18

Dynamic Assignment of Patients to Primary and Secondary Inpatient Units: Is Patience a Virtue?

ONLINE APPENDIX 18.A DESCRIPTION OF IWS IN MCA

TABLE 18.EC.1 *IWs and their size in MCA*

IW name	**Abbrev.**	**Definition**	**Number of beds**
2 West	2W	Intensive care unit (ICU)	30
3 East	3E	Orthopedics and urology surgical services	40
3 West	3W	Medical/surgical organ transplant	36
4 East	4E	Bone marrow transplant, hematology, and oncology	30
4 West	4W	Cardiology and cardiothoracic surgery	36
5 West	5W	Neurosciences and earn, nose, and throat (ENT)	36
7 East	7E	Palliative care and general surgery	36
7 West	7W	Hematology and oncology patients with medical-surgical overflow	24

ONLINE APPENDIX 18.B DEFINITION OF THE FUNCTIONAL OPERATORS

In this part, we define the operators $T^{\underline{u_{ij}}}$, which defines the cost function $J(\cdot)$ after a decision $\underline{u_{ij}}$ is taken at state $\tilde{X}$. Here, we explicitly define this operator for each state $\tilde{X}$.

When the system state is $\tilde{X} = (X_1 \geq 1, X_2 \geq 1, \underline{a_1} \neq 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{(u_{i1}=,u_{i2}=0)}J(\tilde{X}) =& J(X_1, X_2, \underline{a_1} \neq 0, \underline{a_2} = 0),\\
T^{(u_{i1}=,u_{12}=1)}J(\tilde{X}) =& J(X_1 - 1, X_2, \underline{a_1} \neq 0, \underline{a_2} = e_1) + p_{12}\\
T^{(u_{i1}=,u_{22}=1)}J(\tilde{X}) =& J(X_1, X_2 - 1, \underline{a_1} \neq 0, \underline{a_2} = e_2).
\end{aligned}$$

When the state is $\tilde{X} = (X_1 \geq 2, X_2 \geq 2, \underline{a_1} = 0, \underline{a_2} \neq 0)$:

$$\begin{aligned}
T^{(u_{i1}=0,u_{i2}=0)}J(\tilde{X}) =& J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} \neq 0)\\
T^{(u_{11}=1,u_{i2}=0)}J(\tilde{X}) =& J(X_1 - 1, X_2, \underline{a_1} = e_1, \underline{a_2} \neq 0)\\
T^{(u_{21}=1,u_{i2}=0)}J(\tilde{X}) =& J(X_1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} \neq 0) + p_{21}.
\end{aligned}$$

When the system state is $\tilde{X} = (X_1 \geq 2, X_2 \geq 2, \underline{a_1} = 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0}J(\tilde{X}) =& J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} = 0)\\
T^{(u_{11}=1,u_{22}=1)}J(\tilde{X}) =& J(X_1 - 1, X_2 - 1, \underline{a_1} = e_1, \underline{a_2} = e_2)\\
T^{(u_{11}=1,u_{12}=1)}J(\tilde{X}) =& J(X_1 - 2, X_2, \underline{a_1} = e_1, \underline{a_2} = e_1) + p_{12}\\
T^{(u_{11}=1,u_{i2}=0)}J(\tilde{X}) =& J(X_1 - 1, X_2, \underline{a_1} = e_1, \underline{a_2} = 0)\\
T^{(u_{i1}=0,u_{12}=1)}J(\tilde{X}) =& J(X_1 - 1, X_2, \underline{a_1} = 0, \underline{a_2} = e_1) + p_{12}\\
T^{(u_{i1}=0,u_{22}=1)}J(\tilde{X}) =& J(X_1, X_2 - 1, \underline{a_1} = 0, \underline{a_2} = e_2)\\
T^{(u_{21}=1,u_{i2}=0)}J(\tilde{X}) =& J(X_1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = 0) + p_{21}\\
T^{(u_{21}=1,u_{22}=1)}J(\tilde{X}) =& J(X_1, X_2 - 2, \underline{a_1} = e_2, \underline{a_2} = e_2) + p_{21}\\
T^{(u_{12}=1,u_{21}=1)}J(\tilde{X}) =& J(X_1 - 1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = e_1) + p_{12} + p_{21}.
\end{aligned}$$

When the state is $\tilde{X} = (X_1 = 0, X_2 \geq 2, \underline{a_1} = 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0}J(\tilde{X}) =& J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} = 0)\\
T^{(u_{i1}=0,u_{22}=0)}J(\tilde{X}) =& J(X_1, X_2 - 1, \underline{a_1} = 0, \underline{a_2} = e_2)\\
T^{(u_{21}=1,u_{i2}=0)}J(\tilde{X}) =& J(X_1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = 0) + p_{21}\\
T^{(u_{21}=1,u_{22}=1)}J(\tilde{X}) =& J(X_1, X_2 - 2, \underline{a_1} = e_2, \underline{a_2} = e_2) + p_{21}.
\end{aligned}$$

When the state is $\tilde{X} = (X_1 \geq 2, X_2 = 0, \underline{a_1} = 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0}J(\tilde{X}) =& J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} = 0)\\
T^{(u_{11}=1,u_{12}=1)}J(\tilde{X}) =& J(X_1 - 2, X_2, \underline{a_1} = e_1, \underline{a_2} = e_1) + p_{12}\\
T^{(u_{11}=1,u_{i2}=0)}J(\tilde{X}) =& J(X_1 - 1, X_2, \underline{a_1} = e_1, \underline{a_2} = 0)\\
T^{(u_{i1}=0,u_{12}=1)}J(\tilde{X}) =& J(X_1 - 1, X_2, \underline{a_1} = 0, \underline{a_2} = e_1) + p_{12}
\end{aligned}$$

When the state is $\tilde{X} = (X_1 \geq 2, X_2 = 1, \underline{a_1} = 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0} J(\tilde{X}) &= J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} = 0) \\
T^{(u_{11}=1, u_{22}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2 - 1, \underline{a_1} = e_1, \underline{a_2} = e_2) \\
T^{(u_{11}=1, u_{12}=1)} J(\tilde{X}) &= J(X_1 - 2, X_2, \underline{a_1} = e_1, \underline{a_2} = e_1) + p_{12} \\
T^{(u_{11}=1, u_{i2}=0)} J(\tilde{X}) &= J(X_1 - 1, X_2, \underline{a_1} = e_1, \underline{a_2} = 0) \\
T^{(u_{i1}=0, u_{12}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2, \underline{a_1} = 0, \underline{a_2} = e_1) + p_{12} \\
T^{(u_{i1}=0, u_{22}=1)} J(\tilde{X}) &= J(X_1, X_2 - 1, \underline{a_1} = 0, \underline{a_2} = e_2) \\
T^{(u_{21}=1, u_{i2}=0)} J(\tilde{X}) &= J(X_1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = 0) + p_{21} \\
T^{(u_{12}=1, u_{21}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = e_1) + p_{12} + p_{21}.
\end{aligned}$$

When the state is $\tilde{X} = (X_1 = 1, X_2 \geq 2, \underline{a_1} = 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0} J(\tilde{X}) &= J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} = 0) \\
T^{(u_{11}=1, u_{22}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2 - 1, \underline{a_1} = e_1, \underline{a_2} = e_2) \\
T^{(u_{11}=1, u_{i2}=0)} J(\tilde{X}) &= J(X_1 - 1, X_2, \underline{a_1} = e_1, \underline{a_2} = 0) \\
T^{(u_{i1}=0, u_{12}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2, \underline{a_1} = 0, \underline{a_2} = e_1) + p_{12} \\
T^{(u_{i1}=0, u_{22}=1)} J(\tilde{X}) &= J(X_1, X_2 - 1, \underline{a_1} = 0, \underline{a_2} = e_2) \\
T^{(u_{21}=1, u_{i2}=0)} J(\tilde{X}) &= J(X_1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = 0) + p_{21} \\
T^{(u_{21}=1, u_{22}=1)} J(\tilde{X}) &= J(X_1, X_2 - 2, \underline{a_1} = e_2, \underline{a_2} = e_2) + p_{21} \\
T^{(u_{12}=1, u_{21}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} = e_1) + p_{12} + p_{21}.
\end{aligned}$$

When the state is $\tilde{X} = (X_1 \geq 2, X_2 = 2, \underline{a_1} \neq 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0} J(\tilde{X}) &= J(X_1, X_2, \underline{a_1} \neq 0, \underline{a_2} = 0) \\
T^{(u_{i1}, u_{12}=1)} J(\tilde{X}) &= J(X_1 - 1, X_2, \underline{a_1} \neq 0, \underline{a_2} = e_1) + p_{12}.
\end{aligned}$$

When the state is $\tilde{X} = (X_1 = 0, X_2 \geq 2, \underline{a_1} \neq 0, \underline{a_2} = 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0} J(\tilde{X}) &= J(X_1, X_2, \underline{a_1} \neq 0, \underline{a_2} = 0) \\
T^{(u_{i1}, u_{22}=1)} J(\tilde{X}) &= J(X_1, X_2 - 1, \underline{a_1} \neq 0, \underline{a_2} = e_2).
\end{aligned}$$

When the state is $\tilde{X} = (X_1 \geq 2, X_2 = 0, \underline{a_1} = 0, \underline{a_2} \neq 0)$,

$$\begin{aligned}
T^{\underline{u_{ij}}=0} J(\tilde{X}) &= J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} \neq 0) \\
T^{(u_{11}=1, u_{i2}=0)} J(\tilde{X}) &= J(X_1 - 1, X_2, \underline{a_1} = e_1, \underline{a_2} \neq 0).
\end{aligned}$$

When the state is $\tilde{X} = (X_1 = 0, X_2 \geq 2, \underline{a_1} = 0, \underline{a_2} \neq 0)$,

$$\begin{aligned} T^{\underline{u_{ij}}=0} J(\tilde{X}) &= J(X_1, X_2, \underline{a_1} = 0, \underline{a_2} \neq 0) \\ T^{(u_{21}=1, u_{i2}=0)} J(\tilde{X}) &= J(X_1, X_2 - 1, \underline{a_1} = e_2, \underline{a_2} \neq 0) + p_{12}. \end{aligned}$$

ONLINE APPENDIX 18.C PROOFS

18.C.1 Nonidling Policy

In this section, we show that IW j should not be idled when a patient of class j (i.e., a patient whose primary IW is j) is boarded in the ED. We first establish the following monotonicity result.

