

EECS 117

Lecture 15: Biot-Savart Law, Ampere's Law and Magnetic Dipole

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Biot Savart Law and Ampere's Law

● In the last lecture, we have shown that the magnetic force exerted on a small segment of wire flowing a current I with length $d\vec{l}$ is equal to

$$dF_m = Id\vec{l} \times \vec{B}$$

where B is the magnetic flux density, and $\vec{B} = \mu\vec{H}$. The H is called the magnetic field intensity and μ is the permeability of the medium, which is the medium properties just like ϵ .

● The magnetic field intensity H induced by a current carrying wire segment has been shown to be

$$d\vec{H}(r) = \frac{I(r')d\vec{l} \times (\vec{r} - \vec{r}')}{4\pi|\vec{r} - \vec{r}'|^3} = \frac{\vec{J}(r') \times (\vec{r} - \vec{r}')}{4\pi|\vec{r} - \vec{r}'|^3} dv$$

Biot Savart Law and Ampere's Law (cont.)

\vec{r} is a vector pointing towards the point of interest from the origin, and \vec{r}' is a vector towards the wire segment. The second expression is for a non-uniform current distribution in the magnetic field generating system (like a wire).

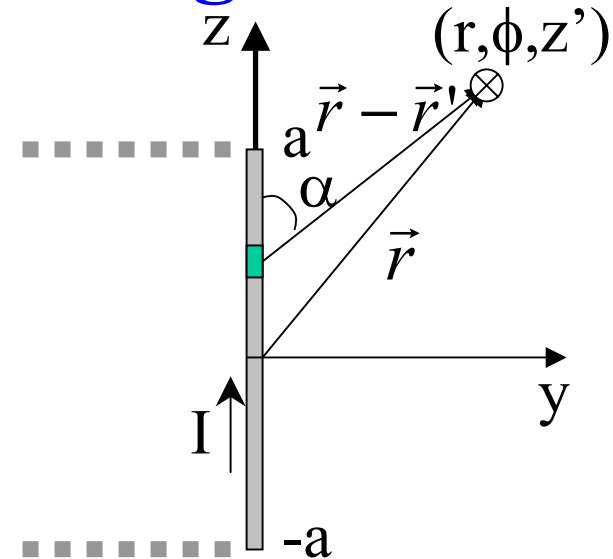
● Ampere's circuital law states that the magnetic field intensity integrated along a closed path is equal to the current flowing through the surface enclosed by the path.

$$\oint \vec{H} \cdot d\vec{l} = I$$

Off-axis Magnetic Field of a Straight Wire

- Consider a segment of a straight wire of length $2a$. Using Biot-Savart Law, the magnetic flux density due to a small segment $d\vec{l}$ is given by

$$\begin{aligned} d\vec{B}(r, \phi, z') &= \hat{\phi} \frac{\mu_0 I d\vec{z} \times (\vec{r} - \vec{r}')}{4\pi |\vec{r} - \vec{r}'|^3} \\ &= \hat{\phi} \frac{\mu_0 I \sin \alpha dz}{4\pi (r^2 + (z - z')^2)} \end{aligned}$$



- The \vec{B} vector is pointing into the paper.

$$\begin{aligned} \vec{B}(r, \phi, z') &= \hat{\phi} \int_{-a}^a \frac{\mu_0 I \sin \alpha dz}{4\pi (r^2 + (z - z')^2)} = \hat{\phi} \int_{-a}^a \frac{\mu_0 I r dz}{4\pi (r^2 + (z - z')^2)^{3/2}} \\ &= \frac{\mu_0 I r (z - z')}{4\pi r^2 (r^2 + (z - z')^2)^{1/2}} \Big|_{-a}^a = \frac{\mu_0 I}{4\pi r} \left[\frac{z' + a}{\sqrt{r^2 + (z' + a)^2}} - \frac{z' - a}{\sqrt{r^2 + (z' - a)^2}} \right] \end{aligned}$$

