

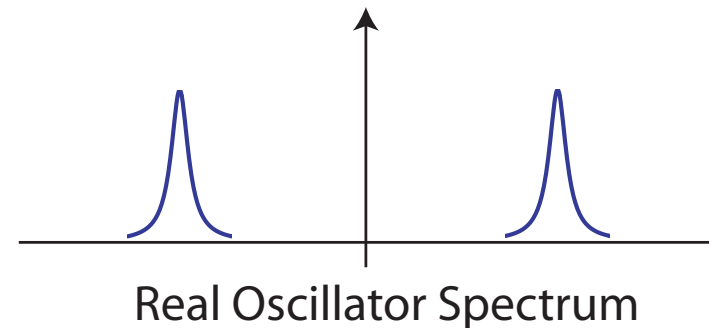
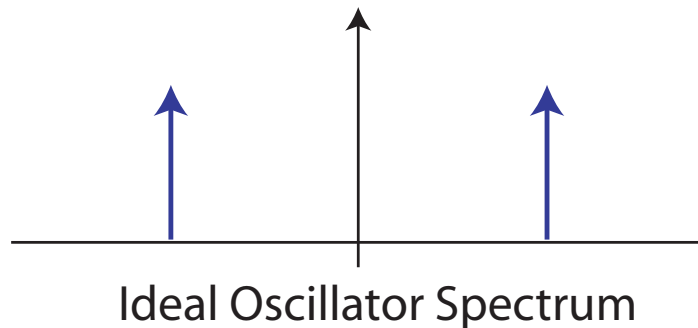
Lecture 24: Oscillator Phase Noise

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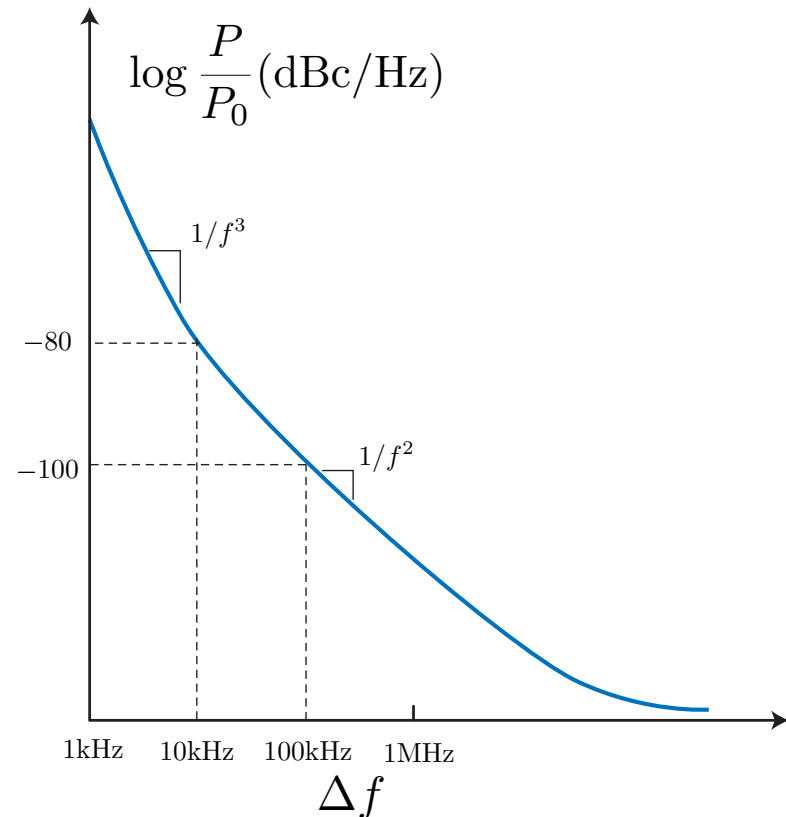
Oscillator Output Spectrum



- The output spectrum of an oscillator is very peaked near the oscillation frequency but not infinitely so.
- If we ignore noise, the closed-loop gain of the system is infinite since $A_l = 1$. But in practice there is noise in any real oscillator.

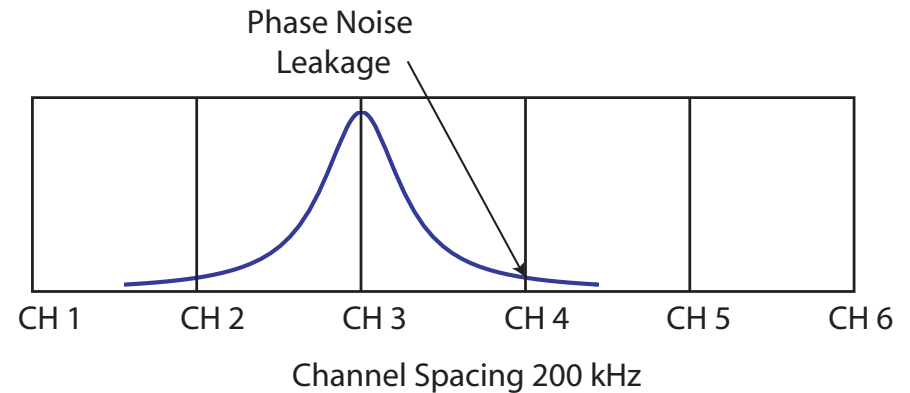
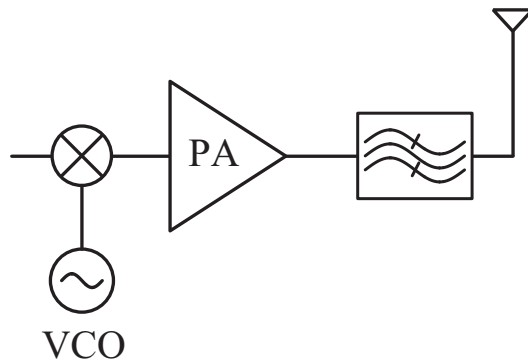
Phase Noise Measurement

- If we zoom into the carrier on a log scale, the relative power at an offset frequency Δf from the carrier drops very rapidly. For the case shown above, at an offset of 100kHz, the power drops to -100dBc .