> **Lemma 18.EC.1** (Monotonicity) *For any $\tilde{X} \in \mathcal{S}$, $n \in \mathbb{Z}^+$, $\beta \in [0, 1)$, and $k \in N_p$: $V_{n,\beta}(\underline{X} + e_k, \underline{a_1}, \underline{a_2}) \geq V_{n,\beta}(\underline{X}, \underline{a_1}, \underline{a_2})$, where $V_{n,\beta}(\cdot)$ represents the n-period discounted cost when the discount factor is β.*

Proof of Lemma 18.EC.1

Similar to Eq. (18.4), the finite-horizon discounted cost optimality equation can be written as

$$\begin{aligned} V_{n+1,\beta}(\tilde{X}) = \frac{1}{\psi}\Bigg[\underline{\theta}\,\underline{X}^T + \beta \min_{u=\underline{u_{ij}} \in \mathcal{U}(\tilde{X})} \Bigg\{ \sum_{i \in N_p} \sum_{j \in N_s} \lambda_i T^{\underline{u_{ij}}} V_{n,\beta}(\underline{X} + e_i, \underline{a_j}) + \sum_{i \in N_p} \sum_{j \in N_s} \sum_{l \in N_p} a_{lj}\mu_l T^{\underline{u_{ij}}} V_{n,\beta}(\underline{X}, \underline{a_j} - e_l) \\ + \left(\psi - \sum_{i \in N_p} \lambda_i - \sum_{k \in N_p} a_{kj}\mu_k\right) \sum_{j \in N_s} V_{n,\beta}(\tilde{X}) \Bigg\}\Bigg], \end{aligned} \tag{18.EC.1}$$

where $V_{n,\beta}(\tilde{X})$ is the optimal cost of the n-period problem starting at state $\tilde{X}$, along with terminal condition $V_{0,\beta}(\tilde{X}) = 0$ for every $\tilde{X} \in \mathcal{S}$. We prove this lemma by induction on n. For $n = 0$, we have $V_{0,\beta}(\tilde{X})$=0. Hence, $V_{0,\beta}(\underline{X} + e_k, \underline{a_1}, \underline{a_2}) = V_{0,\beta}(\underline{X}, \underline{a_1}, \underline{a_2}) = 0$ $(\forall \tilde{X} \in \mathcal{S}, \forall \beta \in [0, 1), \forall k \in N_p)$. Assume that, for some $n \in \mathbb{Z}^+$, the required condition holds: $V_{n,\beta}(\underline{X} + e_k, \underline{a_1}, \underline{a_2}) \geq V_{n,\beta}(\underline{X}, \underline{a_1}, \underline{a_2})$ for any $\tilde{X} \in \mathcal{S}$, $\beta \in [0, 1)$ and $k \in N_p$. We now show that the same condition holds for $n + 1$. From Eq. (18.EC.1), we have

$$\begin{aligned} V_{n+1,\beta}(\underline{X} + e_k, \underline{a_1}, \underline{a_2}) = \frac{1}{\psi}\Bigg[\underline{\theta}(\underline{X} + e_k)^T + \beta \Bigg\{ \sum_{j \in N_s} \left(\lambda_1 T^{\underline{u_{1j}}} V_n(\underline{X} + e_1 + e_k, \underline{a_1}, \underline{a_2}) + \lambda_2 T^{\underline{u_{2j}}} V_n(\underline{X} + e_2 + e_k, \underline{a_1}, \underline{a_2})\right) \\ + \sum_{i \in N_p} \sum_{j \in N_s} \sum_{l \in N_p} a_{jl}\left(p_{ij} + \mu_l V_n(\underline{X} + e_k - e_i, \underline{a_j} - e_l + e_i)\right) + \left(\psi - \lambda_1 - \lambda_2 - \sum_{j \in N_s} \sum_{l \in N_p} a_{jl}\mu_l\right) V_n(\underline{X} + e_k, \underline{a_1}, \underline{a_2}) \Bigg\}\Bigg] \end{aligned} \tag{18.EC.2}$$

If the set of admissible actions are the same for both of the states $(\underline{X} + e_k, \underline{a_1}, \underline{a_2})$ and $(\underline{X}, \underline{a_1}, \underline{a_2})$, the proof is straightforward and follows directly from Eq. (18.EC.2). However, because $\mathcal{U}(\underline{X} + e_k, \underline{a_1}, \underline{a_2})$ can be a larger admissible set than $\mathcal{U}(\underline{X},\underline{a_1},\underline{a_2})$, the optimal action $u^* \in \mathcal{U}(\underline{X}+e_k,\underline{a_1},\underline{a_2})$ may not belong to $\mathcal{U}(\underline{X},\underline{a_1},\underline{a_2})$. If $u^* \notin \mathcal{U}(\underline{X},\underline{a_1},\underline{a_2})$, WLOG assume that $k=1$ and observe that the only possibility for $u^* \notin \mathcal{U}(\underline{X},\underline{a_1},\underline{a_2})$ is that queue 1 is empty at state $(\underline{X},\underline{a_1},\underline{a_2})$. We show that, if the same allocation policy u^* is used at this state but the IW that is assigned to class 1 patients under u^* (say IW 1) is idled, and $\underline{X}+e_1$ is swapped with $\underline{X}$, a lower (or equal) value than $V_{n+1,\beta}(\underline{X}+e_k,\underline{a_1},\underline{a_2})$ can be obtained. That is, following a suboptimal policy at at state $(\underline{X},\underline{a_1},\underline{a_2})$ yields a cost that is not higher than $V_{n+1,\beta}(\underline{X}+e_k,\underline{a_1},\underline{a_2})$. The proof is then established because $V_{n+1,\beta}(\underline{X},\underline{a_1},\underline{a_2})$ is the optimal cost at state $(\underline{X},\underline{a_1},\underline{a_2})$.

First, rewrite the formulation by separating the action related to class 1.

$$\begin{aligned}
V_{n+1,\beta}(\underline{X}+e_1,\underline{a_1},\underline{a_2}) &= \frac{1}{\psi}\Bigg[\underline{\theta}(\underline{X}+e_1)^T+\beta\Bigg\{\sum_{j\in N_s}\Big(\lambda_1 T^{\underline{u_{1j}}}V_n(\underline{X}+e_1+e_1,\underline{a_j})+\lambda_2 T^{\underline{u_{2j}}}V_n(\underline{X}+e_2+e_1,\underline{a_j})\Big)\\
&+\sum_{i\in N_p}\sum_{k\in N_p} a_{1k}\left(p_{i1}+\mu_k(V_n(\underline{X}+e_1-e_i,\underline{a_1}-e_k+e_i,\underline{a_2})-V_n(\underline{X}+e_1,\underline{a_1},\underline{a_2}))\right)\\
&+\sum_{i\in N_p} a_{2l}\left(p_{i2}+\mu_l V_n(\underline{X}+e_1-e_i,\underline{a_1},\underline{a_2}-e_l+e_i)\right)+\left(\psi-\lambda_1-\lambda_2-\sum_{l\in N_p}a_{jl}\mu_l\right)V_n(\underline{X}+e_1,\underline{a_1},\underline{a_2})\Bigg\}\Bigg]
\end{aligned} \tag{18.EC.3}$$

From the induction assumption, we have: $\left(\psi-\lambda_1-\lambda_2-\sum_{l\in N_p}a_{jl}\mu_l\right)\times[V_n(\underline{X}+e_1,\underline{a_1},\underline{a_2})-V_n(\underline{X},\underline{a_1},\underline{a_2})]\geq 0$. Now, subtracting this positive term from Eq. (18.EC.3), we have

$$\begin{aligned}
V_{n+1,\beta}(\underline{X}+e_1,\underline{a_1},\underline{a_2}) &\geq \frac{1}{\psi}\Bigg[\underline{\theta}(\underline{X}+e_1)^T+\beta\Bigg\{\sum_{j\in N_s}\Big(\lambda_1 T^{\underline{u_{1j}}}V_n(\underline{X}+e_1+e_1,\underline{a_j})+\lambda_2 T^{\underline{u_{2j}}}V_n(\underline{X}+e_2+e_1,\underline{a_j})\Big)\\
&+\sum_{i\in N_p}\sum_{k\in N_p} a_{1k}\left(p_{i1}+\mu_k(V_n(\underline{X}+e_1-e_i,\underline{a_1}-e_k+e_i,\underline{a_2})-V_n(\underline{X}+e_1,\underline{a_1},\underline{a_2}))\right)\\
&+\sum_{i\in N_p}\sum_{l\in N_p} a_{2l}\left(p_{i2}+\mu_l V_n(\underline{X}+e_1-e_i,\underline{a_1},\underline{a_2}-e_l+e_i)\right)+\left(\psi-\lambda_1-\lambda_2-\sum_{l\in N_p}a_{jl}\mu_l\right)V_n(\underline{X},\underline{a_1},\underline{a_2})\Bigg\}\Bigg].
\end{aligned} \tag{18.EC.4}$$

