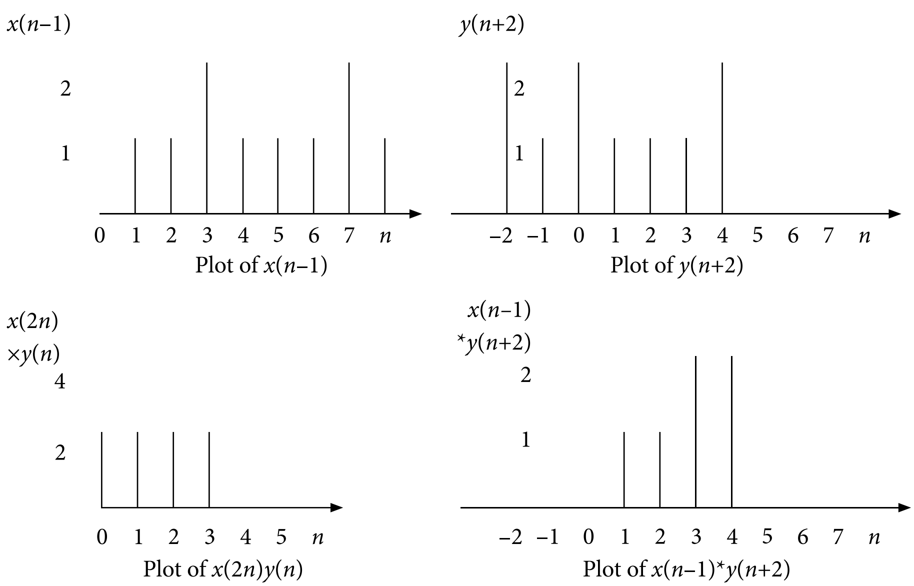


Fig. 2.54 b. Plots of $x(n-1)$, $y(n+2)$, $x(2n)y(n)$ and $x(n-1)*y(n+2)$

P 2.71 Sketch the waveforms given by the following equations, where $u(t)$ is a unit step function and $r(t)$ is a unit ramp function.

i. $x(t) = u(t) - u(t-4)$

ii. $x(t) = u(t+2) - 2u(t) + u(t-2)$

iii. $x(t) = -u(t+2) + 2u(t+1) + u(t-2)$

iv. $x(t) = -r(t+2) - r(t) + r(t-2)$

v. $x(t) = r(t) - r(t-2) - (t-3) + r(t-4)$

Solution

Let us draw each component of $x(t)$ one below the other and then draw $x(t)$.

Consider $x(t) = u(t) - u(t-4)$. It is drawn in Fig. 2.55. $x(t) = u(t+2) - 2u(t) + u(t-2)$ is plotted in Fig. 2.56. $x(t) = -u(t+2) + 2u(t+1) + u(t-2)$ is depicted in Fig. 2.57. $x(t) = -r(t+2) - r(t) + r(t-2)$ is plotted in Fig. 2.58. Note that slope is -1 at $t = -3$ and slope is equal to -2 at $t = 1$, slope is -1 at $t = 2$.

$x(t) = r(t) - r(t-2) - (t-3) + r(t-4)$ is plotted in Fig. 2.58. Note that slope is zero at $t = 2$ and slope is equal to -1 at $t = 3$, slope is zero at $t = 4$.

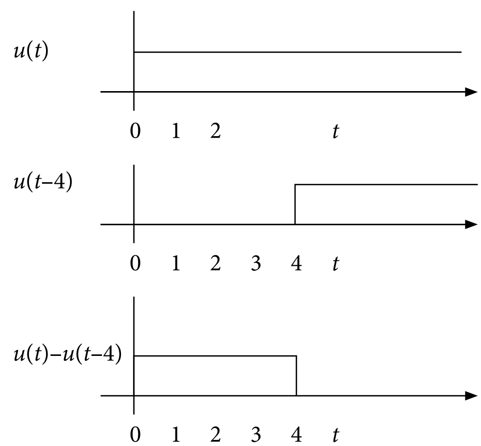


Fig. 2.55 Plot of $x(t)$ in (i)

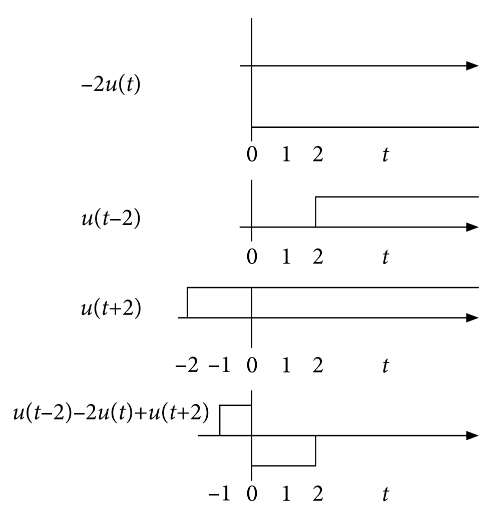


Fig. 2.56 Plot of $x(t)$ in (ii)

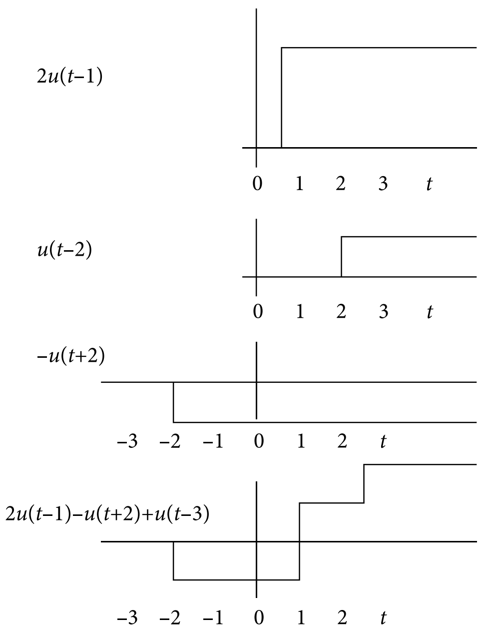


Fig. 2.57 Plot of $x(t)$ in (iii)

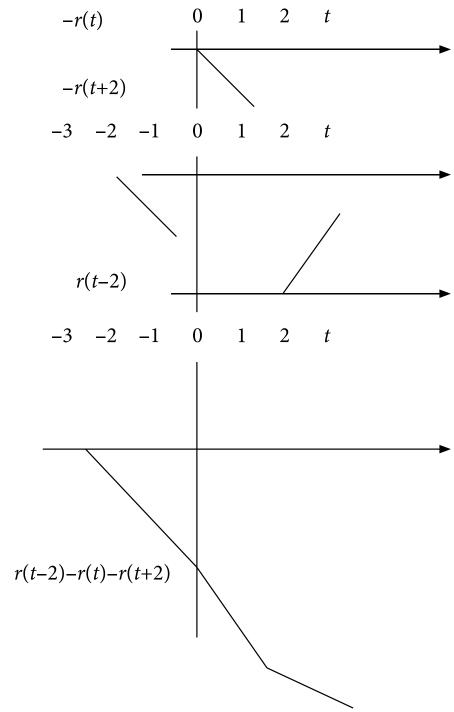


Fig. 2.58 Plot of $x(t)$ in (iv)

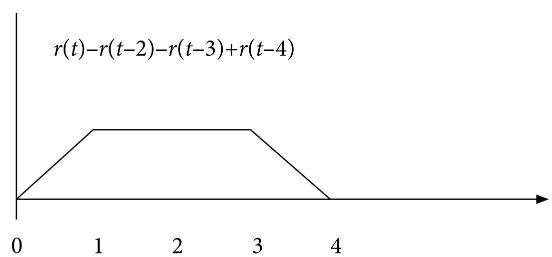


Fig. 2.59 Plot of $x(t)$ in (v)

3

CT and DT Systems

P 3.1 Is the system given by $y[n] = x[-n]$ a linear and shift invariant system?

Solution

Let us first check for linearity.

If the input is scaled, say $ax[-n]$, the output will be $ay[n]$. It is homogeneous. Let us check for additivity.

If the input is $ax_1[-n] + bx_2[-n]$, the output will be $ay_1[n] + by_2[n]$. The system obeys superposition and is linear.

To check for shift invariance, let the input be shifted by k units, say $x[-n + k]$. The output will be $y[n - k]$. The system is shift invariant.

P 3.2 Is the system given by $y(t) = x(t - 2)$ a linear and shift invariant system?

Solution

Let us first check for linearity.

If the input is scaled, say $ax(t)$, the output will be $ax(t - 2)$. It is homogeneous. Let us check for additivity.

If the input is $ax_1(t) + bx_2(t)$, the output will be $ay_1(t) + by_2(t)$. The system obeys superposition and is linear.

To check for shift invariance, let the input be shifted by k time units, say $x(t - k)$. The output will be $y(t - k) = x(t - 2 - k)$. The system is shift invariant.

P 3.3 Verify that the systems given by $y[n] = x[n]\cos(\omega n)$ and $y[n] = nx[n]$ are shift variant.

Solution

Let the input of the first system be shifted by k units. The output is

$$y[n-k]=x[n-k]\cos(\omega n)\neq x[n-k]\cos(\omega(n-k))$$

The system is time variant.

Let the input of the second system be shifted by k units. The output is

$$y[n-k]=nx[n-k]\neq(n-k)x[n-k]$$

The system is time variant.

P 3.4 Check if the systems given by $y(t) = (t - 1)\,x(t)$ and $y(t) = x(t)\cos(\omega t + \pi/4)$ are shift invariant?

Solution

Let the input of the first be shifted by k time units. The output is

$$y(t-k)=(t-1)x(t-k)\neq(t-1-k)x(t-k)$$

The system is time variant.

Let the input of the second system be shifted by k time units, the output is

$$y(t-k)=x(t-k)\cos(\omega t+\pi/4)\neq x(t-k)\cos(\omega(t-k)+\pi/4)$$

The system is time variant.

P 3.5 Find if the following systems are time invariant.

a. $y[n] = x[n] - x[n-1]$. Yes. If the input is shifted by k units, the output is $y[n-k] = x[n-k] - x[n-1-k]$. Thus, the system is time invariant.

b. $y[n] = nx[n-1]$. No. If the input is shifted by k units, the output is $y[n-k] = nx[n-1-k] \neq (n-k)\,x[n-1-k]$. Thus, the system is time variant.

c. $y[n] = x[1-n]$. Yes. If the input is shifted by k units, the output is $y[n-k] = x[1-(n-k)]$. Thus, the system is time invariant.

d. $y[n]=x[n]\sin(\omega n)$. No. If the input is shifted by k units, the output is $y[n-k] = x[n-k]\sin(\omega n) \neq x[n-k]\sin(\omega(n-k))$. Thus, the system is time invariant.

e. $y(t) = x(t) + x(t + 1)$. Yes. If the input is shifted by k time units, say $x(t - k)$, the output will be $x(t - k) + x(t + 1 - k) = y(t - k)$. Thus, the system is time invariant.

f. $y(t) = t^2 x(t)$. No. If the input is shifted by k units, the output is $y(t - k) = t^2 x(t - k) \neq (t - k)^2 x(t - k)$. Thus, the system is time variant.

g. $y(t) = x(4 - t)$. Yes. If the input is shifted by k time units, say $x(t - k)$, the output will be $x(4 - t + k) = y(t - k)$. Thus, the system is time invariant.

h. $y(t) = x(t)\sin(t)$. No. If the input is shifted by k units, the output is $y(t - k) = x(t - k)\sin(t) \neq x(t - k)\sin(t - k)$. Thus, the system is time variant.

P 3.6 Find if the following systems are linear.

a. $y[n] = (n + 1)x[n]$. Yes, linear

b. $y[n] = x[n^2]$. Yes, linear. Let input be $2x[n]$. Let $x[4]$ be $= 4$ and $x[2] = 2$. The output is $2x[n^2] = ay[n]$. If the input is $ax_1[n]+bx_2[n]$, the output is $y[n] = ax_1[n^2]+bx_2[n^2]$.

c. $y[n] = x^3[n]$. No. Not linear. Let input be $ax[n]$. Then, the output is $a^3x^3[n] \neq ay[n]$. If the input is $ax_1[n]+bx_2[n]$, the output is $y[n] = ax_1^3[n]+bx_2^3[n] \neq [ax_1[n]+bx_2[n]]^3$.

d. $y[n] = 2x[n] + 3$. No, not linear. Let the input be = 2 and doubled. Then, the output is initially 7 and after doubling, it is 11. The output is not doubled. The system is non-linear. The graph of the system is linear but not passing through the origin.

e. $y(t) = (t + 2)\,x(t)$. Let the input be doubled. Say for $t = 2$, $x(2) = 4$. $y(2) = (4)(4) = 16$. When input is 8, the output is $4 \times 8 = 32$. The output is also doubled. If the input is $ax_1(t) + bx_2(t)$, the output is $y(t) = (t + 2)(ax_1(t) + bx_2(t))$. Thus, the system is linear.

f. $y(t) = x^3(t)$. No, not linear. Let input be $ax(t)$; the output is $a^3x^3(t) \neq ay(t)$. If the input is $ax_1(t)+bx_2(t)$, the output is $y(t) = ax_1^3(t)+bx_2^3(t) \neq [ax_1(t)+bx_2(t)]^3$.

g. $y(t) = 3x(t) + 1$. No, not linear. Let the input be 2 and doubled; the output is initially 7 and after doubling, it is 13. The output is not doubled. The system is non-linear. The graph of the system is linear but not passing through the origin.

h. $y(t) = \sin(t)\,x(t)$. No, not linear. Let the input be 2 and doubled; the output is initially $2\sin(t_1)$ and after doubling, it is $4\sin(t_2)$. The output is not doubled. The system is non-linear. The graph of the system is linear but not passing through the origin. The sin function is not linear.

P 3.7 Find if the following systems are causal.

a. $y[n] = 5x[n]$. The current output depends only on current input so the system is causal.

b. $y[n] = \sum_{k=-\infty}^{n+1} x(k)$. The current output depends on current and past inputs and also on the next input. So the system is non-causal.

c. $y[n] = x[3 - n]$. Put $n = -2$. The output at $n = -2$ depends on $x[3-(-2)] = x[5]$. This is the next input. So the system is non-causal.

d. $y[n] = x[3n]$. The current output depends on the next input, so the system is non-causal. Put $n = 2$, $y[2]$ depends on $x[6]$.

e. $y(t) = x(t^2)$. The current output depends on the next input, so the system is non-causal. Put $t = 2$, $y(2)$ depends on $x(4)$.

f. $y(t) = x(5 - t)$. The current output depends on the next input, so the system is non–causal. Put $n = 2$, $y(t)$ depends on $x(6)$.

g. $y(t) = x(2t - 2)$. The current output depends on the next input, so the system is non-causal. Put $t = 3$, $y(3)$ depends on $x(4)$.

h. $y(t) = x(-2t)$. The current output depends on the next input, so the system is non-causal. Put $n = -2$, $y(-2)$ depends on $x(4)$.

P 3.8 Find if the following systems are memoryless

a. $y(t) = e^{-2}x(t)$. Yes, memoryless. The current output depends only on current input.

b. $y(t) = \cos(x(t))$. Yes, memoryless. The current output depends only on current input.

c. $y[n] = 5x[n] + 2x[n]u[n]$. Yes, memoryless. The current output depends only on current input.

d. $y(t) = \int_{-\infty}^{t/3} x(\tau)d\tau$. No, the system is with memory. The current output depends on several previous inputs.

e. $y(t) = x(7 - 2t)$. No, the system is with memory. The current output depends on the next input. Put $t = -2$, $y(-2) = x(11)$ or put $t = 2$, $y(2) = x(3)$.

f. $y(t) = x(t/5)$. No, the system is with memory. The current output depends on the previous input. $y(5) = x(1)$

P 3.9 Find if the following systems are stable.

a. $y(t) = \cos(x(t))$. If the input is bounded, the output is also bounded as it is a cos function. The system is stable.

b. $y[n]=\log_{10}(|x[n]|)$. If the input is bounded, the log function is also bounded. So, the system is a stable system.

c. $y[n]=\cos(2\pi x[n])+x[n]$. The function includes an addition of the cos function which has a value less than 1 and the input. If the input is bounded, the output is also bounded. The system is stable.

d. $y(t)=\frac{d}{dt}\left[e^{-t}x(t)\right]$. The function is a differentiation of the exp function which decays to zero as time tends to infinity. The system is stable.

e. $y(t)=x(t/3)$. The system is stable.

f. $y[n]=\sum_{m=-\infty}^{n} x[m+3]$

The output of the system is a sum of infinite terms. The system may diverge.

g. $y[n]=x[n]\sum_{m=-\infty}^{\infty}\delta[n-5m]$

P 3.10 Find if the following systems are invertible.

a. $y[n]=\sum_{m-\infty}^{n} x[m+3]$

The system is non-invertible as the values summed cannot be recovered.

b. $y[n]=x[n-1]+4$. The input can be recovered as $y[n]-4=x[n-1]$. The system is invertible.

c. $y(t) = x^3(t)$. The input can be recovered from the output as the cube root. The system is invertible.

d. $y(t)=x(t/9)$. The input can be recovered from the output. The system is invertible.

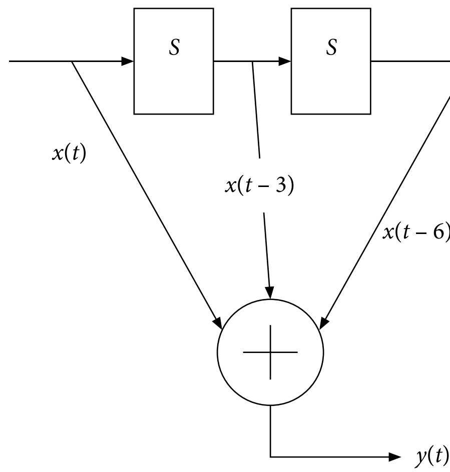
e. $y(t)=\sqrt{x(t)}$. The input can be recovered as the square of the input. But, there is no one to one correspondence. The system is non-invertible.

f. $y[n] = x[2n]$. The input can be obtained by interpolating zeros between alternate samples of $x[2n]$. The system is non-invertible

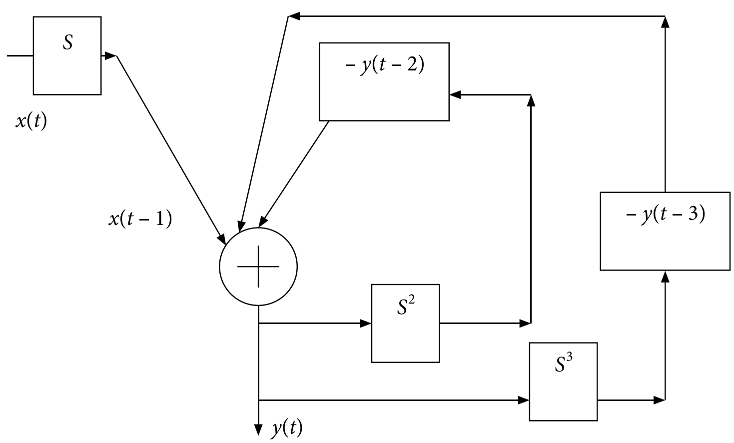
P 3.11 Represent the following systems in terms of interconnection of operators

1. $y(t) = x(t) + x(t - 3) + x(t - 6)$

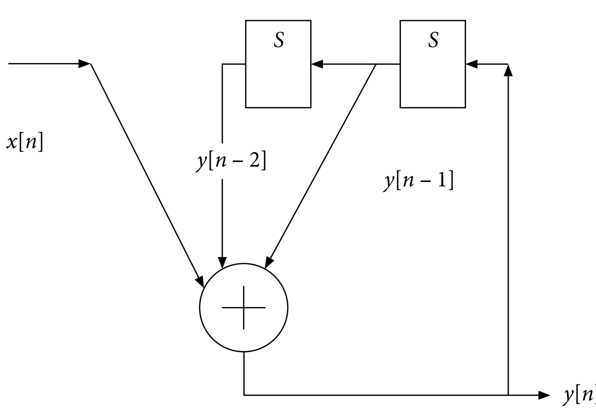
The student may use S to represent a delay of 3 time units.



2. $y(t) = x(t - 1) - y(t - 2) - y(t - 3)$. Let S represent the time delay of 1 unit.

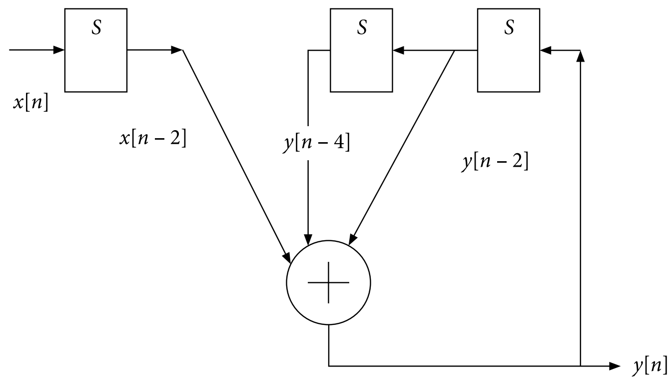


3. $y[n] = x[n] + y[n - 1] + y[n - 2]$



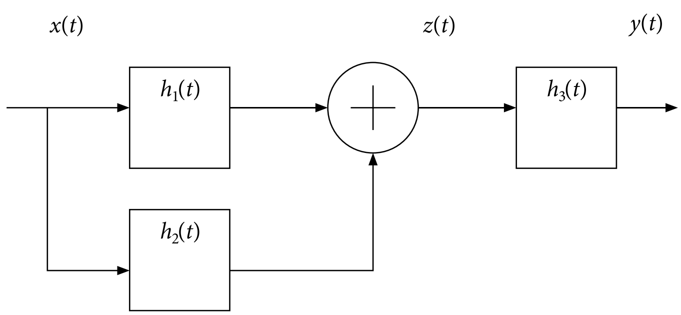
4. $y[n] = x[n-2] + y[n-2] - y[n-4]$

Let S represent the delay of 2 time units.



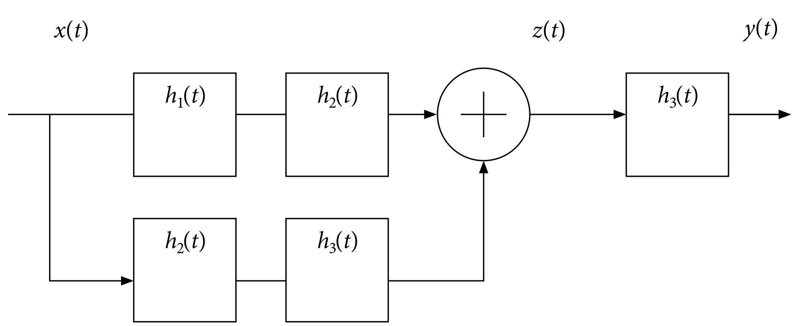
P 3.12 Find the overall impulse response for the interconnection of three systems.

(a)



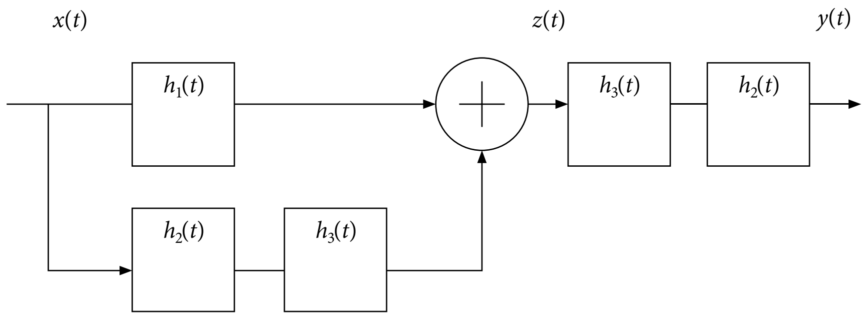
$$h_{\text{overall}}(t) = \{[h_1(t) + h_2(t)] \times h_3(t)$$

(b)



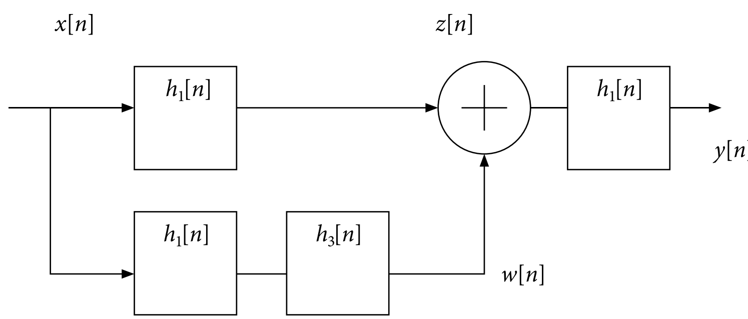
$$h_{\text{overall}}(t) = \{[h_1(t) \times h_2(t)] + [h_2(t) \times h_3(t)]\} \times h_3(t)$$

(c)



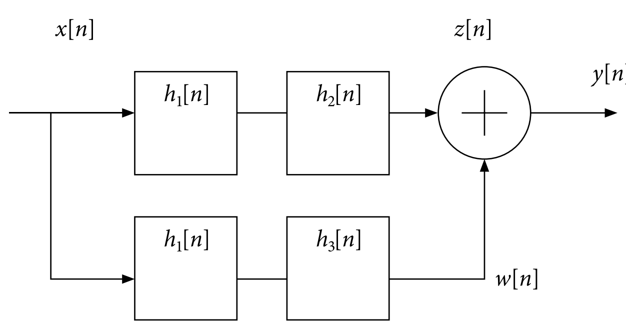
$$h_{\text{overall}}(t) = \left\{ h_1(t) + \left[h_2(t) \times h_3(t) \right] \right\} \times h_3(t) \times h_2(t)$$

(d)



$$h''[n] = \left\{ \left[h_1[n] \times h_2[n] \right] + \left[h_3[n] \times h_1[n] \right] \right\} \times h_1[n]$$

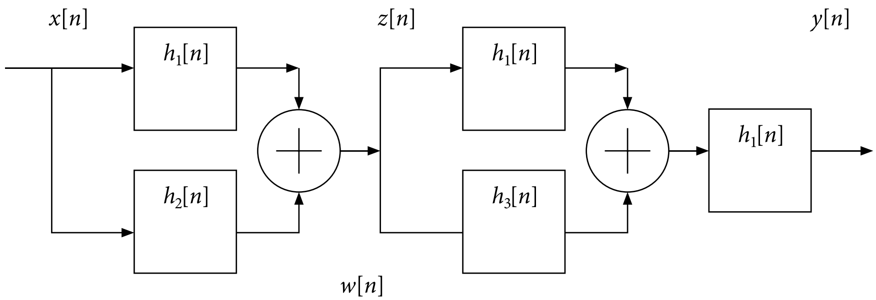
(e)



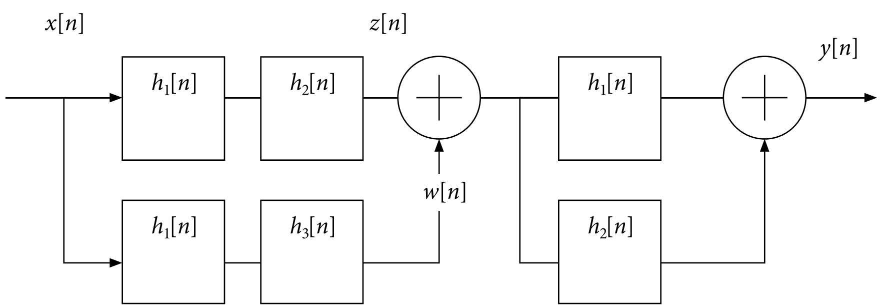
$$h''[n] = \left\{ \left[h_1[n] \times h_2[n] \right] + \left[h_3[n] \times h_1[n] \right] \right\}$$

P 3.13 Find the possible interconnection for the following equation of the overall impulse response of the system.

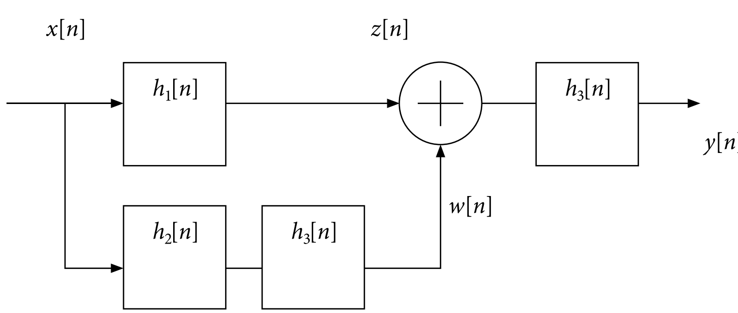
(a) $h_{\text{overall}}[n] = \{[h_1[n] + h_2[n]] \times [h_3[n] + h_1[n]]\} \times h_1[n]$



(b) $h_{\text{overall}}[n] = \{[h_1[n] \times h_2[n]] + [h_3[n] \times h_1[n]]\} \times [h_1[n] + h_2[n]]$



(c) $h_{\text{overall}}(t) = \{h_1(t) + [h_2(t) \times h_3(t)]\} \times h_3(t)$



4

Time Domain Response of CT and DT LTI Systems

P 4.1 Consider a simple second order system with characteristic equation given by $(D^2 + 5D + 6)\ y(t) = 0$. Find the zero input response if the initial conditions are $y(0) = 0$ and $Dy(0) = 5$.

Solution

$$(D^2 + 5D + 6) = 0 \Rightarrow (D + 3)(D + 2) = 0$$

The roots are $D = -3$ and $D = -2$

D is of the form $e^{-\lambda t}$ and e^{-t}. The solution can be written as

$$y(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

This is called the zero input response. We will now apply initial conditions to find the values of c_1 and c_2.

Put $t = 0$ in the solution to get $y(0) = c_1 + c_2 = 0$

Find the derivative of the solution and put $t = 0$ in the equation to get

$$Dy(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t} = 5$$

Put $t = 0$ to get $-3c_1 = 2c_2 + 5$

Put $c_1 = -\ c_2$, $3c_2 = 2c_2 + 5$

$$c_2 = 5 \text{ and } c_1 = -5$$

$$y(t) = (5e^{-t} - 5e^{-2t})u(t)$$

P 4.2 Determine the impulse response for a system given by the differential equation $(D^2+5D+6)y(t)=Dx(t)$.

Solution

Let us first evaluate the characteristic equation of the system and then evaluate the roots of the characteristic equation.
The characteristic equation is given by

$$(D^2 + 5D + 6) = 0 \Rightarrow (D + 3)(D+2)=0$$

The roots are $D = -3$ and $D = -2$ (4.14)

The solution can be written as

$$y(t)=(c_1e^{-3t}+c_2e^{-2t})u(t) \tag{4.15}$$

We need to find the values of the constants. The derivative of $y(t)$ can be written as

$$\bar{y}(t)=-3c_1e^{-2t}-2c_2e^{-t} \tag{4.16}$$

For any system with the denominator polynomial of order n, the initial conditions are given as follows.

$$y(0)=0, Dy(0)=0,\ldots\ldots, D^{n-2}y(0)=0 \text{ and } D^{n-1}y(0)=1; \tag{4.17}$$

We will use the result without going into the proof of the result. The initial conditions for the system with denominator polynomial of order 2, the initial conditions, will be translated as

$$y(0)=0 \quad \text{and} \quad \bar{y}(0)=1$$

where

$\bar{y}(0)=Dy(0)$ is first derivative of y.

Putting values of initial conditions in above equations gives

$$y(0)=c_1+c_2=0 \quad \text{and} \quad \bar{y}(0)=-3c_1-2c_2=1$$

$$\bar{y}(0)=3c_2-2c_2=1$$

$$c_2=1, c_1=-1$$

Solving the equations leads to $-c_1 = c_2 = 1$

$$y(t) = (-e^{-3t} + e^{-2t})u(t).$$

The second order term is zero in the numerator polynomial, i.e., $m < n$. Hence, put $a_0 = 0$ in the impulse response equation. The solution is

$$h(t) = (-e^{-3t} + e^{-2t})u(t)$$

which contains only the characteristic mode terms.

P 4.3 Determine the impulse response for a system given by the differential equation

$$D(D+3)y(t) = (D+2)x(t).$$

Solution

Let us first evaluate the characteristic equation of the system and then evaluate the roots of the characteristic equation.
The characteristic equation is given by

$$D(D+3) = 0 \Rightarrow \text{ roots are } D = -3 \text{ and } D = 0$$

The solution can be written as

$$y(t) = c_1 e^{-3t} + c_2$$

We need to find the values of the constants. The derivative of $y(t)$ can be written as

$$\overline{y}(t) = -3c_1 e^{-3t}$$

For any system with the denominator polynomial of order n, the initial conditions are given as follows.

For the system with denominator polynomial of order 2, the initial conditions will be translated as $y(0) = 0$ and $\overline{y}(0) = 1$, where $\overline{y}(0) = Dy(0)$ is the first derivative of y. Putting values of initial conditions in the previous equations gives

$$y(0) = c_1 + c_2 = 0 \text{ and } \overline{y}(0) = -3c_1 = 1 \Rightarrow c_1 = -1/3$$

Solving the equations leads to $c_1 = -c_2 = -1/3$

$$y(t)=\left(-\frac{1}{3}e^{-2t}+\frac{1}{3}\right)u(t)\,.$$

The second order term is zero in the numerator polynomial, i.e., $m < n$. Hence, put $a_0 = 0$ in the impulse response equation. The solution is

$$h(t)=\left(-\frac{1}{3}e^{-2t}+\frac{1}{3}\right)u(t)$$

which contains only the characteristic mode terms.

P 4.4 Determine the impulse response for a system given by the differential equation

$$(D+3)y(t)=(D+1)x(t)\,.$$

Solution

Let us first evaluate the characteristic equation of the system and then evaluate the roots of the characteristic equation.
The characteristic equation is given by

$$(D + 3) = 0 \;\Rightarrow\; \text{roots are } D = -3$$

The solution can be written as

$$y(t)=a_0\delta(t)+c_1e^{-3t}$$

We need to find the values of the constants. The derivative of $y(t)$ can be written as a_0 is the value of the nth order term in the denominator polynomial and is equal to 1. $\overline{y}(t)=-3c_1e^{-3t}$

For any system with the denominator polynomial of order n, the initial conditions are given as follows.

For the system with denominator polynomial of order 2, the initial conditions will be translated as $y(0)=0$ and $\overline{y}(0)=1$, where $\overline{y}(0)=Dy(0)$ is first derivative y. Putting values of initial conditions in the previous equations gives

$$y(0)=a_0+c_1=0 \text{ and } \overline{y}(0)=-3c_1=1 \Rightarrow c_1=-1/3$$

Solving the equations leads to $a_0 = 1/3$

$$y(t)=\left(-\frac{1}{3}e^{-2t}u(t)+\frac{1}{3}\delta(t)\right).$$

The order of the numerator is the same as that of the denominator polynomial, i.e., $m = n$. Hence, the a_0 term exists in the impulse response equation. The solution is

$$h(t)=\left(-\frac{1}{3}e^{-2t}u(t)+\frac{1}{3}\delta(t)\right)$$

which contains the characteristic mode terms and the response due to the unit impulse at $t = 0$.

P 4.5 Determine the impulse response for a system given by the differential equation $(D+1)y(t)=x(t)$.

Solution

Let us first evaluate the characteristic equation of the system and then evaluate the roots of the characteristic equation.

The characteristic equation is given by

$$(D + 1) = 0 \Rightarrow \text{ roots are } D = -1$$

The solution can be written as

$$y(t)=a_0\delta(t)+c_1e^{-t}$$

We need to find the values of the constants.

$m < n$. Hence, put $a_0 = 0$ in the impulse response equation.

For any system with a denominator polynomial of order n, the initial conditions are given as follows.

For a system with a denominator polynomial of order 1, the initial conditions will be translated as $y(0) = 1$. Putting the values of initial conditions in the previous equations gives $y(0) = c_1 = 1$.

Solving the equations leads to

$$y(t)=(e^{-t}u(t))\cdot$$

The solution is $h(t)=(e^{-t}u(t))$ which contains only the characteristic mode terms.

P 4.6 Let $x(t) = e^{-2t}[u(t) - u(t-3)]$, $h(t) = e^{-t}u(t)$. Find $x(t) \times h(t)$.

Solution

We start with step 1, i.e., drawing $x(\tau)$ and $h(-\tau)$.

Step 1 Let us draw both these waveforms. Plots of $x(\tau)$ and $h(t - \tau)$ for different intervals are shown in Fig. 4.1. We have to shift $h(t - \tau)$ slowly towards the right.

Step 2 Start with time shift t large and negative. Let t vary from minus infinity to zero. We find that until t crosses zero, there is no overlap between the two signals. Hence, the convolution integral has the value of zero from minus infinity to zero. At $t = 0$, the right edge of $h(-\tau)$ touches left edge of $x(\tau)$.

Step 3 Consider the second interval between $t = 0$ to 3. $x(\tau)h(t-\tau) = e^{-2\tau}e^{-(t-\tau)}$. The overlapping interval will be between 0 to t. The output can be calculated as

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau$$

$$= \int_0^t e^{-t}e^{-\tau}d\tau = e^{-t}[-e^{-\tau}]_0^t = e^{-t}[-e^{-t} + 1]$$

$$= e^{-t} - e^{-2t}$$

$$y(2) = e^{-2} - e^{-4}$$

Step 4 The third interval is between 3 to infinity. For $t \geq 3$, the overlapping interval will be 0 to 3. The output is given by

$$y(t) = \int_0^3 x(\tau)h(t-\tau)d\tau = \int_0^3 e^{-2\tau}e^{-(t-\tau)}d\tau$$

$$= \int_0^3 e^{-t}e^{-\tau}d\tau = e^{-t}[-e^{-\tau}]_0^3 = e^{-t}[-e^{-3} + 1]$$

$$= e^{-t}[1 - e^{-3}]$$

$$y(3) = e^{-3} - e^{-9}$$

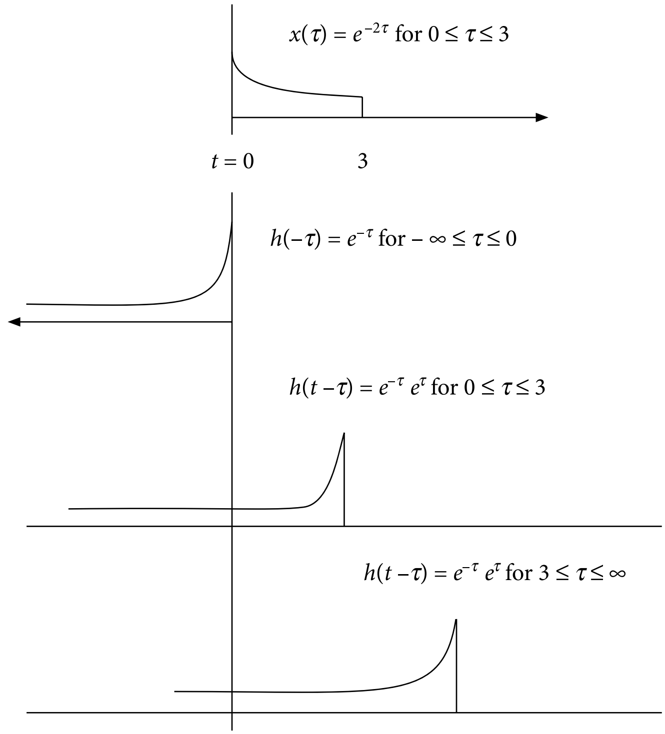


Fig. 4.1 Plots of $x(\tau)$ and $h(t - \tau)$ for various time intervals

The output of the system is shown plotted in Fig. 4.2.

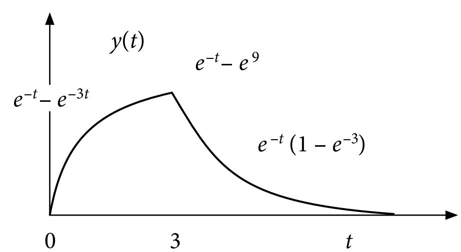


Fig. 4.2 The output of the system

The output can be specified as follows.

$$y(t) = 0 \quad \text{for} \quad t < 0$$

$$= e^{-t} - e^{-3t} \quad \text{for} \quad 0 \le t \le 3$$

$$= e^{-t}(1 - e^{-3}) \quad \text{for} \quad t > 3$$

P 4.7 Let $x(t) = 1 - t$ for $0 \le t \le 1$, $h(t) = e^{-t}u(t)$. Find $x(t) \times h(t)$.

Solution

We start with step 1, i.e., drawing $x(\tau)$ and $h(-\tau)$.

Step 1 Let us draw both these waveforms. Figure. 4.3 shows plots of $x(\tau)$ and $h(-\tau)$.

Step 2 Start with time shift t large and negative. Let t vary from minus infinity to zero. We find that until t crosses zero, there is no overlap between the two signals. Hence, the convolution integral has a value of zero from minus infinity to zero. At $t = 0$, the right edge of $h(-\tau)$ touches the left edge of $x(\tau)$.

Step 3 Consider the second interval between $t = 0$ to 1. $x(\tau)h(t-\tau) = e^{-\tau}(1-t+\tau)$. The overlapping interval will be between 0 to t. The output can be calculated as

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t e^{-\tau}(1-t+\tau)d\tau$$

$$= (1-t)\int_0^t e^{-\tau}d\tau + \int_0^t \tau e^{-\tau}d\tau = (1-t)\left[-e^{-\tau}\right]_0^t + \left[(e^{-\tau}(-\tau-1))\right]_0^t$$

$$= (1-t)[1-e^{-t}] + \left[-te^{-t} - e^{-2t} + 1\right]$$

$$y(t) = 1 - t - e^{-t} + te^{-t} - te^{-t} - e^{-t} + 1$$

$$= [2 - t - 2e^{-t}] \text{ for } 0 \le t \le 1$$

Step 4 The third interval is between 1 to infinity. For $t \ge 1$, the overlapping interval will be $t - 1$ to t. The output is given by

$$y(t) = \int_{t-1}^t x(\tau)h(t-\tau)d\tau = \int_{t-1}^t e^{-\tau}(1-t+\tau)d\tau$$

$$= (1-t)\int_{t-1}^t e^{-\tau}d\tau + \int_{t-1}^t \tau e^{-\tau}d\tau = (1-t)\left[e^{-\tau}\right]_{t-1}^t + \left[\frac{1}{4}e^{-\tau}(-\tau-1)\right]_{t-1}^t$$

$$= (1-t)(e^{-(t-1)} - e^{-t}) - \left[te^{-t} - e^{-t} + te^{-(t-1)} - e^{-(t-1)} - 1\right]$$

$$y(t) = e^{-(t-1)} - 2e^{-t} \text{ for } t > 1$$

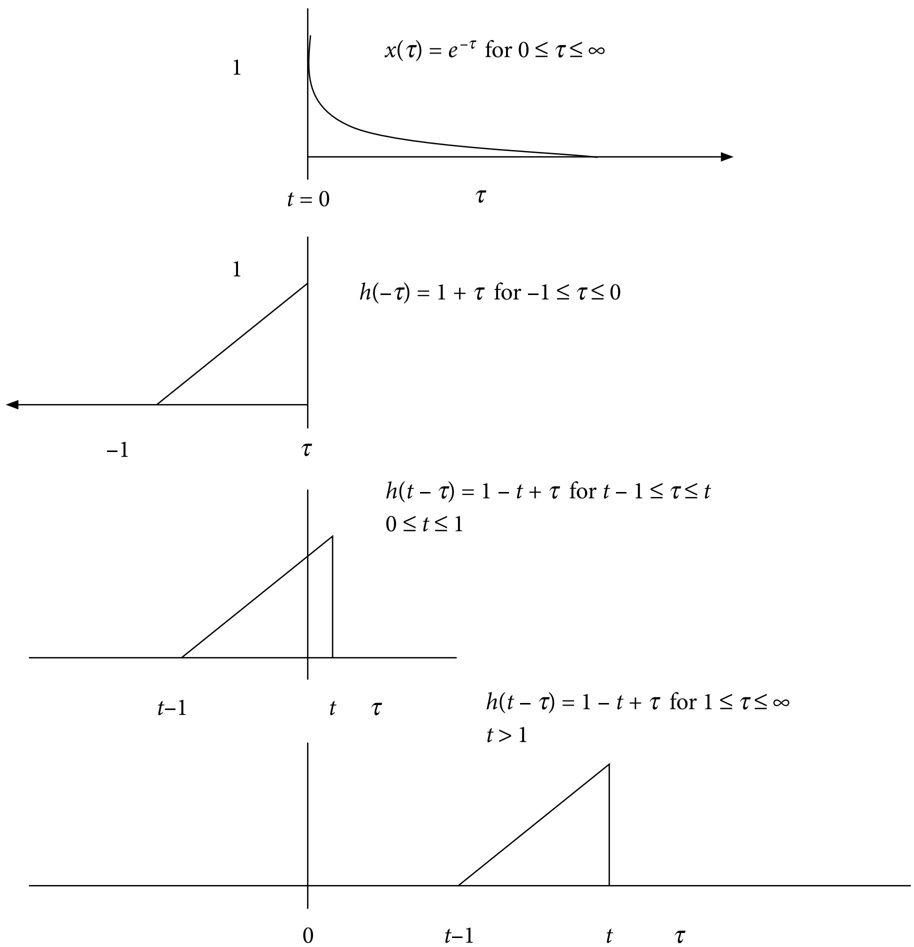


Fig. 4.3 Plot of $x(\tau)$ and $h(t-\tau)$ for different intervals

The output can be specified as follows.

$$
\begin{aligned}
y(t) &= 0 && \text{for } t < 0 \\
&= (2 - t - 2e^{-t}) && \text{for } 0 \le t \le 1 \\
&= e^{-(t-1)} - 2e^{-t} && \text{for } t > 1
\end{aligned}
$$

P 4.8 Let $x(t) = 2$ for $1 \le t \le 2$, $h(t) = 1$ for $0 \le t \le 4$. Find $x(t) \times h(t)$.

Solution

We start with step 1, i.e., drawing $x(\tau)$ and $h(-\tau)$.

Step 1 Let us draw both these waveforms. Figure. 4.4 shows plots of $x(\tau)$ and $h(t - \tau)$ for different intervals.

Step 2 Start with time shift t large and negative. Let t vary from minus infinity to 1. We find that until t crosses 1, there is no overlap between the two signals. Hence, the convolution integral has value of zero from minus infinity to 1. At $t = 1$, the right edge of $h(-\tau)$ touches the left edge of $x(\tau)$.

Step 3 Consider the second interval between $t = 1$ to 2. $x(\tau)h(t-\tau)=2$. The overlapping interval will be between 1 to t. The output can be calculated as

$$y(t)=\int_1^t 2\times 1 d\tau = 2\tau \downarrow_1^t = (2t-2)$$

Step 4 Consider the second interval between $t = 2$ to 5. $x(\tau)h(t-\tau)=2$. The overlapping interval will be from 1 to 2. The output can be calculated as $y(t)=\int_1^2 2\times 1 d\tau = 2\tau \downarrow_1^2 = 2$. The output is constant equal to 4.

Step 5 Consider the second interval between $t = 5$ to 6. $x(\tau)h(t-\tau)=2$. The overlapping interval will be from $t - 3$ to 2. The output can be calculated as

$$y(t)=\int_{t-4}^2 2\times 1 d\tau = 2\tau \downarrow_{t-4}^2 = 2(2-t+4) = 2(6-t)$$

Step 6 Consider the second interval between $t = 5$ to infinity. $x(\tau)h(t-\tau)=2$. No overlapping interval will be there. The output is zero.
The overall output can be summarized as

$$\begin{aligned} y(t) &= 0 && \text{for } t<1 \\ &= 2(t-2) && \text{for } 1\le t\le 2 \\ &= 2 && \text{for } 2\le t\le 5 \\ &= 2(6-t) && \text{for } 5\le t\le 6 \\ &= 0 && \text{for } t>6 \end{aligned}$$

The output $y(t)$ is drawn in Fig. 4.5.

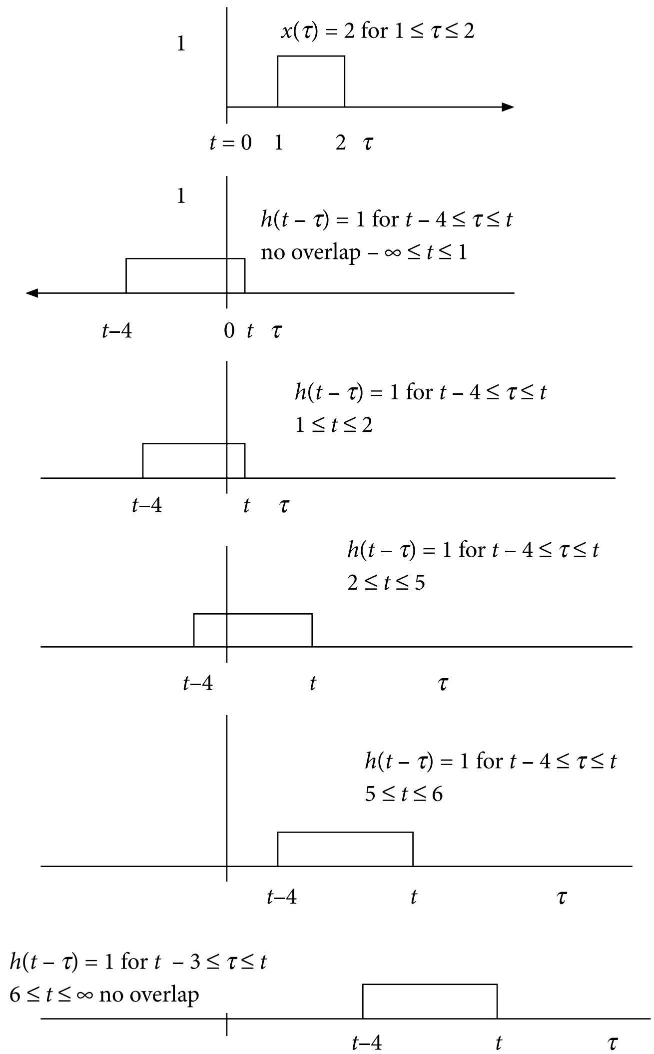


Fig. 4.4 Plots of $x(\tau)$ and $h(t - \tau)$ for different intervals

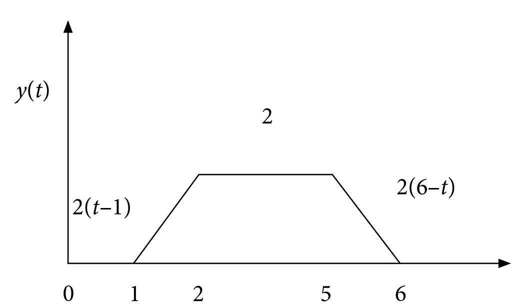


Fig. 4.5 Plot of the output of the system

P 4.9 Let $x(t) = 1$ for $0 \le t \le 2$, $h(t) = t$ for $0 \le t \le 3$. Find $x(t) \times h(t)$.