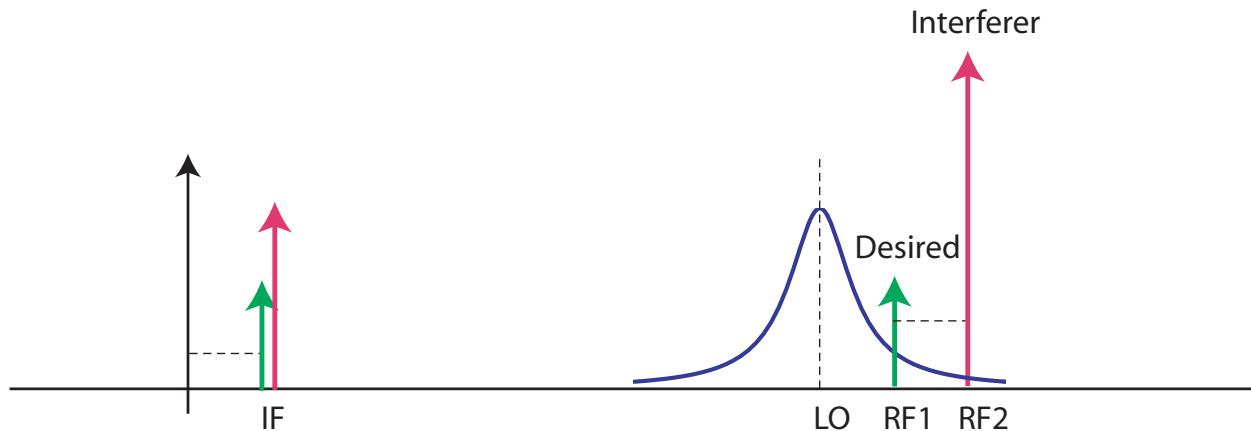
- There is clearly a region where the slope is 20dB/dec. But this range only holds until the noise flattens out. Also, very near the carrier, the slope increases to approximately 30dB/dec.

Phase Noise In TX Chain



- Phase noise in a transmit chain will “leak” power into adjacent channels. Since the power transmitted is large, say about 30dBm, an adjacent channel in a narrowband system may only reside about 200kHz away (GSM), placing a stringent specification on the transmitter spectrum.

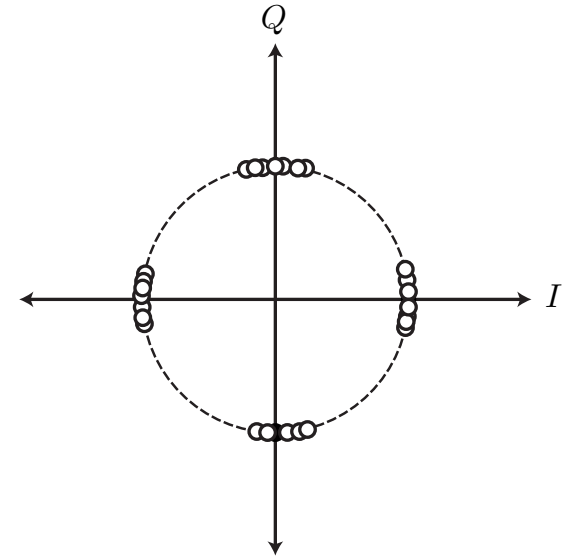
Phase Noise In RX Chain



- In a receive chain, the fact that the LO is not a perfect delta function means that there is a continuum of LO's that can mix with interfering signals and produce energy at the same IF. Here we observe an adjacent channel signal mixing with the “skirt” of the LO and falling on top of the a weak IF signal from the desired channel.

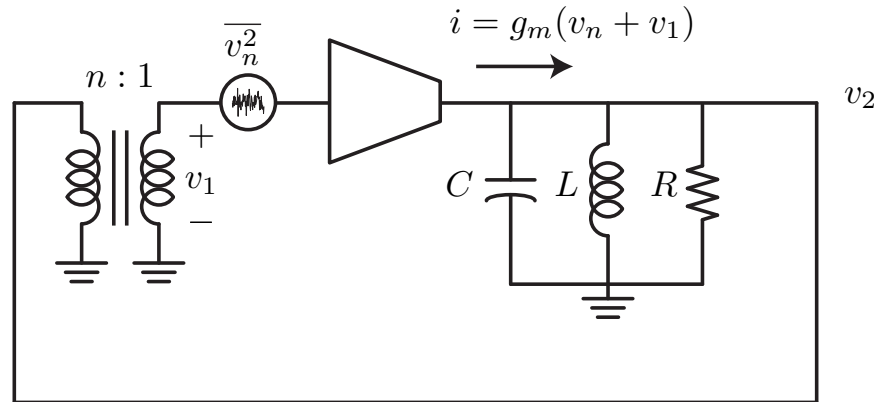
Phase Noise In Digital Communication

- In a digital communication system, phase noise can lead to a lower noise margin. Above, we see that the phase noise causes the constellation of a 4 PSK system to spread out.



- In OFDM systems, a wide bandwidth is split into sub-channels. The phase noise leads to inter carrier interference and a degradation in the digital communication BER.