From the induction assumption, we can write

$$\begin{aligned}
V_{n+1,\beta}(\underline{X}+e_1,\underline{a_1},\underline{a_2}) &\geq \frac{1}{\psi}\Bigg[\underline{\theta}X^T+\beta\Bigg\{\sum_{j\in N_s}\Big(\lambda_1 T^{\underline{u_{1j}}}V_n(\underline{X}+e_1,\underline{a_j})+\lambda_2 T^{\underline{u_{2j}}}V_n(\underline{X}+e_2,\underline{a_j})\Big)\\
&+\sum_{i\in N_p}\sum_{k\in N_p} a_{1k}=1\left(p_{i1}+\mu_k(V_n(\underline{X}-e_i,\underline{a_1}-e_k+e_i,\underline{a_2}))-V_n(\underline{X}+e_1,\underline{a_1},\underline{a_2})\right)\\
&+\sum_{i\in N_p}\sum_{l\in N_p} a_{2l}\left(p_{i2}+\mu_l V_n(\underline{X}-e_i,\underline{a_1},\underline{a_2}-e_l+e_i))\right)+\left(\psi-\lambda_1-\lambda_2-\sum_{l\in N_p}a_{jl}=1\mu_l\right)V_n(\underline{X},\underline{a_1},\underline{a_2})\Bigg\}\Bigg].
\end{aligned} \tag{18.EC.5}$$

Next, we show that the right-hand side of Eq. (18.EC.5) provides an upper bound for $V_{n+1,\beta}(\underline{X},\underline{a_1},\underline{a_2})$. To observe this, consider an admissible (but not necessarily optimal) policy that idles the server allocated to class 1 and use the same allocation as u^* for class 2. This yields

$$
\begin{aligned}
V_{n+1,\beta}(\underline{X}+e_1,\underline{a_1},\underline{a_2}) \geq \frac{1}{\psi}\Big[&\underline{\theta X}^T+\beta\Big\{\sum_{j\in N_s}\left(\lambda_1 T^{u_{1j}} V_n(\underline{X}+e_1,\underline{a_j})+\lambda_2 T^{u_{2j}} V_n(\underline{X}+e_2,\underline{a_j})\right)\\
&+\sum_{i\in N_p}\sum_{k\in N_p} a_{1k}\mu_k\left(V_n(\underline{X},0,\underline{a_2})-V_n(\underline{X}+e_1,\underline{a_1},\underline{a_2})\right)+\sum_{i\in N_p}\sum_{l\in N_p} a_{2l}\left(p_{i2}+\mu_l V_n(\underline{X}-e_i,\underline{a_1},\underline{a_2}-e_l+e_i)\right)\\
&+\left(\psi-\lambda_1-\lambda_2-\sum_{l\in N_p} a_{jl}\mu_l\right)V_n(\underline{X},\underline{a_1},\underline{a_2})\Big\}\Big].
\end{aligned}
\tag{18.EC.6}
$$

Because this policy is an admissible (but not necessarily optimal) policy, it provides an upper bound for $V_{n+1,\beta}(\underline{X},\underline{a_1},\underline{a_2})$, which completes the proof.

> **Proposition 18.EC.1** (Nonidling) *There exists an optimal policy which does not allow idling any IW j when there is a patient of class j boarded in the ED.*

Proof of Proposition 18.EC.1

Let $\pi^{'}$ be a policy that allows idling IW j when IW j is available and the queue of class j patients is not empty. Construct another policy π^* that follows the same allocation strategy as $\pi^{'}$ but assigns patients of class j to IW j whenever IW j is available and the queue of class j patients is not empty. We need to show that cost of policy $\pi^{'}$ is higher than π^*. This requires us to show that the following property holds for every n and every state:

$$
V^{\pi^*}_{n,\beta}(X-e_2,\underline{a_1},\underline{a_2}=e_2) \leq V^{\pi^{'}}_{n,\beta}(X,\underline{a_1},\underline{0}),
\tag{18.EC.7}
$$

or

$$
V^{\pi^*}_{n,\beta}(X-e_1,\underline{a_1}=e_1,\underline{a_2}) \leq V^{\pi^{'}}_{n,\beta}(X,\underline{0},\underline{a_2}).
\tag{18.EC.8}
$$

WLOG assume that $j=1$. Because for $n=0$ we have $V^{\pi^*}_{0,\beta}(\tilde{X})$=$V^{\pi^{'}}_{0,\beta}(\tilde{X})$=0, $V^{\pi^*}_{0,\beta}(\underline{X}-e_1,\underline{a_1}=e_1,\underline{a_2}) = V^{\pi^{'}}_{0,\beta}(\underline{X},0,\underline{a_2}) = 0\ \forall\tilde{X}\in\mathcal{S}$. Assume that, for some $n\in\mathbb{Z}^+$, property (18.EC.8) holds for all $\tilde{X}\in\mathcal{S}$, $\beta\in[0,1)$ and $k\in N_p$. We now show that the same condition holds for n+1. From Eq. (18.EC.1), we have the following equations:

$$
\begin{aligned}
&V^{\pi^*}_{n+1,\beta}(\underline{X}-e_1,\underline{a_1}=e_1,\underline{a_2})\\
&=\frac{1}{\psi}\Big[\underline{\theta}(\underline{X}-e_1)^T+\beta\Big\{\sum_{j\in N_s}\left(\lambda_1 T^{u_{1j}} V^{\pi^*}_n(\underline{X},\underline{a_1}=e_1,\underline{a_2})+\lambda_2 T^{u_{2j}} V^{\pi^*}_n(\underline{X}-e_1+e_2,\underline{a_1}=e_1,\underline{a_2})\right)\\
&\quad+\mu_1 V^{\pi^*}_n(\underline{X}-2e_1,\underline{a_1}=e_1,\underline{a_2})+\sum_{i\in N_p}\sum_{l\in N_p} a_{2l}\left(p_{i2}+\mu_l V^{\pi^*}_n(\underline{X}-e_1-e_i,\underline{a_1}=e_1,\underline{a_2}-e_l+e_i)\right)\\
&\quad+\left(\psi-\lambda_1-\lambda_2-\mu_1-\sum_{l\in N_p} a_{2l}\mu_l\right)V^{\pi^*}_n(\underline{X}-e_1,\underline{a_1}=e_1,\underline{a_2})\Big\}\Big]
\end{aligned}
\tag{18.EC.9}
$$

$$V^{\pi'}_{n+1,\beta}(\underline{X},0,\underline{a_2}) = \frac{1}{\psi}\Bigg[\underline{\theta X}^T + \beta\Bigg\{\sum_{j\in N_s}\Big(\lambda_1 T^{\underline{u_{1j}}} V^{\pi'}_n(\underline{X},0,\underline{a_2}) + \lambda_2 T^{\underline{u_{2j}}} V^{\pi'}_n(\underline{X}+e_2,0,\underline{a_2})\Big)$$

$$+\sum_{i\in N_p}\sum_{l\in N_p} a_{2l}\Big(p_{i2} + \mu_l V^{\pi'}_n(\underline{X}-e_i,0,\underline{a_2}-e_l+e_i)\Big) + \Bigg(\psi - \lambda_1 - \lambda_2 - \sum_{l\in N_p} a_{2l}\mu_l\Bigg) V^{\pi'}_n(\underline{X},0,\underline{a_2})\Bigg\}\Bigg] \tag{18.EC.10}$$

Rewriting (18.EC.9) by separating the actions related to class 1 departure, and subtracting the result from (18.EC.10), we have:

$$\begin{aligned}
&V^{\pi'}_{n+1,\beta}(X,\underline{0},\underline{a_2}) - V^{\pi^*}_{n+1,\beta}(X-e_1,\underline{a_1}=e_1,\underline{a_2}) = \qquad (18.EC.11)\\
&\frac{1}{\psi}\Bigg[\theta_1 + \beta\Bigg\{\lambda_1\Big(V^{\pi'}_n(X_1+1,X_2,0,a_2) - V^{\pi^*}_n(X_1,X_2,\underline{a_1}=e_1,a_2)\Big)\\
&+\lambda_2\Big(V^{\pi'}_n(X_1,X_2+1,0,a_2) - V^{\pi^*}_n(X_1-1,X_2+1,\underline{a_1}=e_1,a_2)\Big)\\
&+\mu_l\Big(V^{\pi'}_n(X_1,X_2-1,0,a_2) - V^{\pi^*}_n(X_1-1,X_2-1,\underline{a_1}=e_1,a_2)\Big)\\
&+\mu_1\Big(V^{\pi^*}_n(X_1-1,X_2,\underline{a_1}=e_1,a_2) - V^{\pi^*}_n(X_1-2,X_2,\underline{a_1}=e_1,a_2)\Big)\\
&+\Bigg(\psi - \lambda_1 - \lambda_2 - \sum_{l\in N_p} a_{jl}\mu_l\Bigg)\\
&\qquad\Big(V^{\pi'}_n(X_1,X_2,0,a_2) - V^{\pi^*}_n(X_1-1,X_2,\underline{a_1}=e_1,a_2)\Big)\Bigg\}\Bigg] \geq 0.
\end{aligned}$$

The inequality follows from the induction assumption as well as the monotonicity of the value function, where

$$\mu_1\Big(V^{\pi^*}_n(X_1-1,X_2,\underline{a_1}=e_1,a_2) - V^{\pi^*}_n(X_1-2,X_2,\underline{a_1}=e_1,a_2)\Big) \geq 0$$

and shows that Eq. (18.EC.8) holds for $n+1$. Note that for the proof we considered the decision upon patient departure. However, a similar proof can be presented for the decisions made upon patient arrival because the differences in service rates are observed when the patient is departing the system.