Solution

We start with step 1, i.e., drawing $x(\tau)$ and $h(-\tau)$.

Step 1 Let us draw both these waveforms. Figure. 4.6 shows plots of $x(\tau)$ and $h(t - \tau)$ for different intervals.

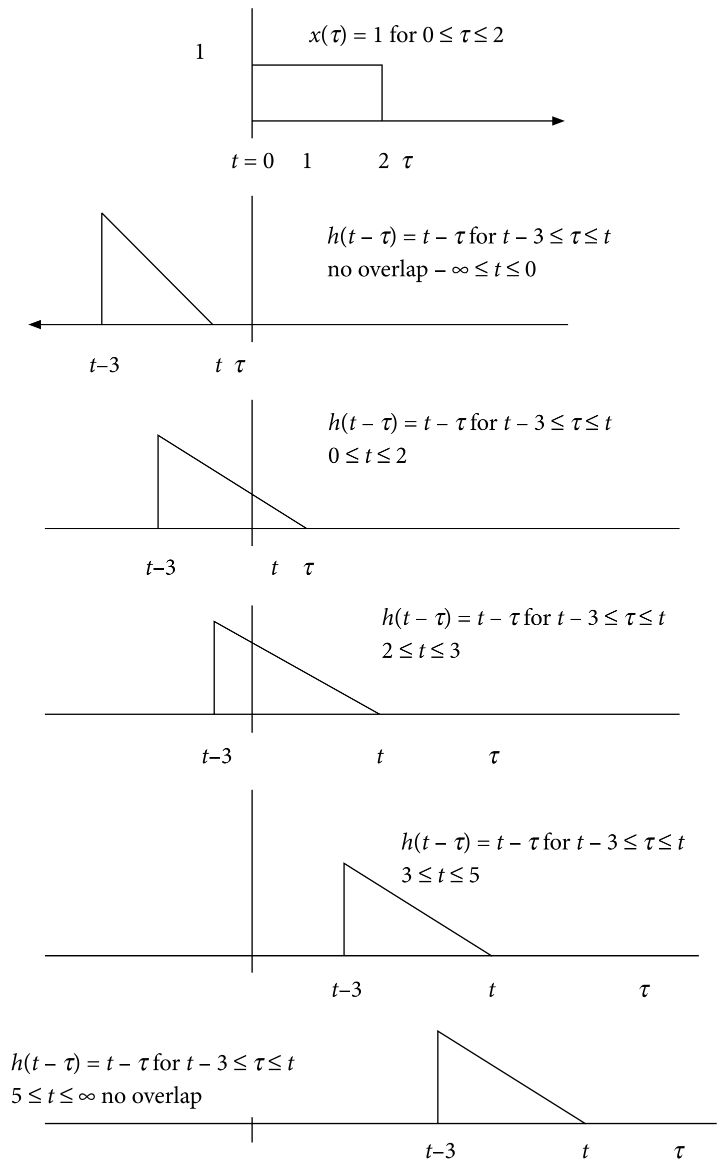


Fig. 4.6 Plots of $x(\tau)$ and $h(t - \tau)$ for different intervals

Step 2 Start with time shift t large and negative. Let t vary from minus infinity to zero. We find that until t crosses zero, there is no overlap between the two signals. Hence, the convolution integral has value of zero from minus infinity to zero. At $t = 0$, the right edge of $h(-\tau)$ touches the left edge of $x(\tau)$.

Step 3 Consider the second interval between $t = 0$ to 2. $x(\tau)h(t-\tau) = 1(t-\tau)$. The overlapping interval will be between 0 to t. The output can be calculated as

$$y(t) = \int_0^t (t-\tau)d\tau = t\tau \downarrow_0^t - \tau^2/2 \downarrow_0^t = (t^2 - t^2/2) = t^2/2$$

as shown in Fig. 4.7.

Step 4 Consider the second interval between $t = 2$ to 3. $x(\tau)h(t-\tau) = (t-\tau)$. The overlapping interval will be from 0 to 2. The output can be calculated as

$$y(t) = \int_0^2 (t-\tau)d\tau = t\tau \downarrow_0^2 - \tau^2/2 \downarrow_0^2 = 2t - 2.$$

The output is as shown in Fig. 4.7.

Step 5 Consider the second interval between $t = 3$ to 5. $x(\tau)h(t-\tau) = (t-\tau)$. The overlapping interval will be from $t - 3$ to 2. The output can be calculated as

$$y(t) = \int_{t-3}^2 (t-\tau)d\tau = t\tau \downarrow_{t-3}^2 - \tau^2/2 \downarrow_{t-3}^2 = t(2-t+3) - \left(2 - \frac{1}{2}(t^2 - 6t + 9)\right)$$

$$= 5t - t^2 - 2 + \frac{1}{2}t^2 - 3t + \frac{9}{2} = 2t - \frac{1}{2}t^2 + \frac{5}{2}$$

as shown in Fig. 4.7.

Step 6 Consider the second interval between $t = 5$ to infinity. $x(\tau)h(t - \tau) = (t - \tau)$.
No overlapping interval will be there. The output is zero.
The overall output can be summarized as

$$y(t) = 0 \quad \text{for} \quad t < 0$$

$$= t^2/2 \quad \text{for} \quad 0 \le t \le 2$$

$$= 2t - 2 \quad \text{for} \quad 2 \le t \le 3$$

$$= 2t - t^2/2 + 5/2 \quad \text{for } 3 \le t \le 5$$

$$= 0 \text{ for } t > 5$$

The overall output can be drawn as shown in Fig. 4.7.

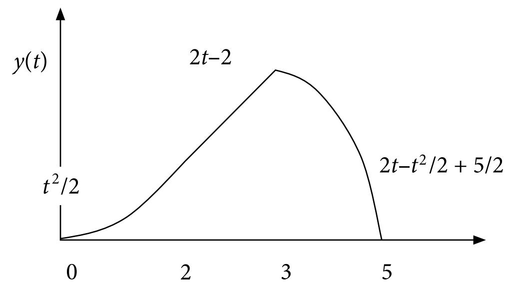


Fig. 4.7 Overall output of the system

P 4.10 Consider a system given by $(D+1)y(t)=x(t)$. The impulse response is calculated as $h(t)=(e^{-t}u(t))$. Let the external input be applied as $x(t)=(e^{-t}u(t))$. Find the response of the system for the applied input.

Solution

We have to find the zero state response by convolving the impulse response with the externally applied input assuming all initial conditions as zero.

We start with step 1, i.e., drawing $x(\tau)$ and $h(-\tau)$.

Step 1 Let us draw both these waveforms. Figure. 4.8 shows plots of $x(\tau)$ and $h(t-\tau)$ for different intervals.

Step 2 Start with time shift t large and negative. Let t vary from minus infinity to zero. We find that until t crosses zero, there is no overlap between the two signals. Hence, the convolution integral has value of zero from minus infinity to zero. At $t = 0$, the right edge of $h(-\tau)$ touches the left edge of $x(\tau)$.

Step 3 Consider the second interval between $t = 0$ to ∞. $x(\tau)h(t-\tau)=e^{-\tau}e^{-(t-\tau)}$. The overlapping interval will be between 0 to t. The output can be calculated as

$$y(t)=\int_0^t x(\tau)h(t-\tau)d\tau=\int_0^t e^{-\tau}e^{-(t-\tau)}d\tau$$

$$=\int_0^t e^{-t}d\tau=e^{-t}[\tau]_0^t=e^{-t}[t]$$

$$=te^{-t}$$

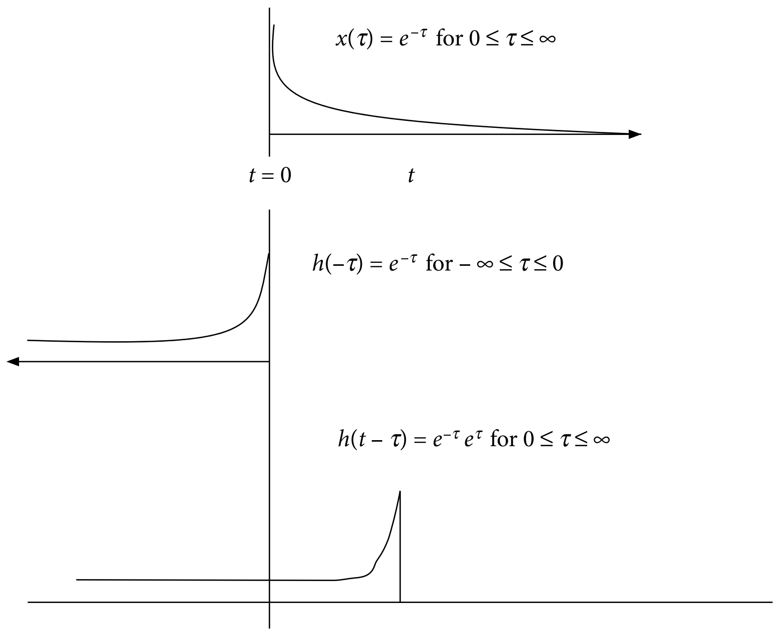


Fig. 4.8 Plots of $x(\tau)$ and $h(t - \tau)$ for various time intervals

The output of the system is shown plotted in Fig. 4.9.

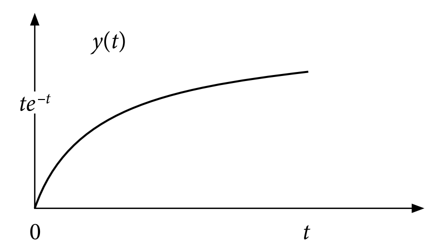


Fig. 4.9 Overall output of the system

The output can be specified as follows.

$$y(t) = 0 \text{ for } t < 0$$

$$= e^{-t} - e^{-2t} \text{ for } 0 \le t \le \infty \tag{4.52}$$

or $\quad y(t) = (e^{-t} - e^{-2t})u(t)$

The output has a term due to external input and a term due to system characteristic equation.

P 4.11 Consider the difference equation

$$y[k] - 0.4\,y[k-1] = f[k]$$

$$y[-1] = 8 \text{ and } f[k] = k^2$$

Find the solution using recursive procedure.

Solution

Put $k = 0$ to get

$$y[0] - 0.4y[-1]=0, y[0]= 3.2$$

Put $k = 1$, $y[1] - 0.4y[0] = f[1]$

$$y[1] = 1.25 + 1 = 2.28 \tag{4.58}$$

Similarly, one can find $y[2]$ and so on.

P 4.12 Consider a simple second order system with characteristic equation given by

$$y[k+2]-0.2y[k+1]-0.15y[k]=5f[k+2] \text{ with } y[-1]=0 \text{ and } y[-2]=5$$

$$f[k]=2^{-k}u[k]$$

Find the zero input response for the system.

Solution

The system equation is given in advance form so that we can write it in operational form. It can be written as $(D^2-0.2D-0.15)y[k]=5D^2 f[k]$.

We will first write the characteristic equation by equating the denominator polynomial to zero when $f[k] = 0$, i.e.,

$$(D^2-0.2D-0.15)=0$$

$$(D-0.5)(D+0.3)=0$$

The solution is $D = 0.5$ and $D = -0.3$.

The zero input response can be written as

$$y[k] = c_1(0.5)^k + c_2(-0.3)^k$$

We will now use initial conditions to find the constants.

$$k = -1,\ y[-1] = c_1(0.5)^{-1} + c_2(-0.3)^{-1} = 2c_1 - \frac{10}{3}c_2 = 0$$

$$k = -2,\ y[-2] = c_1(0.5)^{-2} + c_2(-0.3)^{-2} = 4c_1 + \frac{100}{9}c_2 = 5$$

$$k = -1,\ 6c_1 - 10c_2 = 0,\ 3c_1 - 5c_2 = 0$$

$$k = -2,\ 36c_1 + 100c_2 = 45$$

Solving, we get $c_1 = 15 / 32$, $c_2 = 9 / 32$

The solution can be written as $y[k] = \frac{15}{32}(0.5)^k + \frac{9}{32}(-0.2)^k$

P 4.13 Consider a simple second order system with characteristic equation given by

$$y[k+2] - 0.2y[k+1] - 0.15y[k] = 5f[k+2] \text{ with } y[-1] = 0 \text{ and } y[-2] = 5$$

$$f[k] = 2^{-k}u[k]$$

Find the impulse response of the system.

Solution

The system equation is given in advance form. We can write it by allowing a delay of samples as $y[k] - 0.2y[k-1] - 0.15y[k-2] = 5f[k]$.

We will put $f[k] = \delta[k]$ and $y[k] = h[k]$. The equation becomes

$$h[k] - 0.2h[k-1] - 0.15h[k-2] = 5\delta[k]$$

We will put initial conditions as zero.

$$h[-1] = h[-2] = \ldots\ldots = h[-n] = 0 \text{ for a causal system}$$

Put $k = 0$, $h[0] - 0.2h[-1] - 0.15h[-2] = 5$

$$h[0] = 5$$

Put $k = 1$, $h[1] - 0.2h[0] - 0.15h[-1] = 5 \times 0$

$$h[1] - 0.2 \times 5 = 0 \Rightarrow h[1] = 1$$

To find the closed form expression for $h[n]$, we proceed as follows.
The characteristic equation for the system is

$$(D^2 - 0.2D - 0.15) = 0$$

$$(D - 0.5)(D + 0.3) = 0$$

Solution is $D = 0.5$ and $D = -0.3$
The solution can be written as

$$h[k] = [c_1(0.5)^k + c_2(-0.2)^k]u[k]$$

It is appended by $u[k]$ as it exists only for $k \geq 0$ because the system is causal.

We will now use initial conditions to find the constants.

$$k = 0,\ h[0] = c_1(0.5)^0 + c_2(-0.3)^0 = c_1 + c_2 = 5$$

$$k = 1,\ h[1] = c_1(0.5)^1 + c_2(-0.3)^1 = 0.5c_1 - 0.3c_2 = 2$$

Solving, we get $c_1 = 35 / 8$, $c_2 = 5/8$

The solution can be written as

$$h[k] = \left[\frac{35}{8}(0.6)^k + \frac{5}{8}(-0.2)^k\right]u[k]$$

P 4.14 Consider a simple second order system with impulse response and the input signal given by

$$h[k] = (0.2)^k u[k] \text{ and } f[k] = (0.3)^k u[k]$$

Find the zero state response of the system.

Solution

Here, we have to convolve the impulse response with applied input. The zero state response can be written as

$y[k]=\sum_{m=0}^{k} f[m]h[k-m]$ for a causal system. (Refer to Section 4.8 for LTI causal system impulse response)

$$y[k]=\sum_{m=0}^{k}[(0.3)^{m}]\times(0.2)^{(k-m)}$$

$$y[k]=(0.2)^{k}\sum_{m=0}^{k}\left[\left(\frac{0.3}{0.2}\right)^{m}\right]$$

$$y[k]=(0.2)^{k}\left[\left\{\frac{(0.3)^{k+1}-(0.2)^{k+1}}{(0.2)^{k}(0.3-0.2)}\right\}\right]$$

$$y[k]=10[(0.3)^{k+1}-(0.2)^{k+1}]u[k]$$

P 4.15 Use input side algorithm for convolution to convolve the two sequences $x[n]$ = [1 2 1 1] and $h[n]$ = [1 2 2].

Refer to text for input side algorithm.

P 4.16 Use output side algorithm to convolve the same two sequences.

Convolve the two sequences using conventional method.

P 4.17 An LTI system has an impulse response given by $h[n]=u[n]-u[n-7]$. If the input is $x[n]=u[n-2]-u[n-4]$, find the output of the system.

Solution

Let us first plot the two signals. Figure. 4.10 shows plots of $h[n]$, $x[n]$ and $x[-n]$. Let us find the output $y[n]$ sample by sample using the conventional method by shifting $x[-n]$ towards the right one sample at a time.

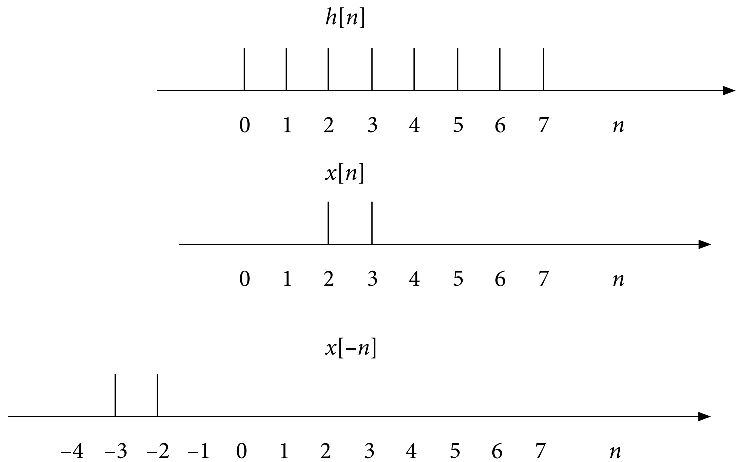


Fig. 4.10 Plots of $h[n]$, $x[n]$ and $x[-n]$

$y[0] = 0, y[1] = 0, y[2] = 1, y[3] = 2, y[4] = 2, y[5] = 2$

$y[6] = 2, y[7] = 2, y[8] = 2, y[9] = 1, y[10] = 0, y[11], y[12] = 0$

We can write $y[n] = \begin{cases} 0 & \text{for } n \le 1 \quad \text{and} \quad n \ge 10 \\ 1 & \text{for } n = 2 \quad \text{and} \quad n = 9 \\ 2 & \text{for } 3 \le n \le 8 \end{cases}$

The plot of the output signal is shown in Fig. 4.11.

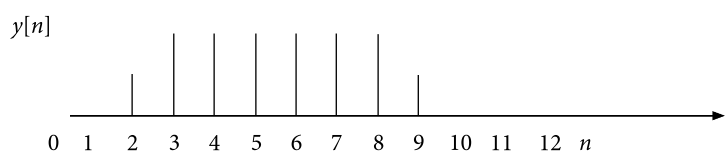


Fig. 4.11 Plot of output signal $y[n]$

P 4.18 An LTI system has the impulse response given by $h[n] = (0.2)n\{u[n] - u[n-2]\}$. If the input is $x[n] = (0.4)^n u[n]$, find the output of the system.

Solution

Let us first plot the two signals. Figure. 4.12 shows plots of $h[n]$, $x[n]$ and $x[-n]$. Let us find the output $y[n]$ sample by sample using the conventional method by shifting $x[-n]$ towards the right one sample at a time.

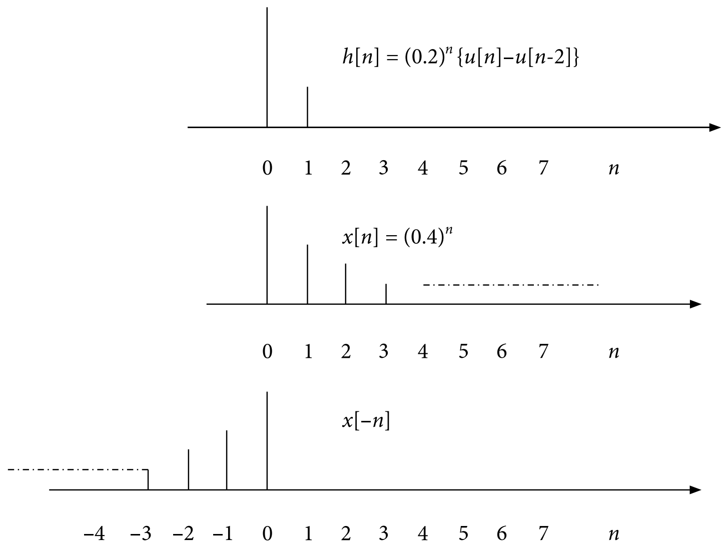


Fig. 4.12 Plots of $h[n]$, $x[n]$ and $x[-n]$

$$y[0] = 1,\ y[1] = (0.2) + (0.4),\ y[2] = (0.2)(0.4) + (0.4)^2,$$

$$y[3] = (0.4)^2(0.2) + (0.4)^3, \ldots\ldots$$

We can write $y[0] = 1$, $y[1] = 0.6$,

$$y[n] = (0.2), (0.4)^{n-1} + (0.4)^n \text{ for n} \geq 2$$

The plot of the output signal is shown in Fig. 4.13.

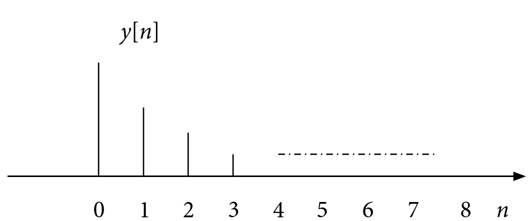


Fig. 4.13 Plot of output signal $y[n]$

P 4.19 An LTI system has the impulse response given by $h[n]=u[n-2]$. If the input is $x[n]=u[n]$, find the output of the system.

Solution

Let us first plot the two signals. Figure. 4.14 shows plots of $h[n]$, $x[n]$ and $x[-n]$. Let us find the output $y[n]$ sample by sample using the conventional method by shifting $x[-n]$ towards the right one sample at a time.

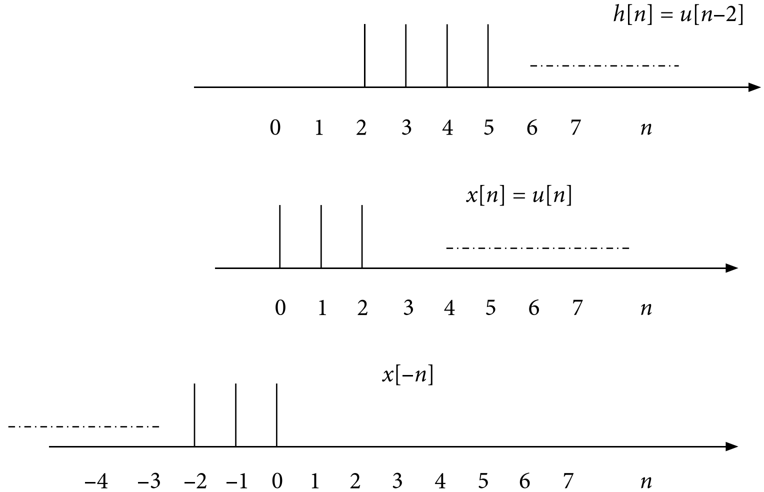


Fig. 4.14 Plots of $h[n]$, $x[n]$ and $x[-n]$

$y[0]=0,\ y[1]=0,\ y[2]=1,\ y[3]=2,\ y[4]=3,\ y[5]=4\ ....$

We can write $y[n]=\begin{cases}0 & \text{for } 0\le n\le 2\\ (n-1) & \text{for all } n\ge 2\end{cases}$

The plot of the output signal is shown in Fig. 4.15.

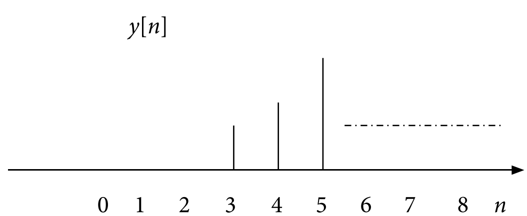


Fig. 4.15 Plot of output signal $y[n]$

P 4.20 An LTI system has the impulse response given by $h[n]$ = 1 for n = 0, –1. If the input is

$$x[n]=\begin{cases}1 & \text{for } n=0,1\\ 3 & \text{for } n=2,3\end{cases},$$

find the output of the system.

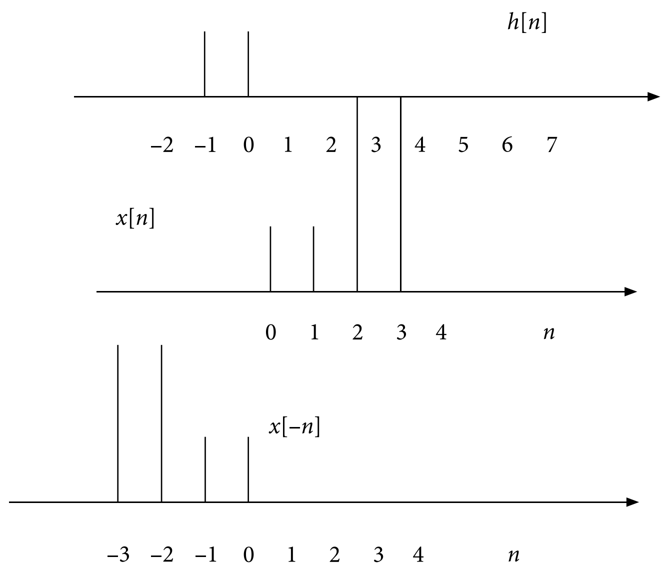


Fig. 4.16 Plot of $h[n]$, $x[n]$ and $x[-n]$

$$y[0]=2,\ y[1]=4,\ y[2]=6,\ y[3]=3,\ y[4]=0,\ y[5]=0 \ldots\ldots.$$

$$y[-1]=1,\ y[-2]=0,\ y[-3]=0, \ldots\ldots$$

We can write $y[n]=\begin{cases}0 & \text{for } n\le -2 \text{ and } n\ge 4\\ 1 & \text{for } n=-1\\ 2 & \text{for } n=0\\ 4 & \text{for } n=1\\ 6 & \text{for } n=2\\ 3 & \text{for } n=3\end{cases}$

The plot of the output signal is shown in Fig. 4.17.

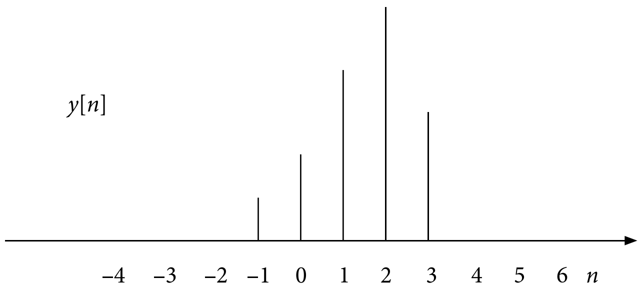


Fig. 4.17 Plot of output signal $y[n]$

P 4.21 An LTI system has the impulse response given by $h[n]=u[n]-u[n-7]$. If the input is $x[n]=u[n-2]-u[n-4]$, find the output of the system. Let us write a MATLAB program for the same. M Check if the following CT systems ATLAB uses 'conv' command to execute convolution. The output of the convolution is shown in Fig. 4.18. Compare Fig. 4.18 with Fig. 4.11.

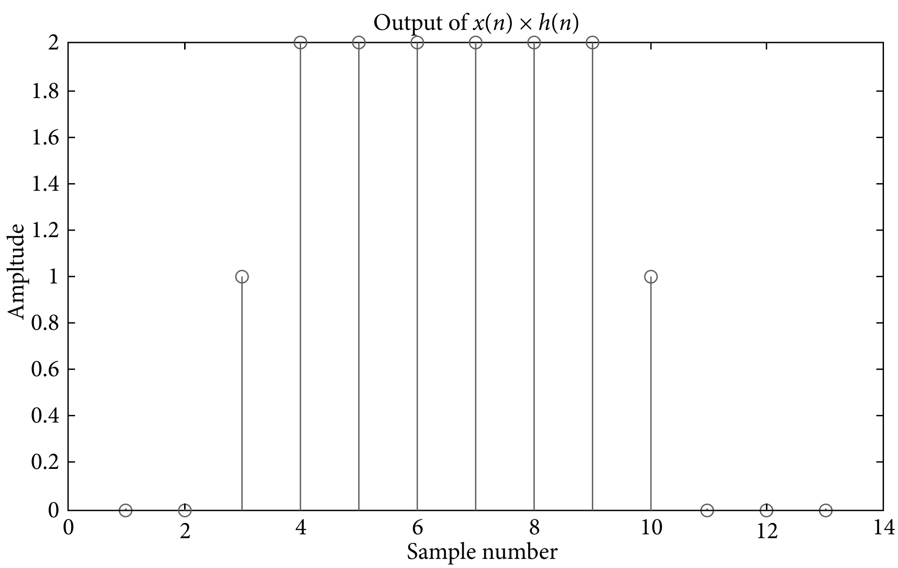


Fig. 4.18 Plot of convolved output for P 4.21

```
clear all;
x=[1,1,1,1,1,1,1];
h=[0,0,1,1,0,0];
b=conv(x,h);
stem(b);title('output of x(n)*h(n)');
xlabel('sample number');ylabel('Ampltude');
```

Note that the index for the output starts at n = 1 rather than zero.

P 4.22 An LTI system has the impulse response given by $h[n]=(0.2)^n\{u[n]-u[n-2]\}$. If the input is $x[n]=(0.4)^n u[n]$, find the output of the system. Let us write a MATLAB program for the same. The output of the convolution is shown in Fig. 4.19. Compare Fig. 4.19 with Fig. 4.13.

```
clear all;
x=[1,0.2,0,0];
h=[1,0.4,(0.4)^2,(0.4)^3,(0.4)^4,(0.4)^5,(0.4)^6,(0
.4)^7,(0.4)^8,(0.4)^9,(0.4)^10];
b=conv(x,h);
disp(b);
stem(b);title('output of x(n)*h(n)');
xlabel('sample number');ylabel('Ampltude');
```

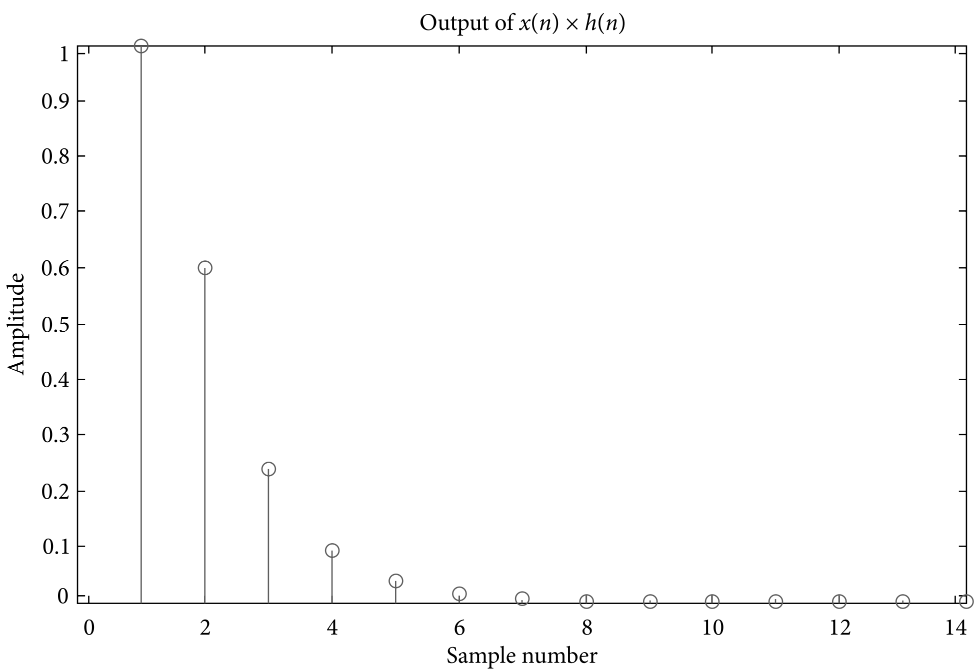


Fig. 4.19 Plot of convolved output for P 4.23

Note that the index for the output starts at n = 1 rather than zero. First 11 values displayed are

Columns 1 through 9

1.0000 0.6000 0.2400 0.0960 0.0384 0.0154 0.0061 0.0025 0.0010

Columns 10 through 11

0.0004 0.0002

P 4.23 An LTI system has the impulse response given by $h[n] = u[n-3]$. If the input is $x[n] = u[n]$, find the output of the system. Let us write a MATLAB program for the same. The output of the convolution is shown in Fig. 4.20. Compare Fig. 4.20 with Fig. 4.15. Here, we have taken only 10 samples of h and x sequence. Hence, the output decreases after sample number 10. Actually, it extends up to infinity. Hence, the actual output after sample number 10 will continuously increase as a function of $(n - 2)$ and will tend to infinity.

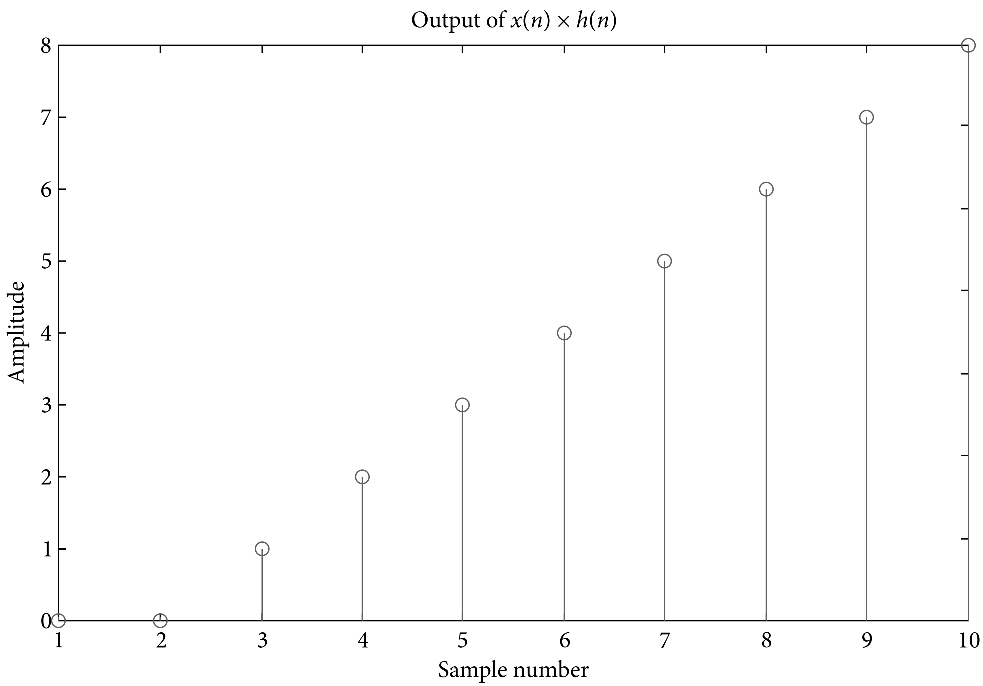


Fig. 4.20 Convolved output for P 4.22

```
clear all;
for i=3:10,
x(i)=1;
end
for i=1:10,
h(i)=1;
end
b=conv(x,h);
for i=1:10,
b1(i)=b(i);
end
```

```
stem(b1);title('output of x(n)*h(n)');
xlabel('sample number');ylabel('Ampltude');
```

P 4.24 An LTI system has the impulse response given by $h[n]$. If the input is $x[n]$ as shown in Fig. 4.16, find the output of the system. The output of the convolution is shown in Fig. 4.21. Compare Fig. 4.21 with Fig. 4.17. Note that we have shifted sample number –2 to 0 and so on. Otherwise the nature of the graph matches.

```
clear all;
x=[1,1,0,0,0,0];
h=[0,1,1,3,3,0];
b=conv(x,h);
stem(b);title('output of x(n)*h(n)');
xlabel('sample number');ylabel('Ampltude');
```

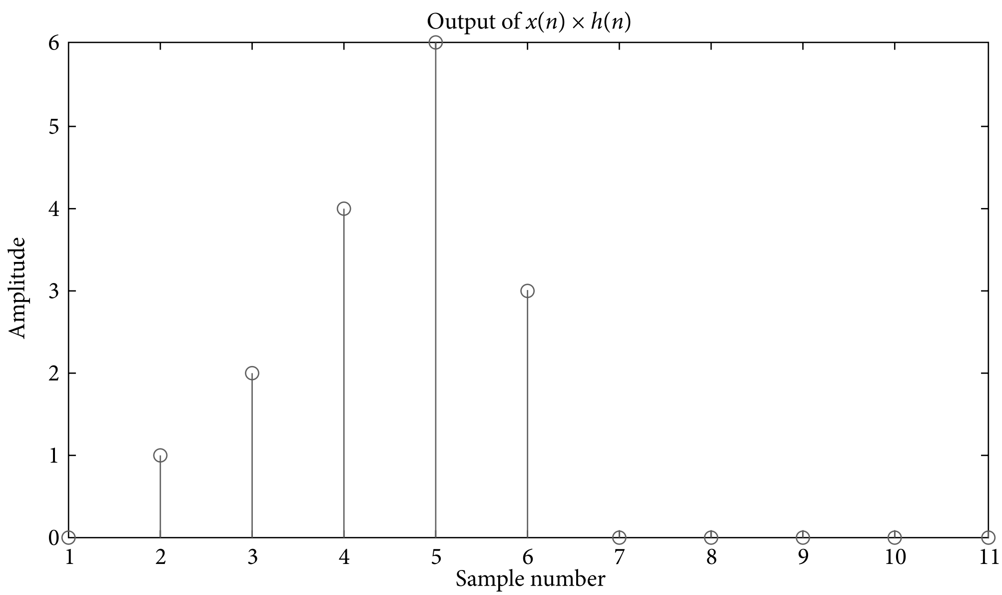


Fig. 4.21 Convolved output for Example 20

P 4.25 Find the step response of the LTI system with impulse response given by $h(t) = (e)^{-t}\, u(t)$.

Solution

We need to find

$$h[n] \times u[n] = (e)^{-t}\, u(t) \times u(t)$$

Let us plot both the signals and $u(t - \tau)$. The plot is shown in Fig. 4.22.

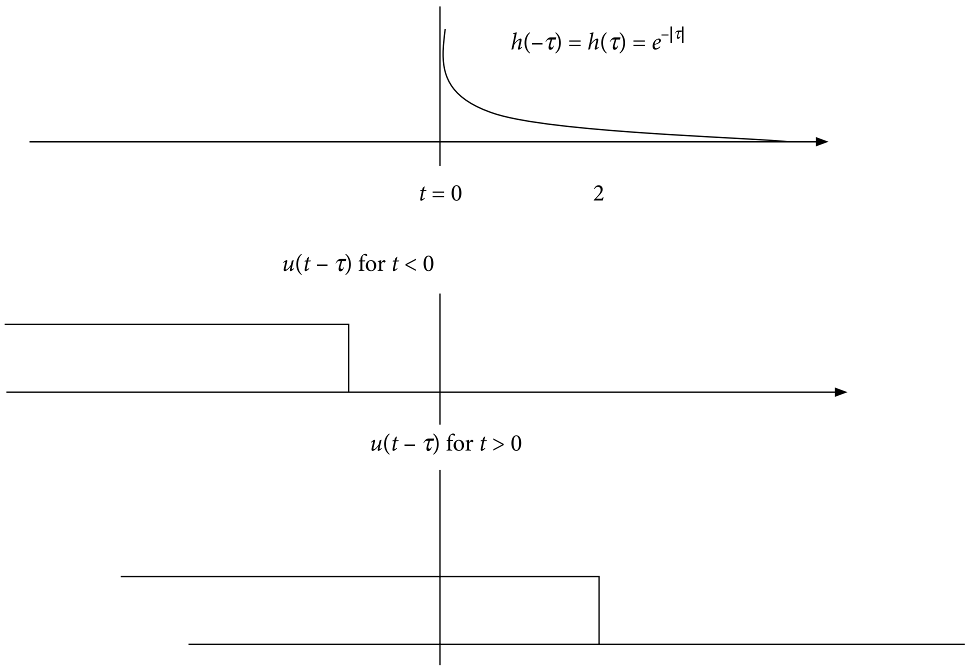


Fig. 4.22 Plot of $h(\tau)$, $u(\tau)$ and $u(t - \tau)$ for different intervals for P 4.22

Consider the first interval for $-\infty < t < 0$. The convolution integral is zero as there is no overlap.
The integral for $t > 0$ can be written as

$$h[n] \times u[n] = (e)^{-t}\, u(t) \times u(t) = \int_0^t e^{-\tau} d\tau = -e^{-\tau} \downarrow_0^t = -e^{-t} - (-1) = 1 - e^{-t}$$

P 4.26 Find the step response of the LTI system with impulse response given by $h(t) = u(t) - u(t-1)$.

Solution

We need to find

$$h[n] \times u[n] = [u(t) - u(t-1)] \times u(t)$$

Let us plot both the signals and $u(t - \tau)$. The plot is shown in Fig. 4.23.

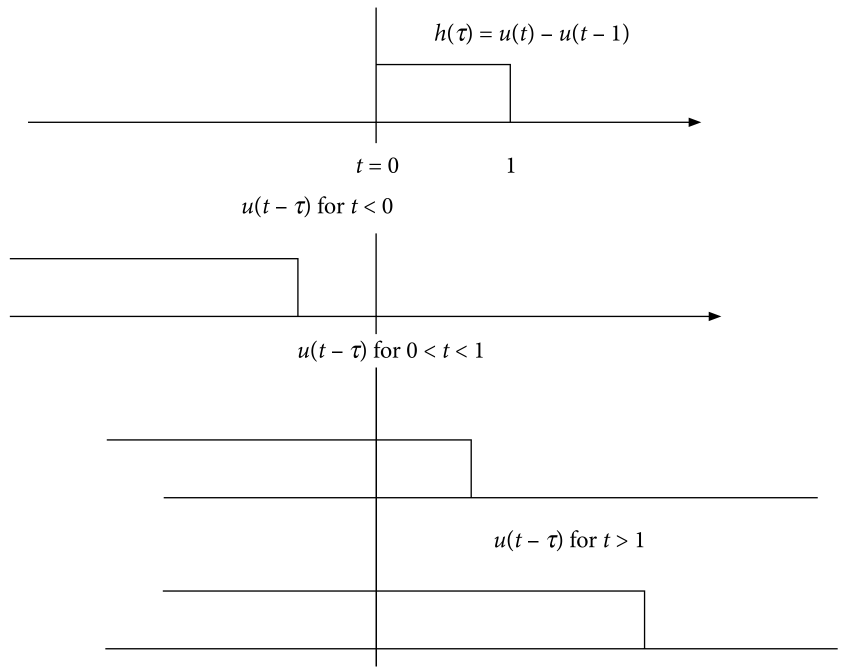


Fig. 4.23 Plot of $h(\tau)$, $u(\tau)$ and $u(t-\tau)$ for different intervals for P 4.26

Consider the first interval for $-\infty < t < 0$. The convolution integral is zero as there is no overlap.
Consider the interval $0 < t < 1$. The convolution integral can be written as

$$h[n] \times u[n] = [u(t) - u(t-2)] \times u(t) = \int_0^t d\tau = \tau \downarrow_0^t = t$$

For interval $1 < t < \infty$, the convolution integral can be written as

$$h[n] \times u[n] = [u(t) - u(t-2)] \times u(t) = \int_0^1 d\tau = \tau \downarrow_0^1 = 1$$

$$y(t) = \begin{cases} t & \text{for } 0 < t \le 1 \\ 1 & \text{for } t > 1 \end{cases}$$

P 4.27 Find the step response of the LTI system with impulse response given by $h(t) = \delta(t) - \delta(t-3)$.

Solution

We need to find

$$h[n] \times u[n] = [\delta(t) - \delta(t-3)] \times u(t)$$

Let us plot both the signals and $u(t-\tau)$. The plot is shown in Fig. 4.24.

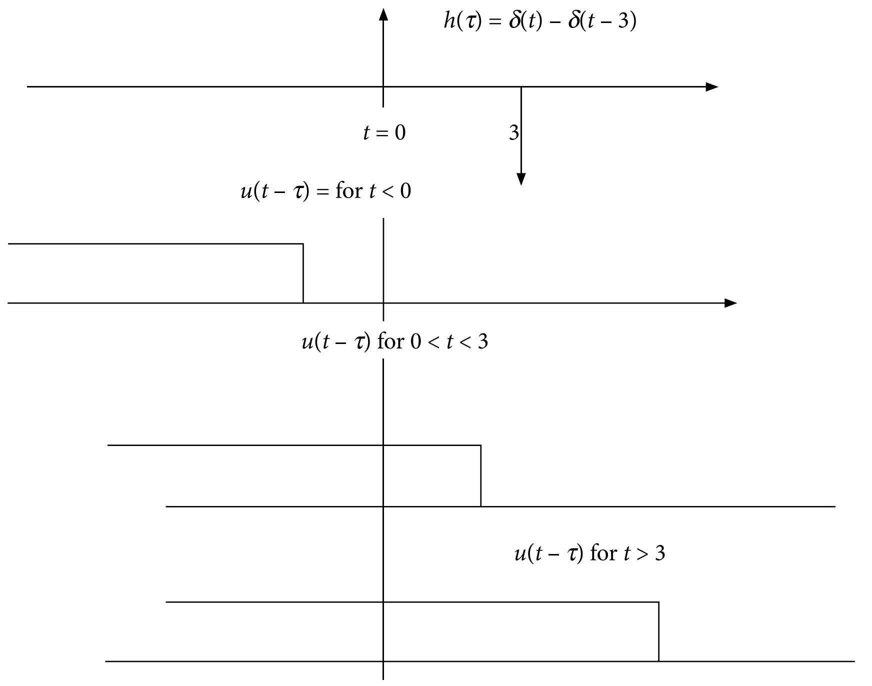


Fig. 4.24 Plot of $h(\tau)$, $u(\tau)$ and $u(t-\tau)$ for different intervals for P 4.27

Consider first interval for $-\infty < t < 0$. The convolution integral is zero as there is no overlap.

Consider the interval $-\infty < t < 3$. The convolution integral can be written as

$$h[n] \times u[n] = [\delta(t) - \delta(t-2)] \times u(t) = \int_{0-}^{t} \delta(t) d\tau = 1$$

For interval $3 < t < \infty$, the convolution integral can be written as

$$h[n] \times u[n] = [\delta(t) - \delta(t-2)] \times u(t) = \int_{-\infty}^{3-} \delta(\tau) d\tau - \int_{3-}^{\infty} \delta(\tau - 3) d\tau = 1 - 1 = 0$$

$$y(t) = \begin{cases} 1 & \text{for } t < 3 \\ 0 & \text{for } t > 3 \end{cases}$$

P 4.28 Find the step response of the LTI system with impulse response given by $h(t) = tu(t-1)$.

Solution

We need to find

$$h[n] \times u[n] = [tu(t-1)] \times u(t)$$

Let us plot both the signals and $u(t - \tau)$. The plot is shown in Fig. 4.25.

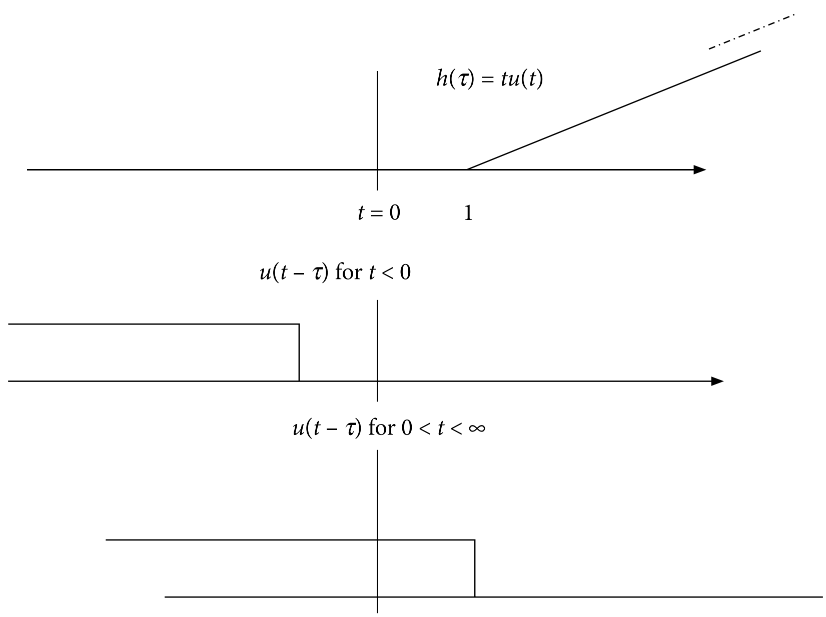


Fig. 4.25 Plot of $h(\tau)$, $u(\tau)$ and $u(t - \tau)$ for different intervals for P 4.28

Consider the first interval for $-\infty < t < 1$. The convolution integral is zero as there is no overlap.

Consider the interval $1 < t < \infty$. The convolution integral can be written as

$$h[n] \times u[n] = [tu(t-1)] \times u(t) = \int_1^t \tau d\tau = \frac{\tau^2}{2} \downarrow_1^t = \frac{(t^2 - 1)}{2}$$

$$y(t) = \left\{ \frac{t^2 - 1}{2} \text{ for } t > 1 \right\}$$

P 4.29 Find the step response of an LTI system with impulse response given by

$$h[n] = \left(\frac{1}{3}\right)^n u[n].$$

Solution

We need to find

$$h[n] \times u[n] = \left(\frac{1}{3}\right)^n u[n] \times u[n]$$

Let us plot both the signals and $u[-n]$. The plot is shown in Fig. 4.26.