LTI Analysis of Oscillator



- Consider a simple LTI analysis of the oscillator with a noise voltage v_n . An active device is assumed to pump energy into the tank through positive feedback. We have

$$v_2 = g_m(v_1 + v_n)Z_T = g_m \left(\frac{v_2}{n} + v_n \right) Z_T$$

Noise Analysis

- Continuing to simplify the above results

$$v_2 = \frac{g_m Z_T v_n}{1 - \frac{g_m Z_T}{n}} = \frac{g_m R v_n}{\frac{R}{Z_T} - \frac{g_m R}{n}}$$

or

$$v_2 = v_n \frac{g_m R}{1 - \frac{g_m R}{n} + j B R}$$

- The reactive term B can be simplified at a small offset $\delta\omega$ from the resonance ω_0

$$B = \frac{1}{j(\omega_0 + \delta\omega)L} + j(\omega_0 + \delta\omega)C$$

Simplification Near Resonance

- Now comes the approximation

$$\begin{aligned} B &\approx \frac{1}{j\omega_0 L} \left(1 - \frac{\delta\omega}{\omega_0} \right) + j(\omega_0 + \delta\omega)C \\ &= j\delta\omega C - \frac{\delta\omega/\omega_0}{j\omega_0 L} = 2j\delta\omega C \end{aligned}$$

- where $\omega_0^2 = 1/(LC)$. Using the notation $A_\ell = g_m R/n$

$$v_2 = v_n \frac{nA_\ell}{(1 - A_\ell) + j2\delta\omega RC}$$

$$v_{2,rms}^2 = \overline{v_n^2} \frac{n^2 A_\ell^2}{(1 - A_\ell)^2 + 4\delta\omega^2 R^2 C^2}$$

Oscillator Power

- If we now observe that the total power of the oscillator is fixed we have

$$P = \frac{v_{2,rms}^2}{R} = \frac{1}{R} \overline{v_n^2} \int_{-\infty}^{\infty} \frac{n^2 A_\ell^2}{(1 - A_\ell)^2 + 4\delta\omega^2 R^2 C^2} d(\delta\omega)$$

- This integral is closed since it's in the known form

$$\int_{-\infty}^{\infty} \frac{dx}{1 + a^2 x^2} = \frac{\pi}{a}$$

$$P = \frac{\overline{v_n^2}}{R} \frac{A_\ell^2}{(1 - A_\ell)^2} \frac{\pi(1 - A_\ell)n^2}{2RC} = \frac{\overline{v_n^2} n^2}{R} \frac{\pi}{2} \frac{1}{RC} \frac{A_\ell^2}{(1 - A_\ell)}$$

- Since $P = P_{osc}$, we can solve for A_ℓ .

Non-unity Loop Gain

- Since P_{osc} is finite, $A_\ell \neq 1$ but it's really close to unity

$$P_{osc}(1 - A_\ell) = \frac{\overline{v_n^2} n^2}{R} \frac{\pi}{2} \frac{1}{RC} A_\ell^2$$

- Since $A_\ell \approx 1$

$$(1 - A_\ell) = \frac{\frac{\overline{v_n^2}}{R}}{P_{osc}} \underbrace{\frac{\pi}{2} \frac{1}{RC}}_{\Delta f_{RC}}$$

- Since we integrated over negative frequencies, the noise voltage is given by

$$\overline{v_n^2} = 2kTR_{eff}$$

Magnitude of A_ℓ

- But since $\overline{v_n^2}/R$ over the equivalent bandwidth Δf_{RC} is much small than P_{osc} , we expect that

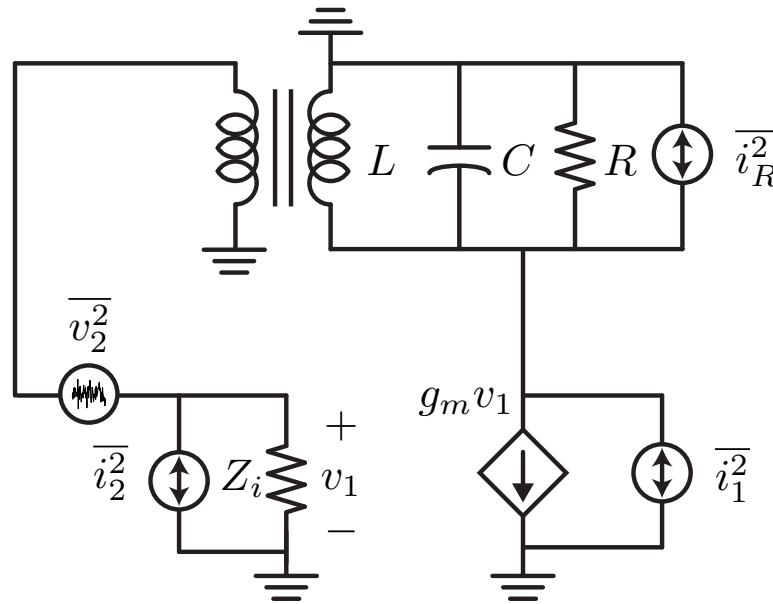
$$(1 - A_\ell) = \frac{\frac{\overline{v_n^2}}{R}}{P_{osc}} \Delta f_{RC} = \epsilon$$