Proof of Proposition 18.1

Consider a primary-secondary patient and IW pair say IW 1 and IW 2. WLOG assume that $\theta_1\mu_1 \geq \theta_2\mu_2$, and $\mu_1 \geq \mu_2$. Consider the case that the expected service time of all patient types are equal to 0 at time $t=1$. Suppose that there are patients of both classes boarding in the ED at $t=1$. Let π be an optimal policy, and we assume that π follows the $\theta\mu$ rule from $t=2$ on. Now suppose that π selects a patient of class 2 at $t=1$. Because the service discipline is nonpreemptive, π may select a of patient class 1 only after the service completion of the patient of class 2. Let $\bar{\pi}$ be the policy that is identical to π except that it interchanges the first time the class 1 and 2 patients are served. The rest of the decisions of $\bar{\pi}$ are the same as those of π. From those defined earlier, when $p_{ij} = 0 \ \forall i \in N_p, j \in N_s$:

$$J_{\pi}(\tilde{X}, T) - J_{\bar{\pi}}(\tilde{X}, T) = \theta_1 \mu_1 - \theta_2 \mu_2. \qquad (18.\text{EC}.12)$$

If $\theta_1 \mu_1 \geq \theta_2 \mu_2$, the equation contradicts the assumption that π is an optimal policy.

Proof of Theorem 18.1

For the case where $X_1 > 0$, the result is straightforward because assigning primary type patients to IW 1 does not incur any penalty cost. From Proposition 18.1, we know that under no penalty cost, it is optimal to follow the $c\mu$ priority policy. Hence, when we include the penalty costs only assignment of class 2 patients becomes more costly. Therefore, it is optimal to assign class 1 patients to IW 1 whenever they are boarded in ED.

To prove the optimality of a threshold policy for IW 1 when $X_1 = 0$, we need to show that the difference $V_n(0, X_2 - 1, \underline{a_1} = e_2, \underline{a_2}) - V_n(0, X_2, \underline{a_1} = 0, \underline{a_2})$ is decreasing in X_2, so that assigning class 2 patients to IW 1 becomes desirable at some level of class 2 queue length, despite the associated penalty cost. Notice that whenever $X_1 > 0$, IW 1 serves class 1 patients. Observe that

$$\begin{aligned} &V_n(0, X_2 - 1, \underline{a_1} = e_2, \underline{a_2}) - V_n(0, X_2, \underline{a_1} = 0, \underline{a_2}) \\ &\quad \geq V_n(0, X_2, \underline{a_1} = e_2, \underline{a_2}) - V_n(0, X_2 + 1, \underline{a_1} = 0, \underline{a_2}). \end{aligned} \qquad (18.\text{EC}.13)$$

We can rewrite the system state by dropping $\underline{a_2}$ because our focus is on times when there is no class 1 patient boarded in the ED and IW 1 is available. Thus, we rewrite the above inequality as

$$V_{n,\beta}(\underline{X}, 0) - V_{n,\beta}(\underline{X} - e_2, e_2) \leq V_{n,\beta}(\underline{X} + e_2, 0) - V_{n,\beta}(\underline{X}, e_2), \qquad (18.\text{EC}.14)$$

which is the same structure that is introduced in Koole (1995)[1]. Following the proof in Koole (1995), we define a set of functions $\mathcal{F}$ that satisfy

$$f(\underline{X}, 0) + f(\underline{X}, e_2) \leq f(\underline{X} + e_2, 0) + f(\underline{X} - e_2, e_2),$$

where $f \in \mathcal{F}$ for all $\underline{X} > \underline{0}$. Now, we assume that $V_{n,\beta} \in \mathcal{F}$ and show that $V_{n+1,\beta} \in \mathcal{F}$ (note that trivially $V_0 \in \mathcal{F}$). Define $min_{u \in \mathcal{U}(\tilde{X})}(T^{u_{i1}} V_{n,\beta}(\tilde{X}))$ as $W_{n,\beta}(\tilde{X})$ and observe that $W_{n,\beta} \in \mathcal{F}$. Assume that the optimal action at state $(\underline{X} + e_2, 0)$ is assigning class 2 to IW 1. We have

$$\begin{aligned} &W_{n,\beta}(\underline{X}, 0) + W_{n,\beta}(\underline{X}, e_2) \\ &\quad \leq V_{n,\beta}(\underline{X} - e_2, e_2) + V_{n,\beta}(\underline{X}, e_2) = W_{n,\beta}(\underline{X} - e_2, 0) + W_{n,\beta}(\underline{X} + e_2, e_2) \end{aligned}$$

[1] Koole, G. (1995). A simple proof of the optimality of a threshold policy in a two-server queueing system. *Systems & Control Letters* 26(5):301–303.

because action of assigning class 2 to IW 1 is suboptimal at state $(\underline{X}, e_2)$. Now assume that the optimal action at state $(\underline{X}+e_2, 0)$ is keeping IW 1 idle. We have

$$\begin{aligned} W_{n,\beta}(\underline{X},0) + W_{n,\beta}(\underline{X},e_2) &\le V_{n,\beta}(\underline{X},0) + V_{n,\beta}(\underline{X},e_2) \\ &\le V_{n,\beta}(\underline{X}+e_2,0) + V_{n,\beta}(\underline{X}-e_2,e_2) \end{aligned}$$

because $V_n \in \mathcal{F}$. Note that because idleness is the optimal action at state$(\underline{X}+e_2,0)$, we can rewrite the last part of the inequality as $W_n(\underline{X}+e_2,0)+W_n(\underline{X}-e_2,e_2)$, which in turn shows that $W_n \in \mathcal{F}$. If we rewrite V_{n+1} as

$$V_{n+1,\beta}(\tilde{X}) = \frac{1}{\psi}\Big[\underline{\theta X}^T + \lambda_1 W_{n,\beta}(X+e_1,\underline{a_1}) + \lambda_2 W_{n,\beta}(X+e_2,\underline{a_1}) + \sum_{k\in N_p}\sum_{i\in N_p} a_{k1}\mu_k W_{n,\beta}(X,\underline{a_1}-e_k) \tag{18.EC.15}$$

$$+ (\psi - \sum_{i\in N_p}\lambda_i - \sum_{k\in N_p} a_{k1}\mu_k) V_{n,\beta}(\tilde{X})\Big],$$

we can conclude that $V_{n+1,\beta} \in \mathcal{F}$ from the induction assumption and the fact that $W_{n,\beta} \in \mathcal{F}$.

Proof of Lemma 18.1

We need to show that if $J \in \mathcal{F}$ then $TJ \in \mathcal{F}$ where $TJ(\underline{Y}) = T_\theta J(\underline{Y}) + \beta\big(T_a J(\underline{Y}) + T_* J(\underline{Y})\big)$. Note that T_θ and T_a trivially satisfy properties (18.14) and (18.15). Thus, it is sufficient to show that operator T_* preserves properties (18.14) and (18.15). Assume $J \in \mathcal{F}$, $\theta_1\mu_1 \ge \theta_2\mu_2$ and $\mu_2 \ge \mu_1$ hold. To show the preservation of property (*i*), we need to examine all possible actions at states (Y), $(Y-e_1)$, $(Y-e_2)$, $(Y+e_1)$, and $(Y+e_1-e_2)$ by using the induction assumption. Notice that there are 2^5 possible cases; however, properties (18.14) and (18.15) restrict several cases, which leave us with the cases shown in Table 18.EC.2. This table also shows the patient class that is assigned to IW 2. We next consider each of the case shown in Table 18.EC.2 separately.