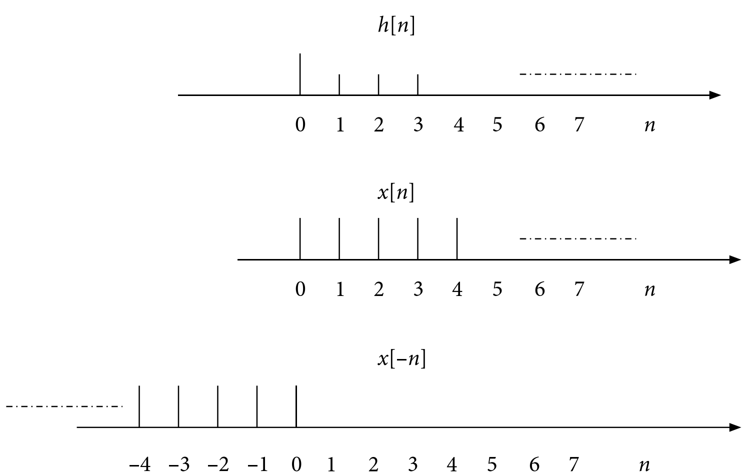


Fig. 4.26 Plot of $h[n]$, $x[n]$ and $x[-n]$

The output of convolution is given by

$$y[0] = 1,\ y[1] = 1 + 1/3,\ y[2] = 1 + 1/3 + 1/9,$$

$$y[n] = 1 + 1/3 + 1/9 + \ldots.. + \left(\frac{1}{3}\right)^n = \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} = \frac{3}{2}\left(1 - (1/3)^{n+1}\right)$$

P 4.30 Find the step response of an LTI system with impulse response given by $h[n] = \delta[n] - \delta[n-2]$.

Solution

We need to find

$$h[n] \times u[n] = \{\delta[n] - \delta[n-2]\} \times u[n]$$

Let us plot both the signals and $u[-n]$. The plot is shown in Fig. 4.27.

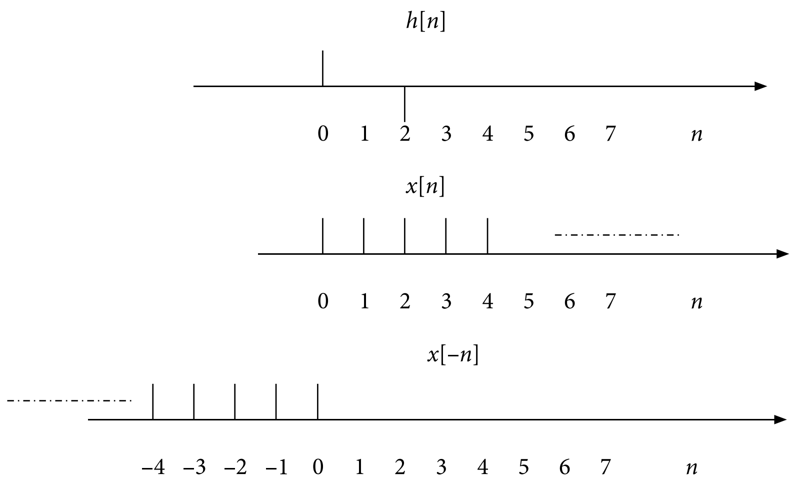


Fig. 4.27 Plot of $h[n]$, $x[n]$ and $x[-n]$

The output of convolution is given by

$$y[0] = 1, y[1] = 1, y[2] = 0, \ldots..$$

$$y[n] = \delta[n] + \delta[n-1]$$

P 4.31 Find the step response of an LTI system with impulse response given by $h[n] = (-1)^n [u[n+1] - u[n-1]]$.

Solution

We need to find

$$h[n] \times u[n] = (-1)^n \{u[n+1] - u[n-1]\} \times u[n]$$

Let us plot both the signals and $u[-n]$. The plot is shown in Fig. 4.28.

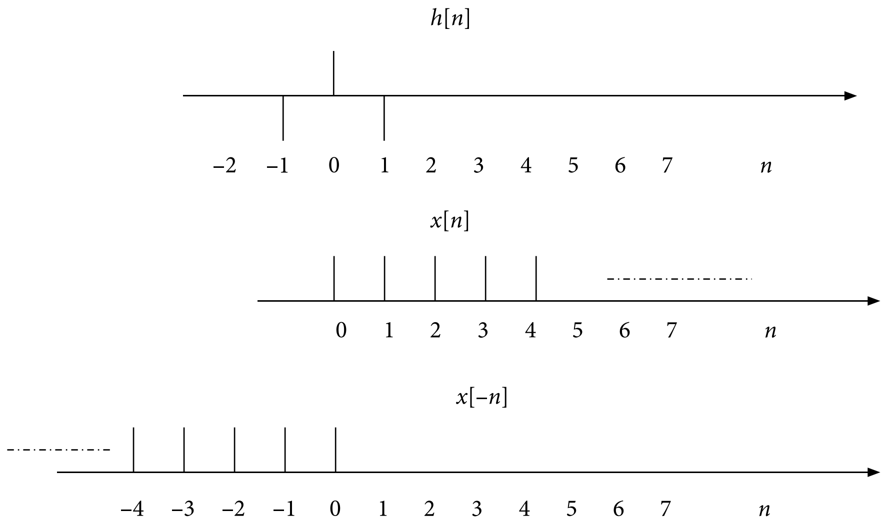


Fig. 4.28 Plot of $h[n]$, $x[n]$ and $x[-n]$

The output of convolution is given by

$$y[0]=0, y[1]=-1, y[2]=-1, \ldots..$$

$$y[n]=\delta[n]-1$$

P 4.32 Find the step response of an LTI system with impulse response given by $h[n]=u[n-1]$.

Solution

We need to find

$$h[n] \times u[n]=u[n-1] \times u[n]$$

Let us plot both the signals and $u[-n]$. The plot is shown in Fig. 4.29.

The output of convolution is given by

$$y[0]=0, y[1]=1, y[2]=2, \ldots..$$

$$y[n]=n-1$$

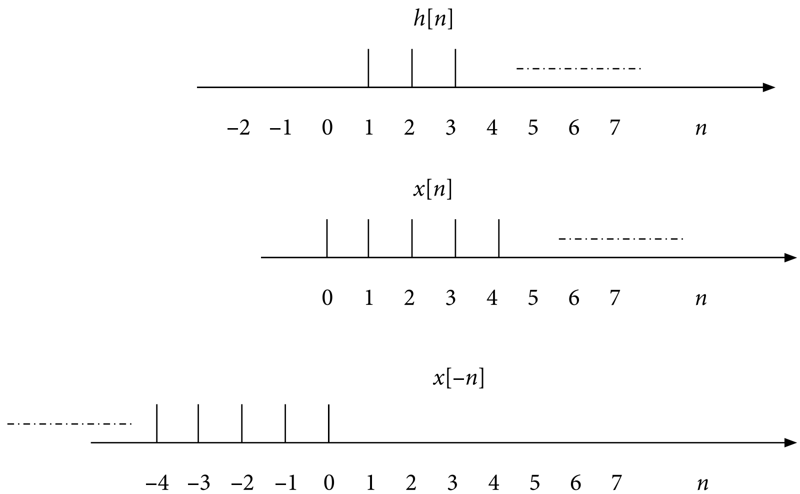


Fig. 4.29 Plot of $h[n]$, $x[n]$ and $x[-n]$

P 4.33 Check if the following CT systems are memoryless.

i. $h(t) = e^{-t}u(t)$

ii. $h(t) = e^{2t}u(t - 2)$

iii. $h(t) = u(t + 4) - 2u(t - 2)$

iv. $h(t) = 2\delta(t)$

v. $h(t) = \sin(5\pi t)u(t)$

Solution

We have to check if the impulse response of the system is a delta function.

i. $h(t) = e^{-t}u(t)$. The impulse response is not a delta function; hence the system is with memory.

ii. $h(t) = e^{2t}u(t-2)$. The impulse response is not a delta function; hence the system is with memory.

iii. $h(t) = u(t+4) - 2u(t-2)$. The impulse response is not a delta function; hence the system is with memory.

iv. $h(t) = 2\delta(t)$. The impulse response is a scaled delta function; hence the system is memoryless.

v. $h(t) = \sin(5\pi t)u(t)$. The impulse response is not a delta function; hence the system is with memory.

P 4.34 Check if the following DT systems are memoryless.

i. $h[n] = 5^n u[-n+1]$

ii. $h[n] = e^{4n}u(n-1)$

iii. $h[n] = \cos\left(\frac{1}{4}\pi n\right)[u([n+1] - u[n-3]$

iv. $h[n] = 2u[n] - 2u[n-1]$

v. $h[n] = \sin(7\pi n)u[n]$

vi. $h[n] = \delta[n] + \cos(2\pi n)$

Solution

We have to check if the impulse response of the system is a delta function.

i. $h[n] = 5^n u[-n+1]$. The impulse response is not a delta function; hence the system is with memory.

ii. $h[n] = e^{4n}u[n-1]$. The impulse response is not a delta function; hence; the system is with memory.

iii. $h[n] = \cos\left(\frac{1}{4}\pi n\right)[u[n1] - u[n-3]]$. The impulse response is not a delta function; hence the system is with memory.

iv. $h[n] = 2u[n] - 2u[n-1] = 2\delta[n]$. The impulse response is a scaled delta function; hence the system is memoryless.

v. $h[n] = \sin(7\pi n)u[n]$. The impulse response is not a delta function; hence the system is with memory.

vi. $h[n] = \delta[n] + \cos(2\pi n)$. The impulse response is not a delta function; hence the system is with memory.

P 4.35 Consider the system represented by

$$y[n] = \sum_{k=-2}^{1} x[k]h[n-k]$$

Is the system casual? If not, explain why.

Solution

$$y[n] = x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1]$$

The current output depends on the next input sample. Hence, the system is non-causal. If $h[n]$ is non-zero for $n < 0$, we can conclude that the system is non-causal.

P 4.36 Check if the following CT systems are causal.

i. $h(t) = e^{-t}u(t)$

ii. $h(t) = e^{2t}u(t - 2)$

iii. $h(t) = u(t + 4) - 2u(t - 2)$

iv. $h(t) = 2\delta(t)$

v. $h(t) = \sin(5\pi t)u(t)$

Solution

We have to check if the impulse response of the system is zero for all $n < 0$.

i. $h(t) = e^{-t}$. The impulse response is zero for all $t < 0$; hence the system is causal.

ii. $h(t) = e^{2t}u(t-2)$. The impulse response exists for all $t \geq 1$ as $u(t - 1) = 1$ for all $t \geq 1$; hence the system is causal.

iii. $h(t) = u(t+4) - 2u(t-2)$. The function $h(t)$ is equal to 1 for $-2 \leq t \leq 1$. The impulse response exists for negative vales of t up to $t = -2$; hence the system is non-causal.

iv. $h(t) = 2\delta(t)$. The impulse response is a scaled delta function and is zero for all negative values of t; hence the system is causal.

v. $h(t) = \sin(5\pi t)u(t)$. The impulse response is zero for all negative values of t; hence the system is causal.

P 4.37 Check if the following DT systems are causal.

i. $h[n] = 5^n u[-n+1]$

ii. $h[n] = e^{4n} u(n-1)$

iii. $h[n] = \cos\left(\frac{1}{4}\pi n\right)[u([n+1] - u[n-3]$

iv. $h[n] = 2u[n] - 2u[n-1]$

v. $h[n] = \sin(7\pi n)u[n]$

vi. $h[n] = \delta[n] + \cos(2\pi n)$

Solution

We have to check if the impulse response of the system is zero for all negative values of n.

i $h[n] = 5^n u[-n+1]$. The impulse response exists for all negative values of n; hence the system is non-causal.

ii. $h[n] = e^{4n} u[n-1]$. The impulse response is zero all negative values of n as it exists for $n \geq 2$; hence the system is causal.

iii. $h[n] = \cos\left(\frac{1}{4}\pi n\right)[u[n+1] - u[n-3]]$. The impulse response exists for $-1 \leq n \leq 2$ and is not zero for all negative values of n; hence the system is non-causal.

iv. $h[n] = 2u[n] - 2u[n-1] = 2\delta[n]$. The impulse response is a scaled delta function and is zero for all negative values of n; hence the system is causal.

v. $h[n] = \sin(7\pi n)u[n]$. The impulse response exists for $n \geq 0$ and is zero for all negative values of n; hence the system is causal.

vi. $h[n] = \delta[n] + \cos(2\pi n)$. The impulse response exists for $n = 0$ and for all values of n; hence the system is non-causal.

P 4.38 Find if a system with the following impulse response $h[n]$ is stable.

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Solution

We have to prove that the impulse response is absolutely summable. That is, $\sum_{k=-\infty}^{\infty} h[k]$ is finite

The impulse response $h[n]$ is a right-handed sequence or is causal. The limit for k will be between zero and infinity. That is,

$$\sum_{k}^{\infty}\left(\frac{1}{2}\right)^{n} = 1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^{2}+\ldots\ldots+\left(\frac{1}{2}\right)^{\infty}$$

This infinite geometric series will converge. The system is stable.

P 4.39 Check if the following CT systems are stable.

i. $h(t) = e^{-t}u(t)$

ii. $h(t) = e^{2t}u(t - 2)$

iii. $h(t) = u(t + 4) - 2u(t - 2)$

iv. $h(t) = 2\delta(t)$

v. $h(t) = \sin(5\pi t)u(t)$

Solution

We have to prove that the impulse response is absolutely summable. That is, $\int_{-\infty}^{\infty} h(t)dt$ is finite.

i. $h(t) = e^{-t}$,

$$\int_{-\infty}^{\infty} h(t)dt = \int_{0}^{\infty} e^{-t}dt = -e^{-t} \downarrow_{0}^{\infty} = 1 \text{ is finite}$$

The impulse response is absolutely summable; hence the system is stable.

ii. $h(t) = e^{2t}u(t-2)$.

$$\int_{-\infty}^{\infty} h(t)dt = \int_{2}^{\infty} e^{2t}dt = \frac{1}{2}e^{2t} \downarrow_{2}^{\infty} = \frac{1}{2}[\infty - e^{4}] \to \infty$$

The impulse response is not absolutely summable; hence the system is not stable.

iii. $h(t) = u(t+4) - 2u(t-2)$

$$\int_{-\infty}^{\infty} h(t)dt = \int_{-4}^{2} dt + \int_{2}^{\infty} -2dt = 6 - \infty \to -\infty$$

The impulse response is not absolutely summable; hence the system is not stable.

iv. $h(t) = 2\delta(t)$.

$$\int_{-\infty}^{\infty} h(t)dt = 2\int_{-\infty}^{\infty} \delta(t)dt = 2\times 1 = 2 \text{ is finite.}$$

The impulse response is absolutely summable; hence the system is stable.

v. $h(t) = \sin(5\pi t)u(t)$

$$\int_{0}^{\infty} h(t)dt = \int_{0}^{\infty} \sin(5\pi t)dt = \lim_{N\to\infty} \frac{N}{5\pi}\left[\cos(5\pi t)\downarrow_{0}^{T}\right]$$

$$= \lim_{N\to\infty} \frac{N}{5\pi}\left[1-(-1)\right] = \infty$$

T represents the period of the cos function. The impulse response is not absolutely summable; hence the system is not stable.

P 4.40 Check if the following DT systems are stable.

i. $h[n] = 5^{n} u[-n-1]$

ii. $h[n] = e^{4n} u(n-1)$

iii. $h[n] = \cos\left(\frac{1}{4}\pi n\right)[u([n+1] - u[n-3]$

iv. $h[n] = 2u[n] - 2u[n-1]$

v. $h[n] = \sin(7\pi n)u[n]$

vi. $h[n] = \delta[n] + \cos(2\pi n)$

Solution

We have to prove that the impulse response is absolutely summable. That is,

$$\sum_{-\infty}^{\infty} h[n] = \text{finite}$$

i. $h[n] = 5^n u[-n-1]$,

$$\sum_{-\infty}^{\infty} h[n] = \sum_{n=-\infty}^{-1} 5^n = \frac{1}{5} + \frac{1}{5^2} + \ldots\ldots = \frac{1}{1-\frac{1}{5}} - 1 = \frac{1}{4} = \text{finite}.$$

The impulse response is absolutely summable; hence the system is stable.

ii. $h[n] = e^{4n} u[n-1]$

$$\sum_{-\infty}^{\infty} h[n] = \sum_{1}^{\infty} e^{4n} = e^4 + e^8 + \ldots\ldots \to \text{infinity}$$

The impulse response is not absolutely summable; hence the system is not stable.

iii. $h[n] = \cos\left(\frac{1}{4}\pi n\right)[u[n+1] - u[n-3]]$

$$\sum_{-1}^{3} h[n] = \sum_{-1}^{3} \cos\left(\frac{1}{4}\pi n\right) = \cos\left(-\frac{1}{4}\pi\right) + 1 + \cos\left(\frac{1}{4}\pi\right) + \cos\left(\frac{1}{2}\pi\right) + \cos\left(\frac{3}{4}\pi\right)$$

$$= \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \text{ is finite}$$

The impulse response is absolutely summable; hence the system is stable

iv. $h[n] = 2u[n] - 2u[n-1] = 2\delta[n]$.

$$2\sum_{-\infty}^{\infty} \delta[n] = 2 \text{ is finite.}$$

The impulse response is absolutely summable; hence the system is stable

v. $h[n] = \sin(7\pi n)u[n]$.

$$\sum_{0}^{\infty} h[n] = \sum_{0}^{\infty} \sin(7\pi n) = \rightarrow \text{infinity}$$

The impulse response is not absolutely summable; hence the system is unstable.

vi. $h[n] = \delta[n] + \cos(2\pi n)$.

$$\sum_{0}^{\infty} h[n] = \delta[n] + \sum_{0}^{\infty} \cos(\pi n) = \rightarrow \text{infinity}$$

The impulse response is not absolutely summable; hence the system is unstable.

P 4.41 Consider the interconnections of the systems as shown in Fig. 4.30. Let the impulse responses be specified as

$$h_1(t) = u(t)$$

$$h_2(t) = \delta(t)$$

$$h_3(t) = \delta(t-2)$$

Find the response of the overall interconnection.

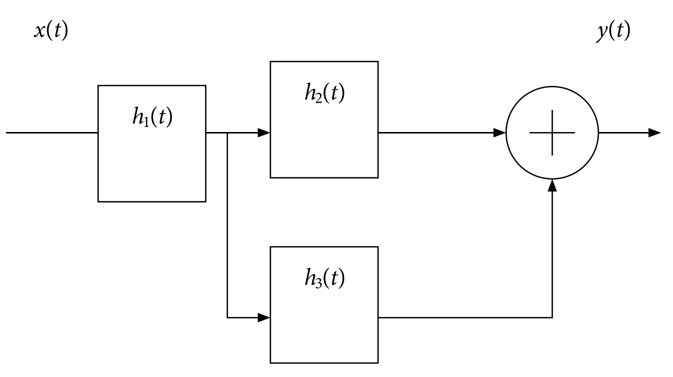


Fig. 4.30 Interconnection of systems for P 4.38

Solution

The impulse response of the overall interconnection can be written as

$$h_1(t) \times \{h_2(t) + h_3(t)\} = \{u(t)\} \times \{\delta(t) + \delta(t-2)\}$$

Figure 4.31 shows a plot of two signals.

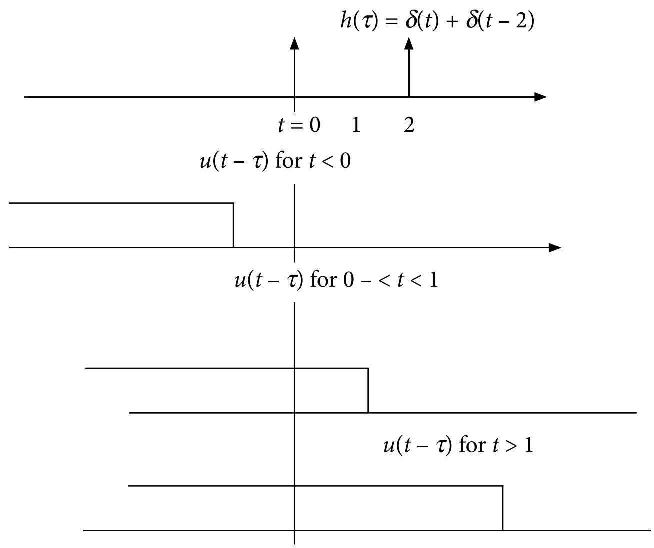


Fig. 4.31 Plot of $h(\tau)$, $u(\tau)$ and $u(t - \tau)$ for different intervals for P 4.38

Consider the first interval for $-\infty < t < 0$. The convolution integral is zero as there is no overlap.
Consider the interval $-\infty < t < 1$. The convolution integral can be written as

$$h[n] \times u[n] = [\delta(t) + \delta(t-2)] \times u(t) = \int_{0-}^{t} \delta(t) d\tau = 1$$

For interval $2+ < t < \infty$, the convolution integral can be written as

$$y[n] = 1 \text{ for } -\infty < t < 2$$

$$= 2 \text{ for } 2 < t < \infty$$

P 4.42 Consider the interconnections of the systems as shown in Fig. 4.32. Let the impulse responses be specified as

$$h_1[n] = u[n]$$

$$h_2[n] = u[n+2] - u[n]$$

$$h_3[n] = \delta[n-2]$$

$$h_4[n] = \left(\frac{1}{2}\right)^n u[n]$$

Find the response of the overall interconnection.

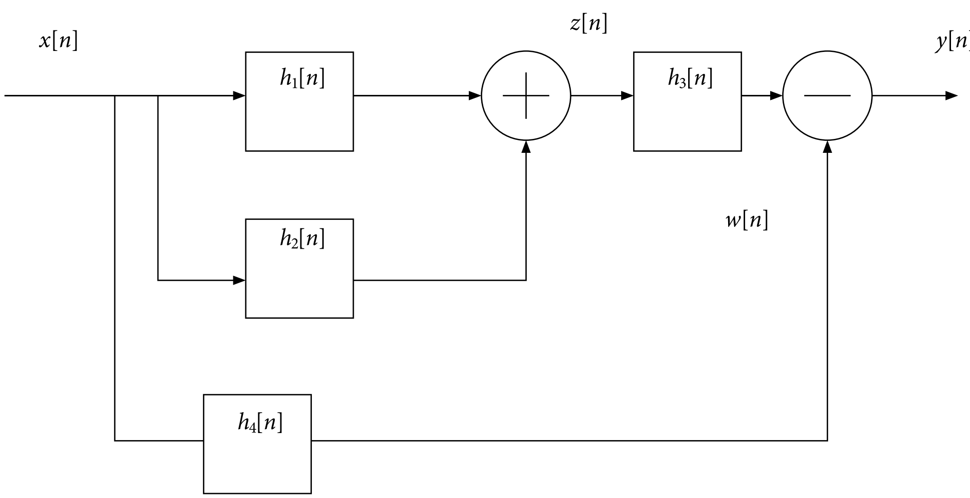


Fig. 4.32 Interconnection of systems for P 4.39

Solution

The impulse response of the overall interconnection can be written as

$$\{(h_1[n] + h_2[n]) \times h_3[n]\} - h_4[n]$$

$$\{(u[n] + u[n+2] - u[n]) \times \delta[n-2]\} - \left(\frac{1}{2}\right)^n u[n]$$

$$= u[n+2] \times \delta[n-2] - \left(\frac{1}{2}\right)^n u[n]$$

$$= u[n] - \left(\frac{1}{2}\right)^n u[n]$$

$$= \left[1 - \left(\frac{1}{2}\right)^n\right] u[n]$$

Note that $u[n+2] \times \delta[n-2] = u[n]$ as is clear from Fig. 4.33.

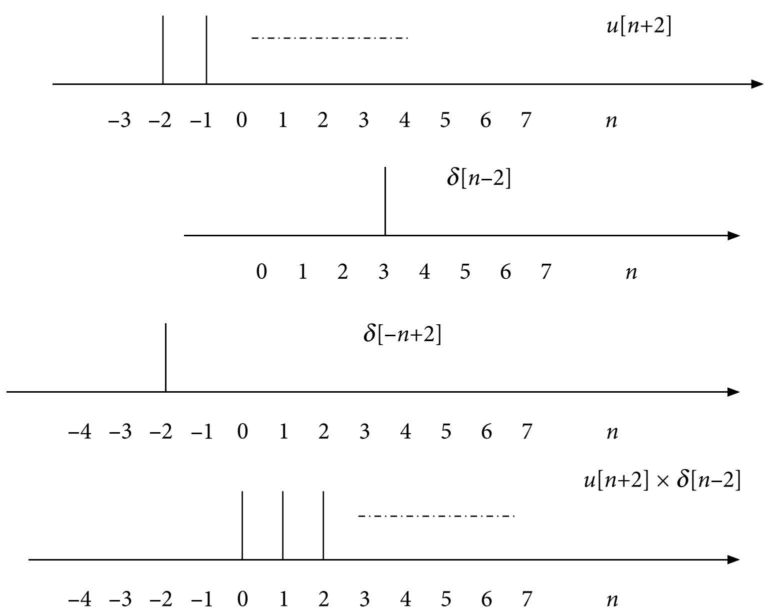


Fig. 4.33 Convolution of $u[n + 3]$ with $\delta[n - 3]$

P 4.43 Consider the interconnections of the systems as shown in Fig. 4.34. Let the impulse responses be specified as

$$h_1[n] = \left(\frac{1}{2}\right)^n [u[n+1] - u[n-1]]$$

$$h_2[n] = \delta[n]$$

$$h_3[n] = u[n-1]$$

Find the response of the overall interconnection.

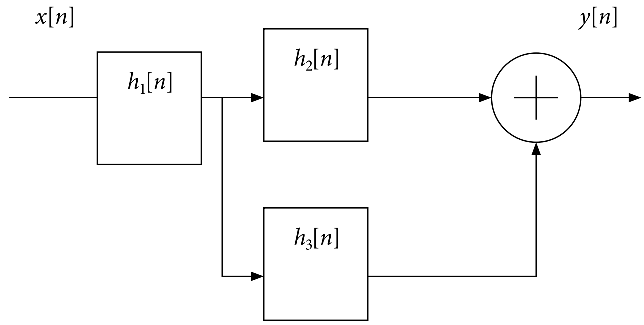


Fig. 4.34 Interconnection of systems for P 4.40

Solution

The impulse response of the overall interconnection can be written as

$$h_1[n] \times \{h_2[n] + h_3[n]\}$$

$$= \left(\frac{1}{2}\right)^n \{u[n+1] - u[n-1]\} \times \{\delta[n] + u[n-1]\}$$

$$= \left(\frac{1}{2}\right)^n \{u[n+1] - u[n-1]\} \times u[n]$$

Figure 4.35 shows a plot of $h_1[n]$, $u[-n]$ and output $y[n]$.

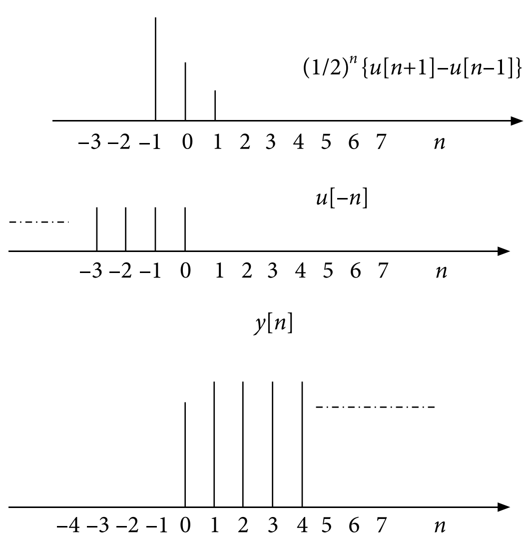


Fig. 4.35 Plot of $h_1[n]$, $u[-n]$ and output $y[n]$

The output $y[n]$ can be calculated as

$$y[0]=[2+1]=3,\ y[1]=[2+1+1/2]=3.5,$$

$$y[2]=3.5,\ y[3]=3.5,\ y[4]=3.5,\ldots\ldots$$

5

Fourier Series Representation of Periodic CT Signals

P 5.1 Prove that $\cos(5\omega t)$ and $\cos(6\omega t)$ are orthogonal to each other.

Solution

Let T represent the period of the cosine function with angular frequency of ω. The period for a cosine function with angular frequency of 5ω will be $T/5$ and the period for a cosine function with angular frequency of 6ω will be $T/6$ and so on. We have to prove that the dot product of the two functions is zero. We know that the integration of cosine function over one period or multiple periods is zero.

$$\int_0^T \cos(5\omega t)\cos(6\omega t)dt = \frac{1}{2}\int_0^T [\cos(11\omega t) + \cos(\omega t)]dt$$

$$= \frac{1}{2}\int_0^T \cos(11\omega t)dt + \frac{1}{2}\int_0^T \cos(\omega t)dt$$

$$= \frac{1}{2}\left[\frac{1}{11\omega}\sin(11\omega t)\right]_0^T + \frac{1}{2}\left[\frac{1}{\omega}\sin(\omega t)\right]_0^T$$

$$= 0$$

P 5.2 Prove that $\exp(j3\omega t)$ and $\exp(j7\omega t)$ are orthogonal to each other.

Solution

Let T represent the period of the cosine function with angular frequency of ω. The period for an exponential function with angular frequency of 3ω will be

$T/3$ and so on. We have to prove that the dot product of the two functions is zero. We know that the integration of cosine function over one period or multiple periods is zero.

$$\int_0^T \exp(j3\omega t)\exp^*(j7\omega t)dt = \int_0^T [\exp(j(3-7)\omega t)]dt$$

$$= \left[\exp\frac{(j(-4)\omega t)}{j(-4)\omega}\right]_0^T$$

$$= \{[\cos(-4\omega t) - j\sin(-4\omega t)]/-4j\omega\}_0^T$$

$$= 0$$

P 5.3 Determine the FS representation for the signal given as

$$x(t) = 3\cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right).$$

Solution

Let us first determine the fundamental period of $x(t)$.

$$\omega_0 = \frac{\pi}{3} = \frac{2\pi}{6} = \frac{2\pi}{T}$$

We can find $T = 6$.

Let us write $x(t)$ as a linear sum of weighted exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\left(\frac{\pi}{3}\right)t}$$

$X(t)$ is already given in terms of a cosine function. So let us write it in terms of exponentials and pull the coefficients.

$$x(t) = 3\cos\left(\frac{\pi}{3}t + \frac{\pi}{2}\right)$$

$$= 3\frac{e^{j\left(\frac{\pi}{3}\right)t + j\frac{\pi}{2}} + e^{-j\left(\frac{\pi}{3}\right)t - j\frac{\pi}{2}}}{2}$$

$$= \frac{3}{2}e^{j\frac{\pi}{2}}e^{j\left(\frac{\pi}{3}\right)t} + \frac{3}{2}e^{-j\frac{\pi}{2}}e^{-j\left(\frac{\pi}{3}\right)t}$$

Referring to the aforementioned equation, we can see that $k = 0$ will give the constant term and $k = 1$ will be the coefficient of the fundamental frequency.

$$X[k]=\begin{cases}\frac{3}{2}e^{-j\left(\frac{\pi}{2}\right)} & \text{for } k=-1\\ \frac{3}{2}e^{j\left(\frac{\pi}{2}\right)} & \text{for } k=1\\ 0 & \text{otherwise}\end{cases}$$

Let us plot the magnitude and phase of $X[k]$. It is shown in Fig. 5.1.

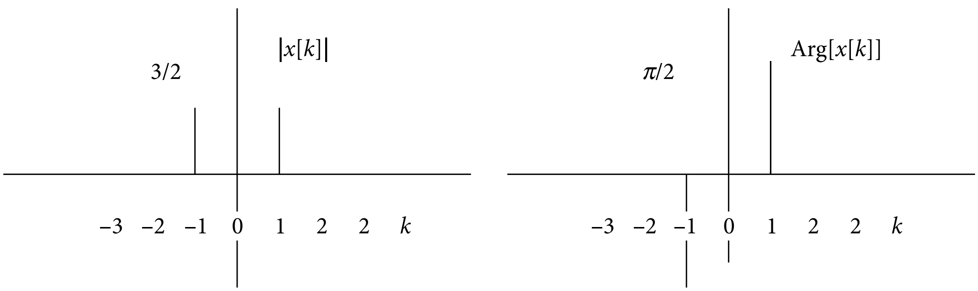


Fig. 5.1 Magnitude and phase plot of $X[k]$

P 5.4 Determine the FS representation for the signal given as $x(t) = 4\cos(2\pi t + 2) + \sin(4\pi t)$.

Solution

Let us first determine the fundamental period of $x(t)$.

For the first term, $\omega_0 = \frac{2\pi}{1} = \frac{2\pi}{T}$. We find $T = 1$.

For the second term, $\omega_0 = \frac{6\pi}{1} = \frac{2\pi}{1/2} = \frac{2\pi}{T}$. we find $T = 1/2$.

The fundamental period is the larger of the two and is equal to 1.

Let us write $x(t)$ as a linear sum of weighted exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk(2\pi)t}$$

is already given in terms of a cosine function. So let us write it in terms of exponentials and pull the coefficients.

$$x(t) = 4\cos(2\pi t + 2) + \sin(4\pi t)$$

$$= 4\frac{1}{2}\left[e^{j(2\pi t+2)} + e^{-j(2\pi t+2)}\right] + \frac{4}{2j}\left[e^{j4\pi t} - e^{-j4\pi t}\right]$$

$$= \frac{2}{1}\left[e^{+2j}e^{j2\pi t} + e^{-2j}e^{-j2\pi t}\right] + \frac{2}{j}\left[e^{j4\pi t} - e^{-j4\pi t}\right]$$

We can see that $k = 0$ will give the constant term; and $k = 1$ will be the coefficient of the fundamental frequency and $k = 2$ will be the coefficient of the second harmonic.

$$X[k] = \begin{cases} 2e^{2j} & \text{for } k = 1 \\ 2e^{-2j} & \text{for } k = -1 \\ 2/j = -2j & \text{for } k = 2 \\ -\frac{2}{j} = 2j & \text{for } k = -2 \\ 0 & \text{otherwise} \end{cases}$$

Let us plot the magnitude and phase of $X[k]$. It is shown in Fig. 5.2.

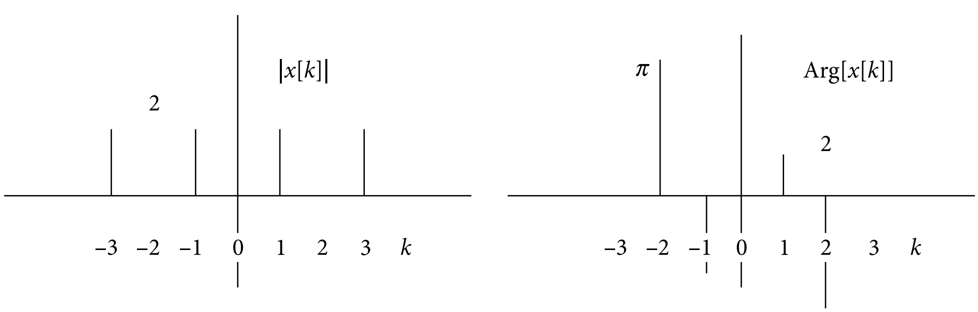


Fig. 5.2 Magnitude and phase plot of $X[k]$

P 5.5 Determine the FS representation for the signal given as $x(t) = \cos(4\pi t) + \cos(6\pi t)$.

Solution

Let us first determine the fundamental period of $x(t)$.

For the first term, $\omega_0 = \frac{2\pi}{1/2} = \frac{2\pi}{T}$. We find $T = 1/2$.

For the second term, $\omega_0 = \frac{2\pi}{1/3} = \frac{2\pi}{T}$. Comparing, we find $T = 1/3$.

The fundamental period is the larger of the two and is equal to $T = 1/2$; the fundamental frequency is 2. The first term has a frequency of 2 which is the second harmonic of 1 Hz and the second term has a frequency of 3 which is the third harmonic of 1 Hz.

Let us write $x(t)$ as a linear sum of weighted exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk(2\pi)t}$$

is already given in terms of a cosine function. So let us write it in terms of exponentials and pull the coefficients.

$$x(t) = \cos(4\pi t) + \cos(6\pi t)$$

$$= \frac{1}{2}\left[e^{j4\pi t} + e^{-j(4\pi t)}\right] + \frac{1}{2}\left[e^{j6\pi t} + e^{-j6\pi t}\right]$$

$$X[k] = \begin{cases} \frac{1}{2} & \text{for } k = 2 \\ \frac{1}{2} & \text{for } k = -2 \\ \frac{1}{2} & \text{for } k = 3 \\ \frac{1}{2} & \text{for } k = -3 \\ 0 & \text{otherwise} \end{cases}$$

Let us plot the magnitude and phase of $X[k]$. It is shown in Fig. 5.3.

Fig. 5.3 Magnitude and phase plot of $X[k]$

P 5.6 Consider a pulse train of rectangular pulses of duration T and period T_0 as shown in Fig. 5.4. Find FS representation.

Solution

The representation of the signal for one period can be written as

$$x_p(t) = \begin{pmatrix} A & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{pmatrix}$$

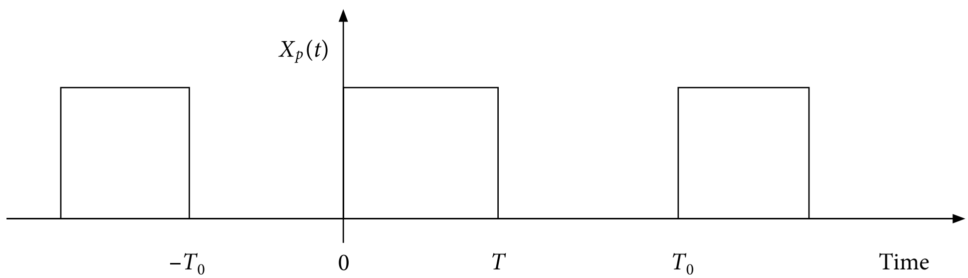


Fig. 5.4 Periodic train of rectangular pulses

We will use the formula for the Fourier series coefficients.

$$c_n = \frac{1}{T_0}\int_0^{T_0} x_P(t)\exp\left(\frac{-j2\pi nt}{T_0}\right)dt \quad n = 0, \pm 1, \pm 2......$$

$$= \frac{1}{T_0}\int_0^{T} A\exp\left(\frac{-j2\pi nt}{T_0}\right)dt$$

$$= \frac{A}{2n\pi}\exp\left(-\frac{j2\pi nt}{T_0}\right)\downarrow_0^T = \frac{A}{2n\pi}\left[\exp\left(-\frac{j2\pi nT}{T_0}\right) - 1\right]$$

A MATLAB program to plot the Fourier series coefficients using the aforementioned equation is given as follows. The number of coefficients plotted is 81. Actually, the sync function extends from minus infinity to plus infinity. The index of the coefficients is from n = –40 to 40.

```
clear all;
T0=4;
T=0.4;
A=1;
for n=1:40,
```

```
   c(n+41)=(1/(2*n*pi))*(exp((-1j*2*n*pi*T)/T0)-1);
 end
c(41)=0.1;
for n=1:40,
    c(n)=c(82-n);
end
 s=-40:1:40;
stem(s,abs(c));
title('plot of magnitude of discrete spectrum');
xlabel('coefficient number');
ylabel('amplitude');
```

The plot of the magnitude response is shown in Fig. 5.5.

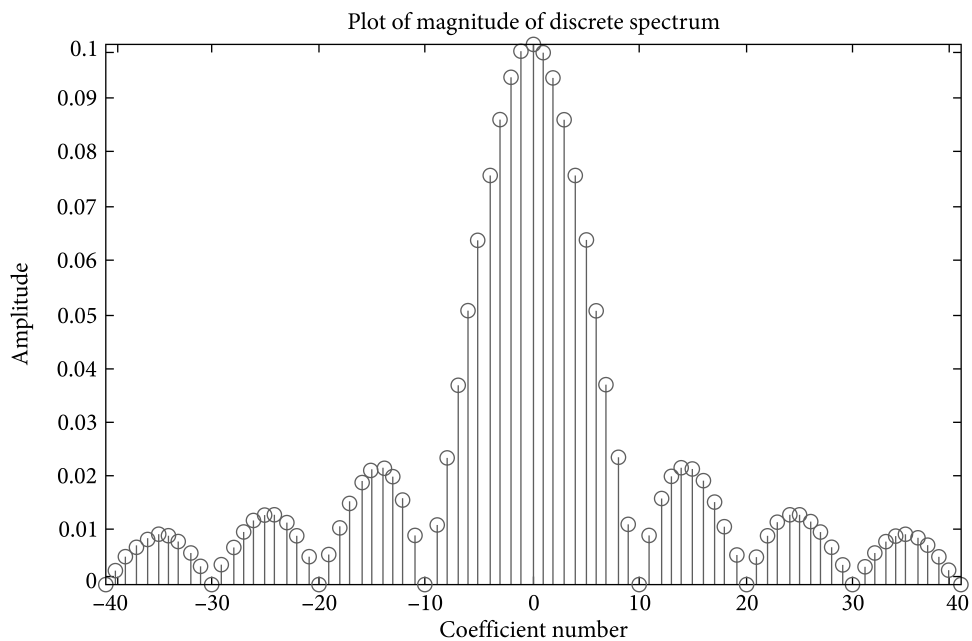


Fig. 5.5 Plot of magnitude response

We will use the property of odd symmetry to find trigonometric FS. Let us first find

$$a_0 = \frac{1}{T_0}\int_0^T A dt = \frac{A}{T_0} t \downarrow_0^T = AT / T_0$$

$$a_n = \frac{2}{T_0}\int_t^{t+T} x_p(t)\cos(n\omega t) dt$$

$$= \frac{2}{T_0}\int_0^T A\cos(n\omega t)dt = \frac{2A\sin(n\omega t)}{T_0 n\omega}\downarrow_0^T = \frac{2A\sin(n\omega T)}{T_0 n\omega}$$

$$b_n = \frac{2}{T_0}\int_t^{t+T} x_p(t)\sin(n\omega t)dt$$

$$= \frac{2}{T_0}\left[\int_0^T A\sin(n\omega t)dt\right]$$

$$= \frac{2}{T_0}\left[\int_0^T A\sin(n\omega t)dt\right] = \frac{2A}{T_0 n\omega}\cos(n\omega t)\downarrow_0^T = \frac{2A}{T_0 n\omega}[\cos(n\omega T)-1]$$

The FS can be written as

$$x(t) = \frac{AT}{T_0} + \sum_{n=-\infty, n\neq 0}^{\infty}\frac{2A\sin(n\omega T)}{T_0 n\omega}\cos(n\omega t) + \infty\sum_{n=-\infty}^{\infty}\frac{2A}{T_0 n\omega}[\cos(n\omega T)-1]\sin(n\omega t)$$

P 5.7 Determine the FS representation for the signal given by $x(t) = |\sin(2t)|$. The periodic wave is shown in Fig. 5.6.

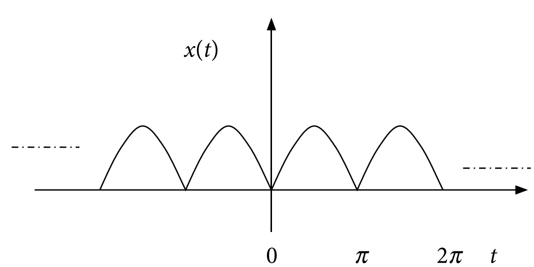


Fig. 5.6 Plot of signal for Problem 5.7

Solution

Step 1 Let us first find the period of the wave. The wave repeats after a time period of π seconds. $\omega = \frac{2\pi}{\pi} = 2$; period of the rectified sine wave is 2 seconds.

Step 2 Let us now find the equation for the sine wave between 0 to π seconds. This is a half part of the sine wave with a period of 2π seconds. So, angular frequency of the sine wave $\omega_0 = 1 = \frac{2\pi}{2\pi}$, and the signal between 0 to π seconds can be written as $x(t) = \sin(t)$.

Step 3 Let us write $x(t)$ as a linear sum of weighted exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk(2)t}$$

We can find $X[k]$ using the formula

$$X[k] = \frac{1}{T}\int_{(T)} x(t)e^{-jk\omega t}\,dt$$

Step 4 Use the formula to find $X[k]$.

$$X[k] = \frac{1}{\pi}\int_0^{\pi} \sin(t)e^{-jk(2)t}\,dt = \frac{1}{2\pi j}\int_0^{\pi} [e^{jt} - e^{-jt}]\, e^{-j2kt}\,dt$$

$$= \frac{1}{2\pi j}\left[\frac{e^{j(1-2k)t}}{(1-2k)j} + \frac{e^{-j(1+2k)t}}{(1+2k)j}\right]_0^{\pi}$$

$$= \frac{1}{2\pi j}\left[\frac{e^{j(1-2k)\pi} - 1}{(1-2k)} + \frac{e^{-j\pi(1+2k)} - 1}{(1+2k)}\right]$$

$$= \frac{1}{2j}\left[\frac{-2(1+2k) - 2(1-2k)}{(1-4k^2)\pi}\right]$$

$$= \frac{-2-4k-2+4k}{2j(1-4k^2)\pi} = \frac{2j}{(1-4k^2)\pi} \text{ for all } k$$

Note that $e^{-j\pi} = e^{j\pi} = (-1)$ and $e^{-j\pi(1-2k)} = e^{j\pi(1+2k)} = (-1)$ for even and odd values of k.

Step 5 Evaluate $X[0]$ by integrating the signal over the period.

$$X[0] = \frac{1}{T}\int_0^1 x(t)dt = \frac{1}{1}\int_0^1 \sin(\pi t)dt$$

$$= -\cos(\pi t)/\pi \downarrow_0^1 = \frac{2}{\pi}$$

The magnitude response can be written as

$$X[k] = \begin{cases} \dfrac{2}{(1-4k^2)\pi} & \text{for all } k \neq 0 \\ \dfrac{2}{\pi} & \text{for } k = 0 \end{cases}$$

Step 6 The exponential Fourier series can be written as

$$x(t) = \frac{2}{\pi} + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{2}{(1-4k^2)\pi} e^{j2\pi kt}$$

Let us find trigonometric FS.

$$a_0 = \frac{1}{1}\int_0^1 \sin(\pi t)dt = -\cos(\pi t)/\pi \downarrow_0^1 = -\frac{(-1)-1}{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{T}\int_t^{t+T} x_p(t)\cos(n\omega t)dt$$

$$= \frac{2}{1}\int_0^1 \sin(\pi t)\cos(2n\pi t)dt = \int_0^1 [\sin(2n+1)\pi t + \sin(1-2n)\pi t]dt$$

$$= [-\cos(2n+1)\pi t/(2n+1)\pi - \cos(1-2n)\pi t/(1-2n)\pi]\downarrow_0^1$$

$$= -\frac{(-1)^{2n+1}-1}{(2n+1)\pi} - \frac{(-1)^{1-2n}-1}{(1-2n)\pi} = \frac{2(1-2n)+2(1+2n)}{(1-4n^2)\pi} = \frac{4}{(1-4n^2)\pi} \text{for } n$$

$$b_n = \frac{2}{T_0}\int_t^{t+T} x_p(t)\sin(n\omega t)dt$$

$$= \frac{2}{1}\left[\int_0^1 \sin(\pi t)\sin(2n\pi t)dt\right] = \int_0^1 \left[\cos(1-2n)\pi t - \cos(1+2n)\pi t\right]dt$$

$$= \left[-\frac{\sin(1+2n)\pi t}{(1+2n)\pi} + \frac{\sin(1-2n)\pi t}{(1-2n)\pi}\right]\downarrow_0^1$$

$$= 0 \text{ for all } n \tag{5.71}$$

The FS can be written as

$$x(t) = \frac{2}{\pi} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{4}{(1-4n^2)\pi} \cos(2n\pi t) \tag{5.72}$$

Let us convert the trigonometric FS coefficients to exponential FS coefficients.

$$c_n = \frac{1}{2}[a_n - jb_n], c_{-n} = \frac{1}{2}[a_n + jb_n], c_0 = a_0$$

$$c_n = \frac{1}{2}\left[\frac{4}{(1-4k^2)\pi}\right] = \frac{2}{(1-4k^2)\pi}$$

$$c_{-n} = \frac{2}{(1-4k^2)\pi}, \tag{5.73}$$

NO sine terms exist as the waveform has even symmetry.
Let us write a MATLAB program to plot the spectrum for the signal.

```
clear all;
t=-5:0.1:5;
x=abs(sin(pi*t));
plot(t,x);title('plot of rectified sine wave');
xlabel('time');ylabel('amplitude');
for k=1:21,
y(k)=2/((1-4*(k-11).*(k-11))*pi);
end
figure;
k1=-10:1:10;
stem(k1,y);title('plot of spectrum of the signal');
xlabel('frequency index');ylabel('amplitude');
```

Figure 5.7 shows the plot of the signal and Fig. 5.8 shows the plot of the magnitude spectrum for the FS representation of the signal.