or

$$A_\ell = 1 - \epsilon$$

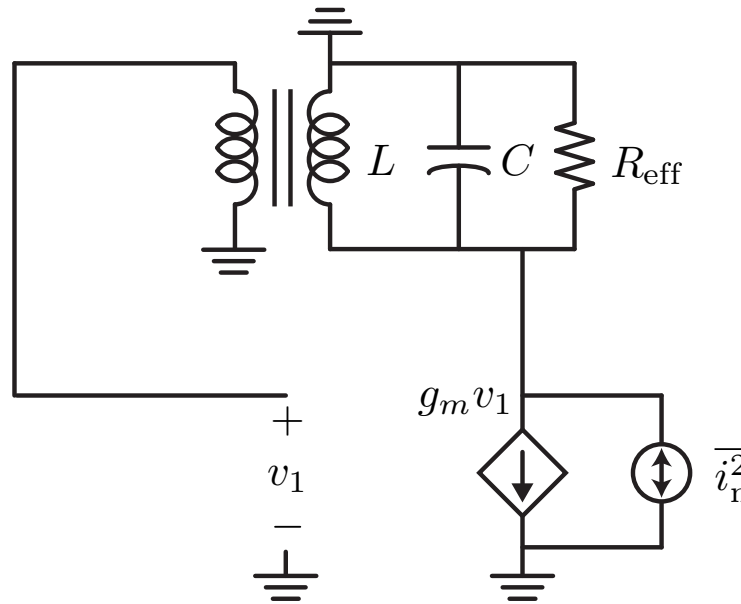
- The LTI interpretation is that the amplifier has positive feedback and it limits on it's own noise. The loop gain is nearly unity but just below so it's “stable”.

Standard Oscillator LTI Analysis



- The above equivalent circuit includes the “drain” noise i_1^2 , the load noise i_R^2 , and an input voltage/current noise v_2^2 and i_2^2 .

Equivalent Noise Model



- All the noise sources can be moved to the output by an appropriate transformation

$$\overline{i_n^2} = \overline{i_1^2} + \frac{\overline{i_2^2}}{n^2} + \overline{v_n^2} \left(g_m - \frac{1}{Z_i} \right)^2 + \overline{i_R^2}$$

LTI Noise Analysis

- The output voltage is given by

$$v_o = -(g_m v_1 + i_n) Z_T$$

since

$$v_1 = \frac{-v_o}{n}$$

we have

$$v_1 = \frac{g_m Z_T}{n} v_o - i_n Z_T$$

$$v_o = \frac{-i_n Z_T}{1 - \frac{g_m Z_T}{n}}$$

Tank Near Resonance

- The tank impedance can be put into this form

$$Z_T = \frac{1}{\frac{1}{R_1} + j\omega C + \frac{1}{j\omega L}} = \frac{R_1}{1 + j\frac{\omega}{\omega_0}Q + \frac{1}{j\omega}\omega_0 Q}$$

- Where the loaded tank $Q = R_1/(\omega_0 L) = \omega_0 R_1 C$

$$Z_T = \frac{R_1}{1 + jQ \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}$$

- If $\omega = \omega_0 + \delta\omega$ and $\delta\omega \ll \omega_0$

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \approx \frac{2\delta\omega}{\omega_0}$$

Transfer Near Resonance

- We now have that

$$Z_T(\omega_0 + \delta\omega) \approx \frac{R_1}{1 + j2Q\frac{\delta\omega}{\omega_0}}$$

- This allows to write the output voltage as

$$v_o = -i_n \frac{Z_T}{1 - \frac{g_m R_1}{n} \frac{1}{1 + j2Q\frac{\delta\omega}{\omega_0}}} = -i_n \frac{R_1}{\left(1 - \frac{g_m R_1}{n}\right) + j2Q\frac{\delta\omega}{\omega_0}}$$

- Now it's time to observe that $A_\ell = \frac{g_m R_1}{n}$ is the initial loop gain. If we assume that $A_\ell \leq 1$, then the circuit is a high gain positive feedback amplifier.

Lorentzian Spectrum

- The power spectrum of v_o is given by

$$v_o^2 = \overline{i_n^2} \frac{R_1^2}{\left(1 - \frac{g_m R_1}{n}\right)^2 + 4Q^2 \frac{\delta\omega^2}{\omega_0^2}}$$

- This has a Lorentzian shape for white noise. For offsets frequencies of interest

$$4Q^2 \frac{\delta\omega^2}{\omega_0^2} \gg \left(1 - \frac{g_m R_1}{n}\right)^2$$

- Thus a characteristic $\delta\omega^2$ roll-off with offset.

Noise at Offsets

- The spectrum normalized to the peak is given by

$$\left(\frac{v_o}{V_o}\right)^2 \approx \frac{\overline{i_n^2} R_1^2}{V_o^2} \left(\frac{\omega_0}{\delta\omega}\right)^2 \frac{1}{4Q^2}$$

- The above equation is in the form of Leeson's Equation. It compactly expresses that the oscillator noise is expressed as noise power over signal power (N/S), divided by Q^2 and dropping like $1/\delta\omega^2$.

Total Noise Power

- We can express the total noise power similar to before

$$\begin{aligned} V_o^2 &= \int_{-\infty}^{\infty} v_o^2 d(\delta\omega) \\ &= \frac{\overline{i_n^2} R_1^2}{(1 - A_\ell)^2} \int_{-\infty}^{\infty} \frac{d(\delta\omega)}{1 + 4Q^2 \left(\frac{\delta\omega}{\omega_0}\right)^2 \frac{1}{(1 - A_\ell)^2}} \\ V_o^2 &= \frac{\overline{i_n^2} R_1^2}{(1 - A_\ell)} \frac{\pi}{2} \frac{f_o}{Q} \end{aligned}$$

Lorentzian Bandwidth

- We again interpret the amplitude of oscillation as the the noise power $\overline{i_n^2} R_1^2$ gained up by the positive feedback

$$(1 - A_\ell) = \frac{\overline{i_n^2} R_1^2}{V_o^2} \frac{\pi f_o}{2Q}$$

- The 3 dB bandwidth of the Lorentzian is found by

$$(1 - A_\ell) = 2Q \frac{f - f_o}{f_o} = \frac{2Q \Delta f}{f_o}$$

$$\Delta f = \frac{f_o}{2Q} (1 - A_\ell) = \frac{\overline{i_n^2} R_1^2}{V_o^2} \frac{\pi}{4Q^2} f_o$$

Example Bandwidth

- For example, take $\overline{i_n^2} = 10^{-22} \text{A}^2/\text{Hz}$, $f_o = 1\text{GHz}$, $R_1 = 300\Omega$, $Q = 10$ and $V_o = 1\text{V}$. This gives a $\Delta f = 0.07\text{Hz}$.
- This is an extremely low bandwidth. This is why on the spectrum analyzer we don't see the peak of the waveform. For even modest offsets of $100 - 1000\text{Hz}$, the $1/\delta\omega^2$ behavior dominates. But we do observe a $1/\delta\omega^3$ region.