On-axis Magnetic Field of a Straight Wire

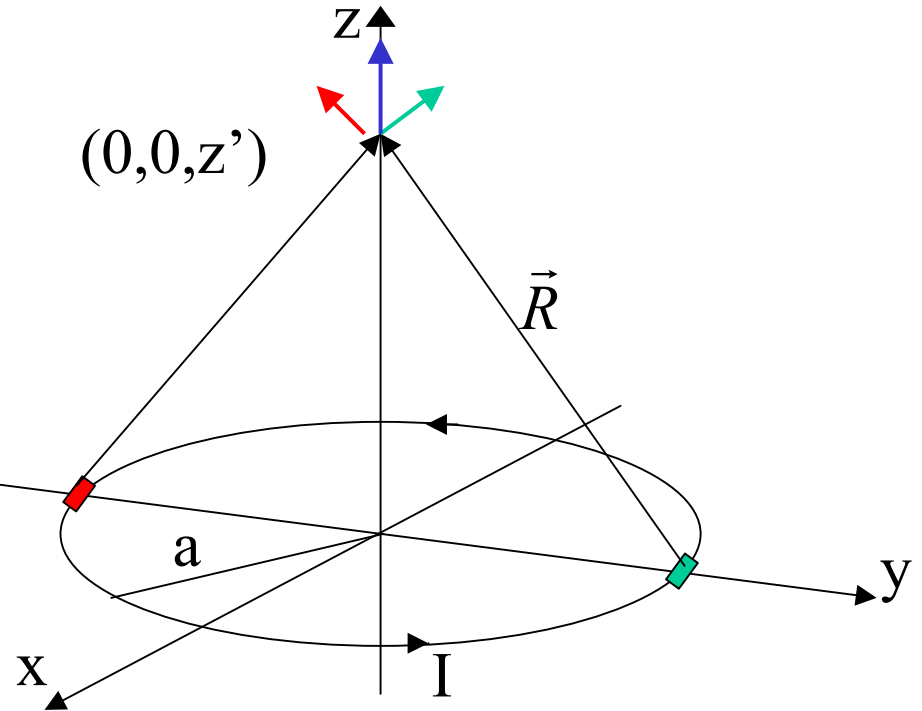
- If the point of interest is at the midpoint of the wire, $z' = 0$, and

$$\vec{B} = \hat{\phi} \frac{\mu_0 I a}{2\pi r \sqrt{r^2 + a^2}}$$

On-axis Magnetic Field of a Circular Loop

• Consider now the magnetic field along the axis of a circular loop wire.

• The green segment induces a magnetic field in green. Because of circular symmetry, the segment (red) on the opposite side create a field (red) of equal magnitude. The resultant field is along the z-axis.



On-axis Magnetic Field of a Circular Loop

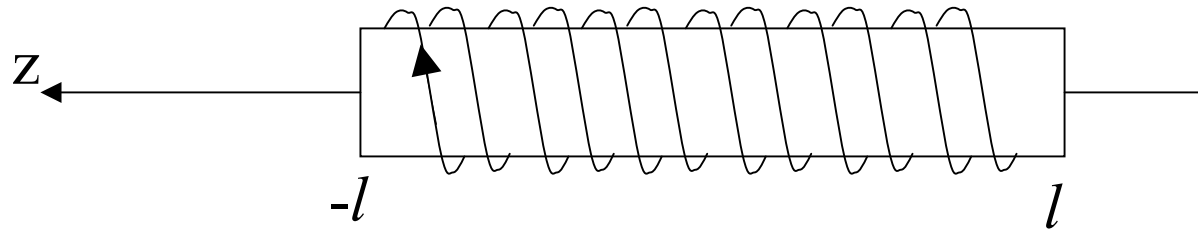
$$d\vec{B}(0,0,z') = \frac{\mu_0 I d\vec{l} \times \hat{R}}{4\pi |\vec{R}|^2}$$

$$dB_z = \frac{\mu_0 I}{4\pi} \frac{a dl}{(a^2 + z^2)^{3/2}} = \frac{\mu_0 I}{4\pi} \frac{a^2 d\phi}{(a^2 + z^2)^{3/2}}$$

$$\vec{B} = \hat{k} \frac{\mu_0 I}{4\pi} \frac{a^2}{(a^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi = \hat{k} \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}$$

• At the center of the loop, $\vec{B} = \frac{\mu_0 I}{2a} \hat{k}$

On-axis Magnetic Field of a Solenoid



- Using the result above, we can calculate the on-axis magnetic field inside the solenoid
- Consider a solenoid with n number of windings per meter. The length of the solenoid is $2l$.