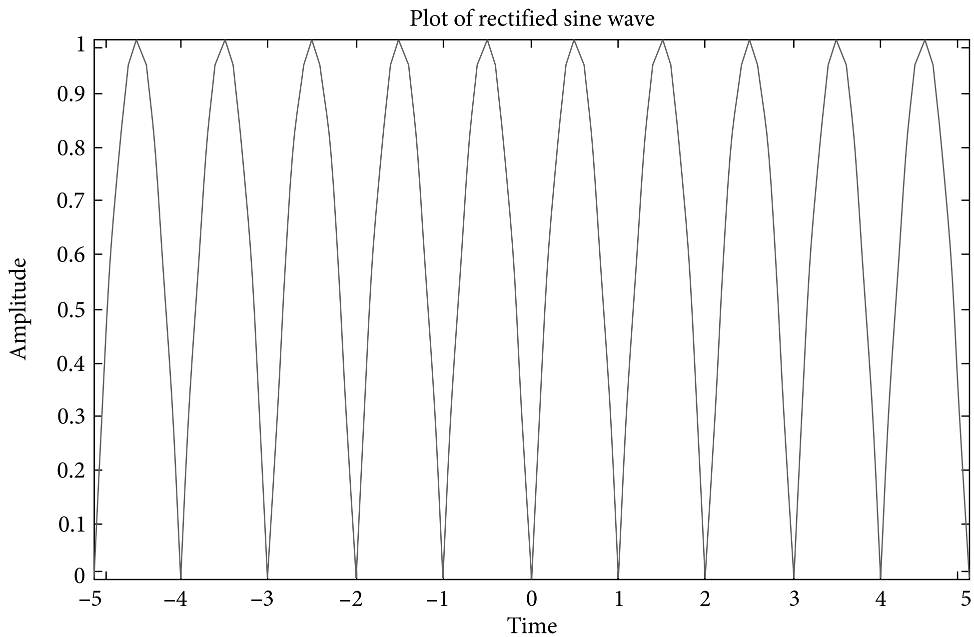


Fig. 5.7 Plot of the full wave rectified sine wave

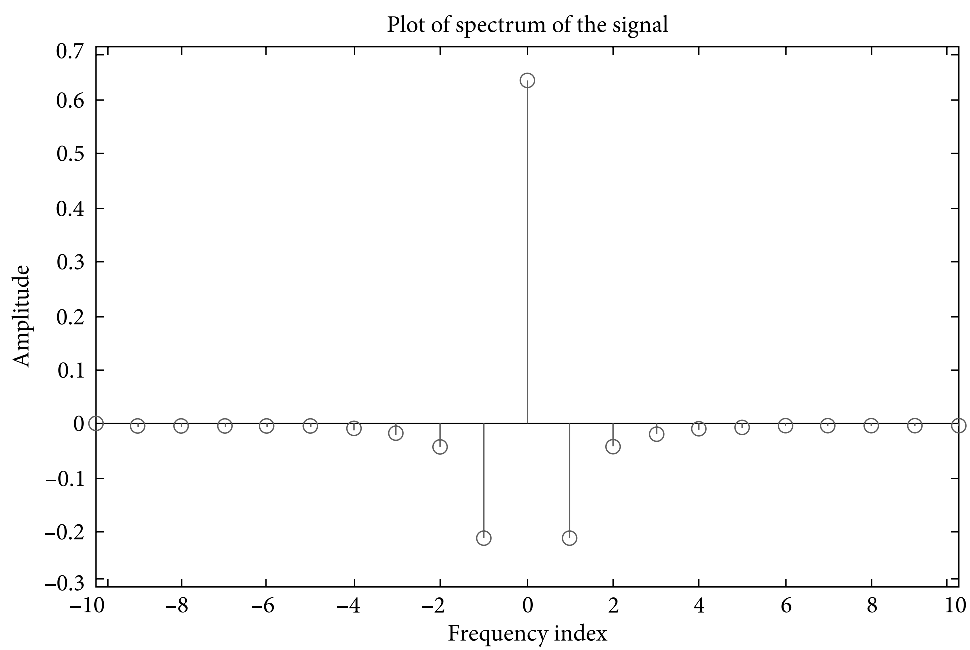


Fig. 5.8 Plot of the FS spectrum for the signal

P 5.8 Determine the FS representation for the signal with the periodic wave as shown in Fig. 5.9. [Note: The signal has no symmetry. It will have sine and cosine terms.]

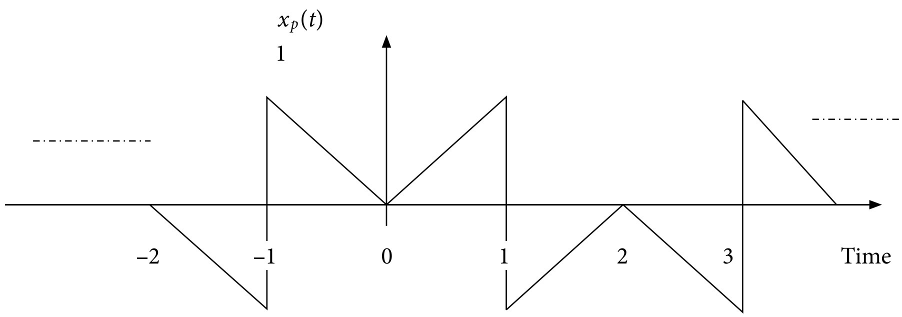


Fig. 5.9 The signal wave for P 5.8

Solution

Step 1 Let us first find the period of the wave. The wave repeats after a time period of 4 seconds. $\omega = \pi = \dfrac{2\pi}{4} = \pi/2$; period of the wave is $\pi/2$ seconds.

Step 2 Let us now find the equation for the wave between 0 to 2 seconds. This is a half part of the wave with a period of 4 seconds. The equation of the signal between 0 to 1 seconds can be written as $x(t) = t$ and that between 1 to 2 can be written as $x(t) = t - 2$.

Step 3 Let us write $x(t)$ as a linear sum of weighted exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk(\pi)t}$$

We can find $X[k]$ using the formula

$$X[k] = \frac{1}{T} \int_{(T)} x(t) e^{-jk\omega t} dt$$

Step 4 Use the formula for FS to find $X[k]$.

$$X[k] = \frac{2}{4} \left[\int_0^1 t e^{-jk(\pi/2)t} dt + \int_1^2 (t-2) e^{-jk(\pi/2)t} dt \right]$$

$$X[k]=\frac{1}{2}\left[\frac{te^{-jk(\pi/2)t}}{-jk\pi/2}\downarrow_0^1-\frac{4}{k^2\pi^2}e^{-jk(\pi/2)t}\downarrow_0^1+\frac{te^{-jk(\pi/2)t}}{-jk\pi/2}\downarrow_1^2\right.$$

$$\left.-\frac{4}{k^2\pi^2}e^{-jk(\pi/2)t}\downarrow_1^2-2\frac{e^{-jk(\pi/2)t}}{-jk\pi/2}\downarrow_1^2\right]$$

$$X[k]=\frac{1}{2}\left[\frac{e^{-jk(\pi/2)}}{-jk\pi/2}-\frac{4}{k^2\pi^2}[e^{-jk(\pi/2)}-1]\right.$$

$$\left.+\frac{2e^{-jk\pi}-e^{-jk(\pi/2)}}{-jk\pi/2}-\frac{4}{k^2\pi^2}[e^{-jk\pi t}-e^{-jk\pi/2}]-2\frac{e^{-jk\pi t}-e^{-jk\pi/2}}{-jk\pi/2}\right]$$

$$X[k]=\frac{1}{2}\left[\frac{(-j)^k}{jk\pi/2}+4\frac{1-(-1)^k}{k^2\pi^2}\right]$$

$$=-\frac{1}{k\pi}+\frac{4}{k^2\pi^2}\text{ for } k \text{ odd}$$

$$=\frac{1}{k\pi}\text{ for } k \text{ even}$$

Note that $e^{-j\pi}=e^{j\pi}=(-1)$ and $e^{-j\pi(1-k)}=e^{j\pi(1+k)}=(-1)$ for even values of k and $e^{-j\pi(1-k)}=e^{j\pi(1+k)}=(1)$ for odd values of k.

Step 5 Evaluate $X[0]$ by integrating the signal over the period.

$$X[0]=0$$

The magnitude response can be written as

$$X[k]=-\frac{1}{k\pi}+\frac{4}{k^2\pi^2}\text{ for } k \text{ odd}$$

$$=\frac{1}{k\pi}\text{ for } k \text{ even}$$

Step 6 The exponential Fourier series can be written as

$$x(t)=\sum_{k=-\infty,k\neq 0}^{\infty}\left[\frac{-2j}{k\pi+2/k^2\pi^2]e^{j\pi kt/2}}\right]\text{for } k \text{ odd}$$

$$x(t) = \sum_{k=-\infty, k\neq 0}^{\infty} [-2j/k\pi]e^{j\pi kt/2} \quad \text{for } k \text{ even}$$

Let us find sine and cosine series using the formula for a_n and b_n.

$$a_0 = 0$$

$$a_n = \frac{2}{4}\left[\int_0^1 t\cos(n\pi t/2)dt + \int_1^2 (t-2)\cos(n\pi t/2)dt\right]$$

$$a_n = 2\times\frac{2}{4}\left[t\sin(n\pi t/2)/n\pi/2 \downarrow_0^1 - \frac{4}{n^2\pi^2}\cos(n\pi t/2)\downarrow_0^1\right.$$

$$\left. + t\sin(n\pi t/2)/n\pi/2 \downarrow_1^2 - \frac{4\cos(n\pi t/2)}{n^2\pi^2}\downarrow_1^2 - 2\frac{\sin(n\pi t/2)}{n\pi/2}\downarrow_1^2\right]$$

$$a_n = \left[\frac{2}{n\pi} + 4\frac{1-(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2}{n\pi} + \frac{8}{n^2\pi^2} \text{ for } n \text{ odd}$$

$$= \frac{2}{n\pi} \text{ for } n \text{ even}$$

Let us derive the exponential series from the trigonometric series.

$$c_n = \frac{1}{2}[a_n - jb_n],\ c_{-n} = \frac{1}{2}[a_n + jb_n],\ c_0 = a_0 = 0$$

$$c_n = \frac{1}{2}\left[\frac{2}{n\pi} + \frac{8}{n^2\pi^2}\right] \text{ for } n \text{ odd}$$

$$= \frac{1}{n\pi} \text{ for } n \text{ even}$$

P 5.9 Determine the FS representation using exponential series for a signal with periodic wave as given in the following equation. Find the FS representation using sine and cosine series.

$$x(t)=\begin{cases}0 & \text{for } -1\le t<0\\ 1-0.5\sin(\pi t) & \text{for } 0\le t<1\end{cases}$$

Solution

Step 1 Let us first find the period of the wave. The wave repeats after 2 seconds. The period is π. $\omega=\frac{2\pi}{2}=\pi$.

Step 2 Find $X[k]$.

$$X[k]=\frac{1}{T}\int_{(T)}x(t)e^{-jk\omega t}dt$$

$$X[k]=\frac{1}{2}\left[\int_0^1(1-0.5\sin(\pi t)e^{-jk(\pi)t}dt\right]$$

$$X[k]=\frac{1}{2}\left\{\left[\frac{1}{jk\pi}e^{-jk\pi t}\right]\downarrow_0^1-\frac{1}{4j}\left[\frac{e^{-j(k-1)\pi t}}{-j(k-1)\pi}-\frac{e^{-j(k+1)\pi t}}{-j(k+1)\pi}\right]\downarrow_0^1\right\}$$

$$X[k]=\frac{1}{2}\left\{\left[\frac{1}{-jk\pi}[e^{-jk\pi}-1]\right]-\frac{1}{4j}\left[\frac{[e^{-j(k-1)\pi}-1]}{-j(k-1)\pi}-\frac{[e^{-j(k+1)\pi}-1]}{-j(k+1)\pi}\right]\right\}$$

$$=\frac{1}{2}\left\{\left[\frac{e^{-jk\pi}-1}{-jk\pi}\right]-\frac{1}{4j}\left[\frac{(k+1)[e^{-(k-1)\pi}-1]-[(k-1)[e^{-j(k+1)\pi}-1]}{-j(k^2-1)\pi}\right]\right\}$$

$$=\frac{1}{2}\left[\frac{-4}{4(k^2-1)\pi}-\frac{(-1)^k-1}{jk\pi}\right]$$

$$=\frac{-1}{2(k^2-1)\pi}+\frac{1}{jk\pi}\text{ for } k \text{ odd}$$

$$=0 \text{ for } k \text{ even}$$

Step 5 Evaluate $X[0]$ by integrating the signal over the period.

$$X[0] = \frac{1}{T}\int_0^1 x(t)dt = \frac{1}{2}\left[\int_0^1 (1-0.5\sin(\pi t))dt\right]$$

$$= \frac{1}{2}[t \downarrow_0^1] - \frac{1}{4}[\cos(\pi t)/\pi]] \downarrow_0^1 = \frac{1}{2} - \frac{(-1)-1}{4\pi} = \frac{1}{2} + \frac{1}{2\pi}$$

Step 6 The exponential Fourier series can be written as

$$x(t) = \frac{1}{2} + \frac{1}{2\pi} + \sum_{k=-\infty, k\neq 0}^{\infty} \left[\frac{1}{jk\pi} + \frac{-k}{2(k^2-1)\pi}\right] e^{jk\pi t} \text{ for } k \text{ odd}$$

Let us find the sine and cosine series using the formula for a_n and b_n.

$$a_n = \frac{2}{T}\int_{(T)} x(t)\cos(nwt)dt$$

$$= \frac{2}{2}\int_0^1 (1-0.5\sin(\pi t)\cos(n\pi t))dt$$

$$= \left[\sin(n\pi t)\downarrow_0^1 - \frac{1}{4}\sin(n+1)\pi t \Big/ (n+1)\pi \downarrow_0^1 + \frac{1}{4}\sin(n-1)\pi t \Big/ (n-1)\pi \downarrow_0^1\right]$$

$$= -\frac{1}{4}\cos(n+1)\pi t \Big/ (n+1)\pi \downarrow_0^1 + \frac{1}{4}\cos(n-1)\pi t \Big/ (n-1)\pi \downarrow_0^1$$

$$= 0 \text{ for } n \text{ even}$$

$$= -\frac{1}{(n^2-1)\pi} \text{ for } n \text{ odd}$$

$$b_n = \frac{2}{T}\int_{(T)} x(t)\sin(nwt)dt$$

$$= \frac{2}{2}\int_0^1 (1-0.5\sin(\pi t)\sin(n\pi t))dt$$

$$= \left[\sin(n\pi t)\downarrow_0^1 - \frac{1}{4}\cos(n-1)\pi t \Big/ (n-1)\pi + \frac{1}{4}\cos(n+1)\pi t \Big/ (n+1)\pi\right]$$

$$= \cos(n\pi t)/n\pi \downarrow_0^1 \frac{1}{4}\left[\sin(n+1)\pi t/(n+1)\pi \downarrow_0^1 - \sin(n-1)\pi t/(n-1)\pi \downarrow_0^1\right]$$

$$= [(-1)^n - 1]/n\pi$$

$$= 0 \text{ for } n \text{ even}$$

$$= -\frac{2}{n\pi} \text{ for } n \text{ odd}$$

Let us derive the exponential series from the trigonometric series.

$$c_n = \frac{1}{2}[a_n - jb_n],\ c_{-n} = \frac{1}{2}[a_n + jb_n],\ c_0 = a_0 = \frac{1}{2} + \frac{1}{2\pi}$$

$$c_n = \frac{1}{2}\left[\frac{1}{(k^2-1)\pi} - j\frac{2}{k\pi}\right] = -\frac{1}{2(k^2-1)\pi} + \frac{2}{jk\pi} \text{ for } n \text{ odd}$$

P 5.10 Find if the following signals satisfy the Dirichlet conditions.

$$x(t) = 2\tan(\pi t)$$

$$x(t) = \sin(0.5\pi / t) \text{ for } 0 < t < 1 \text{ and the signal repeats with a period of 1}$$

Solution

Signal 1 is not absolutely integrable and signal 2 has infinite number of extrema points. So, both the signals do not satisfy Dirichlet conditions.

P 5.11 Consider a train of pulses as shown in Fig. 5.10. Find the FS representation for this periodic signal.

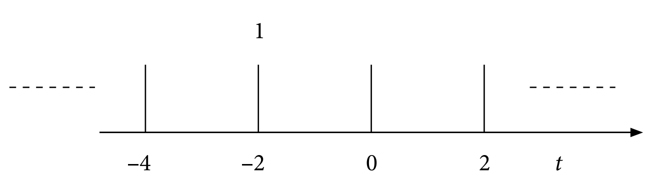


Fig. 5.10 Plot of signal for Problem 5.11

Solution

We will use the formula for exponential FS. The signal has a period of T. Consider the time interval between $-t/2$ and $T/2$.

$$X[k]=\frac{1}{T}\int_{(T)} x(t)e^{-jk\omega t}dt=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk2\pi t/T}dt$$

Put $t = 0$ in the equation,

$$X[k]=c_k=\frac{1}{2},\ \delta(t)=\begin{cases}1 & \text{for } t=0\\ 0 & \text{otherwise}\end{cases}$$

$$x(t)=\sum_{k=-\infty}^{\infty}\frac{1}{2}e^{jk\omega t}$$

The FS is again a train of impulses with a separation of 1/2 as shown in Fig. 5.11.

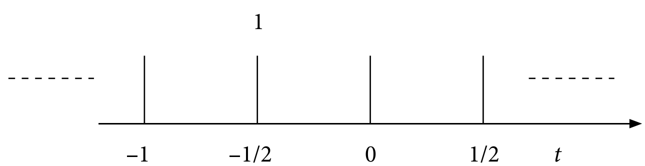


Fig. 5.11 Plot of spectrum for the train of impulse in P 5.11

P 5.12 Use the Fourier series representation for P 5.6 and find the Fourier series representation for the following signal. Use the property of time shifting.

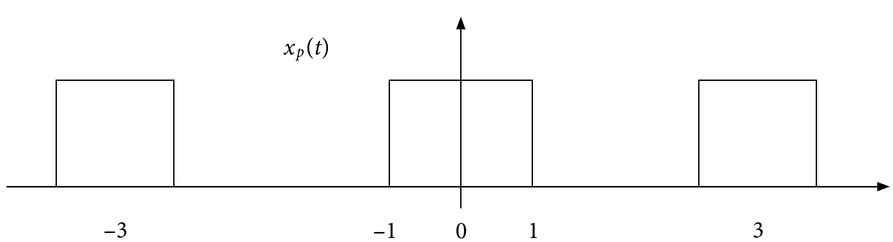


Fig. 5.12 Plot of signal for Problem 5.12

Solution

Here, $T = 2$ and $T_0 = 3$. Time shifting property says that

if $\quad x(t)\leftrightarrow C_n$

then $\quad x(t-t_0) \leftrightarrow e^{-jn\omega t_0} C_n$

Fourier series for Problem 6 is given by

$$c_n = \frac{A}{2n\pi}\left[\exp\left(-\frac{j2\pi nT}{T_0}\right)-1\right]$$

If $\quad x(t) \leftrightarrow C_n$

then $\quad x(t-t_0) \leftrightarrow e^{-jn\omega t_0} C_n$

$$X[k] = \frac{-j}{2\pi k}[e^{-jk(2\pi/3)} - e^{jk(2\pi/3)}]$$

$$= -\frac{1}{\pi k}\sin(2\pi k/3) \text{ for } k \neq 0$$

P 5.13 Use the property of differentiation in time to find the FS representation for the signal shown in Fig. 5.12.

$$C_n = \frac{D_n}{jn\omega} = \frac{2jT_0}{jn2\pi T_0}\sin\left(\frac{2\pi nT}{2T_0}\right) = -\frac{1}{n\pi}\sin\left(\frac{2n\pi}{3}\right), \text{ put } T/2 = 1 \text{ and } T_0 = 3$$

P 5.14 Find the time domain signal with FS coefficients given as follows and with $\omega = \pi$.

$$C_n = j\delta(n-2) - j\delta(n+2) + 4\delta(n-3) + 4\delta(n+3)$$

Solution

$$x(t) = -2\sin(2\pi t) + 8\cos(3\pi t)$$

P 5.15 Determine the time domain signal using its magnitude and phase spectrum given in Fig. 5.13.

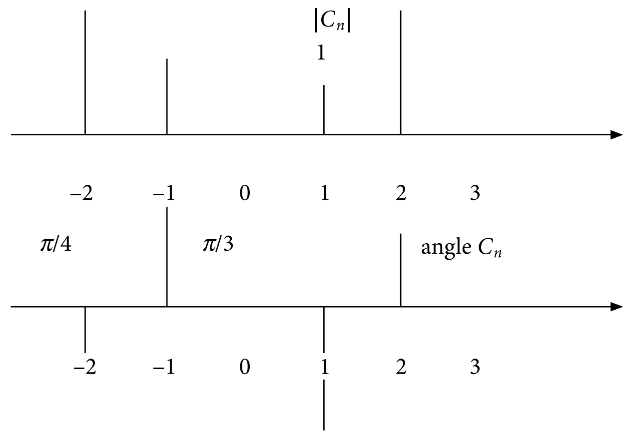


Fig. 5.13 Plot of signal for Problem 5.15

Solution

$$x(t) = 2\cos(2\omega t - \pi / 4) + 4\cos(\omega t + \pi / 3)$$

P 5.16 Find the DTFS coefficients for the signal given by $x[n] = 3\sin\left(\frac{\pi}{4}n + \beta\right)$

Solution

Step 1 Find the fundamental period and fundamental frequency.

$$x[n] = 3\sin\left(\frac{\pi}{4}n + \beta\right) = 3\sin\left(\frac{2\pi n}{8} + \beta\right)$$

Here, fundamental period is $N = 8$; fundamental frequency is 1/8

Step 2 We will write the signal in terms of exponentials.

$$x[n] = 3\sin\left(\frac{\pi}{4}n + \beta\right) = \frac{1}{2j}\left[e^{j2\pi n/8}e^{j\beta} - e^{-j2\pi n/8}e^{-j\beta}\right]$$

The fundamental period is 16. The DTFS will consist of 16 coefficients varying from $k = -3$ to 4.

$$x[n] = \sum_{k=-3}^{4} X[k]e^{jk(\pi/4)n}$$

$$X[k]=\begin{cases} -\dfrac{1}{2j}e^{-j\beta} & \text{for } k=-1 \\ \dfrac{1}{2j}e^{j\beta} & \text{for } k=1 \\ 0 & \text{otherwise} \end{cases}$$

$X[k]$ is also periodic with period of 8. The magnitude and phase plot is shown in Fig. 5.14.

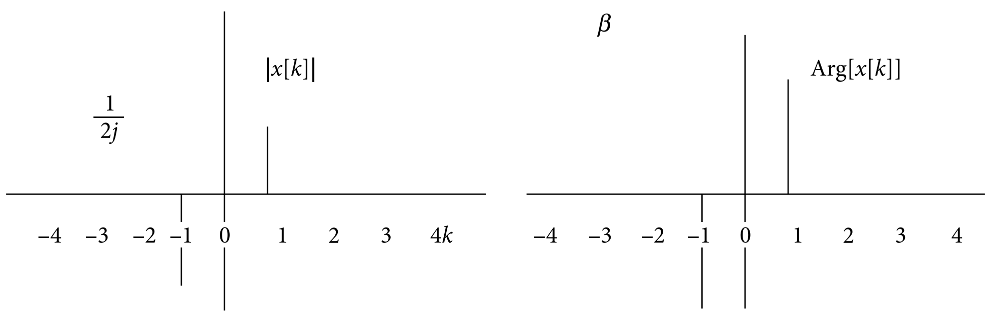


Fig. 5.14 Magnitude and phase plot for the FS in P 5.16

P 5.17 Find the DTFS coefficients for the signal given by

$$x[n]=1/2+\cos\left(\frac{\pi}{5}n+\frac{\pi}{4}\right)$$

Solution

Step 1 Find the fundamental period and fundamental frequency.

$$x[n]=1/2+\cos\left(\frac{\pi}{5}n+\frac{\pi}{4}\right)=1/2+\cos\left(\frac{2\pi n}{10}+\frac{\pi}{4}\right)$$

Here, fundamental period is $N = 10$; fundamental frequency is 1/10

Step 2 We will write the signal in terms of exponentials.

$$x[n]=1/2+\cos\left(\frac{\pi}{5}n+\frac{\pi}{4}\right)=1/2+\frac{1}{2}\left[e^{j2\pi n/10}e^{j\pi/42}+e^{-j2\pi n/10}e^{-j\pi/4}\right]$$

The fundamental period is 20. The DTFS will consist of 20 coefficients varying from $k = -4$ to 5.

$$x[n] = \sum_{k=-4}^{5} X[k] e^{jk(\pi/5)n}$$

$$X[k] = \begin{cases} \frac{1}{2} e^{-j\pi/4} & \text{for } k = -1 \\ \frac{1}{2} e^{j\pi/4} & \text{for } k = 1 \\ \frac{1}{2} & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$X[k]$ is also periodic with a period of 20. The magnitude and phase plot of FS is shown in Fig. 5.15.

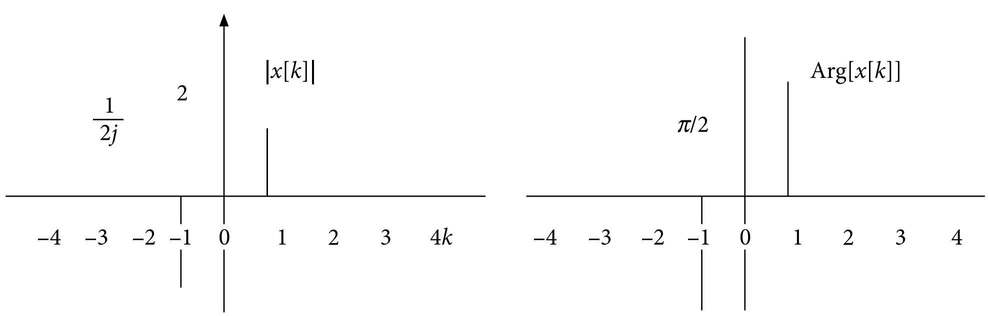


Fig. 5.15 Magnitude and phase plot for the FS in P 5.17

P 5.18 Find the DTFS coefficients for the DT periodic signal shown in Fig. 5.16.

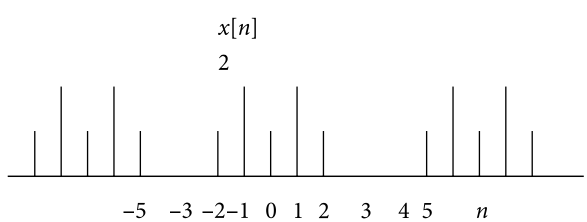


Fig. 5.16 Plot of signal for Problem 5.18

Solution

Step 1 Find the fundamental period and fundamental frequency. The signal is periodic with a period of 6 samples from –2 to 3. We will use the formula for $x[k]$.

$$X[k]=\frac{1}{N}\sum_{n=-2}^{3}x[n]e^{-jkn2\pi/6}=\frac{1}{6}\left[e^{-2j\pi k/3}+e^{2j\pi k/3}+2e^{j\pi k/3}+2e^{-j\pi k/3}+1\right]$$

$$X[k]=\frac{1}{6}+\frac{2}{3}\cos(\pi k/3)+\frac{1}{3}\cos(2\pi k/3)$$

$$X[k]=\begin{Bmatrix}\frac{1}{3}e^{jk\pi/3}+\frac{1}{6}e^{2jk\pi/3} & \text{for } 1\le k\le 3\\ \frac{1}{3}e^{-jk/3}+\frac{1}{6}e^{-2jk\pi/3} & \text{for } -2\le k\le -1\\ \frac{1}{6} & \text{for } k=0\end{Bmatrix}$$

6

Fourier Transform Representation of Aperiodic Signals

P 6.1 Determine the FT representation for the signal given as $x(t) = [2e^{-2t} + 3e^{-4t}]u(t)$. Find the magnitude using manual calculations for 5 points, namely $\omega = 1, 2, 3, 4$ and 5.

Solution

The exponential signal $x(t)$ exists between zero to infinity and is termed as a right-handed signal. We can use Eq. 6.5 to find the FT.

$$X(j\omega) = \int_{-\infty}^{\infty} \left[2e^{-2t} + 3e^{-4t}\right] u(t) e^{-j\omega t}\ dt = \int_{0}^{\infty} \left[2e^{-(2+j\omega)t} + 3e^{-(4+j\omega)t}\right] dt$$

$$X(j\omega) = -\frac{2}{2+j\omega} e^{-(2+j\omega)t} \downarrow_0^\infty - \frac{3}{4+j\omega} e^{-(4+j\omega)t} \downarrow_0^\infty$$

$$= \frac{2}{2+j\omega} + \frac{3}{4+j\omega}$$

We can write the magnitude and phase spectrum as

$$|\,\text{first term}\,| = \left|\frac{2}{2+j\omega} \times \frac{2-j\omega}{2-j\omega}\right| = \left|\frac{2(2-j\omega)}{4+\omega^2}\right|$$

$$= \text{square root (real part}^2 + \text{img part}^2)$$

$$= \text{square root } 2 \times \left[\left(\frac{2}{4+\omega^2} \right)^2 + \left(\frac{j\omega}{4+\omega^2} \right)^2 \right]$$

$$= 2\sqrt{\frac{4+\omega^2}{(4+\omega^2)^2}} = \frac{2}{\sqrt{4+\omega^2}}$$

$$|\text{second term}| = \left| \frac{3}{4+j\omega} \times \frac{4-j\omega}{4-j\omega} \right| = \left| \frac{3(4-j\omega)}{16+\omega^2} \right|$$

$$= \text{square root (real part}^2 + \text{img part}^2)$$

$$= 3 \times \sqrt{\left[\left(\frac{4}{16+\omega^2} \right)^2 + \left(\frac{j\omega}{16+\omega^2} \right)^2 \right]}$$

$$= 3 \times \sqrt{\frac{16+\omega^2}{(16+\omega^2)^2}} = \frac{3}{\sqrt{16+\omega^2}}$$

To evaluate the magnitude of the response, we have to put different values of ω in the equation. We can find the magnitude of FT by finding a square root of the sum of the square of the real part and square of the imaginary part as the equation for FT is a complex quantity. Let us put values of ω as 1, 2, etc. in the equation.

$$\text{If } \omega = 0, |X(j\omega)| = \frac{2}{\sqrt{4}} + \frac{3}{\sqrt{16}} = 1 + 3/4 = 1.75$$

$$\text{If } \omega = \pm 1, |X(j\omega)| = \frac{2}{\sqrt{4+1}} + \frac{3}{\sqrt{16+1}} = 1.6220$$

$$\text{If } \omega = \pm 2, |X(j\omega)| = \frac{2}{\sqrt{4+4}} + \frac{3}{\sqrt{16+4}} = 1.3779$$

$$\text{If } \omega = \pm 3, |X(j\omega)| = \frac{2}{\sqrt{4+9}} + \frac{3}{\sqrt{16+9}} = 1.1547$$

$$\text{If } \omega = \pm 4, |X(j\omega)| = \frac{2}{\sqrt{4+16}} + \frac{3}{\sqrt{16+16}} = 0.9775$$

$$\text{If } \omega = \pm 5, |X(j\omega) = \frac{2}{\sqrt{4+25}} + \frac{3}{\sqrt{16+25}} = 0.8399, \text{ etc}$$

Let us now write a MATLAB program to plot the magnitude and phase response and verify the result of the manual calculations. Figure 6.1 shows the plot of the signal and Fig. 6.2 shows the phase plot of the signal.

```
clear all;
t=0:0.1:40;
x=2*exp(-3*t)+3*exp(-4*t);
plot(t,x);title('plot of exponential
signal');xlabel('time');ylabel('amplitude');
for i=1:20,
                y(i)=abs(2/sqrt((4+(i)*(i)))+abs(3/
sqrt((16+(i)*(i))));
end;
z(21)=1.75;
for i=1:20,
    z(i+21)=y(i);
end
for i=1:20,
    z(i)=y(21-i);
end
figure;
subplot(2,1,1);
s=-20:1:20;
plot(s,z);title('Magnitude plot of Fourier transform
of  exponential  signal');xlabel('frequency');ylabel
('amplitude');
subplot(2,1,2);
for i=1:20,
    y1(i)=angle((2/(4+1j*i))+(3/(16+1j*i)));
end;
z1(21)=0.0;
for i=1:20,
    z1(i+21)=y1(i);
end
for i=1:20,
    z1(i)=-y1(21-i);
end;
plot(s,z1);title('Phase plot of FT of exponential si
gnal');xlabel('frequency');ylabel('angle');
the magnitude of the first 5 frequency points is
1.75   1.6220    1.3779    1.1547    0.9775    0.8399
These values tally with the values found using hand
calculations.
```

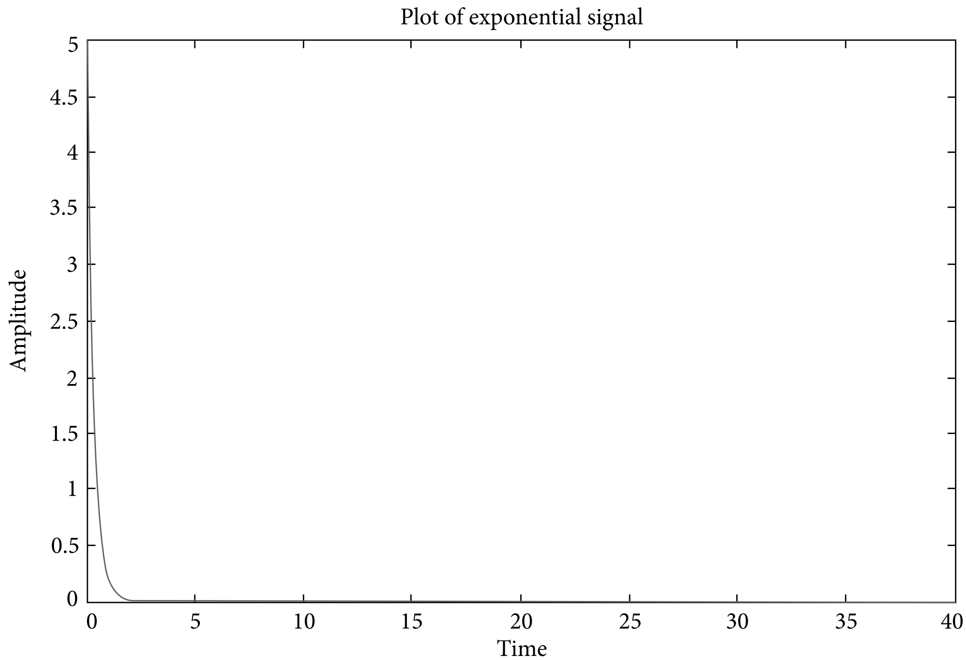


Fig. 6.1 Plot of the exponential signal

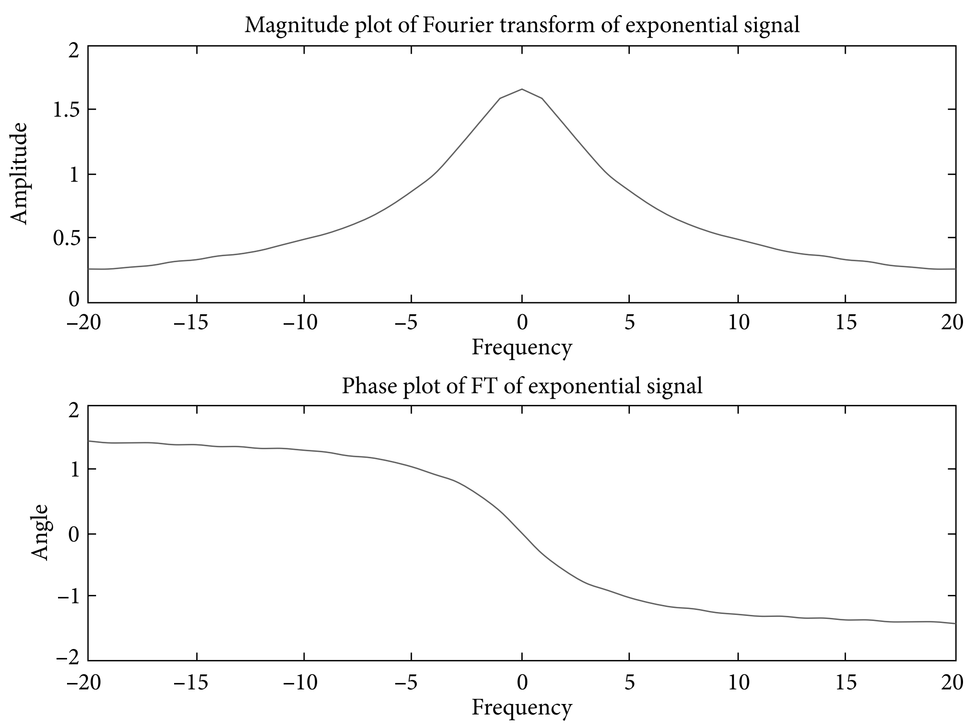


Fig. 6.2 Plot of magnitude and phase of FT of the exponential signal

P 6.2 Consider a rectangular pulse of duration 0.4 s and amplitude 2, as shown in Fig. 6.3 below. Find its FT.

Solution

The rectangular pulse in Fig. 6.3 can be mathematically defined as

$$\mathrm{rect}(t) = \begin{cases} 2 & -0.2 \le t \le 0.2 \\ 0 & |t| > 0.2 \end{cases}$$

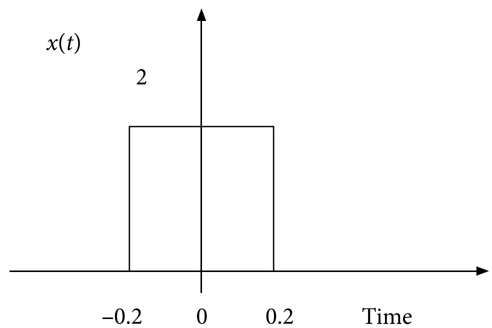


Fig. 6.3 A rectangular pulse of duration 0.4 and amplitude 2

A rectangular pulse of duration 0.4 and amplitude 2 can be written as

$$x(t) = 2\ \mathrm{rect}(t/0.4)$$

A Fourier transform of this rectangular pulse can be written as

$$X(\omega) = \int_{-0.2}^{0.2} 2\exp(-j\omega t)dt$$

$$= -\frac{1}{j\omega} 2e^{-j\omega t} \downarrow_{-0.2}^{0.2} = -\frac{2}{j\omega}[e^{-j0.2\omega} - e^{j0.2\omega}]$$

$$= \frac{2j2\sin(0.2\omega)}{j\omega} = \frac{4\sin(0.2\omega)}{\omega}$$

Let us write a MATLAB program to plot the continuous spectrum. We have used the value of $T = 0.4$. The value of A is kept constant and equal to 2.

```
clear all;
T=0.4;
A=2;
N=50;
for n=1:N,
    c(n+N+1)=4*sin(n*pi*0.2)/(n*pi);
 end
c(N+1)=0.8;
for n=1:N,
    c(n)=c(2*N+2-n);
end
plot(abs(c));
title('Plot of magnitude of continuous spectrum for
T = 4');
xlabel('frequency');
ylabel('amplitude');
```

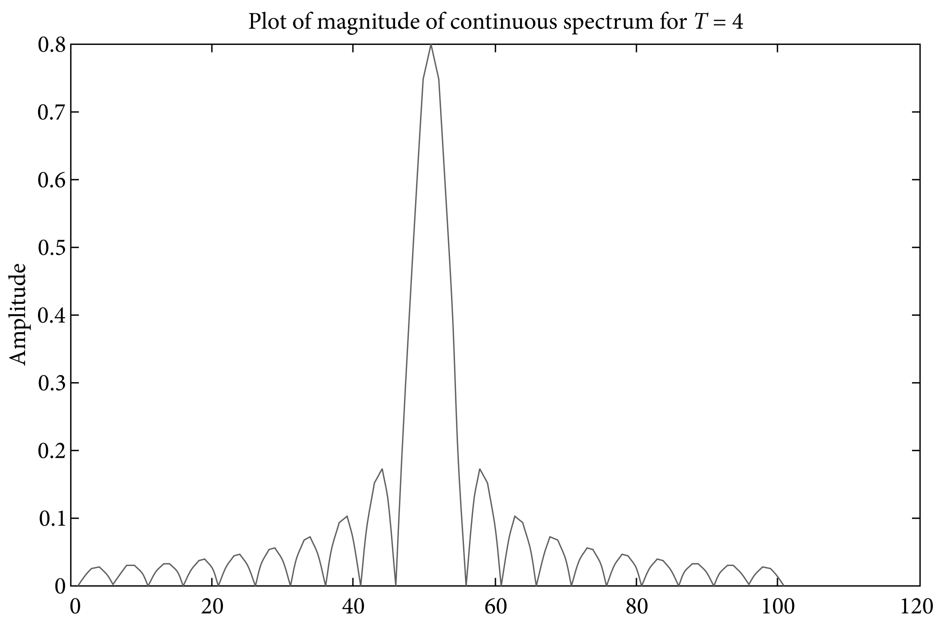



Fig. 6.4 Plot of *t* magnitude of response for FT of rectangular pulse-duration 0.4 sec.

Note: Width of main lobe in frequency domain is 10 units.

P 6.3 Find the IFT of $X(j\omega) = \begin{cases} 1 & -0.1 \le \omega \le 0.1 \\ 0 & |\omega| > 0.1 \end{cases}$ as shown in Fig. 6.5.

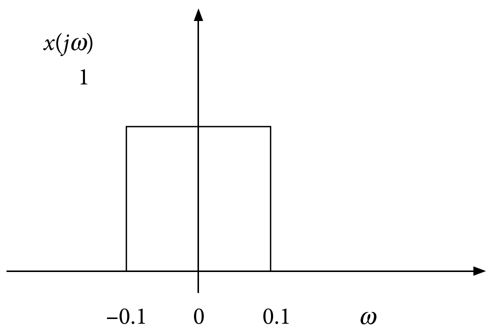


Fig. 6.5 A rectangular frequency domain signal of width 0.1 and amplitude 1

Solution

To find IFT, we will use the equation for IFT

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-0.1}^{0.1}e^{j\omega t}d\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-0.1}^{0.1}e^{j\omega t}d\omega=\frac{1}{2\pi}\left[\frac{1}{jt}e^{j\omega t}\downarrow_{-0.1}^{0.1}\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{jt}\{e^{j0.1t}-e^{-j0.1t}\}\right]$$

$$=\frac{1}{\pi t}[\sin(0.1t)]$$

This is a sinc function. Let us write a MATLAB program for this example. Fig. 6.6 show the magnitude of the response plot of the frequency domain with rectangular pulse of width 0.1.

```
clear all;
w=2;
A=1;
N=50;
for n=1:N,
    X(n+N+1)=sin(n*2)/(pi*n);
```

```
 end
X(N+1)=w/pi;
for n=1:N,
    X(n)=X(2*N+2-n);
end
plot(abs(X));
title('plot of magnitude of IFT of frequency domain
rectangular pulse of width w=2');
xlabel('time');
ylabel('amplitude');
```

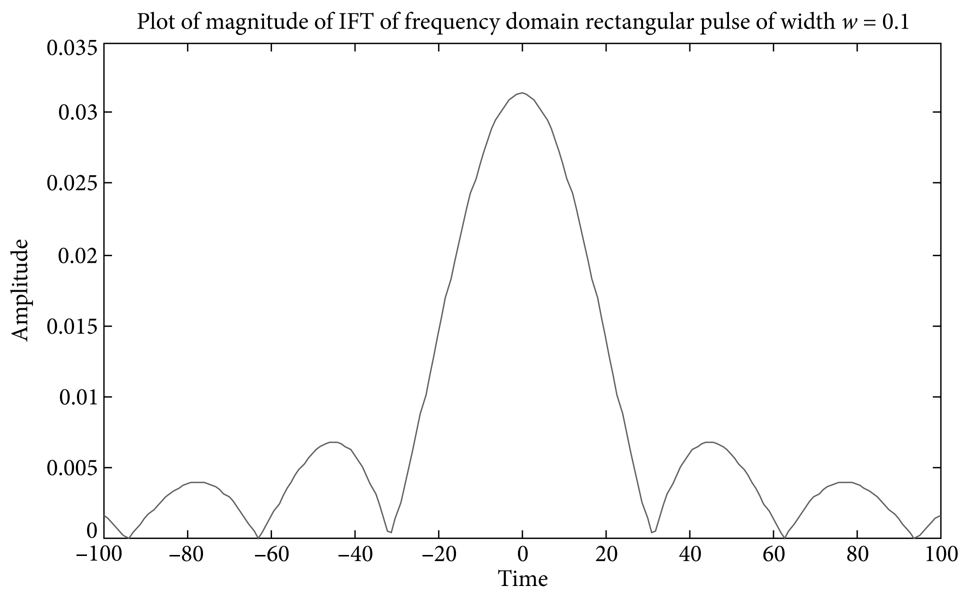


Fig. 6.6 Magnitude plot of the response for frequency domain of width = 0.1

P 6.4 Find FT of the aperiodic signal given by

$$x(t) = \begin{cases} 2t & \text{for } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

FT of $x(t)$ can be written as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{1} 2te^{-j\omega t}dt$$

$$= 2t\int_0^1 e^{-j\omega t}dt - 2\int_0^1\int_0^1 e^{-j\omega t}dt$$

$$= \frac{2te^{-j\omega t}}{-j\omega}\downarrow_0^1 - 2\frac{e^{-j\omega t}}{-\omega^2}\downarrow_0^1$$

$$= \frac{2e^{-j\omega}}{-j\omega} + 2\frac{e^{-j\omega} - 1}{\omega^2}$$

$$= \frac{j2}{\omega}e^{-j\omega} - 2 + 2\frac{e^{-j\omega}}{\omega^2}.$$

Note that we have to use integration by parts to solve the problem.

P 6.5 Find FT of the aperiodic triangular signal given by

$$x(t) = \begin{cases} 1 - t/2 & \text{for } 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Let us plot the signal first. The plot of the signal is shown in Fig. 6.7.

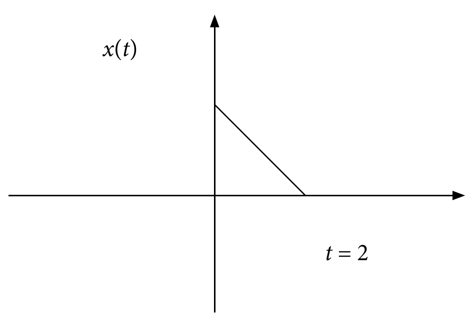


Fig. 6.7 Plot of $x(t)$ for P 6.5

FT of $x(t)$ can be written as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^2 x(t)e^{-j\omega t}dt$$

$$= \int_0^2 (1 - t/2)e^{-j\omega t}dt$$

$$= \frac{e^{-j\omega t}}{-j\omega}\downarrow_0^2 - \frac{1}{2}\left[\frac{te^{-j\omega t}}{-j\omega}\downarrow_0^2 + \frac{e^{-j\omega t}}{\omega^2}\downarrow_0^2\right]$$

$$= \frac{e^{-2j\omega}-1}{-j\omega} - \frac{2e^{-2j\omega}}{2j\omega} - \frac{e^{-2j\omega}-1}{2\omega^2}$$

$$= \frac{-2e^{-2j\omega}+2-2e^{-2j\omega}}{2j\omega} - \frac{e^{-2j\omega}-1}{2\omega^2}$$

$$= \frac{1-2e^{-2j\omega}}{j\omega} - \frac{e^{-2j\omega}-1}{2\omega^2}$$

Note that we have to use integration by parts to solve the problem.

P 6.6 Find FT of aperiodic signal given by $x(t) = e^{-3t}u(t-3)$.

Solution

FT can be written as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{3}^{\infty} e^{-3t}e^{-j\omega t}dt$$

$$= \int_{3}^{\infty} e^{-(3+j\omega)t}dt$$

$$= \frac{e^{-(3+j\omega)t}}{-(3+j\omega)}\downarrow_3^{\infty}$$

$$= \frac{1}{-(3+j\omega)} - \frac{e^{-(3+j\omega)}}{-(3+j\omega)}$$

$$= \frac{e^{-(3+j\omega)}-1}{3+j\omega}$$

P 6.7 Find FT of the aperiodic signal given by $x(t) = e^{-t}\cos(3\pi t)u(t)$. This is a decaying sinusoid.

Solution

FT can be written as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-t}\cos(3\pi t)e^{-j\omega t}dt$$

$$=\frac{1}{2}\int_0^\infty e^{-(1+j\omega)t}(e^{j3\pi t}+e^{-j3\pi t})dt$$

$$=\frac{1}{2}\left[\int_0^\infty e^{-(1+j(\omega-3\pi))t}dt+\int_0^\infty e^{-(1+j(\omega+3\pi))t}dt\right]$$

$$=\frac{1}{2}\left[\frac{e^{-(1+j(\omega-3\pi))t}}{-(1+j(\omega-3\pi))}\downarrow_0^\infty+\frac{e^{-(1+j(\omega+3\pi))t}}{-(1+j(\omega+3\pi))}\downarrow_0^\infty\right]$$

$$=\frac{1}{2}\left[\frac{1}{1+j(\omega-3\pi)}+\frac{1}{1+j(\omega+3\pi)}\right]$$

P 6.8 Find FT of the aperiodic signal given by $x(t) = t^2$ for $|t| < 1$.

Solution

FT can be written as

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-1}^{1}t^2e^{-j\omega t}dt$$

$$=\left[t^2\frac{e^{-j\omega t}}{-j\omega}\downarrow_{-1}^{1}-\int_{-1}^{1}2t\int_{-1}^{1}e^{-j\omega t}dt\right]$$

$$=\frac{e^{-j\omega}-e^{j\omega}}{-j\omega}-\int_{-1}^{1}2t\left[\frac{e^{-j\omega t}}{-j\omega}\right]dt$$

$$=\frac{2j\sin\omega}{j\omega}+\frac{2}{j\omega}\left[te^{-j\omega t}+\frac{e^{-j\omega t}}{\omega^2}\right]\downarrow_{-1}^{1}$$

$$=\frac{2\sin\omega}{\omega}+\frac{2}{j\omega}[e^{-j\omega}+e^{j\omega}]+\frac{2}{j\omega^3}[e^{-j\omega}-e^{j\omega}]$$

$$=\frac{2\sin\omega}{\omega}+\frac{4\cos\omega}{j\omega}-\frac{4\sin\omega}{\omega^3}$$

P 6.9 Find FT of aperiodic signal given by $x(t)=e^{2t}u(-t+1)$.

Solution

FT can be written as

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\infty}^{1}e^{2t}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{1}e^{-(j\omega-2)t}dt$$

$$=\frac{1}{2-j\omega}e^{(2-j\omega)t}\downarrow_{-\infty}^{1}$$

$$=\frac{e^{2-j\omega}}{2-j\omega}$$

P 6.10 Find FT of $x(t)=\delta(t)+u(t)$.

Solution

$$FT[\delta(t)+u(t)]=\int_{-\infty}^{\infty}[\delta(t)+u(t)]\exp(-j2\pi ft)dt$$

$$=1+\int_{0}^{\infty}e^{-j\omega t}dt$$

$$=1+\frac{e^{-j\omega t}}{-j\omega}\downarrow_{0}^{\infty}=1+\frac{1}{j\omega}+\pi\delta(\omega)$$

We know that FT of the unit step function is

$$X(j\omega)=\frac{1}{2}FT[u(t)=1+\text{sgn}(t)]$$

$$=\frac{1}{2}\left[2\pi\delta(\omega)+\frac{2}{j\omega}\right]=\pi\delta(\omega)+\frac{1}{j\omega}$$

P 6.11 Find FT of the signal $x(t)=[3\cos(2\pi t)+2\sin(3\pi t)]$ for all t using the Dirac delta function.