Noise Corner Frequency

- Because the oscillator is really a time-varying system, we should consider the effects of noise folding. For instance, consider any low frequency noise in the system. Due to the pumping action of the oscillator, it will up-convert to the carrier frequency.
- In reality the pumping is not perfectly periodic due to the noise. But we assume that the process is *cyclostationary* to simplify the analysis.
- Since there is always $1/f$ noise in the system, we now see the origin of the $1/f^3$ region in the spectrum.

Noise due to Non-Linear Caps

- Another noise upconversion occurs through non-linear capacitors. This is particularly important on the VCO control line.
- Assume that $C_j = C_0 + K_C \Delta V_c$. Since the frequency is given by

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$

- We see that $f_o = f_{oQ} + K_f \Delta V_c$. $K_f \approx 10 - 100\text{MHz/V}$. The oscillation waveform is given by

$$V(t) = V_o \cos \left(\int 2\pi f_o dt \right)$$

Noise Sidebands

- For ΔV_c a tone at some offset frequency ω_m , we have

$$\Delta V_c = V_m \cos \omega_m t$$

- where $V_m = \sqrt{4kTR_c} \sqrt{2V} / \sqrt{\text{Hz}}$ due to noise. This produces noise sidebands

$$V(t) = V_o \cos \left(\omega_0 t + \sqrt{2} \sqrt{4kTR_c} \frac{K_f 2\pi}{\omega_m} \sin \omega_m t \right)$$

- For small noise

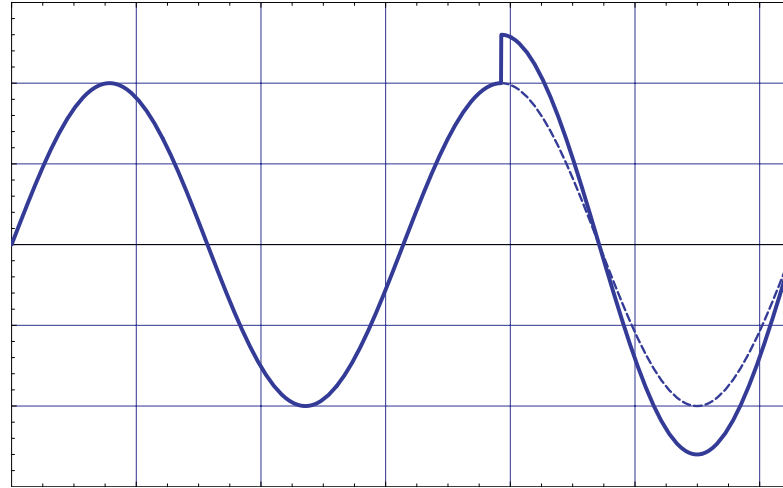
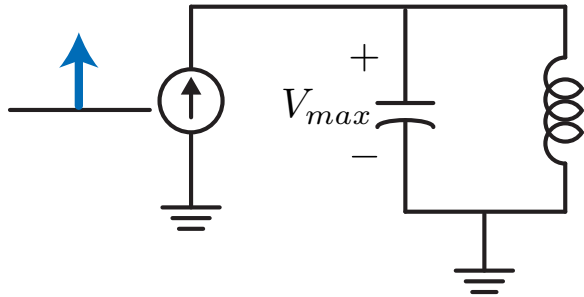
$$V(t) \approx V_o \cos(\omega_0 t) - V_o \sin(\omega_0 t) \sqrt{8kTR_c} \frac{K_f}{\omega_m} \sin(\omega_m t)$$

$$V(t) \approx \frac{V_o}{\omega_m} K_f \sqrt{8kTR_c} \frac{1}{2} \cos(\omega_o \pm \omega_m)$$