$$dB_z = \frac{\mu_0 n I}{2} \frac{a^2 dz}{(a^2 + z^2)^{3/2}}$$

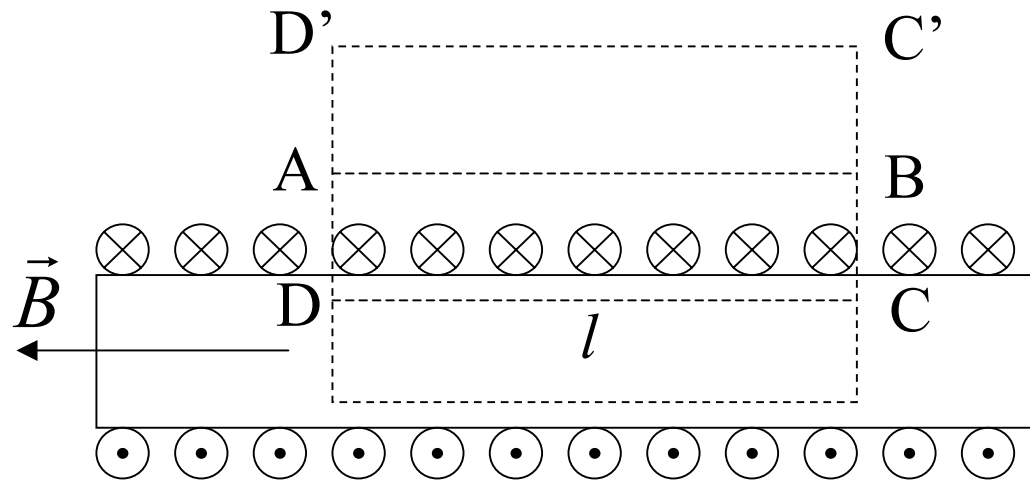
$$B_z = \frac{\mu_0 n I a^2}{2} \int_{-l}^l \frac{dz}{(a^2 + z^2)^{3/2}} = \frac{\mu_0 n I a^2}{2} \frac{z}{a^2 \sqrt{a^2 + z^2}} \Big|_{-l}^l = \frac{\mu_0 N I}{\sqrt{a^2 + l^2}}$$

On-axis Magnetic Field of a Solenoid (cont.)

where $N = 2nl$, the total number of windings in the solenoid.

• For an infinitely long solenoid, $B_z = \mu_0 nI$

Ampere's Circuital Law Version of a Solenoid



The cross section of the solenoid.

- We can obtain the same information, and even learn more by using the Ampere's circuital law.
- The line integral of the B can be broken into the segment parts

$$\oint \vec{B} \cdot d\vec{l} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Solenoid (Ampere's Circuital Law)

• Based on the result we obtained in the previous calculation, \vec{B} inside the solenoid is only in the z direction. The geometry implies that this would be true also for the magnetic field outside the solenoid. So,

$$\int_{BC} = \int_{DA} = 0$$

• If we consider the path ABC'D', the path integrals along BC' and D'A are zero because of the above reason. Because there is not current flowing through the area enclosed by the closed path,

$$\int_{AB} = -\int_{C'D'}$$

This result holds if the hypothetical box ABC'D' is drawn such that the segment C'D' is at infinity. This implies that the magnetic field is uniform *everywhere* outside the solenoid. This contradicts our intuition of a field falling down to zero at infinity and also experimental observation.

Solenoid (Ampere's Circuital Law)

The only solution to this contradiction is that the magnetic field outside of solenoid is zero. This coincides with experimental results.

• The result is:
$$\oint \vec{B} \cdot d\vec{l} = \int_{CD} \vec{B} \cdot d\vec{l} = \mu_0 n l I$$

If the segment CD is far away from the ends of the solenoid, B is uniform along the segment. Thus,

$$\int_{CD} \vec{B} \cdot d\vec{l} = B l = \mu_0 n l I \Rightarrow B = \mu_0 n I$$

This result is the same as the one for the infinite solenoid.

• If we move the segment CD up and down, since the current enclosed remains the same, the path integral along CD is the same. So the magnetic field is uniform *inside* the solenoid. For an infinite solenoid or the region where end effects are negligible, the field *everywhere* inside the solenoid is equal to the above result.