<u>Case 1</u> Note that the set of actions that are defined in case 1 are feasible when $Y_1 \ge 2$. Consider the state $Y_1 = 1, Y_2 \ge 1$, and $u_{22} = 1$ as a feasible (not

TABLE 18.EC.2 *Possible actions*

Cases	$Y+e_1$	Y	$Y-e_2$	$Y-e_1$	$Y+e_1-e_2$
Case 1	1	1	1	1	1
Case 2	2	2	2	2	2
Case 3	1	2	1	2	1
Case 4	1	2	2	2	1
Case 5	2	2	2	2	1
Case 6	2	2	1	2	1
Case 7	1	1	1	2	1

necessarily optimal) action for state $Y-e_1$. See case 7 for the action where patient class 2 is assigned to IW 2 at state $Y-e_1$. We have

$$
\begin{aligned}
&[\tilde{\mu_1}\Delta_1 T_*J(Y)-\tilde{\mu_2}\Delta_2 T_*J(Y+e_1-e_2)]-[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_2)]=\\
&(1-\Lambda-\tilde{\mu_1})\big([\tilde{\mu_1}\Delta_1 T_*J(Y)-\tilde{\mu_2}\Delta_2 T_*J(Y+e_1-e_2)]-[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_2)]\big)\\
&+\tilde{\mu_1}\,[\tilde{\mu_1}\Delta_1 T_*J(Y-2e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_1-e_2)]\geq 0.
\end{aligned}
$$

The inequality in the first line follows from nonnegativity of the term $(1-\Lambda-\tilde{\mu_1})$ and the induction assumptions. The second line follows from the optimality of $c\mu$ rule when $p_{ij}=0$.

<u>Case 2</u> The proof of case 2 can be established similar to that of case 1 by replacing $\tilde{\mu_1}$ by $\tilde{\mu_2}$. Again, similar to case 1, assigning class 2 patients to IW 2 is feasible when $Y_2 \geq 2$. If the state is $Y_1 \geq 1, Y_2 = 1$, the action of assigning class 2 patients to IW 2 is not feasible for the states $Y-e_2$ and $Y+e_1-e_2$. However, the action of assigning class 1 patients to IW 2 is a feasible (not necessarily optimal) at these states (see case 6 for case of assigning class 1 patients at states $Y-e_2$ and $Y+e_1-e_2$).

<u>Case 3</u> Note that

$$
\begin{aligned}
&\tilde{\mu_1}\Delta_1 T_*J(Y)-\tilde{\mu_2}\Delta_2 T_*J(Y+e_1-e_2)\\
&=(1-\Lambda)\,[\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)]-\mu_1\,[\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)-p_{12}]\\
&\geq(1-\Lambda)\,[\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)]-\tilde{\mu_2}\,[\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)-p_{12}]\\
&[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_2)]=\\
&(1-\Lambda)\,[\tilde{\mu_1}\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)]-\tilde{\mu_2}\,[\tilde{\mu_1}\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)-p_{12}].
\end{aligned}
$$

Subtract the last term from the second term, we observe that

$$
(1-\Lambda-\tilde{\mu_2})\big[\big(\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)\big)-\big(\tilde{\mu_1}\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)\big)\big]\geq 0
$$

because $J\in\mathcal{F}$ and the fact that $(1-\Lambda-\tilde{\mu_2})$ is nonnegative.

<u>Case 4</u> Assigning class 2 patients to IW 2 is feasible when $Y_2 \geq 2$. If the state is $Y_1 \geq 1, Y_2 = 1$, the action of assigning class 2 patients to IW 2 is not feasible for the state $Y-e_2$. However, the action of assigning class 1 patients to IW 2 is feasible (not necessarily optimal) at this state (see case 3 for case of assigning class 1 patients at state $Y-e_2$). We have

$$
\begin{aligned}
&\tilde{\mu_1}\Delta_1 T_*J(Y)-\tilde{\mu_2}\Delta_2 T_*J(Y+e_1-e_2)\leq(1-\Lambda-\tilde{\mu_2})\,[\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)]+\tilde{\mu_1}p_{12}\\
&\tilde{\mu_1}\Delta_1 T_*J(Y-e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_2)=\\
&(1-\Lambda-\tilde{\mu_2})\,[\mu_1\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)]+\tilde{\mu_2}\,[\tilde{\mu_1}\Delta_1 J(Y-e_1-e_2)-\tilde{\mu_2}\Delta_2 J(Y-2e_2)-p_{12}]\\
&(1-\Lambda-\mu_2)\,[[\tilde{\mu_1}\Delta_1 J(Y)-\tilde{\mu_2}\Delta_2 J(Y+e_1-e_2)]-[\tilde{\mu_1}\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)]]+p_{12}\tilde{\mu_1}\\
&\geq\mu_2\,[\tilde{\mu_1}\Delta_1 J(Y-e_1-e_2)-\tilde{\mu_2}\Delta_2 J(Y-2e_2)-p_{12}],
\end{aligned}
$$

where the inequality follows from $J \in \mathcal{F}$, nonnegativity of term $p_{12}\mu_1$, and optimality of assigning patient class 2 at state $(Y - e_2)$.

<u>Case 5</u> Assigning class 2 patients to IW 2 is feasible when $Y_2 \geq 2$. If the state is $Y_1 \geq 1, Y_2 = 1$, the action of assigning class 2 patients to IW 2 is not feasible at state $Y - e_2$. However, the action of assigning class 1 patients to IW 2 is a feasible (not necessarily optimal) at this state (see case 6 for case of assigning class 1 patients for the state $Y - e_2$). We have

$$
\begin{aligned}
&\tilde{\mu}_1 \Delta_1 T_* J(Y) - \tilde{\mu}_2 \Delta_2 T_* J(Y + e_1 - e_2) = (1 - \Lambda - \tilde{\mu}_2)\,[\tilde{\mu}_1 \Delta_1 J(Y) - \tilde{\mu}_2 \Delta_2 J(Y + e_1 - e_2)] + \tilde{\mu}_2 p_{12}\\
&\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2) =\\
&(1 - \Lambda - \tilde{\mu}_2)\,[\tilde{\mu}_1 \Delta_1 J(Y - e_1) - \tilde{\mu}_2 \Delta_2 J(Y - e_2)] + \tilde{\mu}_2 (\tilde{\mu}_1 \Delta_1 J(Y - e_1 - e_2) - \tilde{\mu}_2 \Delta_2 J(Y - 2e_2))\\
&[\tilde{\mu}_1 \Delta_1 T_* J(Y) - \tilde{\mu}_2 \Delta_2 T_* J(Y + e_1 - e_2)] - [\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2)]\\
&= (1 - \Lambda - \tilde{\mu}_2)\,[[\tilde{\mu}_1 \Delta_1 J(Y) - \tilde{\mu}_2 \Delta_2 J(Y + e_1 - e_2)] - [\tilde{\mu}_1 \Delta_1 J(Y - e_1) - \tilde{\mu}_2 \Delta_2 J(Y - e_2)]]\\
&- \tilde{\mu}_2\,[\tilde{\mu}_1 \Delta_1 J(Y - e_1 - e_2) - \tilde{\mu}_2 \Delta_2 J(Y - 2e_2) - p_{12}] \geq 0,
\end{aligned}
$$

where the last inequality follows from the optimality of assignment of patient class 2 at state $(Y - e_2)$.

<u>Case 6</u> In this case, we have

$$
\begin{aligned}
&\tilde{\mu}_1 \Delta_1 T_* J(Y) - \tilde{\mu}_2 \Delta_2 T_* J(Y + e_1 - e_2) = (1 - \Lambda - \tilde{\mu}_2)\,[\tilde{\mu}_1 \Delta_1 J(Y) - \tilde{\mu}_2 \Delta_2 J(Y + e_1 - e_2)] + \tilde{\mu}_2 p_{12}\\
&\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2) = (1 - \Lambda - \tilde{\mu}_2)\,[\tilde{\mu}_1 \Delta_1 J(Y - e_1) - \tilde{\mu}_2 \Delta_2 J(Y - e_2)] + \tilde{\mu}_2 p_{12}\\
&[\tilde{\mu}_1 \Delta_1 T_* J(Y) - \tilde{\mu}_2 \Delta_2 T_* J(Y + e_1 - e_2)] - [\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2)]\\
&= (1 - \Lambda - \tilde{\mu}_2)\,[[\tilde{\mu}_1 \Delta_1 J(Y) - \tilde{\mu}_2 \Delta_2 J(Y + e_1 - e_2)] - [\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2)]] \geq 0,
\end{aligned}
$$

because $J \in \mathcal{F}$.

<u>Case 7</u> In this case, we have

$$
\begin{aligned}
&[\tilde{\mu}_1 \Delta_1 T_* J(Y) - \tilde{\mu}_2 \Delta_2 T_* J(Y + e_1 - e_2)] - [\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2)]\\
&= (1 - \Lambda - \tilde{\mu}_1)\,[[\tilde{\mu}_1 \Delta_1 J(Y) - \tilde{\mu}_2 \Delta_2 J(Y + e_1 - e_2)] - [\tilde{\mu}_1 \Delta_1 T_* J(Y - e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y - e_2)]]\\
&+ \tilde{\mu}_1\,[\tilde{\mu}_1 \Delta_1 J(Y - e_1) - \tilde{\mu}_2 \Delta_2 J(Y - e_2) - p_{12}] \geq 0,
\end{aligned}
$$

where the inequality follows from $J \in \mathcal{F}$ and optimality of assigning the IW to class 1 at state (Y).