LTV Phase Noise Model

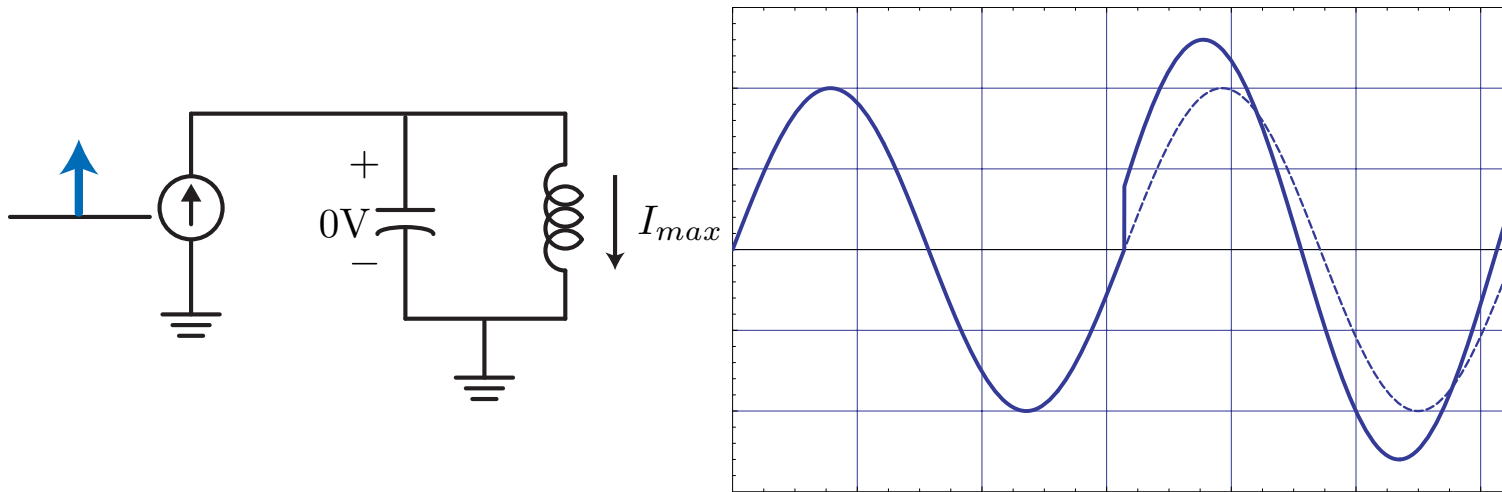
- The noise analysis thus far makes some very bad assumptions. Most importantly, we neglect the time-varying nature of the process. Every oscillator is a quasi-periodic system and the noise analysis should take this into account.
- The following noise model is due to Hajimiri/Lee. It begins with a simple thought experiment.
- Imagine injecting a current impulse into an LC tank at different times. We assume the LC tank is oscillating at the natural frequency.

Injection at Peak Amplitude



- Since the impulse of current “sees” an open circuit across the inductor but a short circuit across the capacitor, all the current will flow into the capacitor, dumping a charge δq onto the capacitor plates.
- Note that if the injection occurs at the peak voltage amplitude, it will change the amplitude of oscillation.
- The phase of oscillation, though, is unaltered.

Injection at Zero-Crossing



- If the injection occurs at the waveform crossing, though, the change in amplitude also changes the phase of the oscillator.
- So we see the *sensitivity* of the oscillator to noise injection is a periodic function of time. There are points of zero sensitivity and points of peak sensitivity.

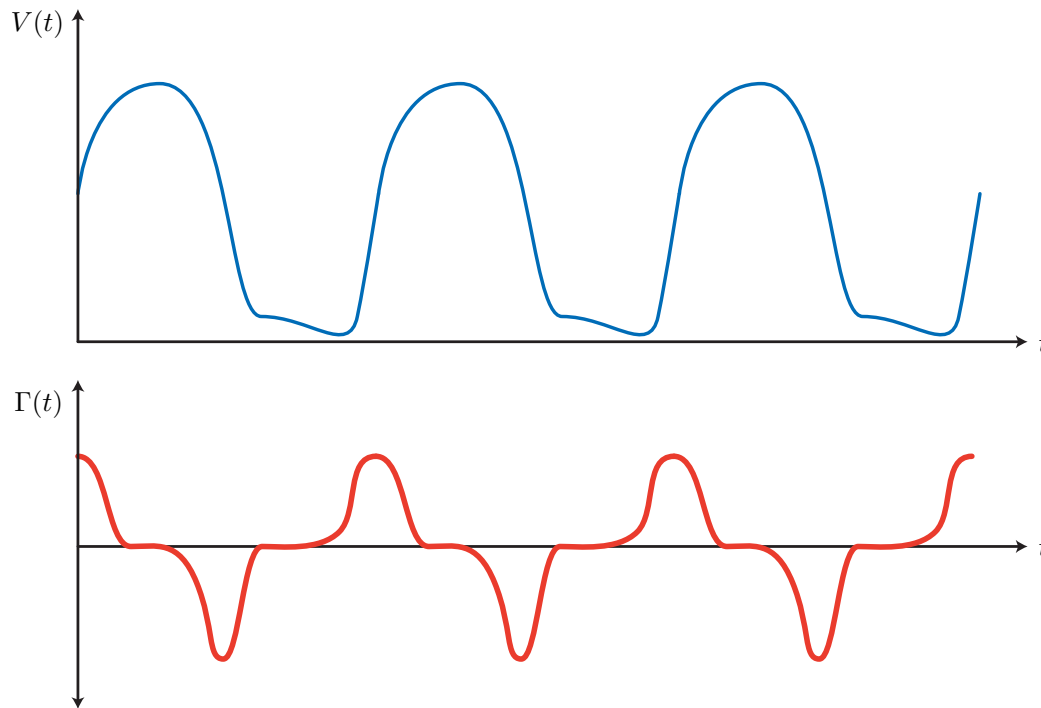
ISF Model

- The key observation (experimentally confirmed) is that the phase change is a linear function of the disturbance injection (for small injections). Therefore we write the impulse response in the following normalized form

$$h_{\phi}(t, \tau) = \frac{\Gamma(\omega_0 \tau)}{q_{max}} u(t - \tau)$$

- The constant $q_{max} = CV_{peak}$ is simply a normalization constant, the peak charge in the oscillator. The response is zero until the system experiences the input (causality), but then it is assumed to occur instantaneously, leading to the step function response. The function $\Gamma(\omega_0 \tau)$, the Impulse Sensitivity Function (ISF), is a periodic function of time, capturing the time varying periodic nature of the system.

Example Waveforms



This ISF can be estimated analytically or calculated from simulation

- Note a hypothetical system with output voltage waveform and ISF. As expected, the ISF peaks during “zero” crossings and is nearly zero at the peak of the waveform.