Vector Potential

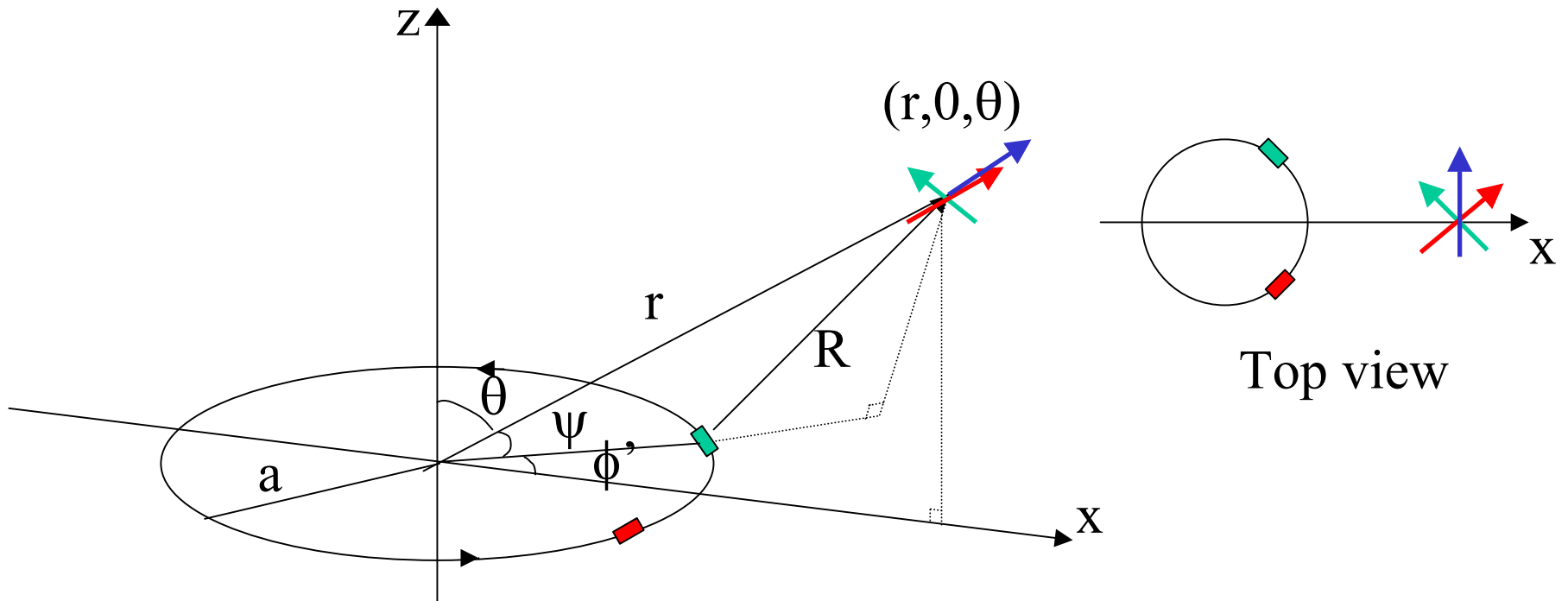
● Since there is no magnetic monopole $\nabla \cdot \vec{B} = 0$, \vec{B} can be expressed as $\nabla \times \vec{A}$, the curl of vector potential.

● For the two different expressions of B, we have

$$\vec{B}(r) = \frac{\mu}{4\pi} \int \frac{I(r') d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\mu}{4\pi} \int \frac{\vec{J}(r') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dv$$

$$\vec{A}(r) = \frac{\mu}{4\pi} \int \frac{I(r') d\vec{l}}{|\vec{r} - \vec{r}'|^2} = \frac{\mu}{4\pi} \int \frac{\vec{J}(r')}{|\vec{r} - \vec{r}'|^2} dv$$

Magnetic Dipole



- For a current loop, the magnetic field off axis can be calculated by finding vector potential.

Magnetic Dipole

• The vector potential due to the green segment has a direction of the green arrow. Because of symmetry, the potential due to the red segment almost cancel the green vector potential and the resultant potential due to these two segments has a direction normal to the xz plane, or in general in $\hat{\phi}$ direction. Without loss of generality, we place the point of interest in the xz plane.

• The ϕ component of A due to a segment is equal to

$$dA_{\phi} = \frac{\mu I dl' \cos \phi'}{4\pi R}$$

• The total is found as the integral around the loop

$$A_{\phi} = \oint \frac{\mu I dl' \cos \phi'}{4\pi R} = \frac{\mu I a}{4\pi} \int_0^{2\pi} \frac{\cos \phi' d\phi'}{R}$$

Magnetic Dipole

where $R^2 = r^2 + a^2 - 2ra \cos \psi = r^2 + a^2 - 2ra \sin \theta \cos \phi'$

- For $r \gg a$

$$R \approx r \sqrt{1 - 2 \frac{a}{r} \sin \theta \cos \phi'} \quad \text{or} \quad \frac{1}{R} \approx \frac{1}{r} \left(1 + \frac{a}{r} \sin \theta \cos \phi' \right)$$

- The total vector potential is

$$A_\phi = \frac{\mu I a}{4\pi} \int_0^{2\pi} \left(\cos \phi' + \frac{a}{r} \sin \theta \cos^2 \phi' \right) d\phi' = \frac{\mu (I \pi a^2) \sin \theta}{4\pi r^2}$$

- \vec{B} are then equal to

$$\begin{aligned} \vec{B} &= \nabla \times A_\phi \hat{\phi} = \frac{\hat{r}}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\hat{\theta}}{r} \frac{\partial}{\partial r} (r A_\phi) \\ &= \frac{\mu (I \pi a^2)}{2\pi r^3} [\hat{r} \cos \theta + \hat{\theta} \sin \theta] \end{aligned}$$

Magnetic Dipole

- The magnitude of the magnetic dipole moment, the counterpart of electric dipole, is defined as

$$m = I\pi a^2$$

The dipole direction is pointing up from the loop for the counter-clockwise I direction (as in this case), and vice versa.

- The vector potential can then be expressed as

$$\vec{A} = -\frac{\mu_0}{4\pi} \vec{m} \times \nabla \left(\frac{1}{r} \right)$$