To show the preservation of the second property, we need to consider all possible actions at the states $(Y), (Y - e_1), (Y + e_2)$. Again properties (18.14) and (18.15) restrict several cases, which leave us with the cases presented in Table 18.EC.3:

TABLE 18.EC.3 *Possible actions*

Cases	$Y+e_2$	Y	$Y-e_1$	$Y-e_1+e_2$	$Y-e_2$
Case 1	1	1	1	1	1
Case 2	2	2	2	2	2
Case 3	1	1	1	2	1
Case 4	1	1	2	2	1
Case 5	2	1	1	2	1
Case 6	2	1	2	2	1
Case 7	2	2	2	2	1

<u>Case 1</u>

Assigning class 1 patients to IW 2 is feasible when $Y_1 \geq 2$. If the state is $Y_{1=1}, Y_2 \geq 1$, the action of assigning class 1 patients to IW 2 is not feasible at states $Y-e_1$ and $Y-e_1+e_2$. However, the action of assigning class 2 patients to IW 2 is a feasible (not necessarily optimal) at these states (see case 4 for case of assigning class 2 patients for the states $Y-e_1$ and $Y-e_1+e_2$). We have

$$[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_2)]-[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1+e_2)-\tilde{\mu_2}\Delta_2 T_*J(Y)]=$$
$$(1-\Lambda-\tilde{\mu_1})\left[\left(\tilde{\mu_1}\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)\right)-\left(\tilde{\mu_1}\Delta_1 J(Y-e_1+e_2)-\tilde{\mu_2}\Delta_2 J(Y)\right)\right]$$
$$+\tilde{\mu_1}\left[\left(\tilde{\mu_1}\Delta_1 J(Y-2e_2)-\tilde{\mu_2}\Delta_2 J(Y-e_1-e_2)\right)-\left(\tilde{\mu_1}\Delta_1 J(Y-2e_1+e_2)-\tilde{\mu_2}\Delta_2 J(Y-e_1)\right)\right]\geq 0.$$

The inequality follows from nonnegativity of the term $(1-\Lambda-\tilde{\mu_1})$ and the fact that $J \in \mathcal{F}$.

<u>Case 2</u>

The proof of case 2 can be shown similar to that of case 1 by replacing $\tilde{\mu_1}$ by $\tilde{\mu_2}$, and by assigning class 2 to the IW. Similar to case 1, assigning class 2 patients to IW 2 is feasible when $Y_2 \geq 2$. If the state is $Y_1 \geq 1, Y_2 = 1$, the action of assigning class 2 patients to IW 2 is not feasible at state $Y-e_2$. However, the action of assigning class 1 patients to IW 2 is feasible (not necessarily optimal) at this state (see case 7 for case of assigning class 1 patients for the state $Y-e_2$).

<u>Case 3</u>

Assigning class 1 patients to IW 2 is feasible when $Y_1 \geq 2$. If the state is $Y_{1=1}, Y_2 \geq 1$, the action of assigning class 1 patients to IW 2 is not feasible at state $Y-e_1$. However, the action of assigning class 2 patients to IW 2 is feasible (not necessarily optimal) in this state (see case 4 for case of assigning class 2 patients for the state $Y-e_1$). We have

$$[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1)-\tilde{\mu_2}\Delta_2 T_*J(Y-e_2)]-[\tilde{\mu_1}\Delta_1 T_*J(Y-e_1+e_2)-\tilde{\mu_2}\Delta_2 T_*J(Y)]\geq$$
$$(1-\Lambda-\tilde{\mu_1})\left[\left(\tilde{\mu_1}\Delta_1 J(Y-e_1)-\tilde{\mu_2}\Delta_2 J(Y-e_2)\right)-\left(\tilde{\mu_1}\Delta_1 J(Y-e_1+e_2)-\tilde{\mu_2}\Delta_2 J(Y)\right)\right]$$
$$+\tilde{\mu_1}\,[\tilde{\mu_1}\Delta_1 J(Y-2e_2)-\tilde{\mu_2}\Delta_2 J(Y-e_1-e_2)-p_{12}]\geq 0,$$

where he inequality follows from the fact that $J \in \mathcal{F}$ and the optimality of assigning class 1 at state $(Y - e_1)$.

Case 4
We have

$$[\mu_1 \Delta_1 T_* J(Y-e_1) - \mu_2 \Delta_2 T_* J(Y-e_2)] - [\mu_1 \Delta_1 T_* J(Y-e_1+e_2) - \mu_2 \Delta_2 T_* J(Y)] \geq$$
$$(\bar{\psi} - \Lambda - \mu_1)\big[\big(\mu_1 \Delta_1 J(Y-e_1) - \mu_2 \Delta_2 J(Y-e_2)\big) - \big(\mu_1 \Delta_1 J(Y-e_1+e_2) - \mu_2 \Delta_2 J(Y)\big)\big] \geq 0.$$

The inequality follows nonnegativity of the term $(1 - \Lambda - \tilde{\mu_1})$ and the fact that $J \in \mathcal{F}$.

Case 5
Assigning class 1 patients to IW 2 is feasible when $Y_1 \geq 2$. If the state is $Y_{1=1}, Y_2 \geq 1$, the action of assigning class 1 patients to IW 2 is not feasible at state $Y - e_1$. However, the action of assigning class 2 patients to IW 2 is feasible (not necessarily optimal) at this state (see case 6 for case of assigning class 2 patients for the state $Y - e_1$). We have

$$[\tilde{\mu_1} \Delta_1 T_* J(Y-e_1) - \tilde{\mu_2} \Delta_2 T_* J(Y-e_2)] - [\tilde{\mu_1} \Delta_1 T_* J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 T_* J(Y)] \geq$$
$$(1-\Lambda)\big[\big(\tilde{\mu_1} \Delta_1 J(Y-e_1) - \tilde{\mu_2} \Delta_2 J(Y-e_2)\big) - \big(\tilde{\mu_1} \Delta_1 J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 J(Y)\big)\big]$$
$$-\tilde{\mu_2}\,[\tilde{\mu_1} \Delta_1 J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 J(Y) - p_{12}] \geq 0,$$

where he inequality follows from the fact that $J \in \mathcal{F}$ and from the optimality of assigning the IW to class 2 at state $(Y + e_2)$.

Case 6
We have

$$[\tilde{\mu_1} \Delta_1 T_* J(Y-e_1) - \tilde{\mu_2} \Delta_2 T_* J(Y-e_2)] - [\tilde{\mu_1} \Delta_1 T_* J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 T_* J(Y)] \geq$$
$$(1-\Lambda-\tilde{\mu_2})\big[\big(\tilde{\mu_1} \Delta_1 J(Y-e_1) - \tilde{\mu_2} \Delta_2 J(Y-e_2)\big) - \big(\tilde{\mu_1} \Delta_1 J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 J(Y)\big)\big] \geq 0,$$

where he inequality follows from nonnegativity of the term $(1 - \Lambda - \tilde{\mu_2})$ and the fact that $J \in \mathcal{F}$.

Case 7
We have

$$[\tilde{\mu_1} \Delta_1 T_* J(Y-e_1) - \tilde{\mu_2} \Delta_2 T_* J(Y-e_2)] - [\tilde{\mu_1} \Delta_1 T_* J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 T_* J(Y)] =$$
$$(1-\Lambda-\tilde{\mu_2})\big[\big(\tilde{\mu_1} \Delta_1 J(Y-e_1) - \tilde{\mu_2} \Delta_2 J(Y-e_2)\big) - \big(\tilde{\mu_1} \Delta_1 J(Y-e_1+e_2) - \tilde{\mu_2} \Delta_2 J(Y)\big)\big]$$
$$-\tilde{\mu_2}\,[\tilde{\mu_1} \Delta_1 J(Y-e_1) - \tilde{\mu_2} \Delta_1 J(Y-e_2) - p_{12}] \geq 0,$$

TABLE 18.EC.4 *Numerical test cases*

Cases	λ_1	λ_2	μ_1	μ_2	θ_1	θ_2	p_1	p_2
1	1	1	2	1	1	1	1	1
2	1	1	2	1	1	1	1000	1000
3	1	1	2	1	1000	1000	1	1
4	1	1	2	1	1	1	10	1
5	1	1	2	1	1	1	1	10

where the inequality follows from the fact that $J \in \mathcal{F}$ and from the optimality of assigning the IW to class 2 at state (Y).
Additionally, we can gain further insights by using Lemma 18.1. These results show that the threshold level depends on the model parameters because the threshold can be defined as $\min\{Y_1 : [\tilde{\mu}_1 \Delta_1 T_* J(Y-e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y-e_2)] \geq p_{12}\}$. Using this, we can identify how the threshold level changes as the model parameters change. From Lemma 18.1, we observe that the difference $[\tilde{\mu}_1 \Delta_1 T_* J(Y-e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y-e_2)]$ is nondecreasing in the number of class 1 patients in the queue. Also, consider $\hat{p}_{12}$ where $\hat{p}_{12} \geq p_{12}$. We can conclude that the threshold level increases as p_{12} increases because

$$\min\{Y_1 : [\tilde{\mu}_1 \Delta_1 T_* J(Y-e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y-e_2)] \geq \hat{p}_{12}\} \geq$$
$$\min\{Y_1 : [\tilde{\mu}_1 \Delta_1 T_* J(Y-e_1) - \tilde{\mu}_2 \Delta_2 T_* J(Y-e_2)] \geq p_{12}\}.$$

Similar to the earlier results, the threshold level also depends the service rates μ_1 and μ_2. It can be observed that the difference $[\mu_1 \Delta_1 T_* J(Y-e_1) - \mu_2 \Delta_2 T_* J(Y-e_2)]$ is nondecreasing in μ_1 (nonincreasing in μ_2), which means that the threshold level increases (decrease) as these parameter increase. Proof of Theorem 18.2 directly follows from Lemma 18.1.