General Response

- For any deterministic input, we have the convolution integral

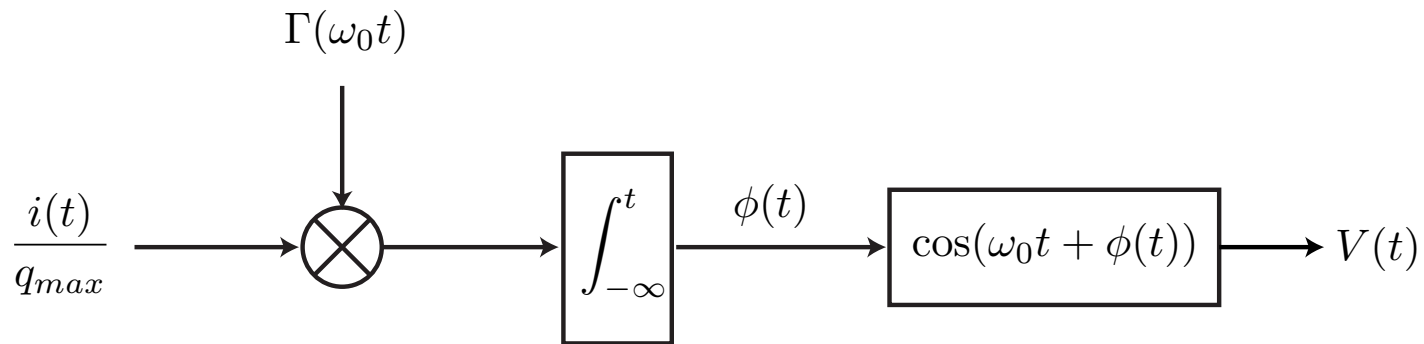
$$\begin{aligned}\phi(t) &= \int_{-\infty}^{\infty} h_{\phi}(t, \tau) i(\tau) d\tau \\ &= \frac{1}{q_{max}} \int_{-\infty}^t \Gamma(\omega_0 \tau) i(\tau) d\tau\end{aligned}$$

- Since the ISF function Γ is periodic

$$\Gamma(\omega_0 \tau) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 \tau + \phi_n)$$

$$\phi(t) = \frac{1}{q_{max}} \left(\frac{c_0}{2} \int_{-\infty}^t i(\tau) d\tau + \sum_{n=1}^{\infty} c_n \int_{-\infty}^t \cos(n\omega_0 \tau) i(\tau) d\tau \right)$$

Phase Deviation to Output



- Graphically, we see that a noise disturbance creates a phase disturbance as shown above, and this in turn modulates the phase of the carrier. This last step is a non-linear process.
- The phase function $\phi(t)$ appears in an oscillator as a phase modulation of the carrier. Note that the phase itself is not (easily) observed directly.

Noise Sidebands

- The noise sidebands due to current noise at an offset $\Delta\omega$ from the m 'th harmonic (including DC) is now calculated.

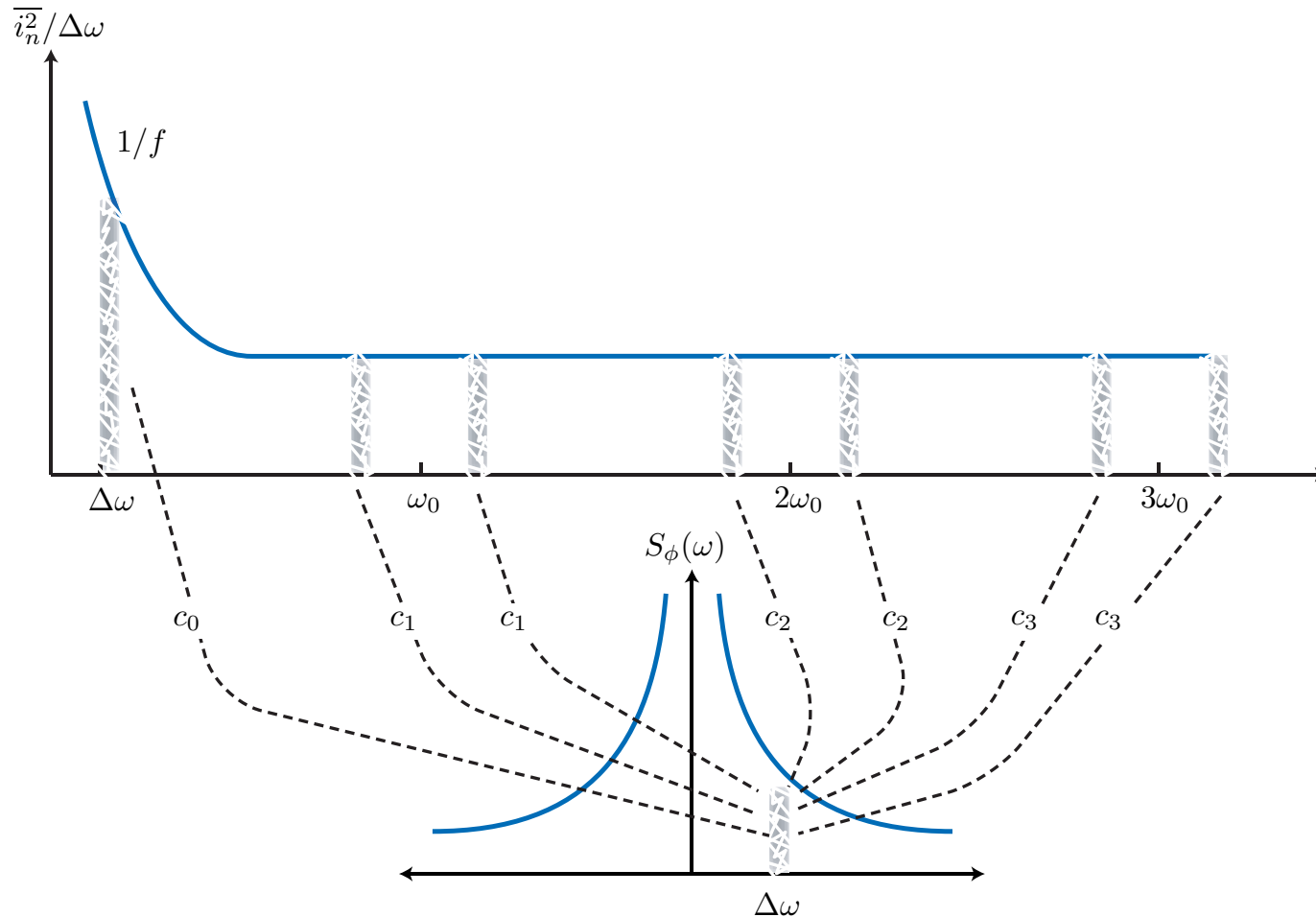
$$\begin{aligned} i(t) &= I_m \cos(m\omega_0 + \Delta\omega)t \\ &= I_m (\cos m\omega_0 t \cos \Delta\omega t - \sin m\omega_0 t \sin \Delta\omega t) \end{aligned}$$