Solution

Let us find the Fourier transform of the exponential signal. Let the signal be given by

$$x(t)=\exp(j2\pi f_c t)\text{ for all } t$$

To find the Fourier transform, we will use the result for the Dirac delta function and the frequency shifting property of Fourier transform, namely

$$\text{If } x(t)\Leftrightarrow X(f)$$

$$\exp(j2\pi f_0 t)x(t)\Leftrightarrow X(f-f_0)$$

$$X(j\omega)=2\pi\delta(\omega)\downarrow_{\omega=\omega-\omega_c}=2\pi\delta(\omega-\omega_c)$$

Let $x(t)$ be a D.C. signal. We know it transforms to

$$A\Leftrightarrow 2\pi\delta(\omega)$$

We will multiply the D.C. signal by the complex exponential to get the signal

$$x(t)=1\times\exp(j2\pi f_c t)\text{ for all } t$$

Now, the Fourier transform of the signal can be found by using the frequency shifting property of the Fourier transform.

$$\text{If } 1\Leftrightarrow 2\pi\delta(\omega)$$

$$\exp(j2\pi f_c t)\times 1\Leftrightarrow 2\pi\delta(\omega-\omega_c)$$

$$x(t)=3\cos(2\pi t)+2\sin(3\pi t)=3/2[e^{j2\pi t}+e^{-j2\pi t}]-j[e^{j3\pi t}-e^{-j3\pi t}]$$

$$X(j\omega)=\frac{3}{2}[2\pi\delta(\omega-\omega_1)+2\pi\delta(\omega+\omega_1)]-j[2\pi\delta(\omega-\omega_2)-2\pi\delta(\omega-\omega_2)]$$

where $\omega_1=2\pi$ and $\omega_2=3\pi$

P 6.12 Find the FT of an ideal sampling function with sampling interval of 4 seconds.

Solution

An ideal sampling function is an infinite sequence of uniformly spaced delta functions. The ideal sampling function can be written as

$$\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

We can recognize the generating function for the ideal sampling function as the delta function $\delta(t)$ with FT of $X\left(\frac{n}{T_0}\right) = 1$ for all n.

$$\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0) \Leftrightarrow \frac{1}{T_0}\sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_0}\right) = \frac{1}{4}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{m}{4}\right)$$

Fig. 6.8 shows a plot of a periodic pulse train and its FT.

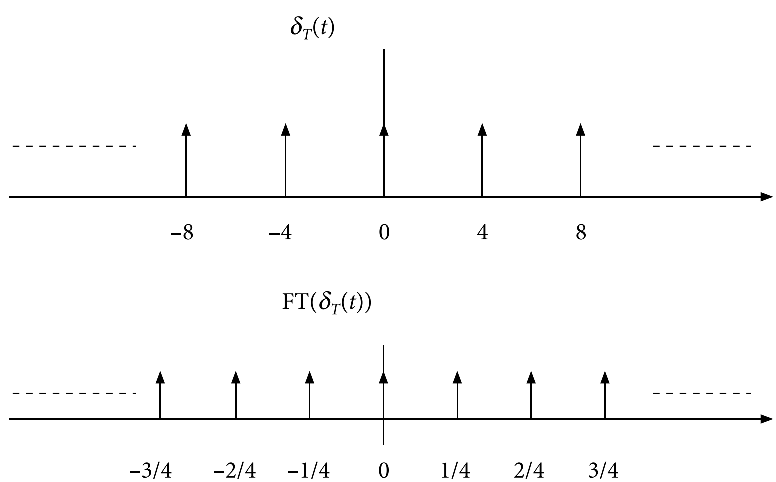


Fig. 6.8 A plot of a periodic pulse train and its FT

P 6.13 Find inverse FT of

$$X(j\omega) = \begin{cases} 4\cos(3\omega) & \text{for } |\omega| \le \pi \\ 0 & \text{for } |\omega| > \pi \end{cases}$$

Solution

To find IFT, we will use the equation for IFT

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

$$x(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}4\cos(3\omega)e^{j\omega t}d\omega$$

$$x(t)=\frac{4}{2\pi}\int_{-\pi}^{\pi}(e^{j3\omega}+e^{-j3\omega})e^{j\omega t}d\omega=\frac{4}{2\pi}\left[\int_{-\pi}^{\pi}e^{j\omega(t+3)}d\omega+\int_{-\pi}^{\pi}e^{j\omega(t-3)}d\omega\right]$$

$$=\frac{4}{2\pi}\left[\frac{1}{j(t+3)}e^{j\omega(t+3)}\downarrow_{-\pi}^{\pi}+\frac{1}{j(t-3)}e^{j\omega(t-3)}\downarrow_{-\pi}^{\pi}\right]$$

$$=\frac{4}{2\pi j}\left[\frac{1}{t+3}\left\{e^{j\pi(t+3)}-e^{-j\pi(t+3)}\right\}+\frac{1}{t-3}\left\{e^{j\pi(t-3)}-e^{-j\pi(t-3)}\right\}\right]$$

$$=\frac{2}{\pi}\left[\frac{1}{t+3}\sin(\pi(t+3))+\frac{1}{t-3}\sin(\pi(t-3))\right]$$

P 6.14 Find the inverse FT using partial fraction expansion of

$$X(j\omega)=\frac{1}{(j\omega)^2+7j\omega+10}$$

Solution

Step 1 We will first decompose the denominator into two factors.

$$X(j\omega)=\frac{1}{(j\omega)^2+7j\omega+10}=\frac{1}{(j\omega+5)(j\omega+2)}$$

Step 2 Decompose the transfer function into component functions using the partial fraction expansion.

$$X(j\omega)=\frac{1}{(j\omega+5)(j\omega+2)}=\frac{k_1}{j\omega+5}+\frac{k_2}{j\omega+2}$$

Find k_1 and k_2

$$k_1 = \frac{1}{j\omega+2}\downarrow_{j\omega=-5} = -1/3$$

$$k_2 = \frac{1}{j\omega+5}\downarrow_{j\omega=-2} = 1/3$$

$$X(j\omega) = \frac{1/3}{j\omega+2} - \frac{1/3}{j\omega+5}$$

Step 3 Find IFT of each component term

$$X(j\omega) = \frac{1/3}{j\omega+2} - \frac{1/3}{j\omega+5}$$

$$x(t) = \frac{1}{3}(e^{-2t} - e^{-5t})u(t)$$

This result is a standard FT pair.

P 6.15 Find the inverse FT using partial fraction expansion of

$$X(j\omega) = \frac{j\omega+1}{(j\omega+2)^2}$$

Solution

Step 1 We will first decompose the denominator into two factors.

$$X(j\omega) = \frac{j\omega+1}{(j\omega+2)^2}$$

Step 2 Decompose the transfer function into component functions using the partial fraction expansion.

$$X(j\omega) = \frac{j\omega+1}{(j\omega+2)^2} = \frac{k_1}{(j\omega+2)^2} + \frac{k_2}{j\omega+2}$$

Find k_1 and k_2

$$k_1 = \frac{j\omega+1}{(j\omega+2)^2}(j\omega+2)^2 \downarrow_{j\omega=-2} = -1$$

$$k_2 = \frac{d}{d\omega}\left(\frac{j\omega+1}{(j\omega+2)^2}(j\omega+2)^2\right)\downarrow_{j\omega=-2} = 1$$

$$X(j\omega) = -\frac{1}{(j\omega+2)^2} + \frac{1}{j\omega+2}$$

Step 3 Find IFT of each component term

$$X(j\omega) = -\frac{1}{(j\omega+2)^2} + \frac{1}{j\omega+2}$$

$$x(t) = (-te^{-2t} + e^{-2t})u(t)$$

P 6.16 Find the inverse FT using partial fraction expansion of

$$X(j\omega) = \frac{1}{-\omega^2 + 3j\omega + 2}$$

Solution

Step 1 We will first decompose the denominator into two factors.

$$X(j\omega) = \frac{1}{-\omega^2 + 3j\omega + 2} = \frac{1}{(j\omega+2)(j\omega+1)}$$

Step 2 Decompose the transfer function into component functions using partial fraction expansion.

$$X(j\omega) = \frac{1}{(j\omega+2)(j\omega+1)} = \frac{k_1}{j\omega+2} + \frac{k_2}{j\omega+1}$$

Find k_1 and k_2

$$k_1 = \frac{1}{j\omega+1}\downarrow_{j\omega=-2} = -1$$

$$k_2 = \frac{1}{j\omega+2}\downarrow_{j\omega=-1} = 1$$

$$X(j\omega) = \left[\frac{1}{j\omega+1} - \frac{1}{j\omega+2}\right]$$

Step 3 Find the IFT of each component term

$$X(j\omega) = \left[\frac{1}{j\omega+1} - \frac{1}{j\omega+2}\right]$$

$$x(t) = (e^{-t} - e^{-2t})u(t)$$

This result is a standard FT pair.

P 6.17 Find the inverse FT using partial fraction expansion.

$$X(j\omega) = \frac{-(j\omega)^2 - 3j\omega - 3}{[(j\omega)^2 + 3\omega + 2](j\omega+3)}$$

Solution

Step 1 We will first decompose the denominator into two factors.

$$X(j\omega) = \frac{-(j\omega)^2 - 3j\omega - 3}{[(j\omega)^2 + 3j\omega + 2](j\omega+3)} = \frac{-(j\omega)^2 - 3j\omega - 3}{(j\omega+2)(j\omega+1)(j\omega+3)}$$

Step 2 Decompose the transfer function into component functions using partial fraction expansion.

$$X(j\omega) = \frac{-(j\omega)^2 - 3j\omega - 3}{(j\omega+2)(j\omega+1)(j\omega+3)} = \frac{k_1}{j\omega+2} + \frac{k_2}{j\omega+1} + \frac{k_3}{j\omega+3}$$

Find k_1, k_2, and k_3

$$k_1 = \frac{-(j\omega)^2 - 3j\omega - 3}{(j\omega+1)(j\omega+3)}\downarrow_{j\omega=-2} = \frac{-4+6-3}{(-1)(1)} = \frac{-1}{-1} = 1$$

$$k_2 = \frac{-(j\omega)^2 - 3j\omega - 3}{(j\omega+2)(j\omega+3)} \downarrow_{j\omega=-1} = \frac{-1+3-3}{(1)(2)} = \frac{-1}{2} = -1/2$$

$$k_3 = \frac{-(j\omega)^2 - 3j\omega - 3}{(j\omega+2)(j\omega+1)} \downarrow_{j\omega=-3} = \frac{-9+9-3}{(-1)(-2)} = \frac{-3}{2} = -3/2$$

$$X(j\omega) = \left[\frac{1}{j\omega+2} - \frac{1/2}{j\omega+1} - \frac{3/2}{j\omega+4}\right]$$

Step 3 Find IFT of each component term

$$X(j\omega) = \left[\frac{1}{j\omega+2} - \frac{1/2}{j\omega+1} - \frac{3/2}{j\omega+3}\right]$$

$$x(t) = \left(e^{-2t} - \frac{1}{2}e^{-t} - \frac{3}{2}e^{-3t}\right)u(t)$$

This result is a standard FT pair.

P 6.18 Find FT of $x[n] = \begin{cases} 1 & \text{for } 0 \le n \le 2 \\ -1 & \text{for } -2 \le n \le -1 \\ 0 & \text{otherwise} \end{cases}$

Solution

Let us use the definition of DTFT

$$X(e^{j\omega n}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-2}^{2} x[n]e^{-j\omega n} = -e^{j2\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$$

Multiply both sides by e^{-jw} and subtract from the first equation

$$X(e^{-j\omega n}) = 1 + e^{-j\omega} - e^{j\omega} + e^{-2j\omega} - e^{2j\omega}$$

$$X(e^{j\omega n}) = 1 - 2j\sin(\omega) - 2j\sin(2\omega)$$

We have to put different values of ω in the equation and find the real and imaginary parts to find the magnitude of response. Magnitude and phase can be calculated using rectangular to polar conversion. Let us write a MATLAB program to plot magnitude and phase response (Figs 6.9 and 6.10). We will use abs and angle command to find magnitude and phase.

```
clear all;
w=0:0.1:20;
w1=w/pi;
x=(1-1j*2*sin(w)-1j*2*sin(2*w);
plot(w1,abs(x));
title('magnitude response of FT');
xlabel('angular  frequency  as  multiple  of  pi');
ylabel('magnitude');
figure;
plot(w1,angle(x));
title('phase response of FT');
xlabel('anglular  frequency  as  multiple  of  pi');
ylabel('phase value');
```

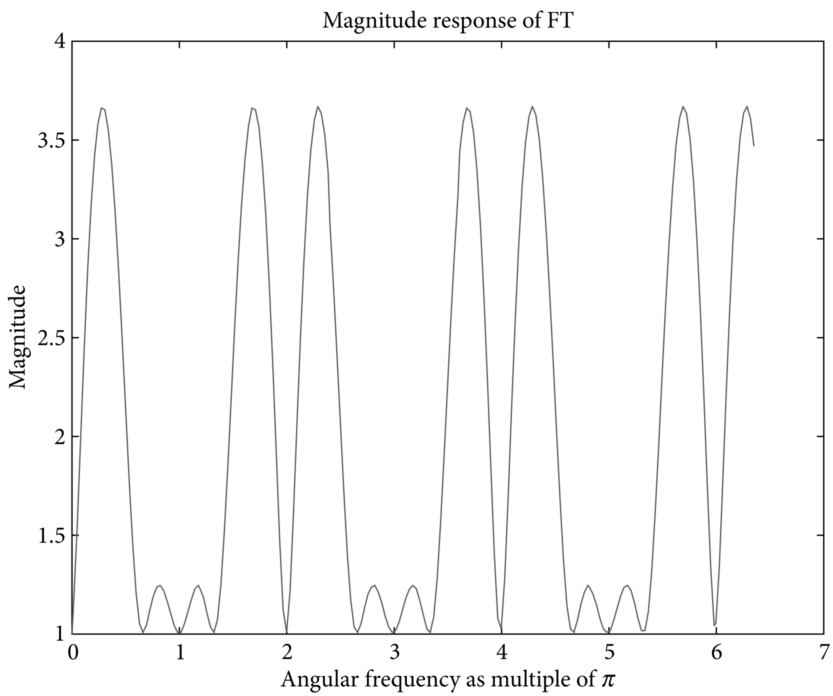


Fig. 6.9 Magnitude response with period 2π

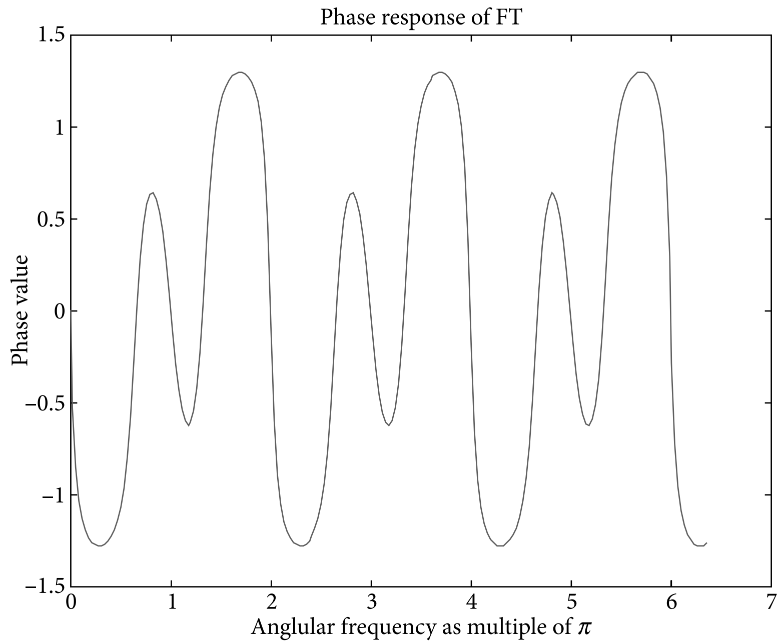


Fig. 6.10 Phase response with period 2π

P 6.19 Find FT of $x[n]=\begin{cases}1/2 & \text{for } 0\le n\le 3\\ 0 & \text{otherwise}\end{cases}$

Solution

Using the definition of DTFT

$$X(e^{j\omega n})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}=\sum_{n=0}^{3}x[n]e^{-j\omega n}=\frac{1}{2}\left[1+e^{-j\omega}+e^{-2j\omega}+e^{-3j\omega}\right]$$

$$=\frac{1}{2}(1+\cos(\omega)+\cos(2\omega)+\cos(3\omega))-\frac{1}{2}j(\sin(\omega)+\sin(2\omega)+\sin(3\omega))$$

To find the magnitude of the response using manual calculations, we have to put different values of ω in the equation and find the real and imaginary parts. Magnitude and phase can be calculated using rectangular to polar conversion. Let us write a MATLAB program to plot magnitude and phase response. We will use abs and angle command to find the magnitude and phase. Figures 6.11 and 6.12 show the magnitude and phase response. We can note that the magnitude response and phase response are both periodic with period equal to 2π.

```
clear all;
w=0:0.1:20;
w1=w/pi;
x=1/2*(1+cos(w)+cos(2*w)+cos(3*w))-1j/2*(sin(w)+sin
(2*w)+sin(3*w));
plot(w1,abs(x));
title('magnitude response of FT');
xlabel('angular frequency as multiple of pi');
ylabel('magnitude');
figure;
plot(w1,angle(x));
title('phase response of FT');
xlabel('anglular frequency as multiple of pi');
ylabel('phase value');
```

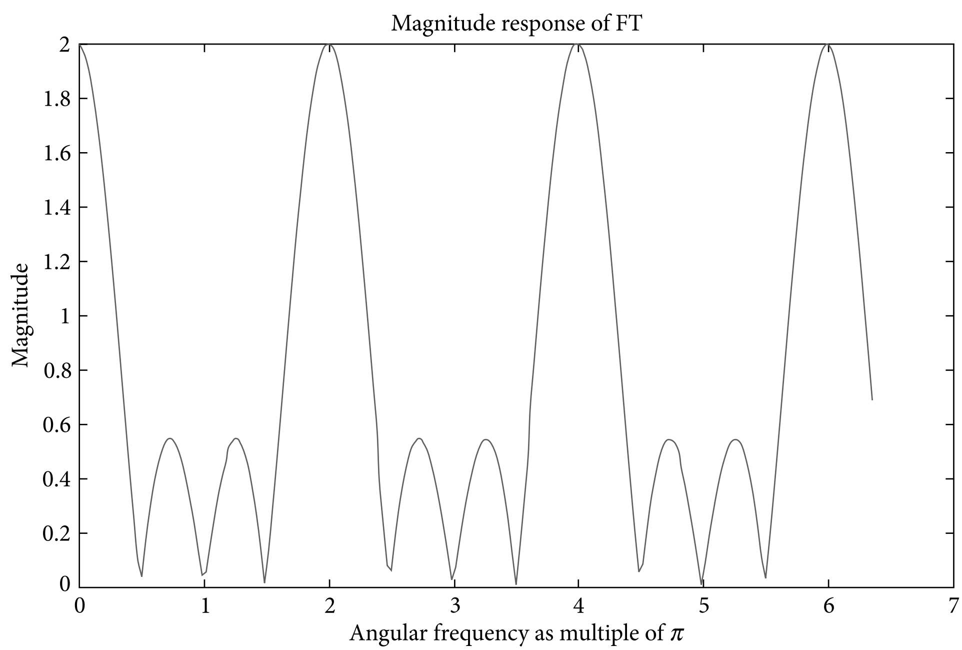


Fig. 6.11 Magnitude response with period 2π

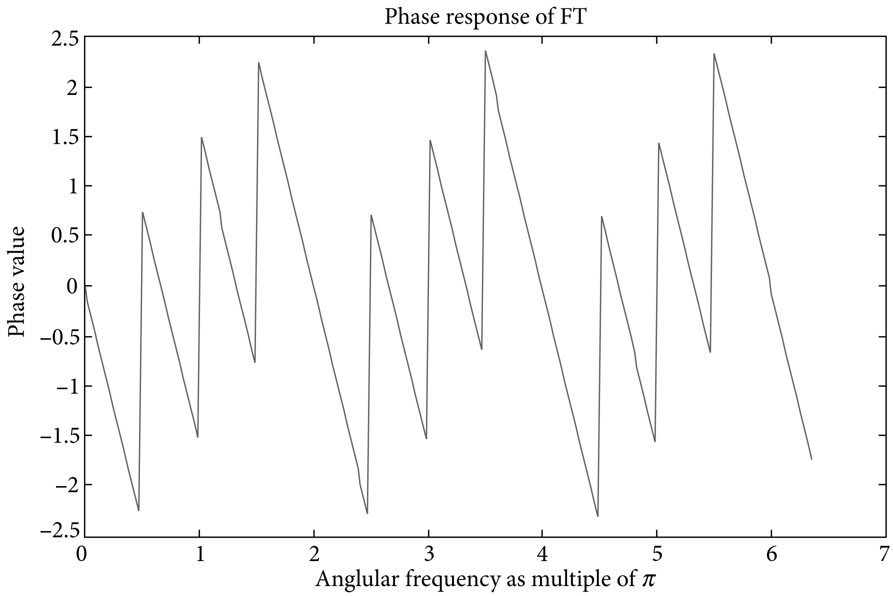


Fig. 6.12 Phase response with period 2π

P 6.20 Find inverse DTFT of $X(e^{j\omega}) = \begin{Bmatrix} 1 & |\Omega| \le \pi/2 \\ 0 & \pi/2 < |\Omega| < \pi \end{Bmatrix}$.

Solution

As DTFT is periodic with period of 2π, it is specified only between $-\pi$ to $+\pi$. We will use the formula for inverse DTFT to find the signal $x[n]$.

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\,dw = \frac{1}{2\pi nj}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right]$$

$$= \frac{\sin(\pi n/2)}{n\pi} \quad \text{for} \quad n \ne 0 \,\&\, x[n] = \frac{\sin(\pi n/2)}{\pi n/2}\frac{\pi/2}{\pi} \to \frac{1}{2} \quad \text{as} \quad n \to 0$$

Figure 6.13 shows the plot of the signal and DTFT of the signal. A MATLAB program to plot inverse DTFT of the periodic rectangular pulse in frequency domain is given here. Inverse DTFT is a DT signal which is a sinc function (aperiodic signal).

```
clear all;
w=pi/2;
A=1;
N=50;
for n=1:N,
    X(n+N+1)=(1/(n*pi))*sin(n*pi/2);
 end
X(N+1)=1/2;
for n=1:N,
    X(n)=X(2*N+2-n);
end
n1=-50:1:50;
stem(n1,X);
title('plot of magnitude of Inverse DTFT of frequency
domain rectangular pulse of width w=0.2');
xlabel('time');ylabel('amplitude');
```

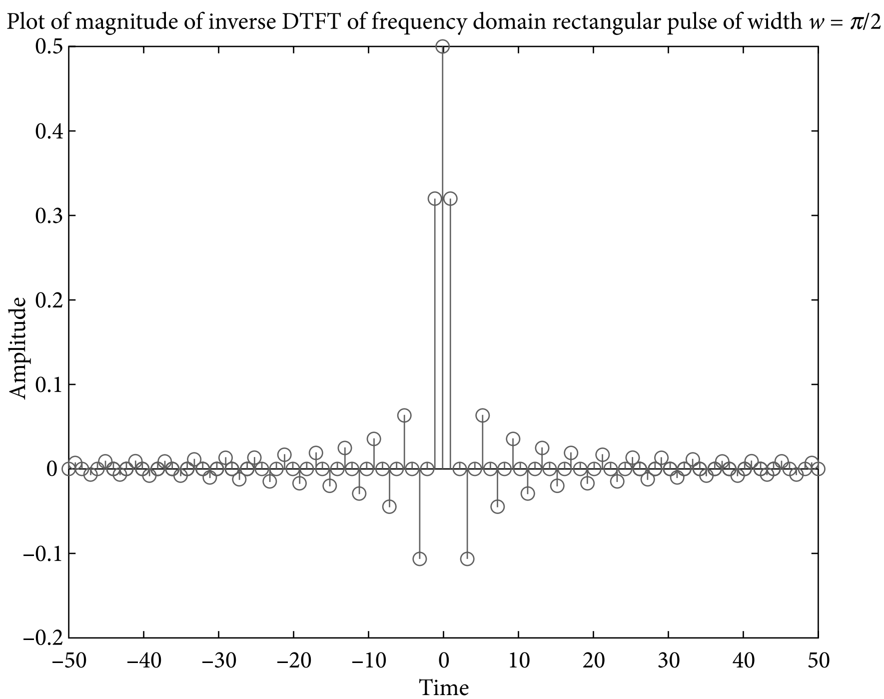


Fig. 6.13 Inverse DTFT of a periodic rectangular pulse

P 6.21 Find DTFT of $x[n] = \delta[n] + \delta[n-1]$.

Solution

We know that the unit impulse is an aperiodic DT signal.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \{\delta[n] + \delta[n-1]\} e^{-j\Omega n} = 1 + e^{-j\Omega}$$

A MATLAB program to plot the inverse DTFT of the rectangular pulse in the frequency domain is given as follows. The plot of the signal and DTFT is shown in Fig. 6.14.

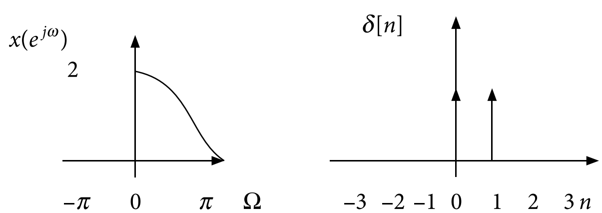


Fig. 6.14 Plot of DTFT and DT signal

```
clear all;
w=0:0.1:pi;
X=1+exp(-1j*w);
stem(w,(X));
title('plot of magnitude of Inverse DTFT of frequency
domain rectangular pulse of width w=pi/2');
xlabel('time');ylabel('amplitude');
```

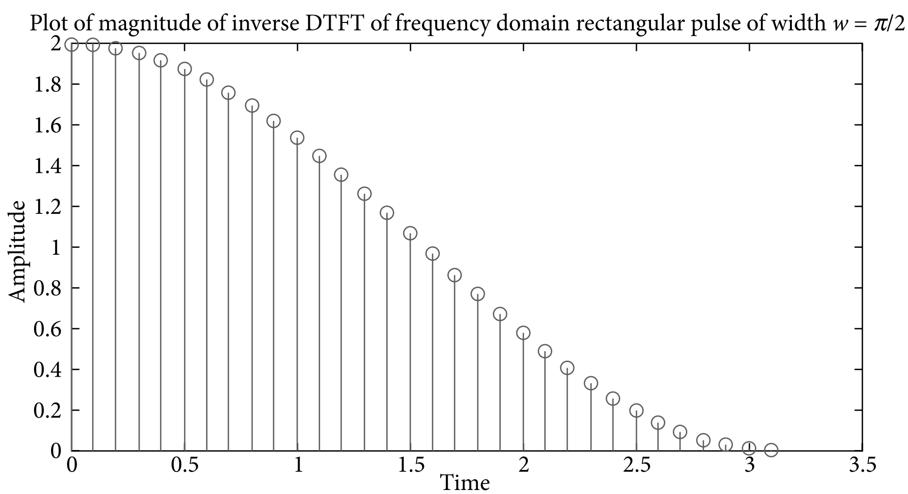


Fig. 6.15 Plot of Inverse DTFT for Problem 6.21

P 6.22 Find inverse DTFT of $X(e^{j\omega}) = \delta(\Omega) + \delta(\Omega - \pi/2)$, for $-\pi < \Omega < \pi$.

Solution

We will use the formula for inverse DTFT.

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega)e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi} + \frac{1}{2\pi}e^{j\pi n/2}$$

A MATLAB program to plot the DT signal in the frequency domain is given as follows. Fig. 6.16 shows a plot of the DT signal

Fig. 6.16 shows a plot of the DT signal.

```
clear all;
n=0:0.1:pi;
x=(1/(2*pi))*(1+exp(1j*n*pi/2));
stem(n,(x));
title('plot of DT signal for frequency domain signal
delta(w)+delta(w-pi/2)');
xlabel('time');ylabel('amplitude');
```

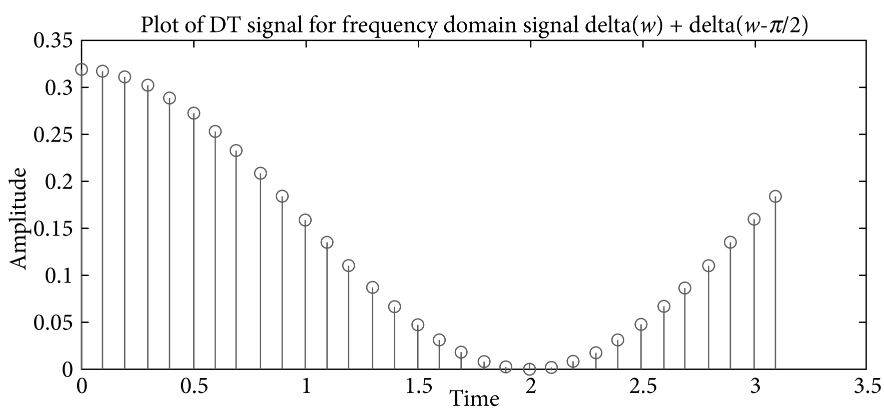


Fig. 6.16 Plot of signal

P 6.23 Find DTFT of the exponential sequence $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$.

Solution

We know that the exponential sequence is an aperiodic DT signal. Let us find DTFT using the formula.

The exponential sequence can be written as

$$x[n]=\left(\frac{1}{3}\right)^n u[n]+\left(\frac{1}{4}\right)^n u[n]$$

$$X(e^{j\Omega})=\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{-j\Omega n}+\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j\Omega n}=\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\Omega}\right)^n+\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\Omega}\right)^n$$

$$=\frac{1}{1-ae^{-j\Omega}}$$

$$X(e^{j\Omega})=\frac{1}{1-\frac{1}{3}e^{-j\Omega}}+\frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

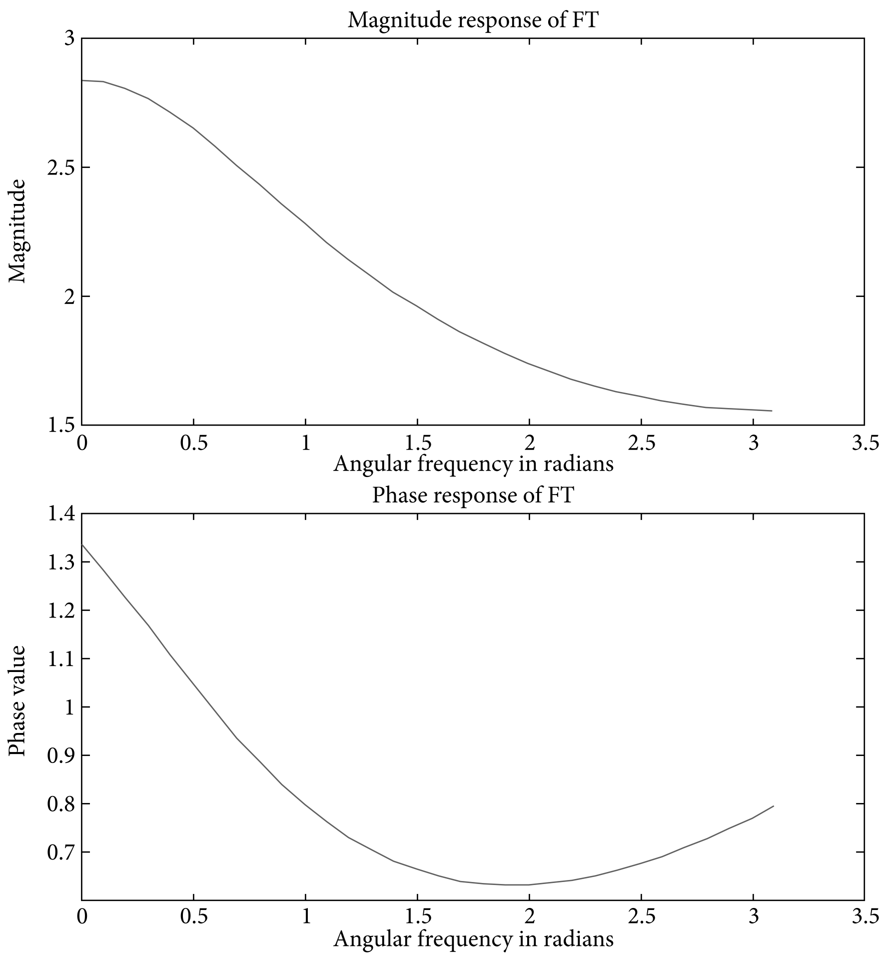


Fig. 6.17 Plot of phase response

The limits are from 0 to infinity as the sequence is appended by $u[n]$.

A MATLAB program to plot the magnitude response and phase response of the FT is given as follows.

```
clear all;
w=0:0.1:pi;
x=abs(1./(1-(1/3)*exp(-1j*w)))+abs(1./
(1-(1/4)*exp(-1j*w)));
plot(w,x);
title('magnitude response of FT');
xlabel('angular frequency in radians');
ylabel('magnitude');
figure;
a=angle(1./(1-(1/3)*exp(-1j*w)))+abs(1./
(1-(1/4)*exp(-1j*w)));
plot(w,a);
title('phase response of FT');
xlabel('anglular frequency in radians'); ylabel('phase
value');
```


P 6.24 Find inverse DTFT of $X(e^{j\Omega}) = 2\cos(3\Omega)$.

Solution

We will use the formula for inverse DTFT.

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}[e^{j3\Omega}e^{j\Omega n} + e^{-j3\Omega}e^{j\Omega n}]d\Omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{j(3+n)\Omega}d\Omega + \int_{-\pi}^{\pi} e^{j(n-3)\Omega}d\Omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(3+n)\Omega}}{j(3+n)} + \frac{e^{j(n-3)\Omega}}{j(n-3)}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{2j\sin(3+n)\pi}{j(3+n)} + \frac{2j\sin(n-3)\pi}{j(n-3)}\right]$$

$$= 2 \text{ for } n = \pm 3$$

and = 0 otherwise

P 6.25 Find inverse DTFT of $X(e^{j\Omega}) = \cos(\Omega/2) + j\sin(\Omega/2)$.

Solution

We will use the formula for inverse DTFT.

$$X(e^{j\Omega}) = \cos(\Omega/2) + j\sin(\Omega/2) = e^{j\Omega/2}$$

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega/2}e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi j(n+1/2)}[e^{j(n+1/2)\Omega}]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi j(n+1/2)}[e^{j(n+1/2)\pi} - e^{-j(n+1/2)\pi}]$$

$$= \frac{\sin(n+1/2)\pi}{(n+1/2)\pi}$$

$$= 1 \text{ for } n = -1/2$$

$$= 0 \text{ otherwise}$$

P 6.26 Find inverse DTFT of $X(e^{j\Omega}) = \begin{cases} e^{\Omega/2} & \text{for } -\pi < \Omega \le 0 \\ e^{-\Omega/2} & \text{for } 0 < \Omega \le \pi \end{cases}$.

Solution

We will use the formula for inverse DTFT.

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} e^{\Omega/2}e^{j\Omega n}d\Omega + \frac{1}{2\pi}\int_{0}^{\pi} e^{-\Omega/2}e^{j\Omega n}d\Omega$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{(jn+1/2)\Omega}d\Omega+\int_{0}^{\pi} e^{(jn-1/2)\Omega}d\Omega\right]$$

$$=\frac{1}{2\pi}\left\{\left[\frac{e^{(jn+1/2)\Omega}}{jn+1/2}\right]_{-\pi}^{0}+\left[\frac{e^{(jn-1/2)\Omega}}{jn-1/2}\right]_{0}^{\pi}\right\}$$

$$=\frac{1}{2\pi}\left[\frac{1-e^{-(jn+1/2)\pi}}{jn+1/2}+\frac{e^{(jn-1/2)\pi}-1}{jn-1/2}\right]$$

A MATLAB program to plot the DTFT and the recovered signal is given as follows. Figures 6.17 and 6.18 show the plot of DTFT and the signal and the plot of the inverse DTFT, i.e., the signal.

```
clear all;
w=-pi:0.1:pi;
x=exp(-abs(w/2));
plot(w,x);title('plot of DTFT of the signal');xlabel('angular frequency');ylabel('amplitude');
figure;
n=-31:1:31;
z=(1./(2.*pi))*((1./(1j.*n+1/2)).*(1.-exp((-(1j.*n)-1/2).*pi)))+(1./(1j.*n-1/2)).*(exp((1j.*n-1/2).*pi)-1.);
stem(n,z);title('plot of the signal');xlabel('time sample');ylabel('amplitude');
```

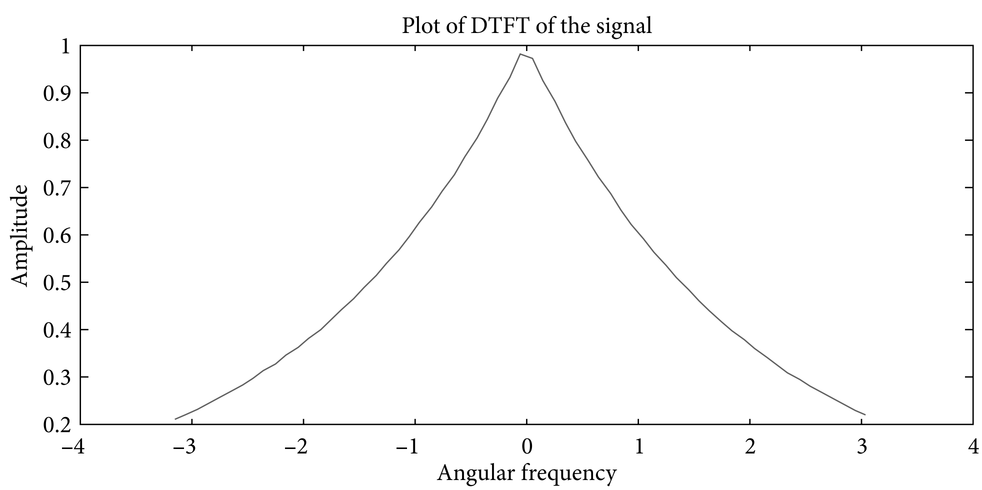


Fig. 6.18 Plot of DTFT

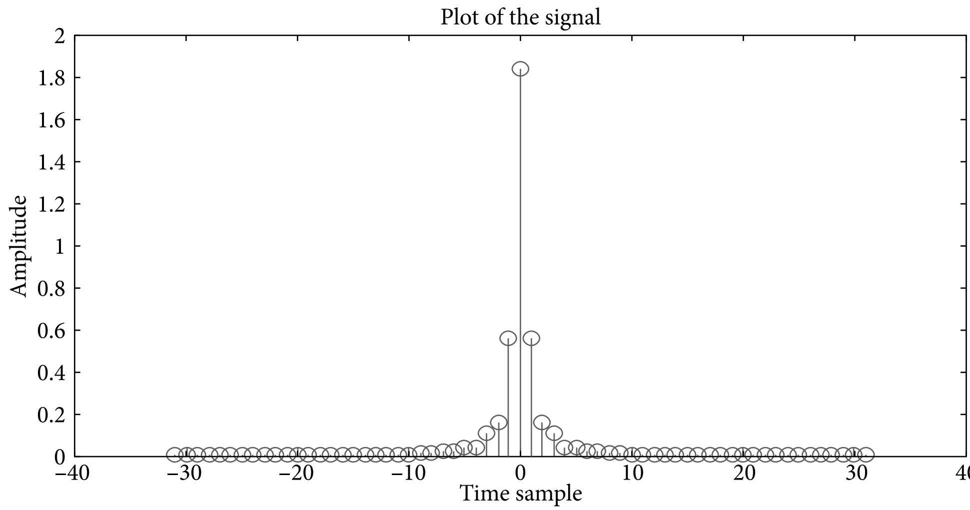


Fig. 6.19 Plot of the signal

P 6.27 Find inverse DTFT of $X(e^{j\Omega}) = \begin{cases} -\sin(3\Omega) & \text{for } -\pi < \Omega \le 0 \\ \sin(3\Omega) & \text{for } 0 < \Omega \le \pi \end{cases}$.

Solution

We will use the formula for inverse DTFT.

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -\sin(3\Omega)e^{j\Omega n}d\Omega + \frac{1}{2\pi}\int_{0}^{\pi}\sin(3\Omega)e^{j\Omega n}d\Omega$$

$$= \frac{1}{4\pi j}\left[-\int_{-\pi}^{0}\left[e^{j(n+3)\Omega} + e^{j(n-3)\Omega}\right]d\Omega + \int_{0}^{\pi}\left[e^{j(n+3)\Omega} + e^{j(n-3)\Omega}\right]d\Omega\right]$$

$$= \frac{1}{4\pi j}\left\{\left[-\frac{e^{j(n+3)\Omega}}{j(n+3)}\right]_{-\pi}^{0} - \left[\frac{e^{j(n-3)\Omega}}{j(n-3)}\right]_{-\pi}^{0} + \left[\frac{e^{j(n+3)\Omega}}{j(n+3)}\right]_{0}^{\pi} + \left[\frac{e^{j(n-3)\Omega}}{j(n-3)}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{4\pi j}\left[-\frac{1-e^{-j(n+3)\pi}}{j(n+3)} - \frac{1-e^{-j(n-3)\pi}}{j(n-3)} + \frac{e^{j(n+3)\pi}-1}{j(n+3)} + \frac{e^{j(n-3)\pi}-1}{j(n-3)}\right]$$

$$= \frac{1}{4\pi j}\left[\frac{e^{j(n+3)\pi} - e^{-j(n+3)\pi}}{j(n+3)} + \frac{e^{j(n-3)\pi} - e^{-j(n-3)\pi}}{j(n-3)}\right]$$

$$= \frac{1}{4\pi j}\left[\frac{2j\sin(n+3)\pi}{j(n+3)} + \frac{2j\sin(n-3)\pi}{j(n-3)}\right]$$

$$= \frac{1}{2\pi j}\left[\frac{\sin(n+3)\pi}{(n+3)} + \frac{\sin(n-3)\pi}{(n-3)}\right]$$

$$= 0 \text{ for } n \neq -3 \text{ and } n \neq 3$$

$$= \frac{1}{2j} \text{ for } n = \pm 3$$

P 6.28 Find inverse FT of the following signal using partial fraction expansion. Use the property of linearity.

$$X(j\omega) = \frac{2(j\omega+3)}{j\omega[(j\omega)^2 + 3j\omega + 2]}$$

Solution

Step 1 We will first decompose the denominator into two factors.

$$X(j\omega) = \frac{2(j\omega+3)}{j\omega[(j\omega)^2 + 3j\omega + 2]} = \frac{2(j\omega+3)}{j\omega(j\omega+1)(j\omega+2)}$$

Step 2 Decompose the transfer function into component functions using partial fraction expansion.

$$X(j\omega) = \frac{k_1}{j\omega} + \frac{k_2}{j\omega+1} + \frac{k_3}{j\omega+2}$$

Find k_1, k_2 and k_3

$$k_1 = \frac{2(j\omega+3)}{(j\omega+2)(j\omega+1)} \downarrow_{j\omega=0} = \frac{6}{2} = 3$$

$$k_2 = \frac{2(j\omega+3)}{j\omega(j\omega+2)}\downarrow_{j\omega=-1} = \frac{2(2)}{-1(1)} = -4$$

$$k_3 = \frac{2(j\omega+3)}{j\omega(j\omega+1)}\downarrow_{j\omega=-2} = \frac{2(-1)}{-2(-1)} = -1$$

$$X(j\omega) = \frac{3}{j\omega} - \frac{4}{j\omega+1} - \frac{1}{j\omega+2}$$

Step 3 Find IFT of each component term

$$X(j\omega) = \frac{3}{j\omega} - \frac{4}{j\omega+1} - \frac{1}{j\omega+2}$$

Using the property of linearity,

$$x(t) = 3u(t) - 4e^{-t}u(t) - e^{-2t}u(t)$$

$$x(t) = (3 - 4e^{-t} - e^{-2t})u(t)$$

P 6.29 Find FT of $x(t) = \cos(3\omega_0 t)u(t)$ using the property of frequency shifting.

Solution

Let us first find FT of $u(t)$.

$$X(j\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{-j\omega t}\,dt = -\frac{e^{-j\omega t}}{j\omega}\downarrow_0^{\infty} = \frac{1}{j\omega}$$

We will now use the frequency shifting property of FT to find FT of $x(t) = \cos(3\omega t)u(t)$

Frequency shifting property of FT states that if $x(t) \leftrightarrow X(j\omega)$ then $e^{j3\omega_0 t}x(t) \leftrightarrow X(j(\omega-\omega_0))$.

$$\cos(3\omega_0 t)u(t) = \frac{1}{2}\left[e^{j3\omega_0 t} + e^{-j3\omega_0 t}\right]u(t) \leftrightarrow \frac{1}{2}\left[\frac{1}{j(\omega-3\omega_0)} + \frac{1}{j(\omega+3\omega_0)}\right]$$

P 6.30 Find FT of $x(t) = e^{3jt}\sin(\omega_0 t)u(t)$ using the property of frequency shifting.

Solution

Frequency shifting property of FT states that if $x(t) \leftrightarrow X(j\omega)$ then $e^{j\omega_0 t}x(t) \leftrightarrow X(j(\omega - \omega_0))$

So, we evaluate FT of $x(t) = \sin(\omega_0 t)u(t)$. We will then use the frequency shifting property to find FT of the signal $x(t)$.

$$\sin(\omega_0 t)u(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]u(t) \leftrightarrow \frac{1}{2j}\left[\frac{1}{j(\omega-\omega_0)} - \frac{1}{j(\omega+\omega_0)}\right]$$

$$e^{3jt}\sin(\omega_0 t)u(t) \leftrightarrow \frac{1}{2j}\left[\frac{1}{j(\omega-\omega_0+3)} - \frac{1}{j(\omega+\omega_0+3)}\right]$$

P 6.31 Find FT of $x(t) = \sin(2\pi f_0(t-5))u(t)$ using the property of time shifting.

Solution

The time shifting property states that if $x(t) \leftrightarrow X(j\omega)$ then $x(t - t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$

We know that

$$\sin(2\pi f_0 t)u(t) \leftrightarrow \frac{\omega_0}{(j\omega)^2 + \omega_0^2}$$

$$\sin(2\pi f_0(t-5))u(t) \leftrightarrow \frac{\omega_0 e^{-j5\omega}}{(j\omega)^2 + \omega_0^2}$$

P 6.32 Use the time shifting property to find FT of the rectangular pulse shown in Fig. 6.20.

Solution

We note that $y(t) = x(t - T)$

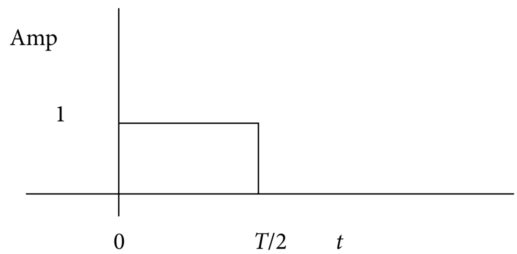


Fig. 6.20 Signal for P 6.32

We use the time shifting property to find $Y(j\omega)$

$$X(j\omega) = \frac{2}{\omega}\sin(\omega T/2)$$

$$Z(j\omega) = e^{-j\omega T/4}\,\frac{2}{\omega}\sin(\omega T/2)$$

P 6.33 Use the frequency shifting property to find inverse DTFT of $Z(j\omega) = \dfrac{1}{1+5e^{j(\Omega+\pi/3)}}$

Solution

We know the following result

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{j\Omega}} \qquad (-5)^n u[n] \leftrightarrow \frac{1}{1+5e^{j\Omega}}$$

We will now use the property of frequency shifting.

$$(-5)^n u[n] \leftrightarrow \frac{1}{1+5e^{j\Omega}} \Rightarrow (-5)^n e^{-j\pi n/3} u[n] \leftrightarrow \frac{1}{1+5e^{j(\Omega+\pi/3)}}$$

P 6.34 Use the frequency differentiation property to find FT of

$x(t) = t\cos(10\pi t)u(t)$.

Solution

The frequency differentiation property states that if $x(t) \leftrightarrow X(j\omega)$ then $tx(t) \leftrightarrow j\dfrac{d}{d\omega}[X(j(\omega))]$

We know that

$$x(t)=\cos(10\pi t)u(t)=\frac{1}{2}\left[e^{j10\pi t}+e^{-j10\pi t}\right]\leftrightarrow\frac{1}{2}\left[\frac{1}{j(\omega-10\pi)}+\frac{1}{j(\omega+10\pi)}\right]$$

We will now use the frequency differentiation property to find FT of $x(t)=t\cos(10\pi t)u(t)$.

$$\cos(10\pi t)u(t)\leftrightarrow\frac{1}{2}\left[\frac{1}{j(\omega-10\pi)}+\frac{1}{j(\omega+10\pi)}\right]$$

$$t\cos(10\pi t)u(t)\leftrightarrow j\frac{d}{d\omega}\left[\frac{1}{2}\left(\frac{j}{j(\omega-10\pi)}+\frac{j}{j(\omega+10\pi)}\right)\right]$$

$$=\left[-\frac{1}{2}\left(\frac{1}{(\omega-10\pi)^2}+\frac{1}{(\omega+10\pi)^2}\right)\right]$$

P 6.35 Use the time differentiation property to find FT of $\frac{d}{dt}(\cos(at)u(t))$.