Computational Results

To gain insights into the structure of the optimal policy, we first generate a set of test cases. Table 18.EC.4 presents these cases. For each case, we solve our MDP numerically using a convergence criteria of 10^{-4} and truncate the queue lengths at $X_1 = X_2 = 70$. To avoid the boundary effects (i.e., when the number in each queue gets close to the boundary of the state space under consideration), we only present the optimal policy for states in which the number in queues are no more than 30.

In the Figures 18.EC.1–18.EC.5 green (dark gray) color represents serving class 1 patients, yellow (light gray) color represent serving class 2 patients, and dark blue (black) represents idling the server. In case 1, the effect of differences in the service rates is analyzed (Figure 18.EC.1). In case 2, the effect of high penalty costs, and in case 3, the effect of high holding costs is analyzed. In cases

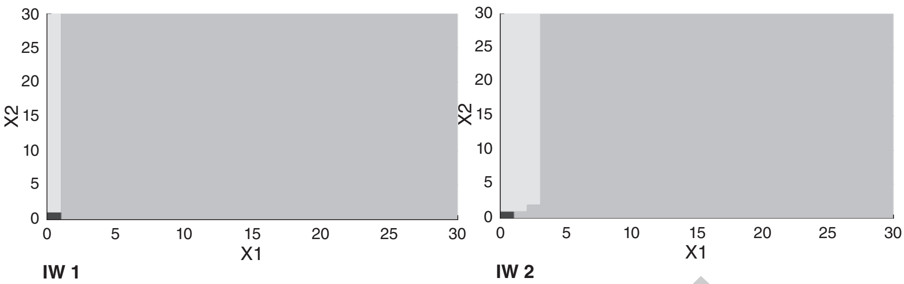

FIGURE 18.EC.1 Case 1

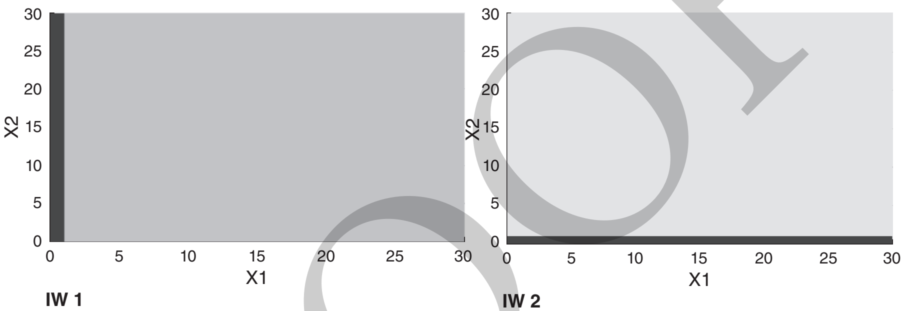

FIGURE 18.EC.2 Case 2

4 and 5 (Figures 18.EC.4–18.EC.5), the effect of the penalty cost on the optimal policy structure is analyzed. Our MDP-based numerical results show that when $\theta_1\mu_1 \geq \theta_2\mu_2$, the structure of the optimal control policy is a state-dependent threshold-type policy, where IW 1 serves class 1 when $X_1 > 0$, and IW 2 performs as a dedicated server, and switches to the $c\mu$ rule after the threshold. When $p_{ij} \gg \theta_i$ $(\forall i \in N_p, j \in N_s)$ (see, e.g., case 2), both of the units start to work as dedicated units. Moreover, when their primary queue is empty, they idle, even if there is a patient in the nonprimary queue waiting for assignment. Under $\theta_i \gg p_{ij}$ $(\forall i \in N_p, j \in N_s)$ (see, e.g., case 3) the well-known $c\mu$ rule becomes the optimal policy. This policy gives strict priority to class 1 patients whenever there is a class 1 patient waiting in the ED. We also observe that, when p_{12} increases, (1) IW 2 delays serving class 1 patients, (2) the threshold level increases, and (3) IW 1 still serves as a dedicated unit when $X_1 > 0$, and switches to serves class 2 patients when $X_1 = 0$. When p_{21} increases IW 1 serves class 2 patients when $X_1 = 0$ and $X_2 > T_2$, where T_2 is some threshold level on number of class 2 patients boarded in the ED. In addition, as the threshold on class 2 patients increase, we observe that the threshold for class 1 patients in IW 2 increases.

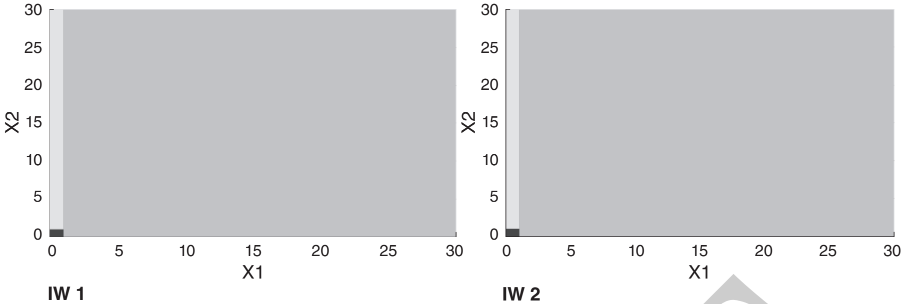

FIGURE 18.EC.3 Case 3

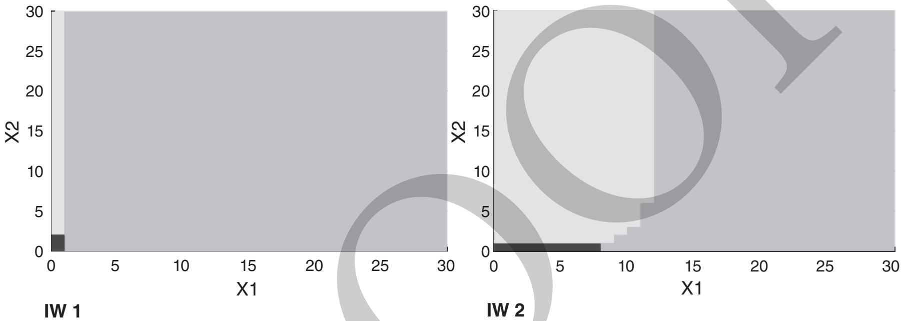

FIGURE 18.EC.4 Case 4

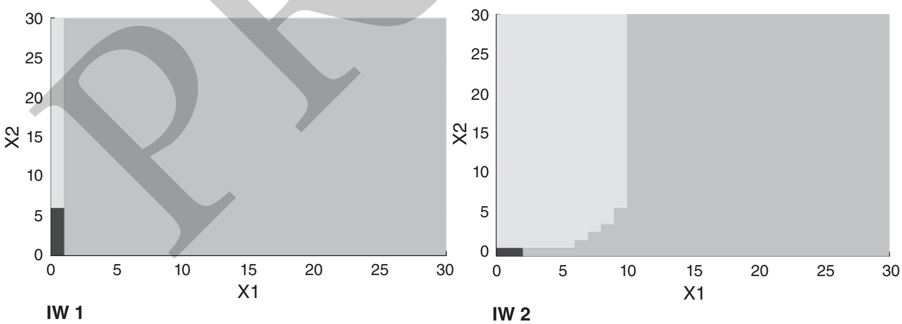

FIGURE 18.EC.5 Case 5

ONLINE APPENDIX 18.D BIRTH-AND-DEATH PROCESSES

In this appendix we present the illustrations of the birth-and-death processes used to construct the BDT heuristic policy discussed in the main chapter.

TABLE 18.EC.5 *ROAE cost (per time unit) combinations in the test suite*

θ_1	θ_2
1	1
2	1
1	2

TABLE 18.EC.6 *Penalty cost parameter combinations in the test suite*

p_{12}	p_{21}
0	0
1	1
10	10
100	100

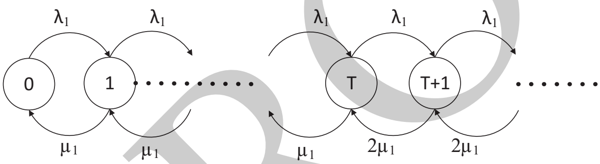

FIGURE 18.EC.6 Birth-and-death process approximation for class 1 patients

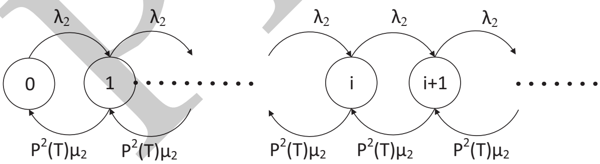

FIGURE 18.EC.7 Birth-and-death process approximation for class 2 patients

ONLINE APPENDIX 18.E NUMERICAL CASES

We generate 216 problem instances. These problem instances cover various cost and arrival rate combinations. Tables 18.EC.5–18.EC.7 provide a summary of the related information. More details are available upon request.