- If we insert this into the above integration, for small offsets $\Delta\omega \ll \omega_0$, we find (approximately) that all terms are orthogonal and integrate to zero except when $n = m$.

- The non-zero term integrates to give

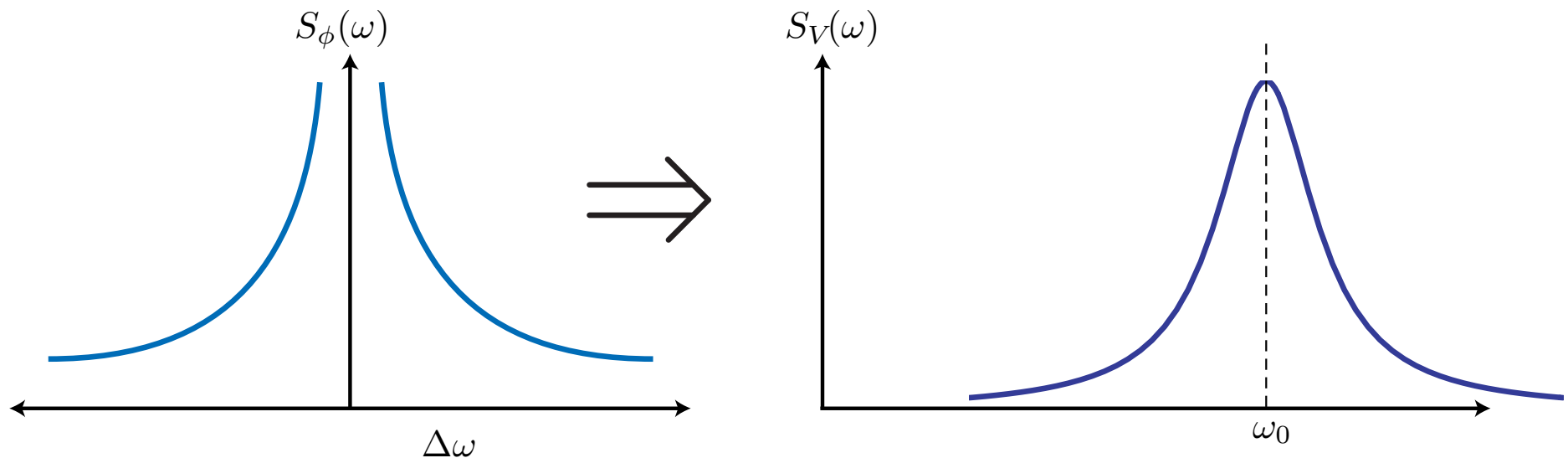
$$\phi(t) \approx \frac{1}{2} c_m \frac{I_m \sin \Delta\omega t}{q_{max} \Delta\omega}$$

Graphical Interpretation



- We see that all noise a distance $\Delta\omega$ around all the harmonics, including DC, contributes to the phase noise. DC $1/f$ noise contributes to the $1/f^3$ region.

Phase Noise versus Voltage Noise



- While the phase noise is *unbounded*, the output voltage is bounded. This is because the sinusoid is a bounded function and so the output voltage spectrum flattens around the carrier. In fact, if we assume that the phase is a Brownian noise process, the spectrum is computed to be a Lorentzian.

White Noise Expression

- We see that the noise power at offset $\Delta\omega$ is given by

$$P_{SBC}(\Delta\omega) \approx 10 \cdot \log \left(\frac{I_m c_m}{2q_{max} \Delta\omega} \right)^2$$

- If the noise is white, then we get equal contribution from all sidebands

$$P_{SBC}(\Delta\omega) \approx 10 \cdot \log \left(\frac{\overline{i_n^2} \sum_{m=0}^{\infty} c_m^2}{4q_{max}^2 \Delta\omega^2} \right)$$

Final Expression

- Parseval taught us that

$$\sum_{m=0}^{\infty} c_m^2 = \frac{1}{\pi} \int_0^{2\pi} |\Gamma(x)|^2 dx = 2\Gamma_{rms}^2$$

- This allows us to write the phase noise in the following form

$$P_{SBC}(\Delta\omega) \approx 10 \cdot \log \left(\frac{\overline{i_n^2} \Gamma_{rms}^2}{2q_{max}^2 \Delta\omega^2} \right)$$

- Thus to minimize the phase noise we must minimize the RMS value of the ISF.