Solution

We know that $e^{-at}u(t)\leftrightarrow\frac{1}{j\omega+a}$. We will now use the time differentiation property

$$\cos(at)u(t)=\frac{1}{2}\left[e^{jat}+e^{-jat}\right]u(t)\leftrightarrow\frac{1}{2}\left[\frac{1}{j(\omega-a)}+\frac{1}{j(\omega+a)}\right]$$

then $\frac{d}{dt}(\cos(at)u(t))\leftrightarrow\frac{1}{2}\left[\frac{j\omega}{j(\omega-a)}+\frac{j\omega}{j(\omega+a)}\right]$

P 6.36 Use the differentiation in frequency, time scaling property to find IFT of

$$X(j\omega)=j\frac{d}{d\omega}\left(\frac{e^{-3j\omega}}{2+j\omega}\right).$$

Solution

We know that using the time shifting property

$$x(t) \leftrightarrow X(j\omega)$$

then $x(t-t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$

$$e^{-at}u(t) \leftrightarrow \frac{1}{j\omega + a}$$

So, $e^{-2t}u(t) \leftrightarrow \dfrac{1}{j\omega + 2}$

We will use time shifting property

$$e^{-2(t-3)}u(t) \leftrightarrow \frac{e^{-3j\omega}}{j\omega + 2}$$

We will now use the frequency differentiation property which states that

if $x(t) \leftrightarrow X(j\omega)$ then $tx(t) \leftrightarrow j\dfrac{d}{d\omega}[X(j(\omega))]$

$$te^{-2(t-3)}u(t) \leftrightarrow j\frac{d}{d\omega}\left(\frac{e^{-3j\omega}}{2 + j\omega}\right)$$

P 6.37 Use the result of FT for a rectangular pulse of amplitude 1 between –1 to 1 and find FT of the scaled rectangular pulse of amplitude 2 between –1/2 to 1/2.

Solution

We know that the FT of a rectangular pulse of width $2T$ (between $-T$ to T) is given by

$$X(j\omega) = \frac{2}{\omega}\sin(\omega T)$$

Put $T = 1$.

$$X(j\omega) = \frac{2}{\omega}\sin(\omega)$$

To find FT of a scaled rectangular pulse with a scaling factor of 2, we will use the property of scaling for FT to get

$$y(t) = x(2t)$$

$$Y(j\omega) = 2X(j\omega/2),$$

$$Y(j\omega) = \frac{8}{\omega}\sin(\omega/2)$$

P 6.38 Find DTFT of the signal $x[n]$ and use the result of the property of scaling to find DTFT of $y[n]$ shown in Fig. 6.21.

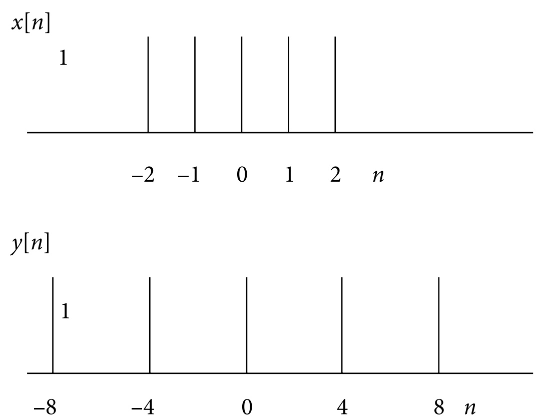


Fig. 6.21 The signal $x[n]$ and its scaled version $y[n]$

Solution

$$X(e^{j\Omega}) = \sum_{n=-M}^{M} x[n]e^{-j\Omega n} = \sum_{n=-2}^{2} x[n]e^{-j\Omega n} = e^{j\Omega} + e^{2j\Omega} + 1 + e^{-j\Omega} + e^{-2j\Omega}$$

$$X(e^{j\Omega}) = 1 + 2\cos(\Omega) + 2\cos(2\Omega)$$

Put $m = M + n$

It can also be written as

$$X(e^{j\Omega}) = \sum_{m=0}^{2M} x[m] e^{-j\Omega(m-M)} = e^{j\Omega M} \sum_{m=0}^{2M} e^{-j\Omega m}$$

$$X(e^{j\Omega}) = e^{j\Omega M}[1 + e^{-j\Omega} + e^{-j2\Omega} .... + e^{-j2M\Omega}]$$

$$(1 - e^{-j\Omega}) X(e^{j\Omega}) = e^{j\Omega M}(1 - e^{-j(2M+1)\Omega})$$

$$X(e^{j\Omega}) = e^{j\Omega M} \frac{1 - e^{-j(2M+1)\Omega}}{1 - e^{-j\Omega}}$$

$$= e^{j\Omega M} \frac{e^{-j(2M+1)\Omega/2} e^{j(2M+1)\Omega/2} - e^{-j(2M+!)\Omega/2} e^{-j(2M+1)\Omega/2}}{e^{-j\Omega/2} e^{j\Omega/2} - e^{-j\Omega/2} e^{-j\Omega/2}}$$

$$= e^{jM\Omega} \frac{e^{-j(2M+1)\Omega/2}(2j\sin((2M+1)\Omega/2)}{e^{-j\Omega/2}(2j\sin(\Omega/2))}$$

$$X(e^{j\Omega}) = \frac{\sin((2M+1)\Omega/2)}{\sin(\Omega/2)} \text{ for } \Omega \neq 0, \pm 2\pi, \pm 4\pi....\text{etc}$$

$$= (2M+1) \text{ for } \Omega = 0, \pm 2\pi, \pm 4\pi....\text{etc}$$

We will now use the property of scaling to obtain DTFT of $y[n]$. Here, scaling factor is ½.

$$x[n] \leftrightarrow 1 + 2\cos(\Omega) + 2\cos(2\Omega)$$

then $y[n] = x[n/4] \leftrightarrow \frac{1}{|a|} X(j\Omega/a) = 4(1 + 2\cos(4\Omega) + 2\cos(8\Omega))$

$$Y(e^{j\Omega}) = 4 + 8\cos(4\Omega) + 8\cos(8\Omega)$$

E 6.39 Show that the DTFT of $x[n] = ne^{j(\pi/4)} a^{n-2} u[n-2]$ is

$$X(e^{j\Omega}) = j\frac{d}{d\Omega}\left\{\frac{e^{-2j(\Omega-\pi/4)}}{1 - ae^{-j(\Omega-\pi/4)}}\right\}$$

Solution

We will use the result

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\Omega}}$$

We will use the time shifting property

$$x[n] = a^n u[n] \leftrightarrow X(e^{j\Omega}) = \frac{1}{1-ae^{-j\Omega}}$$

$$\text{then } x[n-2] = a^{n-2}u[n-2] \leftrightarrow e^{-j2\Omega} \big/ (1-ae^{-j\Omega})$$

We will use frequency shifting property

$$\text{If } a^{n-2}x[n-2] \leftrightarrow \frac{e^{-j2\Omega}}{1-ae^{-j\Omega}} \quad \text{then } e^{-j\pi/4}a^{n-2}x[n-2] \leftrightarrow \frac{e^{-j2(\Omega-\pi/4)}}{1-ae^{-j(\Omega-\pi/4)}}$$

We will use frequency differentiation property to get

$$\text{If } e^{-j\pi/4}a^{n-2}x[n-2] \leftrightarrow \frac{e^{-2j(\Omega-\pi/4)}}{1-ae^{-j(\Omega-\pi/4)}}$$

$$\text{then } ne^{-j\pi/4}a^{n-2}x[n-2] \leftrightarrow j\frac{d}{d\Omega}\left[\frac{e^{-2j(\Omega-\pi/4)}}{1-ae^{-j(\Omega-\pi/4)}}\right]$$

E 6.55 Use the property of convolution in frequency or property of modulation to find inverse DTFT of

$$X(e^{j\Omega}) = \frac{e^{-3j\Omega}}{1+\frac{1}{2}e^{-j\Omega}} * \frac{\sin(13\Omega/2)}{\sin(\Omega/2)}$$

Solution

Property of modulation states that

$$x_1[n] \leftrightarrow X_1(e^{j\Omega}) \text{ and } x_2[n] \leftrightarrow X_2(e^{j\Omega})$$

$$\text{then } x_1[n] \times x_2[n] \leftrightarrow X_1(e^{j\Omega})(*)X_2(e^{j\Omega})$$

(*) denotes circular convolution as DTFT is periodic

We will find inverse DTFT of the two individual terms that are convolved. We will use the result

$$a^n u[n] \leftrightarrow \frac{1}{1-ae^{-j\Omega}}$$

$$\left(-\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{1}{1+\frac{1}{2}e^{-j\Omega}}$$

We will use the time shifting property

$$\left(-\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{1}{1+\frac{1}{2}e^{-j\Omega}}$$

$$\text{then } \left(-\frac{1}{2}\right)^{n-3} u[n-3] \leftrightarrow e^{-j3\Omega} / \left(1+\frac{1}{2}e^{-j\Omega}\right)$$

We will now find inverse DTFT of the second term.

$$x[n] \leftrightarrow X(e^{j\Omega}) = \frac{\sin((2M+1)\Omega/2)}{\sin(\Omega/2)} \text{ for } \Omega \neq 0, \pm 2\pi, \pm 4\pi....\text{etc}$$

$$= (2M+1) \text{ for } \Omega = 0, \pm 2\pi, \pm 4\pi....\text{etc}$$

Here, $x[n]$ is a rectangular wave sequence between $-M$ to $+M$

$$\frac{\sin(13\Omega/2)}{\sin(\Omega/2)} \leftrightarrow y[n] \text{ is a rectangular wave sequence between } -6 \text{ to } +6$$

$$\text{So, } y[n] = \begin{cases} 1 & \text{for } -6 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

The inverse DTFT of the given convolved signal is now the multiplication of their inverse DTFTs.

$$\left(-\frac{1}{2}\right)^{n-3} u[n-3] \leftrightarrow e^{-j3\Omega} / \left(1+\frac{1}{2}e^{-j\Omega}\right)$$

$$y[n] \leftrightarrow \frac{\sin(13\Omega/2)}{\sin(\Omega/2}$$

Inverse DTFT of the convolution of two transforms is

$$\left(-\frac{1}{2}\right)^{n-3} u[n-3] \times \{u[n+6] - u[n-7]\}$$

$$= \left(-\frac{1}{2}\right)^{n-3} \{u[n-3] - u[n-7]\}$$

E 6.56 Use the property of duality to find FT of $x(t) = \dfrac{1}{1+jt}$.

Solution

We know the following result.

$$x(t) = e^{-t}u(t) \leftrightarrow \frac{1}{1+j\omega}$$

Replace ω by t. We get

$$x(jt) = \frac{1}{1+jt} \leftrightarrow 2\pi x(-\omega) = 2\pi e^{\omega} u(-\omega)$$

E 6.57 Use Parseval's theorem to evaluate the sum $\sum_{n=-\infty}^{\infty} \dfrac{\sin^2(4n)}{\pi^2 n^2}$.

Solution

Consider the DT signal

$$x[n] = \frac{\sin(4n)}{\pi n} \leftrightarrow X(e^{j\Omega}) = \begin{cases} 1 & \text{for } |\Omega| \le 4 \\ 0 & \text{for } 4 < |\Omega| < \pi \end{cases}$$

We can write

$$\text{sum} = \sum_{n=\infty}^{\infty} \frac{\sin^2(4n)}{\pi^2 n^2} = \sum_{-\infty}^{\infty} \{x[n]\}^2$$

Using Parseval's theorem

$$\text{sum} = \frac{1}{2\pi}\int_{-4}^{4} |X(e^{j\Omega})|^2\, d\Omega = \frac{1}{2\pi}\Omega \downarrow_{-4}^{4} = 4/\pi$$

E 6.58 Let the impulse response of the system be given by $h(t) = e^{-2t}u(t-2)$ and the input to the system be $x(t) = e^{-4t}u(t)$. Find the output of the system.

Solution

Let us first find FT of the input signal $x(t)$.

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$$

$$x(t) = e^{-4t}u(t) \leftrightarrow \frac{1}{4+j\omega}$$

We will now find FT of the impulse response.

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$$

$$e^{-2t}u(t) \leftrightarrow \frac{1}{2+j\omega}$$

$$e^{-2t}u(t-2) = e^{-4}e^{-2(t-2)}u(t-2) \leftrightarrow \frac{e^{-4}e^{-2j\omega}}{2+j\omega}$$

We have used the time shifting property of FT

Let us multiply the two transforms.

$$\frac{1}{4+j\omega} \times \frac{e^{-4}e^{-2j\omega}}{2+j\omega} = \frac{k_1}{4+j\omega} + \frac{k_2}{2+j\omega}$$

We have used partial fraction expansion

$$k_1 = \frac{e^{-4}e^{-2j\omega}}{2+j\omega} \downarrow_{j\omega=-4} = \frac{e^{-4}e^{8}}{-2} = -\frac{e^{4}}{2}$$

$$k_2 = \frac{e^{-4}e^{-2j\omega}}{4+j\omega}\downarrow_{j\omega=-2} = \frac{e^{-4}e^{4}}{2} = \frac{1}{2}$$

Taking inverse FT of both terms

$$\text{Output} = -\frac{e^4}{2}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t)$$

E 6.59 The output of a system in response to an input $x(t) = e^{-3t}u(t)$ is $y(t) = e^{-t}u(t)$. Find the frequency response and impulse response of the system.

Solution

We will find FT of the input as well as the input.

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$$

$$x(t) = e^{-3t}u(t) \leftrightarrow \frac{1}{3+j\omega}$$

$$y(t) = e^{-t}u(t) \leftrightarrow \frac{1}{1+j\omega}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3+j\omega}{1+j\omega} = 1 + \frac{2}{1+j\omega}$$

The frequency response can be evaluated by putting different values of ω in the equation. Let us now find the inverse FT to get the impulse response of the system.

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3+j\omega}{1+j\omega} = 1 + \frac{2}{1+j\omega}$$

$$h(t) = \delta(t) + 2e^{-t}u(t)$$

E 6.60 If a system is described by the differential equation given by

$$\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=2\frac{d}{dt}x(t)+x(t),$$

find the frequency response and input response of the system.

Solution

To find the frequency response, we will put $\frac{d^2}{dt^2}=(j\omega)^2, \frac{d}{dt}=j\omega$ in the differential equation $\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=2\frac{d}{dt}x(t)+x(t)$ to get

$$(j\omega)^2Y(j\omega)+5(j\omega)Y(j\omega)+6Y(j\omega)=2(j\omega)X(j\omega)+X(j\omega)$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{2j\omega+1}{(j\omega)^2+5j\omega+6}=\frac{k_1}{j\omega+3}+\frac{k_2}{j\omega+2}$$

$$k_1=\frac{2j\omega+1}{j\omega+2}\downarrow_{j\omega=-3}=\frac{-6+1}{-3+2}=5$$

$$k_2=\frac{2j\omega+1}{j\omega+3}\downarrow_{j\omega=-2}=\frac{-4+1}{-2+3}=-3$$

$$H(j\omega)=\frac{5}{j\omega+3}-\frac{3}{j\omega+2}$$

$$h(t)=5e^{-3t}u(t)-3e^{-2t}u(t)$$

E 6.61 If the impulse response of a system is given by $x(t)=\frac{4}{\pi^2t^2}\sin^2(2t)$, find the frequency response.

Solution

We will use the modulation property of FT to get the FT of $x(t)$.

$$x(t)=\frac{4}{\pi^2t^2}\sin^2(2t)=\left(\frac{2}{\pi t}\sin(2t)\right)^2$$

We know the result

$$\frac{\sin(wt)}{\pi t} \leftrightarrow X(j\omega) = \begin{cases} 1 & \text{for } -w \le \omega \le w \\ 0 & \text{for } |\omega| > w \end{cases}$$

$$2\frac{\sin(2t)}{\pi t} = x_1(t) \leftrightarrow X(j\omega) = \begin{cases} 2 & \text{for } -2 \le \omega \le 2 \\ 0 & \text{for } |\omega| > 2 \end{cases}$$

$$x(t) \leftrightarrow X(j\omega) * X(j\omega)$$

We need to convolve the two rectangular window functions.

The transform $X(j\omega)$ is shown plotted in Fig. 6.22. The same signal is convolved with itself to get the result of the convolution as shown in Fig. 6.23.

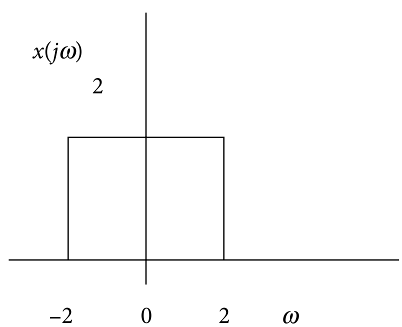


Fig. 6.22 Signal $X(j\omega)$

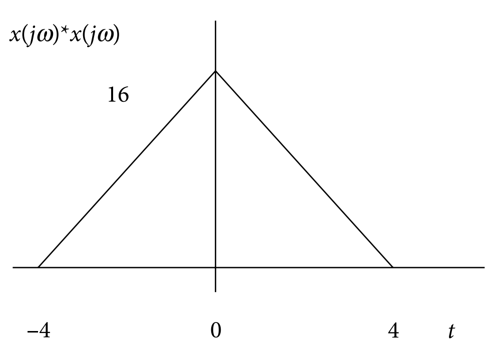


Fig. 6.23 Result of convolution of $X(j\omega)$ with itself

7

Laplace Transform Solution

P 7.1 Find LT of the function $x(t) = t\delta(t)$.

Solution

We will use the property of differentiation in the S domain.

$$\delta(t) \leftrightarrow 1$$

$$t\delta(t) \leftrightarrow -\frac{d}{ds}(1) = 0$$

P 7.2 Find LT of the function $x(t) = (t - 2)u(t)$.

Solution

We will use the definition of LT.

$$X(s) = \int_{-\infty}^{\infty} (t-2)u(t)e^{-st}\,dt = LT(tu(t) - 2u(t) = \frac{1}{s^2} - \frac{2}{s} = \frac{(1-2s)}{s^2} \text{ for all } \mathrm{Re}(s) > 0$$

ROC is the entire right half S plane excluding $s = 0$

P 7.3 Find LT of the function $x(t) = e^{j3t}u(t)$ and $x(t) = e^{-j3t}u(t)$.

Solution

We will use the definition of LT.

$$X(s) = \int_{-\infty}^{\infty} e^{j3t}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{j3t}e^{-st}\,dt = -\frac{e^{-(s-3j)t}}{s-3j}\downarrow_0^{\infty}$$

$$= \frac{1}{s-3j} \text{for all } \mathrm{Re}(s) > 0$$

ROC is the right half S plane with Re(s) > 0

The function has a pole at $s = j3$ which lies on the vertical line passing through zero, i.e., the imaginary axis.

Note: $s = \sigma + j\omega = 0 + j\omega_0$.

We will equate the real parts on both sides and the imaginary parts on both sides. We find that the pole is on the imaginary axis at $j\omega = j3$ and the ROC is the plane on the right side of the imaginary axis.

The pole and the ROC are plotted in Fig. 7.1.

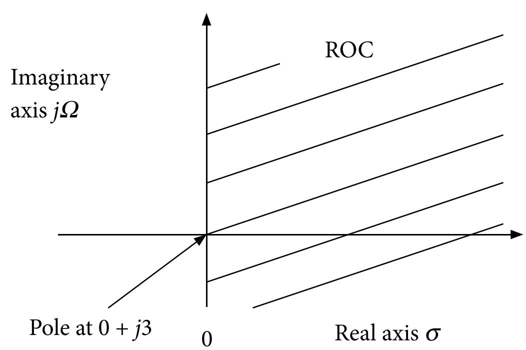


Fig. 7.1 Plot of ROC and a pole for $X(s)$ for $x(t) = e^{j3t}u(t)$ in P 7.3

Let us find LT of the other function.

$$X(s) = \int_{-\infty}^{\infty} e^{-j3t} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-j3t} e^{-st} dt = -\frac{e^{-(s+3j)t}}{s+j\omega_0} \downarrow_0^\infty$$

$$= \frac{1}{s+j3} \text{for all } \mathrm{Re}(s) > 0$$

ROC is the right half S plane with Re(s) > 0

The function has a pole at $s = -j3$ which lies on the vertical line passing through zero.

The pole and the ROC are plotted in Fig. 7.2.

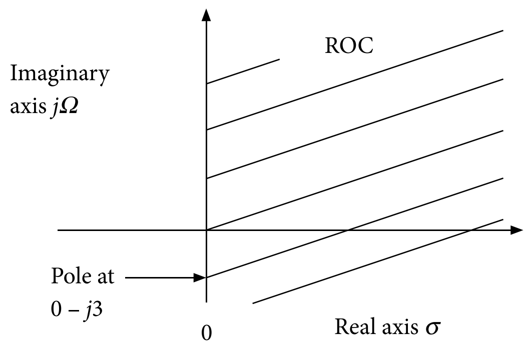


Fig. 7.2 Plot of ROC and a pole for $X(s)$ for $x(t) = e^{-j3t}u(t)$ in P 7.3

Note: Both the functions have same ROC but different LTs.

P 7.4 Find LT of the sine and cosine function $x(t) = \cos(4\pi t)u(t)$ and $x(t) = \sin(4\pi t)u(t)$.

Solution

We will use the definition of LT.

$$X(s) = \int_{-\infty}^{\infty} \cos(4\pi t)u(t)e^{-st}dt = \int_{0}^{\infty} \cos(4\pi t)e^{-st}dt$$

$$= \int_{0}^{\infty} \frac{e^{j4\pi t} + e^{-j4\pi t}}{2} e^{-st}dt = \frac{1}{2}\left[\frac{e^{-[s-j4\pi]t}}{-[(s-j4\pi]}\right]\downarrow_0^\infty + \left[\frac{e^{-[s+j4\pi]t}}{-[s+j4\pi]}\right]\downarrow_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(s-j4\pi)} + \frac{1}{(s+j4\pi)}\right] \text{for all Re}(s) > 0$$

ROC is the right half S plane with Re(s) > 0

$$X(s) = \frac{s}{(s^2 + 16\pi^2)}$$

We will find LT of the other function.

$$X(s) = \int_{-\infty}^{\infty} \sin(4\pi t)u(t)e^{-st}dt = \int_{0}^{\infty} \sin(4\pi t)e^{-st}dt$$

$$= \int_{0}^{\infty} \frac{e^{j4\pi t} - e^{-j4\pi t}}{2j} e^{-st}dt = \frac{1}{2j}\left[\frac{e^{-(s-j4\pi)t}}{-(s-j4\pi)}\right]\downarrow_0^\infty - \left[\frac{e^{-(s+j4\pi)t}}{-(s+j4\pi)}\right]\downarrow_0^\infty$$

$$=\frac{1}{2j}\left[\frac{1}{(s-j4\pi)}-\frac{1}{(s+j4\pi)}\right]\text{for all Re}(s)>0$$

ROC is the right half S plane with Re(s) > 0

$$X(s)=\frac{1}{2j}\left[\frac{s+j4\pi-s+j4\pi}{(s^2+16\pi^2)}\right]=\left[\frac{4\pi}{(s^2+16\pi^2)}\right]$$

P 7.5 Find LT of the damped sine and cosine function $x(t)=e^{-2t}\sin(2t)u(t)$ and $x(t)=e^{-2t}\cos(2t)u(t)$.

Solution

We will use the definition of LT.

$$X(s)=\int_{-\infty}^{\infty}e^{-2t}\sin(2t)u(t)e^{-st}dt=\int_{0}^{\infty}e^{-2t}\sin(2t)e^{-st}dt$$

$$=\int_{0}^{\infty}e^{-2t}\frac{e^{j2t}-e^{-j2t}}{2j}e^{-st}dt=\frac{1}{2j}\left\{\left[\frac{e^{-[(s+2)-j2]t}}{-[(s+2)-j2]}\right]\downarrow_0^\infty-\left[\frac{e^{-[(s+2)+j2]t}}{-[(s+2)+j2]}\right]\right\}_0^\infty$$

$$=\frac{1}{2j}\left[\frac{1}{(s+2)-j2}-\frac{1}{(s+2)+j2}\right]\text{for all Re}(s)>-2$$

ROC is the right half S plane with Re(s) >–2

$$X(s)=\frac{1}{2j}\left[\frac{s+2+j2-s-2+j2}{(s+a)^2+\omega_0^2}\right]=\left[\frac{2}{(s+2)^2+4}\right]$$

There are two poles. One at $s=-2+j2$ and the other at $s=-2-j2$. The ROC and the pole plot is shown in Fig. 7.3.

Let us find LT of the other function.

$$X(s)=\int_{-\infty}^{\infty}e^{-2t}\cos(2t)u(t)e^{-st}dt=\int_{0}^{\infty}e^{-2t}\cos(2t)e^{-st}dt$$

$$=\int_{0}^{\infty}e^{-2t}\frac{e^{j2t}+e^{-j2t}}{2}e^{-st}dt=\frac{1}{2}\left\{\left[\frac{e^{-[(s+2)-j2]t}}{-[(s+2)-j2]}\right]\downarrow_0^\infty+\left[\frac{e^{-[(s+2)+j2]t}}{-[(s+2)+j2]}\right]\right\}_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(s+2)-j2} + \frac{1}{(s+2)+j2}\right] \text{ for all } \operatorname{Re}(s) > -2$$

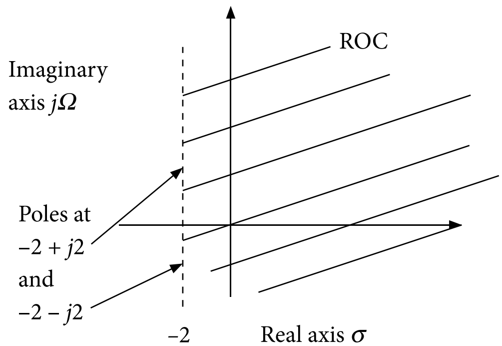


Fig. 7.3 Plot of ROC and poles for LT of the damped sine function

ROC is the right half S plane with Re(s) > −2

$$X(s) = \frac{(s+2)}{(s+2)^2 + 4}$$

The ROC and the pole plot are shown in Fig. 7.4.

Note that ROCs and the pole zero plots for damped sin and cosine functions are the same but they have different LTs.

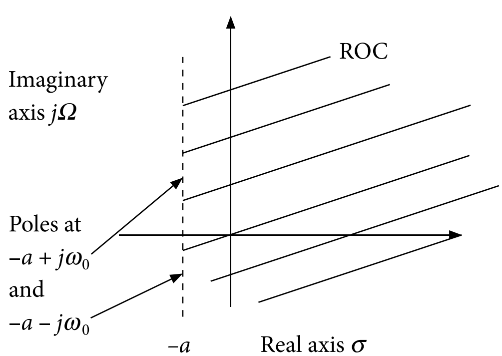


Fig. 7.4 Plot of ROC and poles for LT of the damped cosine function

P 7.6 Find LT of the both-sided signal $x(t) = e^{-5t}u(t) + e^{-2t}u(-t)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S)=\int_{-\infty}^{\infty} e^{-5t}u(t)e^{-st}dt+\int_{-\infty}^{\infty} e^{-2t}u(-t)e^{-st}dt$$

$$=\int_{0}^{\infty} e^{-5t}e^{-st}dt+\int_{-\infty}^{0} e^{-2t}e^{-st}dt$$

$$=-\frac{e^{-(s+5)t}}{s+5}\downarrow_{0}^{\infty}-\frac{e^{-(s+2)t}}{s+2}\downarrow_{-\infty}^{0}$$

$$=\frac{1}{s+5}-\frac{1}{s+2}=\frac{s+2-s-5}{s^2+7s+10}=\frac{-3}{s^2+7s+10}$$

The first term converges for all $s + 5 > 0$, i.e., $\mathrm{Re}(s) > -5$

ROC is the right half S plane with $\mathrm{Re}(s) > -5$

The second term converges for all $-s -2 > 0$, i.e., $\mathrm{Re}(s) > -2$

ROC is the left half S plane with $\mathrm{Re}(s) < -2$

The common area of convergence exists from –5 to –2. ROC is shown plotted in Fig. 7.5. Note that ROC is a strip parallel to the imaginary axis.

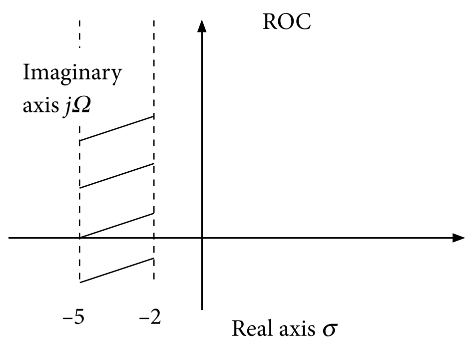


Fig. 7.5 ROC of $x(t)=e^{-5t}u(t)+e^{-2t}u(-t)$

P 7.7 Find LT of the signal $x(t) = u(u - 7)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S)=\int_{-\infty}^{\infty} u(t-4)e^{-st}dt=\int_{7}^{\infty} e^{-st}dt=\frac{e^{-st}}{-s}\downarrow_{7}^{\infty}$$

$$= \frac{e^{-7s}}{s} \text{ for all } \mathrm{Re}(s) > 0$$

ROC is the right half S plane with Re(s) > 0

Note: When $s < 0$, the integral → ∞

There is a pole at $s = 0$.

Note: The LT function has a pole at $s = 0$ and a zero at $s = \infty$

P 7.8 Find LT of $x(t) = e^{-5t}u(t-5)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} e^{-5t}u(t-5)e^{-st}\,dt = \int_{5}^{\infty} e^{-5t}e^{-st}\,dt = \frac{e^{-(s+5)t}}{-(s+5)} \downarrow_{5}^{\infty}$$

$$= \frac{e^{-5(s+5)}}{(s+5)} \text{ for all } \mathrm{Re}(s) > -5$$

ROC is the right half S plane with Re(s) > –5

Note: When $s < -5$, the integral → ∞

There is a pole at $s = 0$. Pole and ROC are shown plotted in Fig. 7.6.

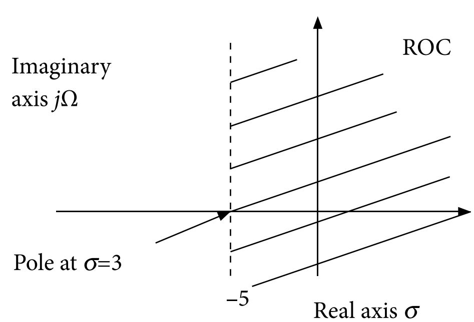


Fig. 7.6 Plot of ROC and pole of the function for P 7.8

Note: The LT function has a pole at $s = -5$.

P 7.9 Find LT of $x(t) = \delta(t+3)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} \delta(t+3)e^{-st}\,dt = e^{-st} \downarrow_{t=-3}$$

$= e^{3s}$ for all s

ROC is the right half S plane with Re(s) < 0

Note: When $s > 0$, the integral $\to \infty$

There is a pole at $s = 0$. Pole and ROC are shown plotted in Fig. 7.7. The function has a zero at $s = \infty$.

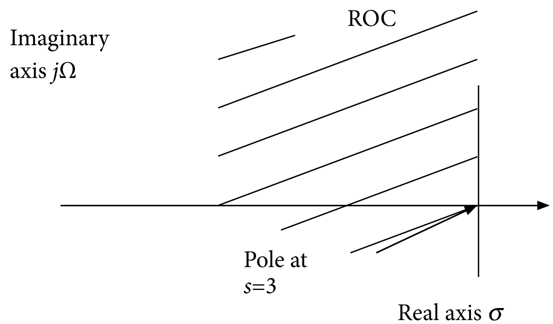


Fig. 7.7 Plot of ROC and pole of the function for P 7.9

Note: The LT function has a pole at $s = 0$.

P 7.10 Find LT of $x(t) = \sin(3t)u(t-5)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} \sin(3t)u(t-5)e^{-st}\,dt = \frac{1}{2j}\int_{5}^{\infty}[e^{3t} - e^{-3t}]e^{-st}\,dt$$

$$= \frac{1}{2j}\left[\frac{e^{-(s-3)t}}{-(s-3)}\downarrow_{3}^{\infty} - \frac{e^{-(s+3)t}}{-(s+3)}\downarrow_{5}^{\infty}\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-5(s-3)}}{(s-3)} - \frac{e^{-5(s+3)}}{(s+3)}\right]$$

= ROC is the right half S plane with Re(s) > 3 for the first term and ROC is the right half S plane with Re(s) > –3 for the second term

The common area of convergence is the right half S plane with Re(s) > 3.

There is a pole at $s = 3$ and $s = -3$. Pole and ROC are shown plotted in Fig. 7.8.

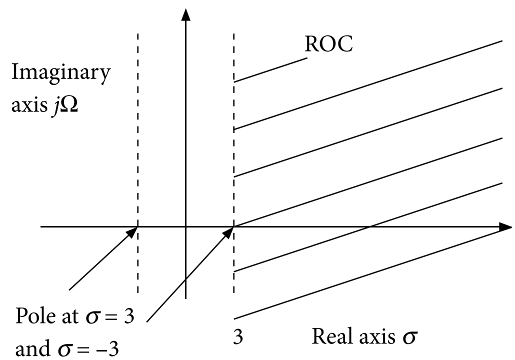


Fig. 7.8 Plot of ROC and pole of the function for P 7.10

Note: The LT function has a pole at $s = 2$ and $s = -2$.

P 7.11 Find LT of $x(t) = e^{-7|t|}$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} e^{-7|t|} e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-7t} e^{-st}\,dt + \int_{-\infty}^{0} e^{7t} e^{-st}\,dt$$

$$= -\frac{e^{-(s+7)t}}{s+7}\downarrow_{0}^{\infty} - \frac{e^{-(s-7)t}}{s-7}\downarrow_{-\infty}^{0}$$

$$= \frac{1}{s+7} - \frac{1}{s-7} = \frac{14}{s^2 - 49}$$

The first term converges for all $s + 7 > 0$, i.e., $\mathrm{Re}(s) > -7$

ROC is the right half S plane with $\mathrm{Re}(s) > -7$

The second term converges for all $-s + 7 > 0$, i.e., $\mathrm{Re}(s) < 7$

ROC is the left half S plane with $\mathrm{Re}(s) < 7$

ROC $-7 < \mathrm{Re}(s) < 7$

The plot of ROC and LT are depicted in Fig. 7.9.

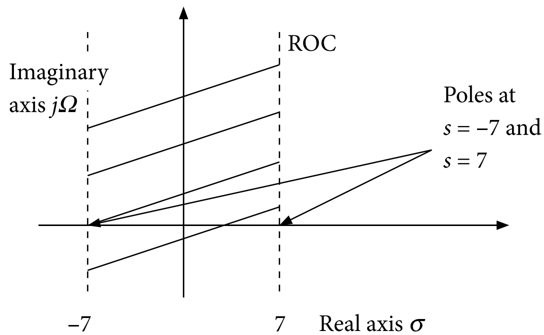


Fig. 7.9 Plot of poles and ROC for $x(t) = e^{-7|t|}$

P 7.12 Find LT of $x(t) = e^{-3t}\cos(7t)u(t)$ and find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} e^{-3t}\cos(7t)u(t)e^{-st}\,dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-3t}e^{-st}e^{j7t}dt + \int_0^{\infty} e^{-3t}e^{-st}e^{-j7t}dt\right]$$

$$= \frac{1}{2}\left[-\frac{e^{-(s+3-j7)t}}{s+3-j7}\downarrow_0^{\infty} - \frac{e^{-(s+3+j7)t}}{s+3+j7}\downarrow_0^{\infty}\right] \tag{7.61}$$

$$= \frac{1}{2}\left[\frac{1}{s+3-j7} + \frac{1}{s+3+j7}\right]$$

$$= \frac{s+3}{(s+3)^2+49}$$

The first term converges for all $s + 3 - j7 > 0$, i.e., $\mathrm{Re}(s) > -3$

ROC is the right half S plane with $\mathrm{Re}(s) > -3$

Note: If $\mathrm{Re}(a) = s > -3, (s + 3) > 0,\ e^{-(s+3)\infty} \rightarrow 0$

If $\mathrm{Re}(a) = s < -3,\ (s + 3) < 0,\ e^{-(s+3)\infty} \rightarrow \infty$

The second term converges for all $s + 3 + j\omega_0 > 0$, i.e., $\mathrm{Re}(s) > -3$

ROC is the right half S plane with $\mathrm{Re}(s) > -3$

The poles of the function are at $s = -3 + j7$ and $s = -3 - j7$. The plot of ROC and LT are depicted in Fig. 7.10.

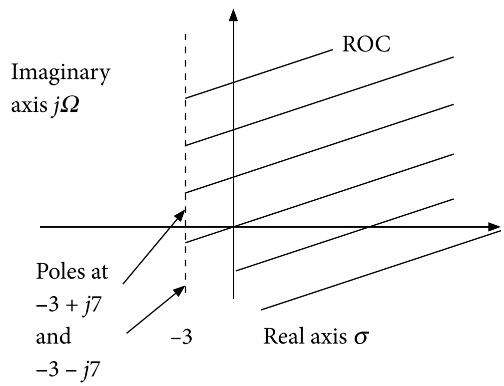


Fig. 7.10 Plot of poles and ROC for $x(t) = e^{-3t}\cos(7t)u(t)$

P 7.13 Find LT of $x(t) = e^{-3t}u(t) + e^{8t}u(t)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_0^\infty e^{-3t}e^{-st}dt + \int_0^\infty e^{8t}e^{-st}dt$$

$$= -\frac{e^{-(s+3)t}}{s+3}\downarrow_0^\infty - \frac{e^{-(s-8)t}}{s-8}\downarrow_0^\infty$$

$$= \frac{1}{s+3} + \frac{1}{s-8} = \frac{2s-5}{s^2-5s-24}$$

The first term converges for all $s + 1 > 0$, i.e., Re(s) > –3

ROC is the right half S plane with Re(s) > –3

The second term converges for all $s - 2 > 0$, i.e., Re(s) > 8

ROC is the right half S plane with Re(s) > 8

The common area of convergence is the area with Re(s) > 8. ROC and poles are plotted in Fig. 7.11.

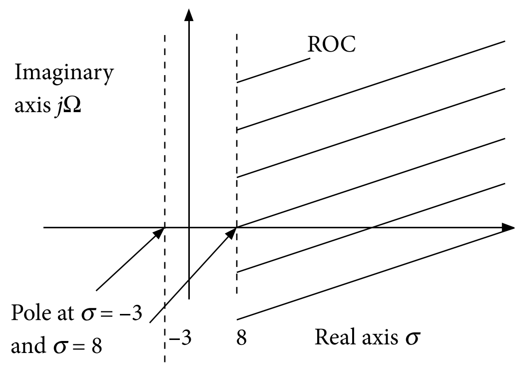


Fig. 7.11 ROC and poles for P 7.13

P 7.14 Find LT of $x(t) = e^{-2t}u(-t) + e^{5t}u(-t)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{0} e^{-2t}e^{-st}dt + \int_{-\infty}^{0} e^{5t}e^{-st}dt$$

$$= -\frac{e^{-(s+2)t}}{s+2}\downarrow_{-\infty}^{0} - \frac{e^{-(s-5)t}}{s-5}\downarrow_{-\infty}^{0}$$

$$= -\frac{1}{s+2} - \frac{1}{s-5} = -\frac{2s-3}{s^2-3s-10} \tag{7.63}$$

The first term converges for all $s + 2 < 0$, i.e., Re(s) < –2

ROC is the left half S plane with Re(s) > –2

The second term converges for all $s - 5 < 0$, i.e., Re(s) < 5

ROC is the left half S plane with Re(s) < 5

The common area of convergence is the area with Re(s) < –2. ROC and poles are plotted in Fig. 7.12.

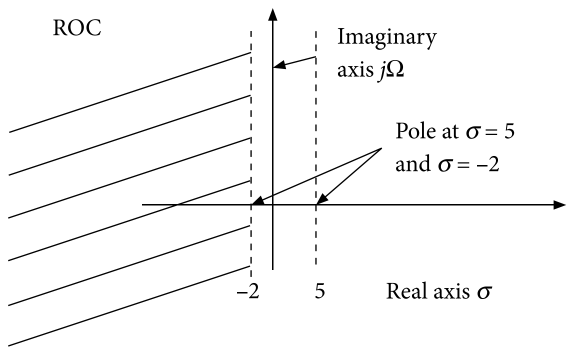


Fig. 7.12 ROC and poles for P 7.14

P 7.15 Find LT of $x(t) = e^{2t}\sin(3t)u(t)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \frac{1}{2j}\left[\int_0^\infty e^{2t}e^{j3t}e^{-st}dt - \int_0^\infty e^{2t}e^{-j3t}e^{-st}dt\right]$$

$$= \frac{1}{2j}\left[-\frac{e^{-(s-2-j3)t}}{s-2-j3}\downarrow_0^\infty + \frac{e^{-(s-2+j3)t}}{s-2+j3}\downarrow_0^\infty\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-2-j3} - \frac{1}{s-2+j3}\right] = \frac{1}{2j}\left[\frac{s-2+j3-s+2+j3}{(s-2)^2+9}\right]$$

$$= \frac{3}{(s-2)^2+9} \tag{7.64}$$

The first term converges for all $s - 2 > 0$, i.e., $\text{Re}(s) > 2$

ROC is the right half S plane with $\text{Re}(s) > 2$

The second term converges for all $s - 2$ again, i.e., $\text{Re}(s) > 2$

The common area of convergence is $\text{Re}(s) > 2$. The ROC and the poles are plotted in Fig. 7.13.

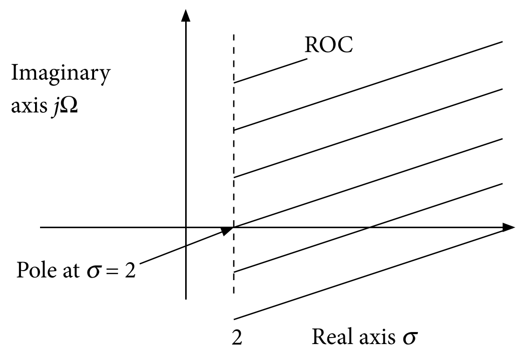


Fig. 7.13 Plot of ROC and poles for P 7.15

P 7.16 Find LT of $x(t) = \cos^2(6t)u(t)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} \cos^2(6t)u(t)e^{-st}\,dt = \frac{1}{2}[\int_0^{\infty}(1+\cos 12t)e^{-st}\,dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-st}\,dt + \int_0^{\infty} e^{12jt}e^{-st}\,dt + \int_0^{\infty} e^{-12jt}e^{-st}\,dt\right]$$

$$= \left[\frac{1}{2}\left[\frac{e^{-st}}{-s}\right]\downarrow_0^{\infty} - \frac{e^{-(s-12j)t}}{s-12j}\downarrow_0^{\infty} - \frac{e^{-(s+12j)t}}{s+12j}\downarrow_0^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{1}{s-12j} + \frac{1}{s+12j}\right] = \frac{1}{2}\left[\frac{s^2+144+s^2+12js+s^2-12js}{s(s^2+144)}\right]$$

$$= \frac{3s^2+144}{s(s^2+144)}$$

The second term converges for all $s - 12j > 0$, i.e., $\mathrm{Re}(s) > 0$

ROC is the right half S plane with $\mathrm{Re}(s) > 0$

The third term converges for all $s + 12j > 0$ again, i.e., $\mathrm{Re}(s) > 0$

The common area of convergence is $\mathrm{Re}(s) > 0$. The ROC and the poles are plotted in Fig. 7.14.

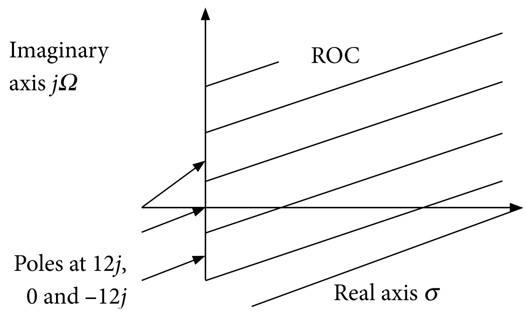


Fig. 7.14 Plot of ROC and poles for P 7.16

P 7.17 Find LT of $x(t) = (1 + \sin 3t \cos 3t)u(t)$. Also find ROC.

Solution

We will use the definition of LT.

$$X(S) = \int_{-\infty}^{\infty} (1 + \sin 3t \cos 3t)u(t)e^{-st}\,dt = \int_0^{\infty}\left(1 + \frac{1}{2}\sin 6t\right)e^{-st}\,dt$$

$$= \int_0^{\infty} e^{-st}\,dt + \frac{1}{4j}\left[\int_0^{\infty} e^{6jt}e^{-st}\,dt - \int_0^{\infty} e^{-6jt}e^{-st}\,dt\right]$$

$$= \left[\frac{e^{-st}}{-s}\right]\downarrow_0^{\infty} - \frac{1}{4j}\left[\frac{e^{-(s-6j)t}}{s-6j}\downarrow_0^{\infty} + \frac{e^{-(s+6j)t}}{s+6j}\downarrow_0^{\infty}\right]$$

$$= \left[\frac{1}{s} + \frac{1}{4j}\left[\frac{1}{s-6j} - \frac{1}{s+6j}\right]\right] = \left[\frac{1}{s} + \frac{s+6j-s+6j}{4j(s^2+36)}\right]$$

$$= \frac{1}{s} + \frac{1}{(s^2+36)} = \frac{s^2+s+36}{s(s^2+36)}$$

The second term converges for all $s - 6j > 0$, i.e. $\mathrm{Re}(s) > 0$

ROC is the right half S plane with $\mathrm{Re}(s) > 0$

The third term converges for all $s + 6j > 0$ again, i.e., $\mathrm{Re}(s) > 0$

The common area of convergence is $\mathrm{Re}(s) > 0$. The ROC and the poles are plotted in Fig. 7.15.

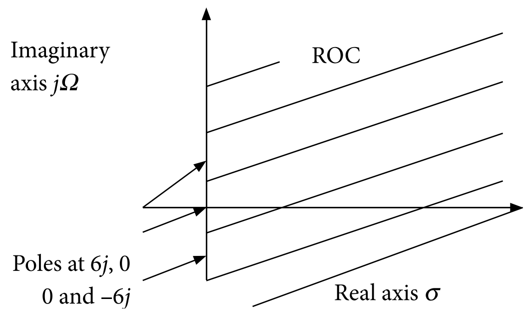


Fig. 7.15 Plot of ROC and poles for P 7.17

P 7.18 Find LT of the signal drawn in Fig. 7.16. Also find ROC.

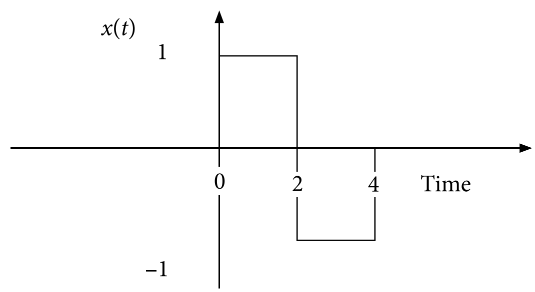


Fig. 7.16 Plot of signal for P 7.18

Solution

We will use the definition of LT.

$$X(S) = \int_0^2 e^{-st}\,dt - \int_2^4 e^{-st}\,dt$$

$$= -\frac{e^{-st}}{s}\downarrow_0^2 + \frac{e^{-st}}{s}\downarrow_2^4$$

$$= -\frac{1}{s}[e^{-2s} - 1] + \frac{1}{s}[e^{-4s} - e^{-2s}]$$

$$= \frac{1}{s}[1 - 2e^{-2s} + e^{-s4}] = \frac{1}{s}[1 - e^{-2s}]^2$$

Both the terms converge for all $s > 0$, i.e., $\mathrm{Re}(s) > 0$

ROC is the right half S plane with $\mathrm{Re}(s) > 0$

Note: If $\mathrm{Re}(a) = \sigma > 0$, $e^{-\infty} \to 0$

If $\mathrm{Re}(a) = s < 0$, $e^{-\infty} \to 0$

The ROC and poles/zeros are plotted in Fig. 7.17. There is a double zero at $s = 0$

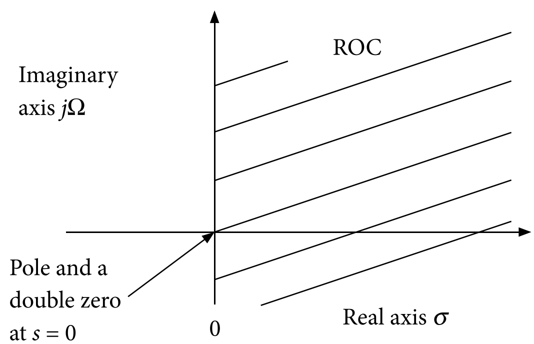


Fig. 7.17 Plot of ROC and poles for P 7.18

P 7.19 Find LT of the signal drawn in Fig. 7.18. Also find ROC.

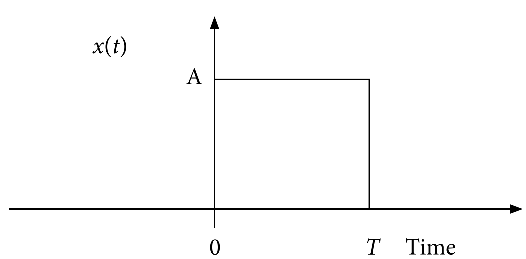


Fig. 7.18 Plot of signal for P 7.19

Solution

We will use the definition of LT.

$$X(S) = \int_0^T A \times e^{-st} dt$$

$$= -\frac{Ae^{-st}}{s} \downarrow_0^T$$

$$= -\frac{A}{s}[e^{-sT} - 1]$$

$$= \frac{A}{s}[1 - e^{-sT}]$$

The term converges for all $s > 0$, i.e., $\mathrm{Re}(s) > 0$

ROC is the right half S plane with $\mathrm{Re}(s) > 0$

The ROC and poles/zeros are plotted in Fig. 7.19. There is a zero at $s = 0$

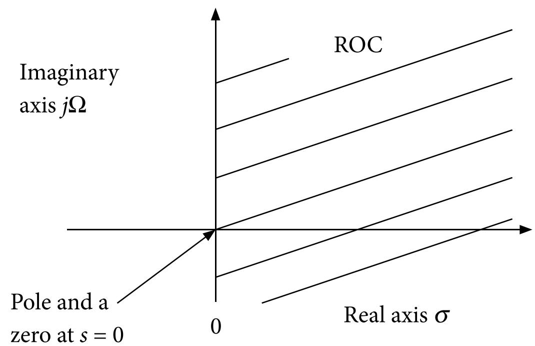


Fig. 7.19 Plot of ROC and pole and zero for P 7.19

P 7.20 Find LT of the signal drawn in Fig. 7.20. Also find ROC.