TABLE 18.EC.7 *Arrival rate combinations in the test suite*

λ_1/μ_1	λ_2	μ_2
0.1-0.9	0.4	1
0.1-0.9	0.8	1

TABLE 18.EC.8 *p-values for comparison on the equality of means of service times for primary and secondary IWs*

Patient Type	**p-value**
Type 1-CP	0.750
Type 2-CP	0.216
Type 1-CHF	0.601
Type 2-CHF	0.218

TABLE 18.EC.9 *Average service time (in days) for patients in each IW from different admission sources*

IW	**ED admits**	**Direct admits**	**OR admits**
4 West	3.57	7.41	4.05
5 West	3.60	4.38	2.93

ONLINE APPENDIX 18.F DATA ANALYSES

In Tables 18.EC.8–18.EC.9, we present a summary of some of the main results from our data analyses. More details are available upon request.

ONLINE APPENDIX 18.G SIMULATION MODEL

18.G.1 Cost Cases for Simulation

In our simulation model, we assume that the cost associated with the risk of adverse events that may occur while a patient is boarded in the ED is the same for both patient classes (patients requiring a bed from 4 West or 5 West). The reason behind this assumption is the similarity between the ESI distribution among 4 West and 5 West ED admit patients. Our data analyses show that 30% of ED patients that require a bed from 4 West are ESI 2 patients, and 69% of them are ESI 3 patients, while these proportions for 5 West patients are 28% and 70%, respectively. Because patients with similar severity are subject to similar levels of adverse events, we assume that the cost associated with the

TABLE 18.EC.10 *ROAE cost (per hour) cases used in the simulation model*

Cases	θ
Case 1	1
Case 2	5
Case 3	10

TABLE 18.EC.11 *Penalty cost parameters used in the simulation model*

Cases	**Type 1**	**Type 2**
Case 1	1	0.5
Case 2	5	2.5
Case 3	10	5

TABLE 18.EC.12 *Improvement in performance measures due to a trigger-based policy*

Performance measure	**Improvement (%)**
Average boarding time	6.8
Average number of patients boarded	10.3
Overflow proportion	−56
2-hour boarding rate	3.2

risk of adverse events are the same for 4 West and 5 West patients admitted through the ED.

18.G.2 Performance of a Pure Trigger-Based Policy

We analyze a static pure trigger-based policy that assigns patients to their secondary IWs only when patients' boarding time exceeds a certain trigger level (and when there is a bed available in the secondary IW). We use a two-hour trigger time in our analyses because a two-hour boarding rate is an important performance measure for EDs that we also use in other parts of our analyses. Note that, although this trigger based policy imposes an upper bound on the boarding time, it cannot fully eliminate boarding times that are over two hours because patients can be assigned to their secondary IWs only when there is a bed available in that IW.

From our results, we observe that the trigger-based policy reduces the average boarding time (compared to the current practice in which it is not used). However, it increases overflow proportions and, hence, penalty costs incurred due to secondary unit assignments. Moreover, because this policy is not an adaptive policy and does not change based on system parameters, under the cases where the penalty costs are high, it can lead to large increases in the total cost (which we observe to be as high as 18% in some cases, compared to the current practice). Overall, our results indicate that the LEWC-p policy proposed in the main body results in improvements that are both larger and more robust compared to those under a pure trigger-based policy.

ONLINE APPENDIX 18.H EXTENDED SIMULATION MODEL

In the simulation model that is described in Section 18.6, we use the data that is obtained from our partner hospital. Due to limitations in data, we cannot have the exact patient flow that is described in Figure 18.3 and instead model the patient flow as described in Figure 18.6. In this section, we generate a simulation model that does not use the exact hospital data we have collected but models the patient flow that is described in Figure 18.3 with additional characteristics that is obtained from the data analyses. Additionally, we model 5 p.m. discharge rounds to include a well-known concept used in some hospitals.

In the extended model, we relax our assumptions on stationary arrival rates and consider time-dependent bed request rates. We also consider lognormal LOS distributions for each patient type and consider multiple bed availabilities in each IW. Additionally, we model seven IWs and seven patient classes and model a general number of (unpooled) beds in each IW. We only focus on patients admitted to the hospital through the ED in the extended simulation model.

We consider the patient flow depicted in Figure 18.EC.8. In this figure, the solid lines represent primary bed assignments, dashed lines represents ideal secondary bed assignments, and dotted lines represents the backup secondary IW assignments. When a patient is assigned to a backup secondary IW, the patient experiences greater reduction in quality of care compared to assignment to an ideal secondary IW. We additionally relax the paired primary-secondary IW assumption and use a more general hospital network flow in the extended model. We also include 5 p.m. discharge rounds in our simulation setting.

Our objective in using an extended simulation model is to test the performance of LEWC-p compared to alternative policies in complex hospital networks. To this end, we consider alternative patient flow policies including a "primary-only" where patients are served only in their primary IWs (policy 1) and a "both-primary-and-secondary" policy where IW beds are shared (policy 2). We additionally consider serving the longest queue (policy 3) and assigning a fixed proportion of patients to ideal secondary and backup secondary IWs (Policy 4, 40% and 30%; policy 5. 30% and 20%; policy 6, 20% and 20%). Lastly, we consider an overflow trigger policy (policy 7) where patients are overflowed to their secondary IWs (if there is no capacity available in the

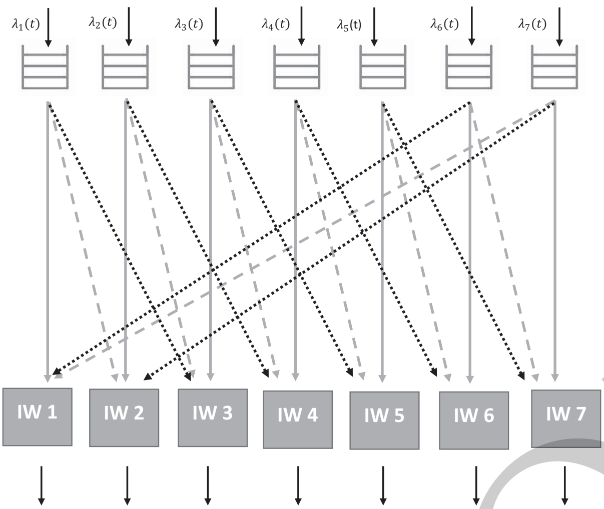

FIGURE 18.EC.8 Patient flow in extended simulation model

ideal secondary IW, and then assign to the back-up secondary IW) if their boarding time in ED exceeds two hours and if there is available capacity in the IW.

We compare the total cost metric of the proposed LEWC-p policy with the previsouly mentioned alternative patient flow policies under various cost combinations. In Table 18.EC.13, we report the relative difference in the total cost between LEWC-p and the alternative policy in each column, where rows indicate the cost parameter combination considered.

We allow for a total of 15 alternative cost combinations to observe the performance of the proposed policy under various cost parameters. In each combination, we use a higher cost parameter for backup secondary assignments than for the ideal secondary assignments. In cases 1–9, the cost of backup overflow is twice the cost of ideal overflows (penalty costs (2,4), (5,10), (50,100) are used when cost associated with ROAE per unit time is fixed at one, penalty costs (10,20), (25,50), (250,500) are used when cost associated with ROAE per unit time is fixed at five, and penalty costs (20,40), (50,100), (500,1000) are used when cost associated with ROAE per unit time is fixed at 10). In cases 10–15, cost of ideal overflow is 10 fold less than the back-up overflows (penalty costs are set as (15,25), (20,30), (25,35), (30,40), (35,45), (40,50) when cost associated with ROAE per unit time is fixed at five).

Our results indicate that in all cost parameter settings, LEWC-p performs better than alternative policies with respect to the total cost metric. This is because the LEWC-p policy effectively takes into account the trade-off between ROAE and quality of care: It is not as conservative as the primary-only policy in secondary IW assignments and yet not as aggressive as the both-primary-

TABLE 18.EC.13 *Relative difference in total cost between LEWC-p and alternative policies*

Cost Case	**Policy 1**	**Policy 2**	**Policy 3**	**Policy 4**	**Policy 5**	**Policy 6**	**Policy 7**
Case 1	4.20	0.46	0.43	0.36	0.25	0.52	1.95
Case 2	1.74	0.91	0.74	0.58	0.25	0.79	0.63
Case 3	0.45	2.84	2.37	1.93	1.05	2.36	0.42
Case 4	4.29	0.49	0.45	0.39	0.28	0.54	2.00
Case 5	1.91	1.03	0.86	0.68	0.33	0.90	0.74
Case 6	0.14	3.11	2.61	2.15	1.19	2.61	0.24
Case 7	4.23	0.47	0.44	0.37	0.26	0.53	1.97
Case 8	3.79	2.33	2.05	1.76	1.19	2.13	1.86
Case 9	0.09	6.54	5.62	4.76	3.02	5.61	0.14
Case 10	4.10	0.57	0.43	0.37	0.28	0.46	1.89
Case 11	3.37	0.76	0.55	0.45	0.30	0.54	1.51
Case 12	2.87	0.92	0.65	0.53	0.33	0.63	1.24
Case 13	2.19	0.89	0.60	0.47	0.24	0.56	0.87
Case 14	1.86	0.96	0.64	0.50	0.25	0.59	0.69
Case 15	1.59	1.01	0.68	0.52	0.25	0.61	0.55

and-secondary policy (where beds are fully shared) in making use of such assignments.

The extended simulation model suggests that our proposed policy still performs well in more complex systems. Hence, it might be an effective policy for hospital-wide implementation.