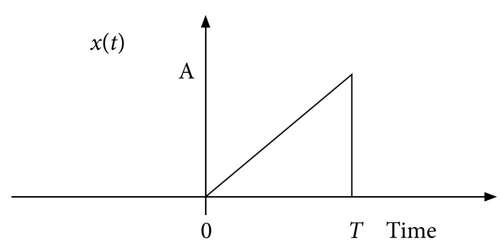


Fig. 7.20 Plot of signal for P 7.20

Solution

We will use the definition of LT. The equation of the straight line is $x(t) = t/2$.

$$X(S) = A\int_0^T t/T \times e^{-st}\, dt$$

$$=\frac{A}{T}\left[\frac{te^{-st}}{-s}\downarrow_0^T-\int_0^T\frac{e^{-st}}{-s}dt\right]$$

$$=\frac{A}{T}\left[\frac{Te^{-Ts}}{-s}-\frac{e^{-st}}{s^2}\downarrow_0^T\right]$$

$$=\frac{A}{T}\left[\frac{Te^{-Ts}}{-s}-\frac{e^{-Ts}-1}{s^2}\right]=\frac{A}{T}\left[\frac{1-sTe^{-Ts}-e^{-Ts}}{s^2}\right]$$

The term converges for all $s>0$, i.e., Re(s) > 0

ROC is the right half S plane with Re(s) > 0

The ROC and poles/zeros are plotted in Fig. 7.21. There is a double pole at $s = 0$

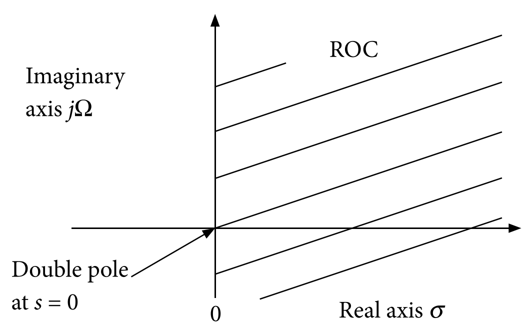


Fig. 7.21 Plot of ROC and pole and zero for P 7.20

P 7.21 Find LT of the signal drawn in Fig. 7.22. Also find ROC.

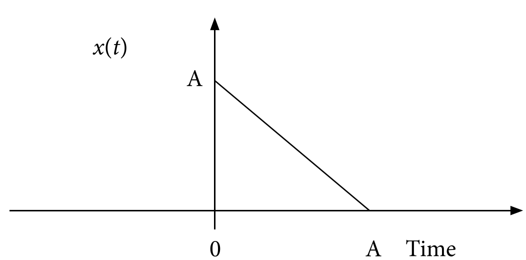


Fig. 7.22 Plot of signal for P 7.21

Solution

We will use the definition of LT. We have to find equation of the straight line.

$$Y = mx + c.$$

Points (0, A) and (A, 0) are on the line. These coordinates must satisfy the equation.

$$A = c \text{ and } 0 = m + c$$

Hence $m = -c = -1$; The equation becomes $y = -x + A$. We have to write the signal as $x(t) = -t + A$.

$$X(S) = \int_0^A (A-t)e^{-st}\,dt$$

$$= A\left[\frac{e^{-st}}{-s}\downarrow_0^A - \frac{te^{-st}}{-s}\downarrow_0^A + \int_0^A \frac{e^{-st}}{-s}dt\right]$$

$$= \left[A\frac{e^{-sA}-1}{-s} - \frac{e^{-sA}}{-s} + \frac{e^{-st}}{s^2}\downarrow_0^A\right]$$

$$= \left[A\frac{e^{-sA}-1}{-s} + \frac{e^{-sA}}{s} + \frac{e^{-sA}-1}{s^2}\right]$$

ROC: Re(s) > 0

The ROC and poles/zeros are plotted in Fig. 7.23. There is a double pole at $s = 0$

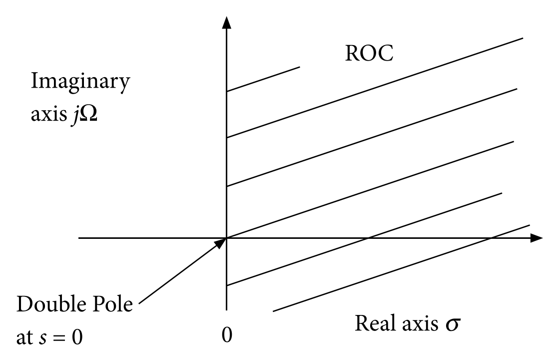


Fig. 7.23 Plot of ROC and pole and zero for P 7.21

P 7.22 Find LT of the signal drawn in Fig. 7.24. Also find ROC.

$$x(t) = \sin t, 0 < t \le 1$$
$$= 0 \quad \text{elsewhere}$$

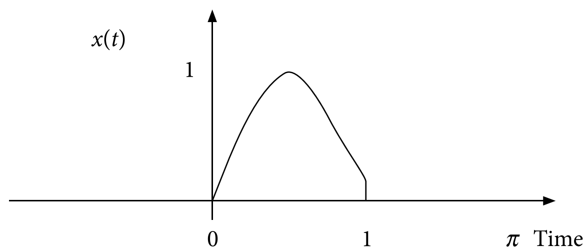


Fig. 7.24 Plot of signal for P 7.22

Solution

We will use the definition of LT.

$$X(S) = \int_0^1 \sin t \times e^{-st} dt = \frac{1}{2j}\left[\int_0^1 e^{jt} e^{-st} dt - \int_0^1 e^{-jt} e^{-st} dt\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)}\downarrow_0^1 - \left[\frac{e^{-(s+j)t}}{-(s+j)}\right]\downarrow_0^1\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-s}+1}{s-j} - \frac{e^{-s}+1}{s+j}\right]$$

$$= \frac{1}{2j}\left[\frac{se^{-s}+s+je^{-s}+j-se^{-s}-s+je^{-s}+j}{s^2+1}\right] = \frac{(e^{-s}+1)}{s^2+1}$$

The term converges for all $s > 0$, i.e., $\mathrm{Re}(s) > 0$

ROC is the right half S plane with $\mathrm{Re}(s) > 0$

The ROC and poles/zeros are plotted in Fig. 7.25. There is a double pole at $s = 0$

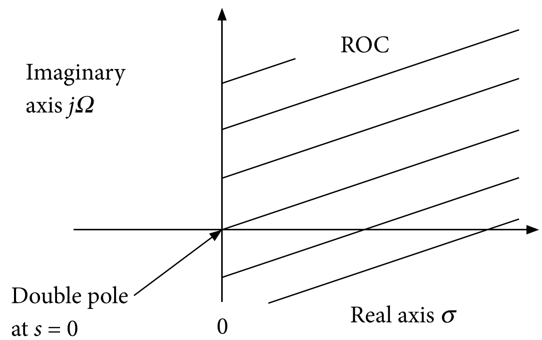


Fig. 7.25 Plot of ROC and pole and zero for P 7.22

P 7.23 Find LT of the signal drawn in Fig. 7.26.

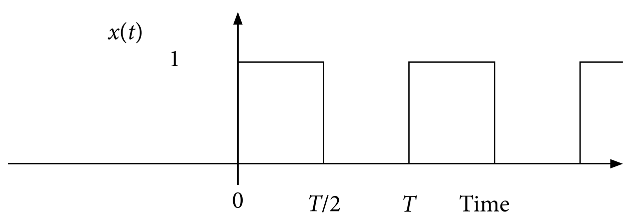


Fig. 7.26 Plot of signal for P 7.23

Solution

We will use the periodicity property of LT.

$$X(S) = \frac{1}{1-e^{-sT}} \int_0^T x(t)e^{-st}\,dt = \frac{1}{1-e^{-sT}}[\text{transform of one period}]$$

Let us find the transform of one period.

$$X(S) = \int_0^{T/2} e^{-st}\,dt$$

$$= -\frac{e^{-st}}{s}\downarrow_0^{T/2}$$

$$= -\frac{1}{s}[e^{-sT/2} - 1]$$

$$= \frac{1}{s}[1 - e^{-sT/2}]$$

$$X(S)_{\text{periodic signal}} = \frac{(1-e^{-sT/2})}{s(1-e^{-sT})} = \frac{1}{s(1+e^{-sT/2})}$$

P 7.24 Find LT of the signal drawn in Fig. 7.27.

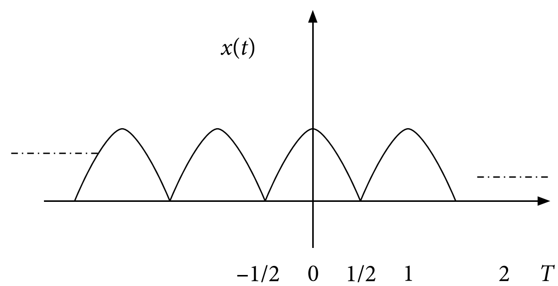


Fig. 7.27 Plot of signal for P 7.24

Solution

We will find LT of one period first.

$$X(S) = \int_{-1/2}^{1/2} \cos \pi t \times e^{-st} dt = \frac{1}{2}\left[\int_{-1/2}^{1/2} e^{j\pi t} e^{-st} dt + \int_{-1/2}^{1/2} e^{-j\pi t} e^{-st} dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(s-j\pi)t}}{-(s-j\pi)}\downarrow_{-1/2}^{1/2} + \left[\frac{e^{-(s+j\pi)t}}{-(s+j\pi)}\right]\downarrow_{-1/2}^{1/2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{(s-j\pi)1/2} - e^{-(s-j\pi)1/2}}{s-j\pi} + \frac{e^{(s+j\pi)1/2} - e^{-(s+j\pi)1/2}}{s+j\pi}\right] \qquad \text{[Note } e^{j\pi/2} = e^{-j\pi/2} = 1]$$

$$= \frac{1}{2}\left[\frac{se^{s(1/2)} - se^{-(s)1/2} + j\pi e^{(s)1/2} - j\pi e^{-(s)1/2} + se^{(s)1/2} - se^{-(s)1/2} - j\pi e^{(s)1/2} + j\pi e^{-(s)1/2}}{s^2+\pi^2}\right]$$

$$= \frac{(se^{s/2} - se^{-s/2})}{s^2+\pi^2}$$

$$X(S)_{\text{periodic signal}} = \frac{s(e^{s/2} - e^{-s/2})}{(s^2+\pi^2)(1-e^{-S})}, \text{ period} = 1$$

P 7.25 Find LT of the signal drawn in Fig. 7.28.

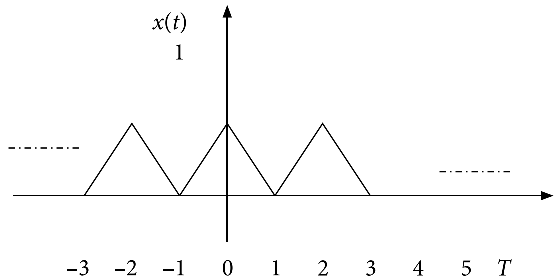


Fig. 7.28 Plot of signal for P 7.25

Solution

Let us find LT of one period. It is same as that for P 7.24.

$$X(S)=\int_{-1}^{0}(-t)\times e^{-st}dt+\int_{0}^{1}(1-t)\times e^{-st}dt$$

$$=\left[-\frac{te^{-st}}{-s}\downarrow_{-1}^{0}+\int_{-1}^{0}\frac{e^{-st}}{-s}dt\right]+\left[\frac{e^{-st}}{-s}\right]\downarrow_{0}^{1}-\left[\frac{te^{-st}}{-s}\right]\downarrow_{0}^{1}+\int_{0}^{1}\frac{e^{-st}}{-s}dt$$

$$=\frac{e^{-s}}{s}+\frac{e^{-st}}{s^2}\downarrow_{-1}^{0}+\frac{e^{-s}-1}{-s}-\frac{e^{-s}}{-s}+\frac{e^{-st}}{s^2}\downarrow_{0}^{1}$$

$$=\frac{e^{-s}}{s}+\frac{1-e^{s}}{s^2}-\frac{e^{-s}}{s}+\frac{1}{s}+\frac{e^{-s}}{s}+\frac{e^{-s}-1}{s^2}$$

$$=\frac{1-e^{s}+s+se^{-s}+e^{-s}-1}{s^2}$$

$$=\frac{s(1+e^{-s})+(e^{-s}-e^{s})}{s^2}$$

The term converges for all $s>0$, i.e., $\mathrm{Re}(s)>0$, double pole at $s=0$

ROC is the right half S plane with $\mathrm{Re}(s)>0$

$$X(s)_{\text{periodic signal}}=\frac{s(1+e^{-s})+(e^{-s}-e^{s})}{s^2(1-e^{-2s})},\ \text{period}=2$$

P 7.26 Find LT of the following signals using properties of LT.

1. $x(t) = t^2 e^{-2t} u(t)$ 2. $x(t) = te^{-3t}\sin(t)u(t)$ 3. $x(t) = e^{5t}u(-t)$

4. $x(t) = e^{-5t}\cos(t)u(t)$

Solution

1. $LT(tu(t)) = \dfrac{1}{s^2}$ (we will use property of differentiation in frequency.)

$$LT(t^2u(t)) = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$LT(t^2e^{-2t}u(t) = \frac{2}{(s+2)^3}$$

2. We have to find LT ($x(t) = te^{-3t}\sin(t)u(t)$); we know that $LT(tu(t)) = \dfrac{1}{s^2}$.

To find LT of $LT(te^{-t}u(t))$, we will use the property of frequency shifting.

$$\text{If } tu(t) \leftrightarrow \frac{1}{s^2} \quad \text{then } e^{-3t}tu(t) \leftrightarrow X(s+3) = \frac{1}{(s+3)^2}$$

We can write $\sin(t) = \dfrac{e^{jt} - e^{-jt}}{2j}$ and again use the property of frequency shifting.

$$\text{If } te^{-3t}u(t) \leftrightarrow \frac{1}{(s+3)^2} \quad \text{then}$$

$$\frac{1}{2j}[e^{-jt}te^{-3t}u(t)] \leftrightarrow X(s+3+j) = \frac{1}{2j}\left[\frac{1}{(s+3+j)^2}\right]$$

$$te^{-3t}\sin(t)u(t) \leftrightarrow \frac{1}{2j}\left[\frac{1}{(s+3-j)^2} - \frac{1}{(s+3+j)^2}\right]$$

$$= \frac{1}{2j}\left[\frac{4(s+3)j}{(s+3-j)^2(s+3+j)^2}\right] = \frac{2(s+3)}{((s+3)^2+1)^2}$$

3. We have to find LT of $x(t) = e^{5t}u(-t)$.

We know that $x(t) = e^{5t}u(t) \leftrightarrow \frac{1}{s-5}$; we will use property of time reversal.

$$\text{If } x(t) = e^{5t}u(t) \leftrightarrow X(s) = \frac{1}{s-5} \quad \text{then } x(-t) = e^{5t}u(-t) \leftrightarrow X(-s) = -\frac{1}{s+5}$$

We have to find LT of $x(t) = e^{-5t}\cos(t)u(t)$.

We know that $LT(\cos(t)u(t)) = \frac{s}{s^2+1}$.

We will use the property of frequency shifting to find

$$LT(e^{-5t}\cos(t)u(t)) = \frac{s+5}{(s+5)^2+1}$$

P 7.27 Find LT of the following signals using properties of LT.

1. $x(t) = (t^2 - 3t)u(t-2)$ 2. $x(t) = (t-4)u(t-4)$

3. $x(t) = (2tu(t) - 3(t-5)u(t-5))$

Solution

1. We have to find LT of $x(t) = (t^2 - 3t)u(t-2)$

$$x(t) = (t^2 - 3t)u(t-2) = t^2u(t-2) - 3tu(t-2)$$

$$LT(t^2u(t)) = \frac{2}{s^3};\ LT(t^2u(t-2)) = \frac{2e^{-2s}}{s^3}$$

$$LT(-3tu(t-2)) = \frac{-3e^{-2s}}{s^2}$$

$$LT(x(t)) = \frac{e^{-2s}}{s^3}(2-3s)$$

We have used the time shifting property for all the signals.

2. We have to find LT of $LT(x(t)) = LT(tu(t)) \leftrightarrow \frac{1}{s^2}$

$$LT(x(t-4)) = LT((t-4)u(t-4)) = \frac{e^{-4s}}{s^2}$$

3. We have to find LT of $LT(x(t)) = LT(2tu(t) - 3(t-5)u(t-5))$

$$= \frac{2}{s^2} - \frac{3e^{-5s}}{s^2} = \frac{2-3e^{-5s}}{s^2}$$

P 7.28 LTs of some signals are given here. Find the initial and final value of the signal using initial and final value theorem.

1. $X(s) = \frac{1}{s(s-5)}$
2. $X(s) = \frac{s+2}{s^2+2s+1}$
3. $X(s) = \frac{3s+5}{s(2s+3)}$
4. $X(s) = \frac{s+2}{s^2(s+3)}$

Solution

1. Initial value is given by

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{1}{s-5} = 0$$

Final value is given by

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{1}{s-5} = -\frac{1}{5}$$

2. Initial value is given by

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{s^2+2s}{s^2+2s+1} = \lim_{s\to\infty} \frac{1+2/s}{1+2/s+1/s^2} = 1$$

Final value is given by

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s^2+2s}{s^2+2s+1} = 0$$

3. Initial value is given by

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{3s+5}{2s+3} = \lim_{s\to\infty} \frac{3+5/s}{2+3/s} = 3/2$$

Final value is given by

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{3s+5}{2s+3} = 5/3$$

4. Initial value is given by

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{s+2}{s(s+3)} = \lim_{s\to\infty} \frac{1/s+2/s^2}{1+3/s} = 0$$

Final value is given by

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s+2}{s(s+3)} = \infty$$

P 7.29 A signal is given by $x(t) = e^{-(t-7)/3}u(t-7)$. Find LT.

Solution

We know that

$$LT(y(t) = e^{-t}u(t)) = \frac{1}{s+1}$$

We will use the time shifting property to find

$$LT(y(t-7)) = LT(e^{-(t-7)}u(t-7)) = \frac{e^{-7s}}{s+1}$$

We will now use the time scaling property to find

$$LT(y(t-7)/3)) = LT(e^{-(t-7)/3}u(t-7)) = \frac{3e^{-7s}}{s+1}$$

P 7.30 A signal is given by $x(t)=\sin(2t)\cos(3t)u(t)$. Find LT.

Solution

$$x(t)=\sin(2t)\cos(3t)u(t)=\frac{1}{2}[\sin(2+3)t+\sin(2-3)t]u(t)$$

We will use the property of linearity to find LT of the two terms.

$$LT(x(t))=\frac{1}{2}\left[\frac{5}{s^2+25}-\frac{1}{s^2+1}\right]$$

P 7.31 Find LT of $x(t)=e^{3t}u(t)*t^2u(t)$

Solution

We know that

$$LT(e^{3t}u(t))=\frac{1}{s-3}$$

$$LT(u(t))=\frac{1}{s};$$

$$LT(t^2u(t))=-\frac{d}{ds}\left(\frac{1}{s^2}\right)=\frac{2}{s^3}$$

Note: We have used the property of differentiation in the *s* domain.

$$LT(e^{2t}u(t)*t^2u(t))=\frac{1}{s-3}\times\frac{2}{s^3}=\frac{2}{s^3(s-3)}$$

Note: We have used the convolution property.

P 7.32 Find LT of $x(t)=3e^{3t}u(t)$ using the property of differentiation in time domain.

Solution

We know that

$$LT(e^{3t}u(t))=\frac{1}{s-3}$$

$$LT(3e^{3t}u(t)) = \frac{3}{s-3}$$

We will now apply the property of differentiation in time for $x_1(t) = e^{3t}u(t)$

$$LT(e^{3t}u(t)) = \frac{1}{s-3};$$

$$LT\left(\frac{d}{dt}e^{3t}u(t)\right) = LT(3e^{3t}u(t)) = sX_1(s) - x(0^+)$$

$$= \frac{s}{s-3} - 1 = \frac{s-s+3}{s-3} = \frac{3}{s-3}$$

Let us find the initial value using initial value theorem.

$$x(0^+) = \lim_{s\to\infty} sX_1(s) = \lim_{s\to\infty} \frac{s}{s-3} = \lim_{s\to\infty} \frac{1}{1-3/s} = 1$$

P 7.33 Find the inverse LT of $X(s) = \frac{1}{(s+2)(s^2+s+1)}$ with ROC given by Re(s) > –0.5. Plot poles and zeros.

Solution

The denominator is already in the factored form. So Step 1 is over.

Step 2 We will use partial fraction expansion and decompose the function into three terms.

$$X(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{k_1}{s+2} + \frac{k_2 s + k_3}{s^2+s+1}$$

Find k_1, k_2 and k_3

$$k_1 = (s+2)X(s)\downarrow_{s=0} = \frac{1}{(s^2+s+1)}\downarrow_{s=-2} = \frac{1}{3}$$

$$1 = \frac{1}{3}[s^2+s+1] + s^2(k_2) + (2k_2+k_3)s + 2k_3$$

$$1 = s^2(1/3 + k_2) + s(1/3 + 2k_2 + k_3) + (1/3 + 2k_3)$$

$$1/3 + 2k_3 = 1 \Rightarrow k_3 = 1/3$$

$$1/3 + k_2 = 0 \Rightarrow k_2 = -1/3$$

We have to study ROC to find the inverse LT. If the pole for the term is on the left-hand side of the imaginary axis, like at $s = -1$, we have recover the right-handed signal. If the pole for the term is on the right-hand side of the imaginary axis, like at $s = 1$, we have to recover the left-handed signal. Here, the common area of convergence is Re(s) > 0. We will recover all the signals as right-handed signals.

$$X(s) = \frac{1/3}{s+2} + \frac{-1/3s + 1/3}{s^2 + s + 1}$$

$$X(s) = \frac{1/3}{s+2} + \frac{1}{3}\frac{s+0.5}{(s+0.5)^2 + 0866^2} + .577\frac{0.866}{(s+0.5)^2 + 0.866^2}$$

Taking ILT, we get

$$x(t) = \left(\frac{1}{3}e^{-2t}u(t) + \frac{1}{3}e^{-0.5t}\cos(0.866t)u(t) + 0.577e^{-0.5t}\sin(0.866t)u(t)\right)$$

Let us draw the pole zero plot using the following MATLAB program. Figure 7.29 shows the pole zero plot.

```
clear all;
clc;
s = tf('s');
b=[1 3 3 2];
a=[1];
f=tf(a,b);
pzmap(f);
```

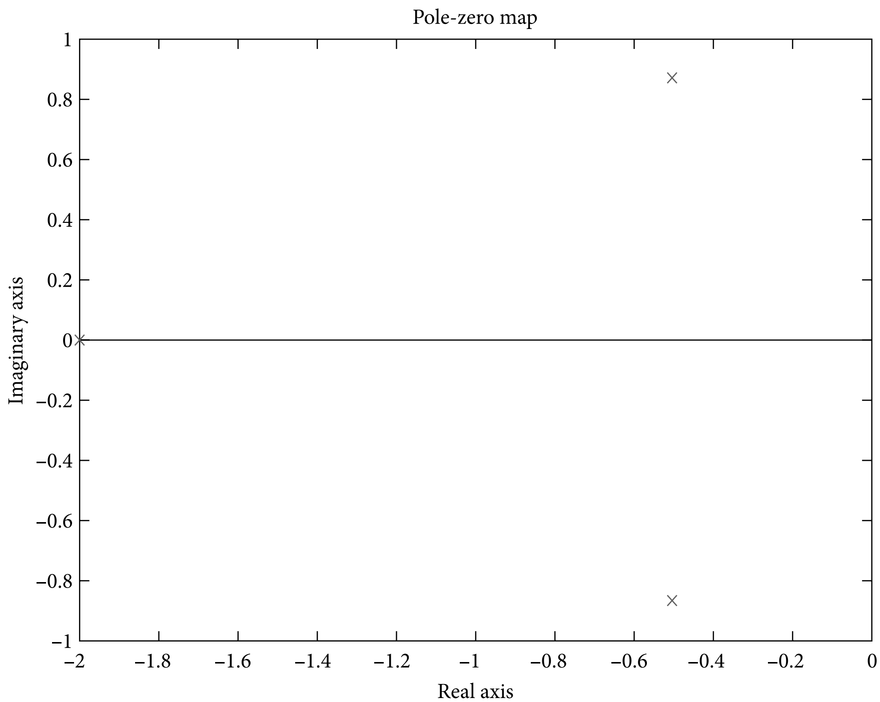


Fig. 7.29 Plot of poles and zeros for the transfer function for P 7.33

P 7.34 Find the inverse LT of $X(s) = \dfrac{2}{s(s+1)(s+2)}$ with ROC given by $-1 < \text{Re}(s) < 0$.

Solution

The denominator is already in the factored form. So Step 1 is over.

Step 2 We will use partial fraction expansion and decompose the function into three terms.

$$X(s) = \frac{2}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

Find k_1, k_2 and k_3

$$k_1 = (s)X(s)\downarrow_{s=0} = \frac{2}{(s+1)(s+2)}\downarrow_{s=0} = \frac{2}{2\times 1} = 1$$

$$k_2 = (s+1)X(s)\downarrow_{s=-1} = \frac{2}{(s)(s+2)}\downarrow_{s=1} = \frac{2}{(-1)\times 1} = -2$$

$$k_3 = (s+2)X(s)\downarrow_{s=-2} = \frac{2}{(s+1)(s)}\downarrow_{s=-2} = \frac{2}{-1\times -2} = 1$$

We have to study ROC to find the inverse LT. The common area of convergence is $-1 < \mathrm{Re}(s) < 0$. We will recover the term with pole at zero as a left-handed signal and terms with poles at -1 and -2 as the right-handed signals.

$$X(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}$$

Taking ILT, we get

$$x(t) = (u(-t) - 2(e^{t})u(t) + e^{-2t}u(t))$$

We have to study ROC to find the inverse LT.

If the ROC is specified as $-2 < \mathrm{Re}(s) < -1$, we will recover the terms with poles at zero and -1 as the left-handed signals and terms with a pole at -2 as a right-handed signal.

$$x(t) = (u(-t) + 2(e^{t})u(-t) + e^{-2t}u(t))$$

P 7.35 Find the inverse LT of $X(s) = \dfrac{3s^2 + 8s + 23}{(s+3)(s^2+2s+10)}$.

Solution

Step 1 The denominator is in the factored form.

Let us draw the pole zero plot using the following MATLAB program. Figure 7.30 shows the pole zero plot.

```
clear all;
clc;
s = tf('s');
b=[1 5 16 30];
a=[3 8 23];
f=tf(a,b);
pzmap(f);
```

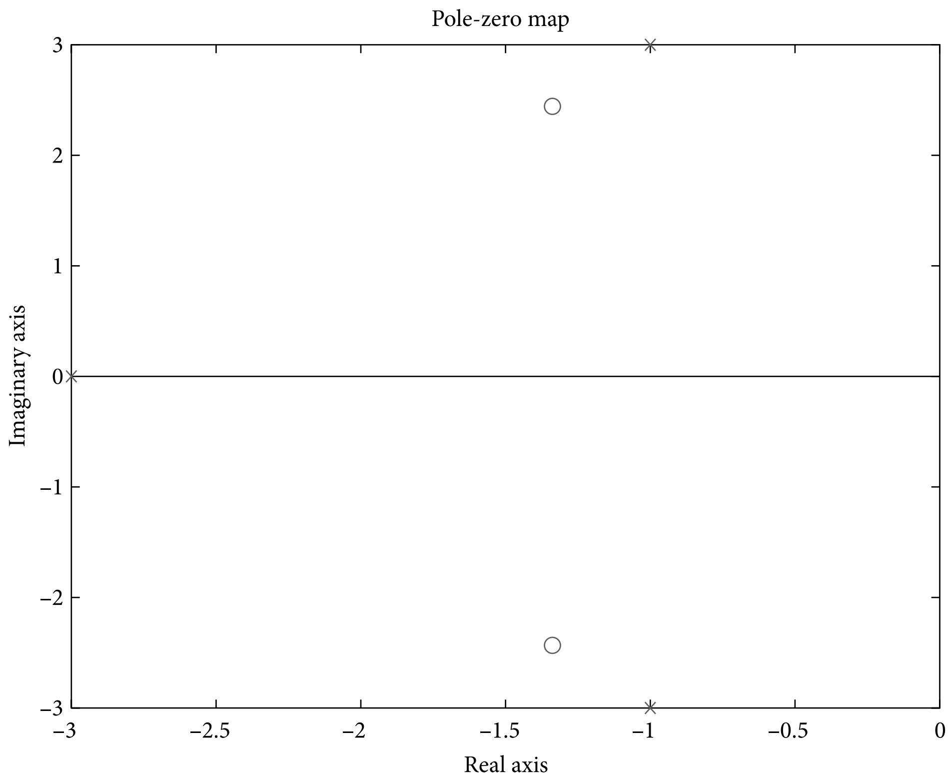


Fig. 7.30 Pole zero plot for P 7.35

Step 2 We will use partial fraction expansion and decompose the function into three terms.

$$X(s)=\frac{3s^2+8s+23}{(s+3)(s^2+2s+10)}=\frac{k_1}{s+3}+\frac{k_2s+k_3}{s^2+2s+10}$$

Find k_1, k_2 and k_3

$$k_1=(s+3)X(s)\downarrow_{s=-1}=\frac{3s^2+8s+23}{(s^2+2s+10)}\downarrow_{s=-3}=\frac{26}{13}=2$$

Put the value of k_1 in the equation and equate the numerators on both sides; we get

$$3s^2+8s+23=2(s^2+2s+10)+k_2s^2+(3k_2+k_3)s+3k_3$$

$$3s^2+8s+23=(k_2+2)s^2+(3k_2+k_3+4)s+(3k_3+20)$$

Equate the coefficients of s^2, s and a constant term.

$$k_2 + 2 = 3 \Rightarrow k_2 = 1$$

$$3k_3 + 20 = 23 \Rightarrow k_3 = 1$$

putting values of the constants $k1$, $k2$ and $k3$ we get

$$X(s) = \frac{3s^2 + 8s + 23}{(s+3)(s^2 + 2s + 10)} = \frac{2}{s+3} + \frac{s+1}{s^2 + 2s + 10}$$

$$X(s) = \frac{2}{s+3} + \frac{(s+1)}{(s+1)^2 + 9}$$

Take ILT.

$$x(t) = 2e^{-3t}u(t) + e^{-t}\cos(3t)u(t)$$

P 7.36 Find the inverse LT of $X(s) = \dfrac{4}{(s+2)(s+4)}$ with ROC $-4 < \mathrm{Re}(s) < -2$.

Solution

Step 1 The denominator is to be in the factored form.

$$X(s) = \frac{4}{(s+2)(s+4)}$$

Step 2 We will use partial fraction expansion and decompose the function into three terms.

$$X(s) = \frac{4}{(s+2)(s+4)} = \frac{k_1}{s+2} + \frac{k_2}{s+4}$$

Find k_1, $k_{2\,3}$

$$k_1 = (s+2)X(s)\downarrow_{s=-2} = \frac{4}{s+4}\downarrow_{s=-2} = \frac{4}{2} = 2$$

$$k_2 = (s+4)X(s)\downarrow_{s=-4} = \frac{4}{s+2}\downarrow_{s=-4} = \frac{4}{-2} = -2$$

$$X(s) = \frac{4}{(s+2)(s+4)} = \frac{2}{s+2} - \frac{2}{s+4}$$

$$x(t) = -2e^{-2t}u(-t) - 2e^{-4t}u(t)$$

Note that the term with pole at –2 is recovered as the left-handed signal and the term with pole at –4 is recovered as the right-handed signal so that the common area of convergence is same as the ROC specified.

P 7.37 Find the inverse LT of $X(s) = \dfrac{s+2}{s^2(s+1)}$ with ROC $-1 < \mathrm{Re}(s) < 0$ and ROC $\mathrm{Re}(s) > 0$

Solution

Step 1 The denominator is already in the factored form.

Step 2 We will use partial fraction expansion and decompose the function in three terms.

$$X(s) = \frac{s+2}{s^2(s+1)} = \frac{k_1}{s} + \frac{k_2}{s^2} + \frac{k_3}{(s+1)}$$

Find k_1, k_2 and k_3

$$k_1 = \frac{d}{ds}\left[(s^2)X(s)\downarrow_{s=0}\right] = \frac{d}{ds}\left(\frac{s+2}{(s+1)}\right)\downarrow_{s=0}$$

$$= \frac{s+1-s-2}{(s+1)^2} = \frac{-1}{(s+1)^2}\downarrow_{s=0} = \frac{-1}{1} = -1$$

$$k_2 = (s)^2 X(s)\downarrow_{s=0} = \frac{s+2}{(s+1)}\downarrow_{s=0} = \frac{2}{1} = 2$$

$$k_3 = (s+1)X(s)\downarrow_{s=-1} = \frac{s+2}{s^2}\downarrow_{s=-1} = 1$$

$$X(s)=\frac{s+2}{(s^2)(s+1)}=\frac{-1}{s}+\frac{2}{(s)^2}+\frac{1}{(s+1)} \tag{7.146}$$

We will now find ILT by inspection. If ROC is Re(s) > 0,

$$x(t)=-u(t)+2tu(t)+e^{-t}u(t)$$

Note that we have recovered all signals as causal signals.

If ROC is –1 < Re(s) < 0, we have to recover the term with pole at 0 as a left-handed signal and terms with pole at –1 as the right-handed signals.

$$x(t)=u(-t)-2tu(-t)+e^{-t}u(t)$$

P 7.38 Find the inverse LT of $X(s)=\frac{s^2+5s+5}{s^2+3s+2}$ with ROC Re(s) > –1

Solution

Step 1 Factorize the denominator. Note that the degree of the numerator is the same as the degree of the denominator. So, we have to bring it into the fraction form so that we can apply partial fraction expansion.

$$X(s)=\frac{s^2+5s+4}{s^2+3s+2}=1+\frac{2s+3}{(s+2)(s+1)}$$

Step 2 We will use partial fraction expansion and decompose the function into three terms.

$$X(s)=1+\frac{2s+3}{(s+2)(s+1)}=1+\frac{k_1}{s+2}+\frac{k_2}{s+1}$$

Find k_1, and k_2

$$k_1=[(s+2)X(s)\downarrow_{s=-2}]=\left(\frac{2s+3}{(s+1)}\right)\downarrow_{s=-2}=1$$

$$k_2=(s+1)X(s)\downarrow_{s=-1}=\frac{2s+3}{(s+2)}\downarrow_{s=-1}=1$$

$$X(s) = 1 + \frac{1}{s+2} + \frac{1}{s+1}$$

We will now find ILT by inspection. If ROC is Re(s) > –1,

$$x(t) = \delta(t) + e^{-t}u(t) + e^{-2t}u(t)$$

Note that we have recovered all signals as causal signals.

P 7.39 Find the inverse LT of $X(s) = \dfrac{s^3 + 5s^2 + 13s + 9}{s^2 + 4s + 8}$ with ROC Re(s) > –2

Solution

Step 1 Factorize the denominator. Note that the degree of the numerator is same as the degree of the denominator. So, we have to bring it into the fraction form so that we can apply partial fraction expansion.

$$X(s) = \frac{s^3 + 5s^2 + 13s + 9}{s^2 + 4s + 8} = s + 1 + \frac{s+1}{s^2 + 4s + 8}$$

$$= s + 1 + \frac{s+2}{(s+2)^4 + 2^2} - \frac{1}{(s+2)^2 + 2^2} \tag{7.154}$$

We will now find ILT by inspection. If ROC is Re(s) > –2,

$$x(t) = \delta'(t) + \delta(t) + e^{-2t}\cos(2t)u(t) - \frac{1}{2}e^{-2t}\sin(2t)u(t)$$

Note that we have recovered all signals as causal signals. $\delta'(t)$ stands for the derivative of $\delta(t)$. We have used the standard formula for cos and sin functions.

P 7.40 Find the inverse LT of $X(s) = \dfrac{s^2 - 3s + 1}{s^2 + 2s + 1}$ with ROC Re(s) > –1

Solution

Step 1 Factorize the denominator. Note that the degree of the numerator is same as the degree of the denominator. So, we have to bring it into the fraction form so that we can apply partial fraction expansion.

$$X(s) = \frac{s^2 - 3s + 1}{s^2 + 2s + 1} = 1 + \frac{-5s}{s^2 + 2s + 1}$$

$$=1-\frac{5(s+1)}{(s+1)^2}+\frac{5}{(s+1)^2}$$

We will now find ILT by inspection. If ROC is Re(s) > −1,

$$x(t)=\delta(t)-5e^{-t}u(t)+5te^{-t}u(t)$$

Note that we have recovered all signals as causal signals.

P 7.41 Use LT to find the output of the system if the system is described by the differential equation $\frac{d}{dt}y(t)+3y(t)=x(t)$ with input given by $x(t)=e^{-4t}u(t)$ and initial condition is $y(0^+) = -2$. Draw the response of the system using a MATLAB program.

Solution

Take LT of the given differential equation.

$$\frac{d}{dt}y(t)+3y(t)=x(t)$$

LT gives

$$sY(s)-y(0^+)+3Y(s)=X(s)$$

$$sY(s)+2+3Y(s)=\frac{1}{s+4}$$

$$(s+3)Y(s)=\frac{1}{s+4}-2$$

$$Y(s)=\frac{1}{(s+4)(s+3)}-\frac{2}{(s+3)}$$

A MATLAB program to plot the response of the system to the given input with and without initial conditions is given as follow. The responses are shown in Figs 7.31 and 7.32.

```
clear all;
clc;
s = tf('s');
```

```
b=[1 7 12];
a=[1];
f=tf(a,b);
c=impulse(f);
plot(c);title('response to the input');xlabel('time');
ylabel('amplitude');

clear all;
clc;
s = tf('s');
b=[1 7 12];
a=[-2 -7];
f=tf(a,b);
c=impulse(f);
plot(c);title('response to the input with initial co
nditions');xlabel('time');ylabel('amplitude');
```

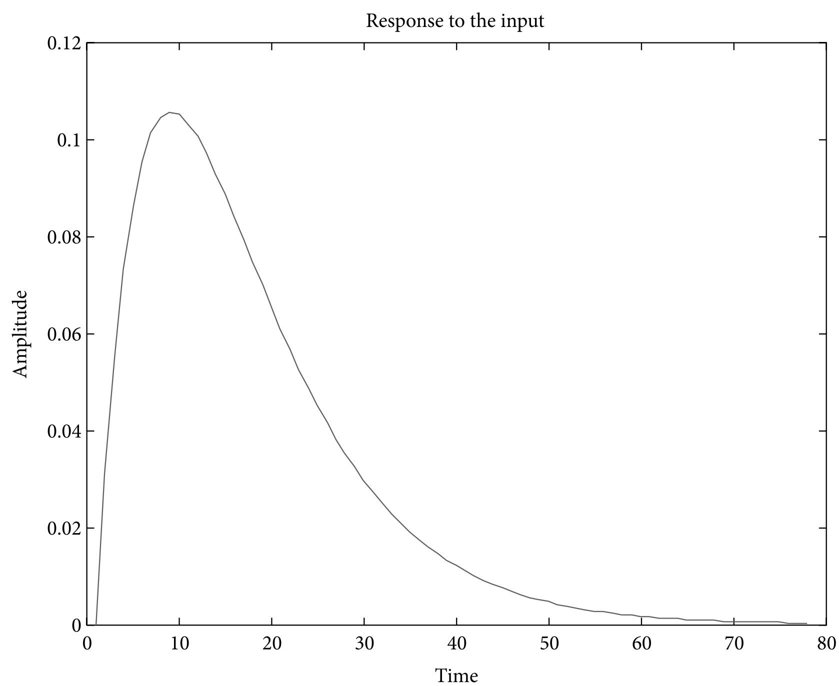



Fig. 7.31 Response of the system to the given input

We will use partial fraction expansion.

$$Y(s) = \frac{1}{(s+4)(s+3)} - \frac{2}{(s+3)}$$

$$Y(s) = \frac{1}{s+3} - \frac{1}{s+4} - \frac{2}{s+3}$$

$$Y(s) = -\frac{1}{s+4} - \frac{1}{s+3}$$

Take ILT

$$y(t) = -e^{-4t}u(t) - e^{-3t}u(t)$$

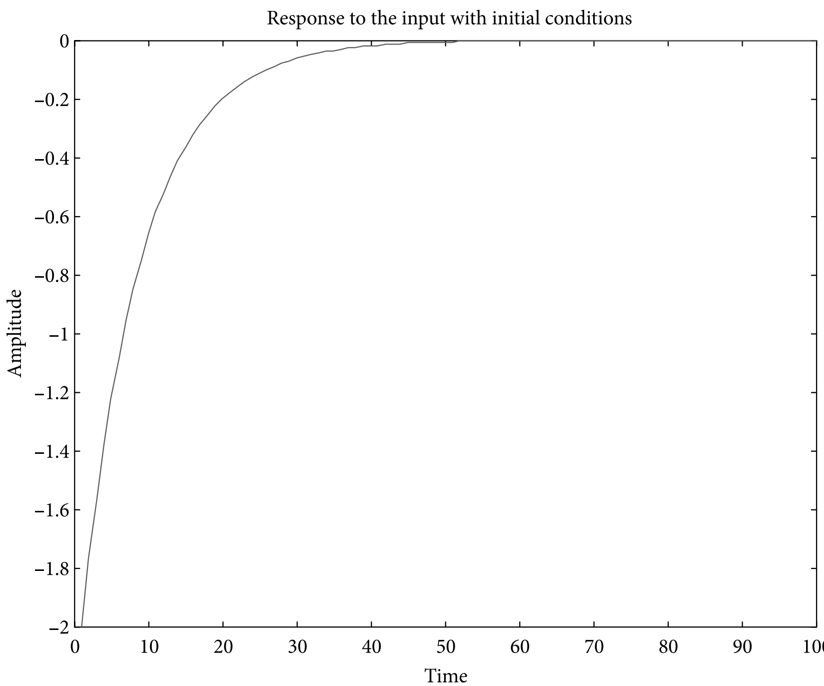


Fig. 7.32 Response of the system to the given input with initial conditions

P 7.42 Use LT to find the transfer function and the impulse response of the system, if the system is described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 10y(t) = x(t).$$

Write a MATLAB program to draw the impulse response.

Solution

Take LT of the given differential equation to find the transfer function.

$$\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+10y(t)=x(t)$$

$$s^2Y(s)+4sY(s)+10Y(s)=X(s)$$

$$H(s)=\frac{1}{s^2+4s+10}$$

$$=\frac{1}{(s+2)^2+6}$$

$$h(t)=\frac{1}{\sqrt{6}}e^{-2t}\sin(\sqrt{6}t)u(t)$$

A MATLAB program to plot the impulse response of the system is given as follows. The response is shown in Fig. 7.33.

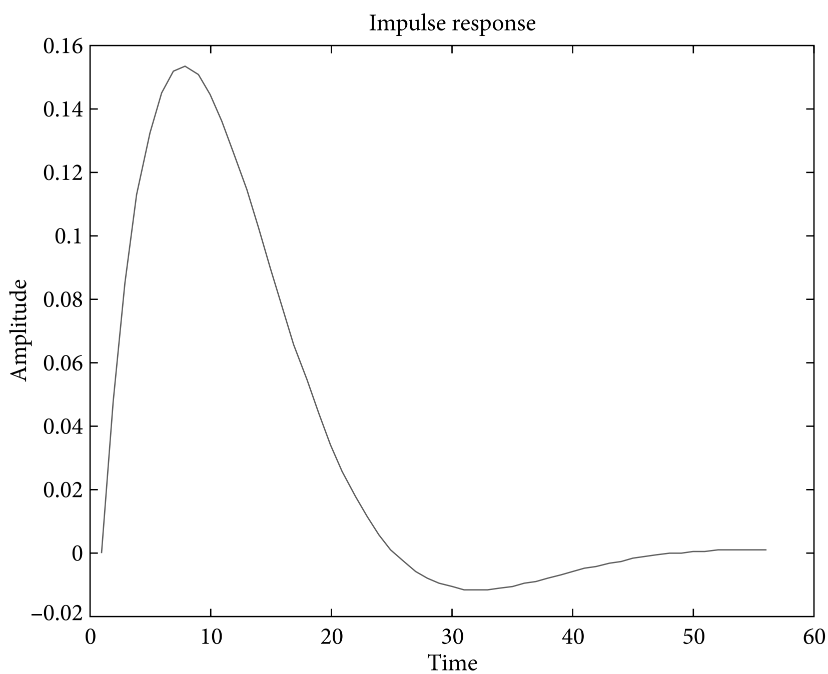


Fig. 7.33 Impulse response for P 7.42

```
clear all;
clc;
s = tf('s');
b=[1 4 10];
a=[1];
f=tf(a,b);
c=impulse(f);
plot(c);title('impulse  response');xlabel('time');yl
abel('amplitude');
```

P 7.43 Use LT to find the transfer function and the impulse response of the causal and stable system if the system is described by the differential equation

$$\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)=\frac{d^2}{dt^2}x(t)+6\frac{d}{dt}x(t)+7x(t)$$

Solution

Take LT of the given differential equation.

$$\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)=\frac{d^2}{dt^2}x(t)+6\frac{d}{dt}x(t)+7x(t)$$

$$s^2Y(s)+3sY(s)+2Y(s)=s^2X(s)+6sX(s)+7X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s^2+6s+7}{s^2+3s+2}$$

$$H(s)=\frac{s^2+6s+7}{s^2+3s+2}=1+\frac{3s+5}{(s+2)(s+1)}$$

Use partial fraction expansion

$$H(s)=1+\frac{1}{s+2}+\frac{2}{s+1}$$

Take ILT

$$h(t)=\delta(t)+e^{-2t}u(t)+2e^{-t}u(t)$$

A MATLAB program to plot the impulse response of the system is given as follows. The response is shown in Fig. 7.34.

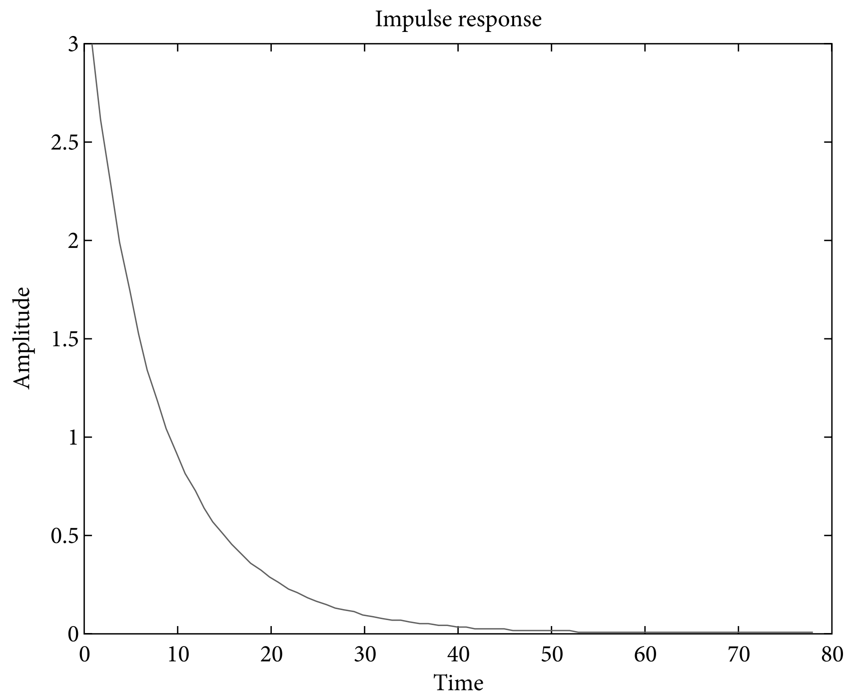


Fig. 7.34 Impulse response for P 7.43


```
clear all;
clc;
s = tf('s');
b=[1 3 2];
a=[1 6 7];
f=tf(a,b);
c=impulse(f);
plot(c);title('impulse  response');xlabel('time');yl
abel('amplitude');
```

P 7.44 Find the forced response of the system with differential equation given by

$$\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)=x(t)$$

to the input given by $x(t)=e^{-3t}u(t)$. Write a MATLAB program to plot the impulse response and the forced response.

Solution

We will first find the transfer function of the system. Take the LT of the given differential equation.

$$\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)=x(t)$$

$$(s^2+3s+2)Y(s)=X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{(s^2+3s+2)}$$

A MATLAB program to plot the impulse response of the system is given as follows. The response is shown in Fig. 7.35.

```
clear all;
clc;
s = tf('s');
b=[1 3 2];
a=[1];
f=tf(a,b);
c=impulse(f);
plot(c);title('impulse  response');xlabel('time');yl
abel('amplitude');
```

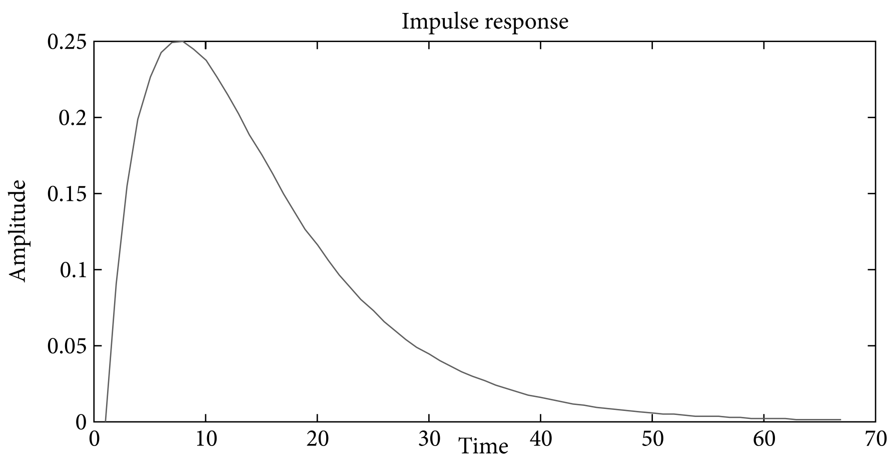



Fig. 7.35 Impulse response for P 7.44

To find the forced response, we will apply the input $x(t) = e^{-3t}u(t)$

$$\frac{Y(s)}{X(s)} = \frac{1}{(s^2 + 3s + 2)}$$

$$Y(s) = \frac{1}{(s+2)(s+1)} X(s) = \frac{1}{(s+2)(s+3)(s+1)}$$

We will use partial fraction expansion.

$$Y(s) = \frac{1}{(s+2)(s+3)(s+1)} = -\frac{1}{s+2} + \frac{1/2}{s+3} + \frac{1/2}{s+1}$$

Teaser: The reader is encouraged to verify the coefficients.

Take ILT to find the solution. The response due to input is the steady state response and response due to the poles of the system is the transient response.

$$Y(s) = \frac{1/2}{s+1} + \frac{1/2}{s+3} - \frac{1}{s+2}$$

$$y(t) = \left(\frac{1}{2}e^{-3t}u(t)\right) + \left(\frac{1}{2}e^{-t}u(t) - e^{-2t}u(t)\right)$$

Forced resonse = Steady state response + Transient response

A MATLAB program to plot the forced response of the system is given as follows. The response is shown in Fig. 7.36.

```
clear all;
clc;
s = tf('s');
b=[1 6 11 6];
a=[1];
f=tf(a,b);
c=impulse(f);
plot(c);title('forced  response');xlabel('time');yla
bel('amplitude');
```

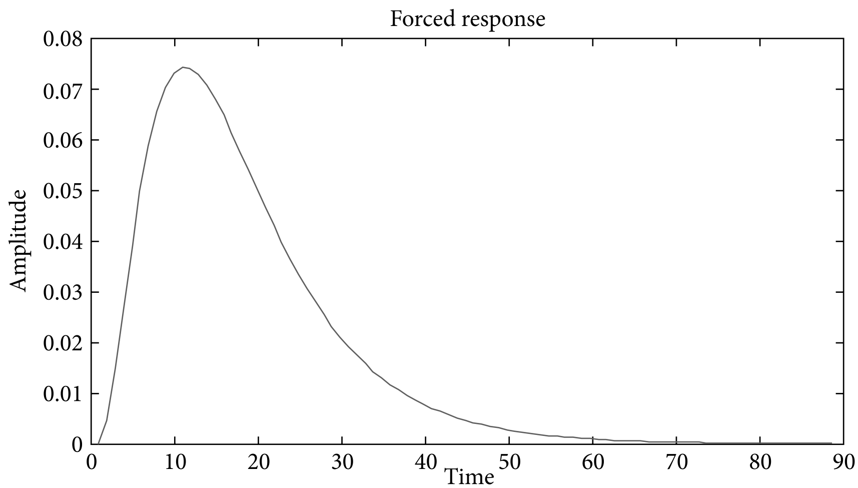


Fig. 7.36 Forced response of the system for P 7.44

P 7.45 Find the natural response, forced response and total response of the system with a differential equation given by

$$\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)=x(t)$$

to the input given by $x(t) = u(t)$. The initial conditions are $\frac{d}{dt}y(0)=3,\ y(0)=1.$ Write a MATLAB program to draw the natural response and forced response.

Solution

We will first find the transfer function of the system. Take the LT of the given differential equation and apply initial conditions.

$$\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)-\frac{d}{dt}y(0)-y(0)=x(t)$$

$$(s^2Y(s)-sy(0)-\frac{d}{dt}y(o)+3(sY(s)-y(0)+2)=X(s)$$

$$s^2Y(s)-(s+3)y(0)-\frac{d}{dt}y(0)+3sY(s)+2Y(s)=X(s)$$

$$Y(s)(s^2+3s+2) = X(s) + (s+3)\times 1 + 3$$

$$Y(s) = \frac{s+6}{s^2+3s+2} + \frac{X(s)}{s^2+3s+2}$$

The response due to the first term is due to the initial conditions and hence is the natural response.

$$Y(s) = \frac{s+6}{s^2+3s+2}$$

$$Y(s) = \frac{5}{s+1} - \frac{4}{s+2}$$

$$y(t) = 5e^{-t}u(t) - 4e^{-2t}u(t)$$

A MATLAB program to plot the natural response is given as follows. The response is plotted in Fig. 7.37.

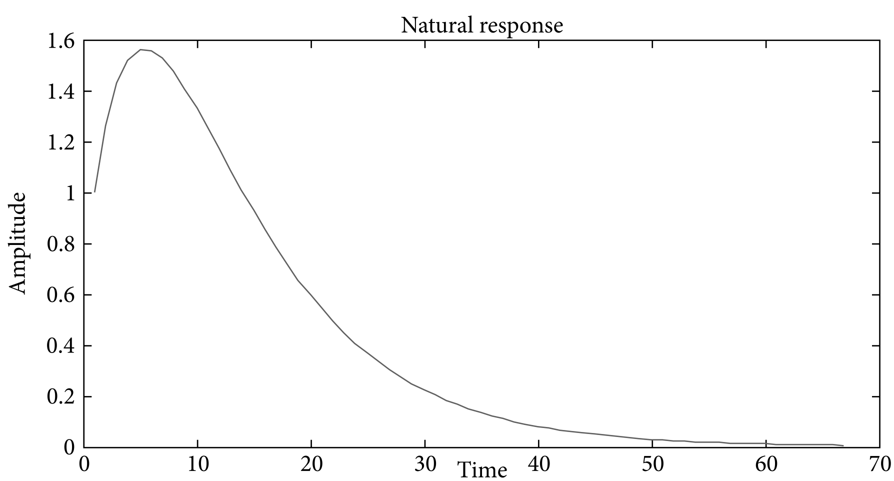


Fig. 7.37 Plot of natural response for P 7.45

```
clear all;
clc;
s = tf('s');
b=[1 3 2];
a=[1 6];
f=tf(a,b);
```

```
b=impulse(f);
plot(b);title('natural  response');xlabel('time');yl
abel('amplitude');
```

The response due to the second term is the forced response. To find the forced response, we will apply the input $x(t)=e^{-t}u(t)$

$$\frac{Y(s)}{X(s)}=\frac{1}{(s^2+3s+2)}$$

$$Y(s)=\frac{1}{(s+2)(s+1)}X(s)=\frac{1}{s(s+2)(s+1)}$$

We will use partial fraction expansion.

$$Y(s)=\frac{1}{s(s+2)(s+1)}=\frac{1/3}{s}-\frac{1}{s+1}+\frac{1/2}{s+2}$$

Teaser: The reader is encouraged to verify the coefficients.

Take ILT to find the solution. The forced response is the sum of the steady state response and the transient response. The response due to the input is the steady state response and the response due to the poles of the system is the transient response.

$$Y(s)=\frac{1/3}{s}-\frac{1}{s+1}+\frac{1/2}{s+2}$$

$$y(t)=\left(\frac{1}{3}u(t)\right)-\left(e^{-t}u(t)-\frac{1}{2}e^{-2t}u(t)\right)$$

A MATLAB program to plot the forced response is given as follows. The response is plotted in Fig. 7.38.

```
clear all;
clc;
s = tf('s');
b=[1 3 2 0];
a=[1];
f=tf(a,b);
b=impulse(f);
plot(b);title('forced  response');xlabel('time');yla
bel('amplitude');
```

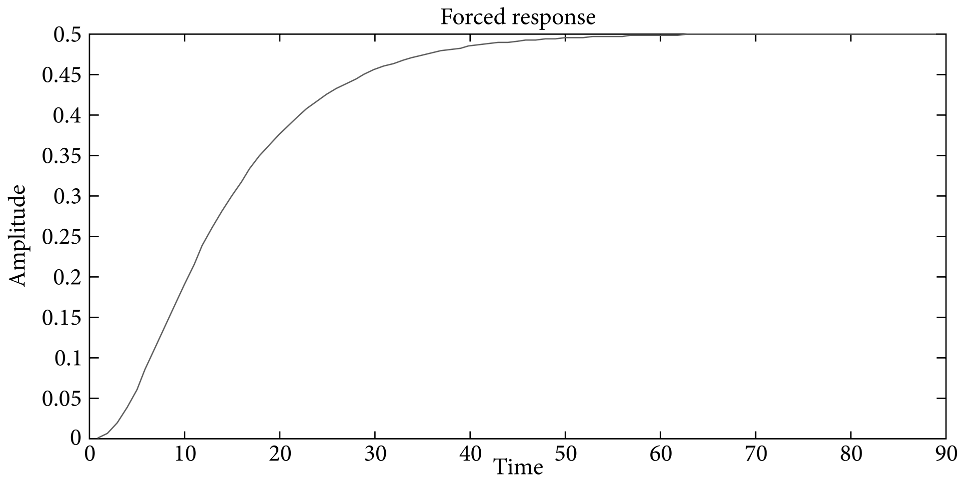


Fig. 7.38 Plot of forced response for P 7.45

The total response is given by the addition of the natural response and the forced response.

$$y(t) = 5e^{-t}u(t) - 4e^{-2t}u(t) + \left(\frac{1}{3}u(t)\right) - \left(e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t)\right)$$

$$y(t) = \frac{1}{3}u(t) + 4e^{-t}u(t) - \frac{9}{2}e^{-2t}u(t)$$

8

Z Transform

P 8.1 Find ZT of the following sequences

a. $f(n) = 2e^{-6n} - 2e^{-3n} + 24ne^{-6n}$

b. $f(n) = (1+n)U(n)$

c. $f(n) = \cos(n\omega T)U(n)$

d. $f(n) = na^n \sin(n\omega T)U(n)$

e. $f(n) = n^2U(n)$

f. $f(n) = \cos(n\pi / 3)U(n)$

Solution

a. $f(n) = 2e^{-6n} - 2e^{-3n} + 24ne^{-6n}$

We will assume that the sequence exists for $n \geq 0$. We will use the basic formula for ZT.

$$F(Z) = \sum_{n=0}^{\infty} f(n)Z^{-n}$$

Let us put the value of $f(n)$ in the equation to get

$$F(Z) = \sum_{n=0}^{\infty} 2e^{-6n}Z^{-n} - \sum_{n=0}^{\infty} 2e^{-3n}Z^{-n} + \sum_{n=0}^{\infty} 24ne^{-6n}Z^{-n} \tag{1}$$

$$F(Z)=2\sum_{n=0}^{\infty}(e^{-6}Z^{-1})^n-2\sum_{n=0}^{\infty}(e^{-3}Z^{-1})^n+24\sum_{n=0}^{\infty}n(e^{-6}Z^{-1})^n \tag{2}$$

We will use the closed form expression for the infinite sum to get

$$F(Z)=2\frac{1}{1-e^{-6}Z^{-1}}-2\frac{1}{1-e^{-3}Z^{-1}}+24\frac{e^{-6}Z^{-1}}{(1-e^{-6}Z^{-1})^2} \tag{3}$$

Note that we have used the property of differentiation in the Z domain to get the ZT of the last term.

To calculate the ROC, we have to equate the terms in bracket in Eq. (2) less than 1. ROC is thus

$$|e^{-6}Z^{-1}|<1\Rightarrow|Z|>|e^{-6}|$$

for the first and the last terms. For the second term, the condition is

$$|e^{-3}Z^{-1}|<1\Rightarrow|Z|>|e^{-3}|.$$

Thus, the ROC is the radius of the larger circle, that is, $|Z|>|e^{-6}|$.

b. $f(n)=(1+n)U(n)$

The sequence exists for $n\geq 0$ as it is appended by $U(n)$. We will use the basic formula for ZT.

$$F(Z)=\sum_{n=0}^{\infty}f(n)Z^{-n}$$

Let us put the value of $f(n)$ in the equation to get

$$F(Z)=\sum_{n=0}^{\infty}(1+n)Z^{-n}=\sum_{n=0}^{\infty}Z^{-n}+\sum_{n=0}^{\infty}nZ^{-n}$$

Therefore, the ZT is found as

$$F(Z)=\frac{1}{1-Z^{-1}}+\frac{Z^{-1}}{(1-Z^{-1})^2}=\frac{1-Z^{-1}+Z^{-1}}{(1-Z^{-1})^2}=\frac{1}{(1-Z^{-1})^2}$$

We can find ROC by putting $|Z^{-1}|<1\Rightarrow|Z|>1$

c. $f(n)=\cos(n\omega T)U(n)$

The sequence exists for $n \geq 0$ as it is appended by $U(n)$. We will use the basic formula for ZT.

$$F(Z)=\sum_{n=0}^{\infty} f(n)Z^{-n}$$

Let us put the value of $f(n)$ in this equation to get

$$F(Z)=\sum_{n=0}^{\infty}\cos(n\omega T)Z^{-n}=\frac{1}{2}\left[\sum_{n=0}^{\infty}e^{jn\omega T}Z^{-n}+\sum_{n=0}^{\infty}e^{-jn\omega T}Z^{-n}\right]$$

$$F(Z)=\frac{1}{2}\left[\frac{1}{1-e^{j\omega T}Z^{-1}}+\frac{1}{1-e^{-j\omega T}Z^{-1}}\right]=\frac{1}{2}\left[\frac{2-(e^{j\omega T}+e^{-j\omega T})Z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})Z^{-1}+Z^{-2}}\right]$$

$$F(Z)=\frac{1}{2}\left[\frac{2-2\cos(\omega T)Z^{-1}}{1-2\cos(\omega T)Z^{-1}+Z^{-2}}\right]$$

$$F(Z)=\frac{1-\cos(\omega T)Z^{-1}}{1-2\cos(\omega T)Z^{-1}+Z^{-2}}$$

We can find ROC by putting $|e^{j\omega T}Z^{-1}|<1 \Rightarrow |Z|>1$

and $|e^{-j\omega T}Z^{-1}|<1 \Rightarrow |Z|>1$ as $|e^{j\omega T}|=1$ and $|e^{-j\omega T}|=1$

d. $f(n)=na^{n}\sin(n\omega T)U(n)$

The sequence exists for $n \geq 0$ as it is appended by $U(n)$. We will use a basic formula for ZT and calculate ZT of $f(n) = \sin(n\omega T)U(n)$.

$$F(Z)=\sum_{n=0}^{\infty} f(n)Z^{-n}$$

Let us put the value of $f(n)$ in this equation to get

$$F(Z)=\sum_{n=0}^{\infty}\sin(n\omega T)Z^{-n}=\frac{1}{2j}\left[\sum_{n=0}^{\infty}e^{jn\omega T}Z^{-n}-\sum_{n=0}^{\infty}e^{-jn\omega T}Z^{-n}\right]$$

$$F(Z)=\frac{1}{2j}\left[\frac{1}{1-e^{j\omega T}Z^{-1}}-\frac{1}{1-e^{-j\omega T}Z^{-1}}\right]=\frac{1}{2j}\left[\frac{(e^{j\omega T}-e^{-j\omega T})Z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})Z^{-1}+Z^{-2}}\right]$$

$$F(Z)=\frac{1}{2j}\left[\frac{2j\sin(\omega T)Z^{-1}}{1-2\cos(\omega T)Z^{-1}+Z^{-2}}\right]$$

Now, we will calculate ZT of $f(n)=a^n\sin(n\omega T)U(n)$. We have to use the property of scaling in the Z domain to get ZT of $f(n)=a^n\sin(n\omega T)U(n)$.

$$F(Z)=\frac{\sin(\omega T)Z^{-1}}{1-2\cos(\omega T)Z^{-1}+Z^{-2}}$$

$$F(Z)=\frac{\sin(\omega T)aZ^{-1}}{1-2\cos(\omega T)aZ^{-1}+a^2Z^{-2}}$$

Note that we have replaced Z^{-1} by aZ^{-1}. We will use the property of differentiation in the Z domain to get ZT of $f(n) = na^n \sin(n\omega T)U(n)$. We have to calculate $-Zd(F(Z))/dZ$. The reader is encouraged to find the result. ROC is $|e^{j\omega T}aZ^{-1}|<1\Rightarrow|Z|>a$ and $|e^{-j\omega T}aZ^{-1}|<1\Rightarrow|Z|>a$ as $|e^{j\omega T}|=1$ and $|e^{-j\omega T}|=1$.

e. $f(n)=n^2U(n)$

The sequence exists for $n\geq 0$ as it is appended by $U(n)$. We will use a basic formula for ZT and calculate ZT of $U(n)$.

$$F(Z)=\sum_{n=0}^{\infty}f(n)Z^{-n}$$

Let us put the value of $f(n)$ in the equation to get

$$F(Z)=\sum_{n=0}^{\infty}Z^{-n}$$

$$F(Z)=\frac{1}{1-Z^{-1}}$$

We will use the property of differentiation in the Z domain two times to get ZT of $f(n) = n^2U(n)$.

$$F(Z)=\frac{1}{2!}\frac{d}{dZ}\left[\frac{d}{dZ}\left(\frac{1}{1-Z^{-1}}\right)\right]=\frac{1}{2}\frac{d}{dZ}\left(\frac{-Z^{-2}}{(1-Z^{-1})^2}\right)=\frac{1}{2}\frac{(1-Z^{-1})^2 2Z^{-3}+Z^{-2}(2Z^{-2}-2Z^{-3})}{(1-Z^{-1})^4}$$

ROC $|Z|>1$ and double zero is introduced at the origin or a pole at infinity. This indicates that ROC excludes infinity.

f. $f(n)=\cos(n\pi/3)U(n)$

The sequence exists for $n \geq 0$ as it is appended by $U(n)$. We will use a basic formula for ZT and calculate ZT of $U(n)$.

$$F(Z)=\sum_{n=0}^{\infty} f(n)Z^{-n}$$

Let us put the value of $f(n)$ in this equation to get

$$F(Z)=\sum_{n=0}^{\infty}\cos(n\pi/3)Z^{-n}=\frac{1}{2}\left[\sum_{n=0}^{\infty}e^{jn\pi/3}Z^{-n}+\sum_{n=0}^{\infty}e^{-jn\pi/3}Z^{-n}\right]$$

$$F(Z)=\frac{1}{2}\left[\frac{1}{1-e^{j\pi/3}Z^{-1}}+\frac{1}{1-e^{-j\pi/3}Z^{-1}}\right]=\frac{1}{2}\left[\frac{2-(e^{j\pi/3}+e^{-j\pi/3})Z^{-1}}{1-(e^{j\pi/3}+e^{-j\pi/3})Z^{-1}+Z^{-2}}\right]$$

$$F(Z)=\frac{1}{2}\left[\frac{2-2\cos(\pi/3)Z^{-1}}{1-2\cos(\pi/3)Z^{-1}+Z^{-2}}\right]$$

$$F(Z)=\frac{1-\cos(\pi/3)Z^{-1}}{1-2\cos(\pi/3)Z^{-1}+Z^{-2}}$$

We will find ROC by putting $|e^{j\pi/3}Z^{-1}|<1 \Rightarrow |Z|>1$ and $|e^{-j\pi/3}Z^{-1}|<1 \Rightarrow |Z|>1$ as $|e^{j\pi/3}|=1$ and $|e^{-j\pi/3}|=1$.

P 8.2 Given a 6-periodic sequence,

$$F(n)=\{1,1,1,-1,-1,-1,1,1,1,-1\}$$

show that

$$F(Z)=\frac{Z(Z^2+Z+1)}{Z^3+1}$$

Solution

We will use a basic formula for ZT and calculate ZT

$$F(Z) = 1 + Z^{-1} + Z^{-2} - Z^{-3} + Z^{-4} + Z^{-5} + \dots$$

Let us put the value of $f(n)$ in the equation to get

$$F(Z) = \left[1 + Z^{-1} + Z^{-2}\right] \times \left[1 - Z^{-3} + Z^{-6} - Z^{-9} \dots\right]$$

$$= \left[1 + Z^{-1} + Z^{-2}\right] \sum_{n=0}^{\infty} (-1)^n Z^{-3n}$$

$$= \left[1 + Z^{-1} + Z^{-2}\right] \frac{1}{1 + Z^{-3}}$$

$$= \left[\frac{Z^2 + Z + 1}{Z^2}\right] \times \left[\frac{Z^3}{Z^3 + 1}\right]$$

$$= \frac{Z(Z^2 + Z + 1)}{Z^3 + 1}$$

P 8.3 Express the Z transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

in terms of $X(Z)$.
Hint: Find the difference $y(n) - y(n-1)$.

Solution

The difference $y(n) - y(n-1)$ is found out as

$$y(n) - y(n-1) = \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x(k)$$

$$y(n) - y(n-1) = x(n)$$

Taking Z transforms on both sides, we get

$$ZT(y(n) - y(n-1)) = ZT(x(n))$$

$$Y(Z) - Z^{-1}Y(Z) = X(Z)$$

Therefore, the Z transform of $y(n)$ in terms of $X(Z)$ is obtained as

$$Y(Z) = \frac{X(Z)}{1 - Z^{-1}}$$

P 8.4 Find IZT of the following Z domain functions

a. $X(Z) = \dfrac{1 + 3Z^{-1}}{1 + 3Z^{-1} + 2Z^{-2}}$

b. $X(Z) = \dfrac{1 + 2Z^{-2}}{1 + Z^{-2}}$

c. $X(Z) = \dfrac{1 - aZ^{-1}}{Z^{-1} - a}$

d. $X(Z) = \dfrac{Z^{-6} + Z^{-7}}{1 - Z^{-1}}$

Solution

a. $X(Z) = \dfrac{1 + 3Z^{-1}}{1 + 3Z^{-1} + 2Z^{-2}}$

We will use the partial fraction method to calculate IZT.

Step 1 We will first decompose the denominator polynomial into a number of factors.

$$X(Z) = \frac{Z^2 + 3Z}{Z^2 + 3Z + 2} = \frac{Z^2 + 3Z}{(Z+1)(Z+2)}$$

Step 2 We express $X(Z)/Z$ as the sum of two terms

$$\frac{X(Z)}{Z} = \frac{Z^2 + 3Z}{Z^2 + 3Z + 2} = \frac{Z^2 + 3Z}{(Z+1)(Z+2)}$$

$$\frac{X(Z)}{Z} = \frac{k_1}{Z+1} + \frac{k_2}{Z+2}$$

Let us calculate k_1 and k_2

$$k_1 = \left.\frac{Z+3}{Z+2}\right|_{Z=-1} = \frac{-1+3}{-1+2} = 2$$

$$k_2 = \left.\frac{Z+3}{Z+1}\right|_{Z=-2} = \frac{-2+3}{-2+1} = -1$$

Step 3 We put these values in the equation for $X(Z)/Z$

$$\frac{X(Z)}{Z} = \frac{2}{Z+1} - \frac{1}{Z+2}$$

$$X(Z) = \frac{2}{1+Z^{-1}} - \frac{1}{1+2Z^{-1}}$$

Step 4 We find IZT

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n)$$

b. $$X(Z) = \frac{1+2Z^{-2}}{1+Z^{-2}}$$

We will use the residue method to find IZT

Step 1 Express $X(Z)/Z$ as a sum of two terms

$$\frac{X(Z)}{Z} = \frac{Z^2+2}{Z(Z^2+1)} = \frac{Z^2+2}{Z(Z+j)(Z-j)}$$

$$\frac{X(Z)}{Z} = \frac{k_1}{Z} + \frac{k_2}{Z+j} + \frac{k_3}{Z-j}$$

Let us calculate k_1, k_2 and k_3

$$k_1 = \left.\frac{Z^2+2}{Z^2+1}\right|_{Z=0} = \frac{2}{1} = 2$$

$$k_2 = \left.\frac{Z^2+2}{Z(Z-j)}\right|_{Z=-j} = \frac{-1+2}{-j(-j-j)} = \frac{1}{-2}$$

$$k_3 = \left.\frac{Z^2+2}{Z(Z+j)}\right|_{Z=j} = \frac{-1+2}{j(j+j)} = \frac{1}{-2}$$

Note: $(j)^2 = -1$

Step 3 Substitute values of k_1, k_2 and k_3 to get

$$\frac{X(Z)}{Z} = \frac{2}{Z} + \frac{-1/2}{Z+j} + \frac{-1/2}{Z-j}$$

$$X(Z) = 2 + \frac{(-1/2)Z}{Z+j} + \frac{(-1/2)Z}{Z-j}$$

Taking IZT we get,

$$x(n) = 2\partial(n) - \frac{1}{2}(-j)^n u(n) - \frac{1}{2}(j)^n u(n)$$

c. $X(Z) = \dfrac{1-aZ^{-1}}{Z^{-1}-a}$

We can write $X(Z)$ as

$$X(Z) = \frac{Z-a}{-aZ+1} = -\frac{1}{a}\left(\frac{Z-a}{Z-1/a}\right)$$

Let us calculate $X(Z)/Z$

$$\frac{X(Z)}{Z} = -\frac{1}{a}\left(\frac{Z-a}{Z(Z-1/a)}\right)$$

Step 1 Express $X(Z)/Z$ as a sum of two terms

$$\frac{X(Z)}{Z} = -\frac{Z-a}{aZ(Z-1/a)} = \frac{k_1}{Z} + \frac{k_2}{Z-1/a}$$

Step 2 Calculate k_1 and k_2

$$k_1 = -\frac{Z-a}{a(Z-1/a)}\bigg|_{Z=0} = \frac{a}{a(-1/a)} = -a$$

$$X(S) = \int_{-\infty}^{\infty} (1+\sin 3t \cos 3t)u(t)e^{-st}dt = \int_0^{\infty} \left(1+\frac{1}{2}\sin 6t\right) e^{-st}dt$$

Step 3 Substitute the values of k_1 and k_2; we get

$$\frac{X(Z)}{Z} = -\frac{a}{Z} + \frac{(a^2-1)/a}{Z-1/a}$$

$$X(Z) = -a + \frac{a^2-1}{a}\frac{Z}{Z-1/a}$$

Step 4 Taking IZT, we get

$$x(n) = -a\partial(n) + \frac{a^2-1}{a}\left(\frac{1}{a}\right)^n u(n)$$

d. $$X(Z) = \frac{Z^{-6}+Z^{-7}}{1-Z^{-1}}$$

We can write $X(Z)$ as

$$X(Z) = Z^{-6}\frac{1+Z^{-1}}{1-Z^{-1}} = Z^{-6}\frac{Z+1}{Z-1}$$

Step 1 Calculate $X(Z)/Z$.

$$\frac{X(Z)}{Z} = Z^{-6}\frac{Z+1}{Z(Z-1)}$$

We will take aside Z^{-6} and calculate IZT of the rest of the terms

$$\frac{X(Z)}{Z} = \frac{Z+1}{Z(Z-1)} = \frac{k_1}{Z} + \frac{k_2}{Z-1}$$

Step 2 Calculate k_1 and k_2

$$k_1 = \left.\frac{Z+1}{Z-1}\right|_{Z=0} = -1$$

$$k_2 = \left.\frac{Z+1}{Z}\right|_{Z=1} = 2$$

Step 3 Substitute the values of k_1 and k_2 and find IZT

$$X(Z) = -1 + \frac{2Z}{Z-1}$$

$$x(n) = -\partial(n) + 2u(n)$$

Taking Z^{-6} into consideration, we have to introduce the delay of 6 samples in the signal.

$$x(n) = -\partial(n-6) + 2u(n-6)$$

P 8.5 Use PFE to find IZT of the following ZTs:

a. $F(Z) = \dfrac{Z}{(Z-e^{-a})(Z-e^{-b})}$

where a and b are positive constants.

b. $F(Z) = \dfrac{Z^2}{(Z-1)(Z-0.8)}$

Solution

a. $$F(Z) = \frac{Z}{(Z - e^{-a})(Z - e^{-b})}$$

where a and b are positive constants.

Step 1 Find $F(Z)/Z$

$$\frac{F(Z)}{Z} = \frac{1}{(Z - e^{-a})(Z - e^{-b})} = \frac{k_1}{Z - e^{-a}} + \frac{k_2}{Z - e^{-b}}$$

Step 2 Find k_1 and k_2.

$$k_1 = \left.\frac{1}{Z - e^{-b}}\right|_{Z=e^{-a}} = \frac{1}{e^{-a} - e^{-b}}$$

$$k_2 = \left.\frac{1}{Z - e^{-a}}\right|_{Z=e^{-b}} = -\frac{1}{e^{-a} - e^{-b}}$$

Step 3 Substitute values of k_1 and k_2 and find $F(Z)$

$$F(Z) = (e^{-a} - e^{-b})\left[\frac{Z}{Z - e^{-a}} - \frac{Z}{Z - e^{-b}}\right]$$

Step 4 Find IZT

$$f(n) = \left[\frac{1}{e^{-a} - e^{-b}}\right]\left[(e^{-an} - e^{-bn})u(n)\right]$$

b. $$F(Z) = \frac{Z^2}{(Z - 1)(Z - 0.8)}$$

Step 1 Find $F(Z)/Z$

$$\frac{F(Z)}{Z} = \frac{Z}{(Z - 1)(Z - 0.8)} = \frac{k_1}{Z - 1} + \frac{k_2}{Z - 0.8}$$

Step 2 Find k_1 and k_2.

$$k_1 = \left.\frac{Z}{Z-0.8}\right|_{Z=1} = \frac{1}{1-0.8} = 5$$

$$k_2 = \left.\frac{Z}{Z-1}\right|_{Z=0.8} = -\frac{0.8}{0.8-1} = -4$$

Step 3 Substitute values of k_1 and k_2 and find $F(Z)$

$$F(Z) = \frac{5Z}{Z-1} - \frac{4Z}{Z-0.8}$$

Step 4 Find IZT

$$f(n) = \left[5(1)^n + 4(0.8)^n\right]u(n)$$

P 8.6 Using partial fraction expansion, find IZT of $F(Z)$ and verify it using the long division method.

$$F(Z) = \frac{1+2Z^1}{1-0.4Z^{-1}-0.12Z^{-2}}$$

if $f(n)$ is causal.

Solution

Given that

$$F(Z) = \frac{1+2Z^1}{1-0.4Z^{-1}-0.12Z^{-2}}$$

We can also write $F(Z)$ as

$$F(Z) = \frac{Z^2+2Z}{Z^2-0.4Z-0.12}$$

Step 1 Find $F(Z)/Z$.

$$\frac{F(Z)}{Z}=\frac{Z+2}{(Z-0.6)(Z+0.2)}=\frac{k_1}{Z-0.6}+\frac{k_2}{Z+0.2}$$

Step 2 Find k_1 and k_2.

$$k_1=\left.\frac{Z+2}{Z+0.2}\right|_{Z=0.6}=\frac{0.6+2}{0.6+0.2}=\frac{2.6}{0.8}=3.25$$

$$k_2=\left.\frac{Z+2}{Z-0.6}\right|_{Z=-0.2}=\frac{-0.2+2}{-0.2-0.6}=\frac{-1.8}{0.8}=-2.25$$

Step 3 Substitute values of k_1 and k_2 and find $F(Z)$.

$$F(Z)=\frac{3.25Z}{Z-0.6}-\frac{2.25Z}{Z+0.2}$$

Step 4 Find IZT.

$$f(n)=\left[3.25(0.6)^n-2.25(-0.2)^n\right]u(n)$$

We will also solve the problem using long division method.

$$F(Z)=\frac{1+2Z^1}{1-0.4Z^{-1}-0.12Z^{-2}}=1+2.4Z^{-1}+1.08Z^{-2}+\ldots$$

The reader is encouraged to verify that the results are the same.

P 8.7 Use residue method to find IZT of the following ZTs:

a. $F(Z)=\dfrac{Z^2+3Z}{(Z-0.5)^3}$

b. $F(Z)=\dfrac{1}{(Z-1)^2(Z-0.5)}$

Solution

a. $F(Z) = \dfrac{Z^2 + 3Z}{(Z-0.5)^3}$

Step 1 Find $G(Z) = F(Z)Z^{n-1}$

$$G(Z) = \frac{Z^{n+1} + 3Z^n}{(Z-0.5)^3}$$

Let $n \geq 1$.

Step 2 There is a double pole at $Z = 0.5$. Find the residue at Z = 0.5.

$$R[G(Z)]_{Z=0.5} = \frac{1}{2}\frac{d^2}{dZ^2}(Z^{n+1} + 3Z^n)$$

$$= \frac{1}{2}\frac{d}{dZ}\left[(n+1)Z^n + 3(n)Z^{n-1}\right]$$

$$= \frac{1}{2}\left[(n+1)nZ^{n-1} + 3n(n-1)Z^{n-2}\right]_{Z=0.5}$$

$$= \frac{1}{2}\left[(n+1)n(0.5)^{n-1} + 3n(n-1)(0.5)^{n-2}\right]$$

$$= n(n+1)(0.5)^n + 3n(n-1)(0.5)^{n-1}$$

Step 3 Find IZT.

$$f(n) = (0.5)^{n-1}\left[\frac{1}{2}n(n+1) + 3n(n-1)\right]u(n)$$

b. $F(Z) = \dfrac{1}{(Z-1)^2(Z-0.5)}$

Step 1 Find $G(Z) = F(Z)Z^{n-1}$

$$G(Z) = \frac{Z^{n-1}}{(Z-1)^2(Z-0.5)}$$

Let $n \geq 1$.

Step 2 There is a double pole at $Z = 1$ and a pole at $Z = 0.5$. Find the residue at Z = 1 and Z = 0.5.

$$R[G(Z)]\downarrow_{Z=0.5} = \frac{Z^{n-1}}{(Z-1)^2}\downarrow_{Z=0.5} = \frac{(0.5)^{n-1}}{(-0.5)^2} = 4(0.5)^{n-1}$$

$$R[G(Z)]\downarrow_{Z=-1} = \frac{d}{dZ}\left[\frac{Z^{n-1}}{(Z-0.5)}\right]\downarrow_{Z=1} = \frac{(Z-0.5)(n-1)Z^{n-2} - Z^{n-1}}{(Z-0.5)^2}\downarrow_{Z=1}$$

$$= \frac{(0.5)(n-1)(1)^{n-2} - (1)^{n-1}}{(0.5)^2} = \frac{(n-1)(0.5)-1}{(0.5)^2} = 2(n-1)-4$$

Step 3 Find IZT

$$f(n) = 4(0.5)^{n-1} + 2(n-1) - 4 \text{ for } n \geq 1$$

P 8.8 Given the difference equation

$$y(n) + b^2 y(n-2) = 0 \text{ for } n \geq 0 \text{ and } |b| < 1$$

with initial conditions $y(-1) = 0$ and $y(-2) = -1$, show that

$$y(n) = b^{n+2}\cos\left(\frac{n\pi}{2}\right)$$

Solution

Take ZT of each term

$$Y(Z) + b^2\left[Z^{-2}Y(Z) + Z^{-1}y(-1) + y(-2)\right] = 0$$

Putting values of $y(-1)$ and $y(-2)$, we get

$$Y(Z)(1 + b^2Z^{-2}) = b^2$$

$$Y(Z) = \frac{b^2Z^2}{Z^2 + b^2}$$

We will use the residue method to find IZT

$$G(Z)=\frac{b^2Z^{n+1}}{(Z+jb)(Z-jb)}$$

Now $y(n)$ can be written as

$$y(n)=R_{Z=jb}+R_{Z=-jb}$$

where

$$R_{Z=jb}=\frac{b^2b^{n+1}(j)^{n+1}}{2jb}=\frac{b^{n+2}(j)^n}{2}$$

As $j = e^{j\pi/2}$, on putting this value we get $R_{Z=jb}$ as

$$R_{Z=jb}=\frac{b^2b^{n+1}(j)^{n+1}}{2jb}=\frac{b^{n+2}(e)^{jn\pi/2}}{2}$$

and

$$R_{Z=-jb}=\frac{b^2b^{n+1}(-j)^{n+1}}{2(-j)b}=\frac{b^{n+2}(e)^{-jn\pi/2}}{2}$$

Taking IZT, we get

$$y(n)=\frac{b^{n+2}}{2}\left[e^{jn\pi/2}+e^{-jn\pi/2}\right]=b^{n+2}\cos\left(\frac{n\pi}{2}\right)$$

P 8.9 Find $f(n)$ corresponding to the difference equation

$$f(n-2)-2f(n-1)+f(n)=1 \text{ for } n\geq 0$$

with initial conditions $f(-1) = -0.5$ and $f(-2) = 0$. Show that

$$f(n)=(0.5)n^2+n \text{ for } n\geq 0$$

Solution

Given that

$$f(n-2)-2f(n-1)+f(n)=1 \text{ for } n \geq 0$$

Taking ZT, we get

$$z^{-2}F(Z)+Z^{-1}f(-1)+f(-2)-2Z^{-1}F(Z)-2f(-1)+F(Z)=\frac{Z}{Z-1}$$

Putting values of $f(-1)$ and $f(-2)$, we get

$$F(Z)=\frac{Z^2}{(Z-1)^3}+\frac{0.5Z}{(Z-1)^2}=\frac{1.5Z^2-0.5Z}{(Z-1)^3}$$

We can find $f(n)$ using the residue method where

$$G(Z)=\frac{1.5Z^{n+1}-0.5Z^n}{(Z-1)^3}$$

and IZT is found to be

$$f(n)=\frac{1}{2}\frac{d^2}{dZ^2}[1.5Z^{n+1}-0.5Z^n]=0.5n^2+n \text{ for } n \geq 0$$

9

Random Signals and Processes

P 9.1 Write the question as 'Let there be 80 balls in a box, all of same size and shape. There are 20 balls of red color, 30 balls of blue color and 30 balls of green color. Let us consider the event of drawing one ball from the box. Find the probability of drawing a red ball, blue ball and a green ball.

$$P(\text{red ball}) = {}^{20}\!/_{80} = 25\%\,,\ P(\text{blue ball}) = {}^{30}\!/_{80} = 37.5\%\,,$$

$$P(\text{green ball}) = {}^{30}\!/_{80} = 37.5\%$$

Total probability sums to one.

P 9.2 Let the event A be drawing a red ball in problem 1 and event B be drawing a blue card on second draw without replacing the first ball drawn. Find. P(A intersection B)

$$P(A) = 2/8 \text{ and } P(B/A) = \frac{30}{79}$$

$$\text{So, } P(A \cap B) = P(B/A) \times P(A) = \frac{30}{79} \times \frac{2}{8} = \frac{60}{632}$$

P 9.3 Consider a binary symmetric channel with a priori probabilities. A priori probabilities indicate the probability of transmission of symbols 0 and 1 before the experiment is performed i.e. before transmission takes place. The conditional probabilities are given by. Find received symbol probabilities. Find the transmission probabilities for correct transmission and transmission with error.

Received symbol probabilities can be calculated as

$$P(A_1) = P(A_1 / B_1) \times P(B_1) + P(A_1 / B_2) \times P(B_2)$$

$$= 0.8 \times 0.8 + 0.2 \times 0.2$$

$$= 0.68$$

$$P(A_2) = P(A_2 / B_1) \times P(B_1) + P(A_2 / B_2) \times P(B_2)$$

$$= 0.2 \times 0.8 + 0.8 \times 0.2$$

$$= 0.32$$

The probabilities for correct symbol transmission is given by

$$P(B_1 / A_1) = \frac{P(A_1 / B_1) \times P(B_1)}{P(A_1)} = \frac{0.8 \times 0.8}{0.68} \approx 0.94$$

$$P(B_2 / A_2) = \frac{P(A_2 / B_2) \times P(B_2)}{P(A_2)} = \frac{0.8 \times 0.2}{0.32} \approx 0.5$$

Now, let us calculate the probabilities of error

$$P(B_1 / A_2) = \frac{P(A_2 / B_1) \times P(B_1)}{P(A_2)} = \frac{0.2 \times 0.8}{0.32} \approx 0.5$$

$$P(B_2 / A_1) = \frac{P(A_1 / B_2) \times P(B_2)}{P(A_1)} = \frac{0.2 \times 0.2}{0.68} \approx 0.06$$

P 9.4 Probability of having HIV is P(H) = 0.2. Probability of not having HIV is, probability for getting test positive given that person has HIV is and probability for getting test positive given that person is not having HIV is find the probability that person has HIV given that the test is positive.

$$P(H / \text{Pos}) = \frac{P(\text{Pos} / H) \times P(H)}{P(H) \times P(\text{Pos} / H) + P(\bar{H}) \times P(\text{Pos} / \bar{H})}$$

$$= \frac{0.2 \times 0.95}{0.2 \times 0.95 + .8 \times 0.02} = 0.9223$$

P 9.5 Consider a pair of dice. Find the probability of getting the sum of the dots on the two faces as less than 5. Also find the probability for getting sum of faces equal to 9.

We understand that the probability of getting any particular combination of faces is 1/6 × 1/6 = 1/36.

Table 9.1 Possible outcomes for throw of pair of dice

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Total outcomes are 36. Probability of getting the sum of faces as less than 5: count such combinations for which sum of faces is less than or equal to 5. Probability is 10/36. The probability for getting sum of faces equal to 8: such combinations include (4,4), (3,5), (5,3), (2,6), (6,2). The probability is 5/36.

P 9.6 A random variable has a distribution function given by

$$\begin{aligned} F_X(x) &= 0 \quad -\infty < x \leq -8 \\ &= \frac{1}{6} \quad -8 \leq x \leq -5 \\ &= \frac{x}{15} + \frac{x}{2} \quad -5 < x < 5 \\ &= \frac{5}{6} \quad 5 \leq x < 8 \\ &= 1 \quad 8 \leq x < \infty \end{aligned}$$

Draw the CDF. Find $P(X \leq 4)$ and $P(-5 \leq X \leq 5)$.

Plot the pdf for a CDF specified in problem 9.5.

Let us find $P(X \leq 4) = 4/15 + 1/2 = 23/30 = F_X(4)$.

$$P(-5 \leq X \leq 4) = F_X(4) - F_X(-5) = 23/30 - 1/6 = 18/30$$

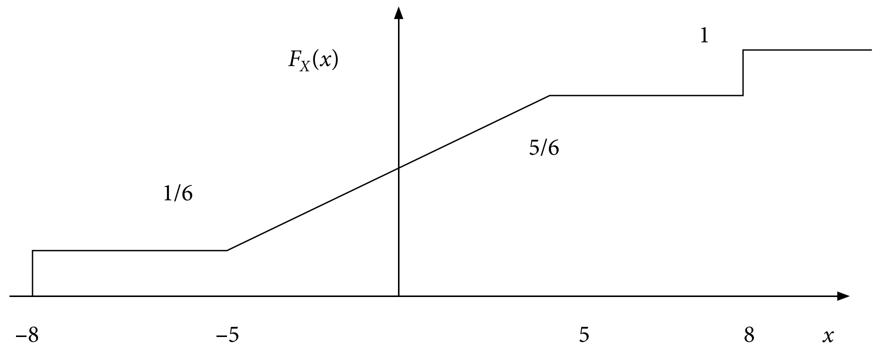


Fig. 9.1 Plot of CDF for Problem 9.6

P 9.7 Consider a random variable X with a probability density function shown in Fig. 1. Find A, mean value of X and variance of X.

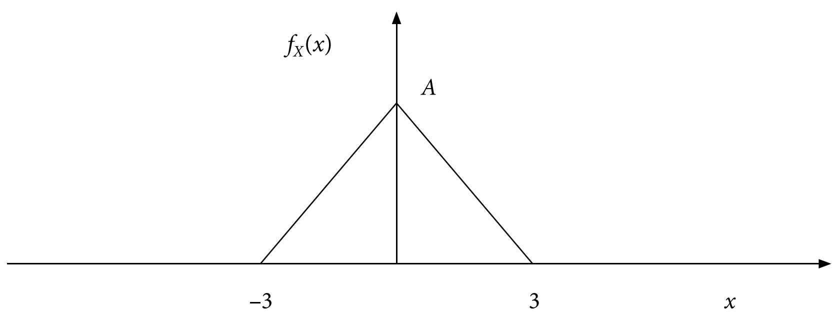


Fig. 9.2 Pdf for a random variable x

Solution

1. $A = 1/3$

   To find the mean value = 0

   To find the variance = 7.8333

P 9.8 If a uniform random variable is defined as

$$f_X(x) = K \text{ if } 3 \le x \le 7$$
$$= 0 \text{ otherwise}$$

Find K, mean value and variance.

K = ¼.

$\mu = 5$.

Variance = 4/3

P 9.9 Find the probability of the event $(X \le 4)$ for a Gaussian variable having a mean value of 2 and variance of 1.

$$y = (4-2)/1 = 2/1 = 2$$

$$F_Y(y) = P(y \le 2) = 0.5793$$

We will now use the table for a normalized distribution to find the value of CDF. Referring to the normal distribution function table, we can read $F(4) = P(Y \le 2) = 0.5793$.

P 9.10 Find the total number of combinations of 5 things taken from 10 things.

$$\text{Number of possible combinations } = \begin{pmatrix} n \\ r \end{pmatrix} = P_r^n / P_r^r$$

$$= n(n-i)(n-2)....(n-r+1)/r! = \frac{n!}{(n-r)!r!} = \frac{10!}{5!5!} = \frac{3628800}{14400} = 252$$

P 9.11 Find the total number of permutations of 5 things taken from 10 things.

$$\text{Number of possible outcomes} = P_r^n = (n(n-i)(n-2)....(n-r+1)$$

$$\text{Permutations} = \frac{n!}{(n-r)!} = \frac{10!}{5!} = \frac{3628800}{120} = 30240$$

P 9.12 Write a MATLAB program to generate a Gaussian random variable. Use a rand command to generate 12 random variables with uniform distribution. (use Central limit theorem)

A MATLAB program is given as follows.

```
clear all;
x1=rand(100);
x2=rand(100);
x3=rand(100);
x4=rand(100);
x5=rand(100);
x6=rand(100);
x7=rand(100);
x8=rand(100);
x9=rand(100);
```

```
x10=rand(100);
x11=rand(100);
x12=rand(100);
x13=rand(100);
x14=rand(100);
y=(1/sqrt(14))*(x1+x2+x3+x4+x5+x6+x7+x8+x9+x10+
x11+x12);
z=hist(y,10);
bar(z);title('plot of a sum variable generated
using   central   limit   theorem');xlabel('bin
value');ylabel('amplitude');
```

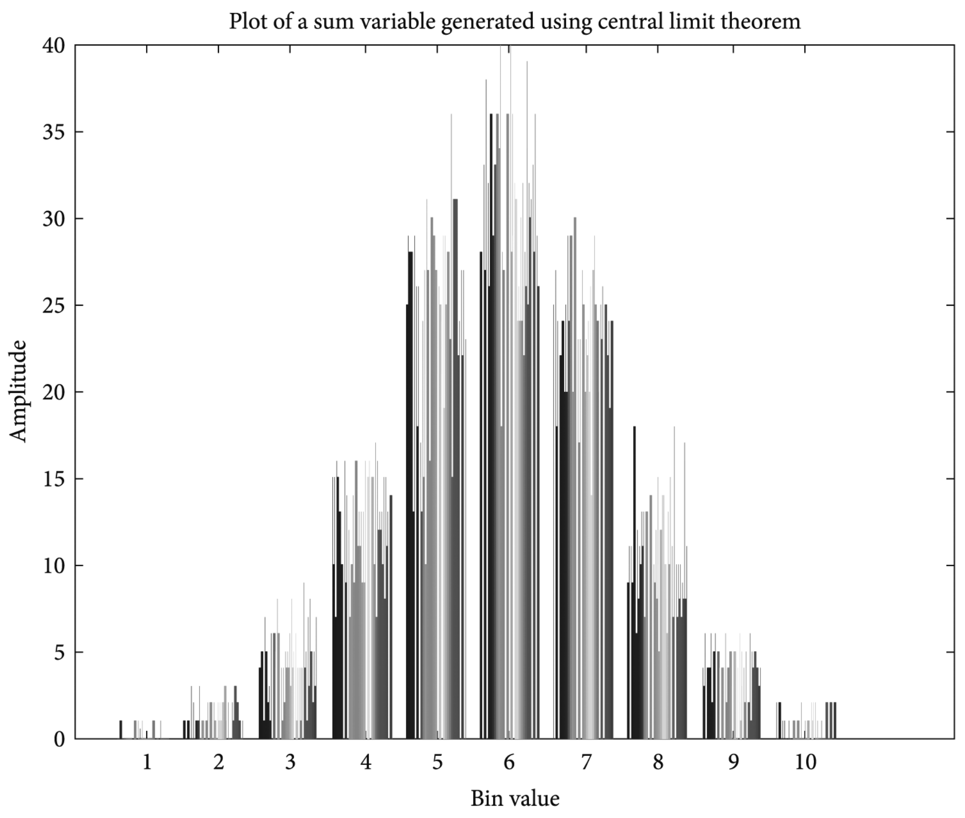


Fig. 9.3 Plot of sum of variables for P 9.12

P 9.13 The experiment is conducted with a fair die. The die is tossed 240 times, for example. The data obtained are shown in the following table. Find the test statistic to check if it obeys a uniform distribution.

Face No.	1	2	3	4	5	6
Observed frequency	36	40	44	38	42	40

$$\chi^2 = \sum_{i=1}^{N} \frac{(g_i - f_i)^2}{f_i}$$

$$= \frac{(36-40)^2}{40} + \frac{(40-40)^2}{40} + \frac{(44-40)^2}{40} + \frac{(38-40)^2}{40} + \frac{(42-40)^2}{40} + \frac{(40-40)^2}{40}$$

$$= 1.0$$

This value of test statistic is less than the 5 percent level equal to $240 \times 0.05 = 1.2$. Hence, the distribution is uniform.

P 9.14 Consider a random process given by $x(t) = 10\cos(2\pi t + \theta)$. Prove that the process is wide sense stationary if ϑ is a uniformly distributed random variable on the interval 0 to π.

Let us find the mean value of a process.

$$E[x(t)] = \frac{1}{2\pi} \int_0^{2\pi} 10\cos(2\pi t + \vartheta) d\vartheta = 0$$

Let us find the autocorrelation function.

$$R_{xx}(\tau) = E[x(t)x(t+\tau)] = E[A\cos(\omega_0 t + \vartheta) A\cos(\omega_0 t + \omega_0 \tau + \vartheta)]$$

$$= \frac{A^2}{2}[\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\vartheta)]$$

$$= \frac{A^2}{2}\cos(\omega_0 \tau) = \frac{100}{2}\cos(2\pi\tau)$$

We find that the autocorrelation is a function of time difference; the mean value is constant. The process is a wide sense stationary process.