

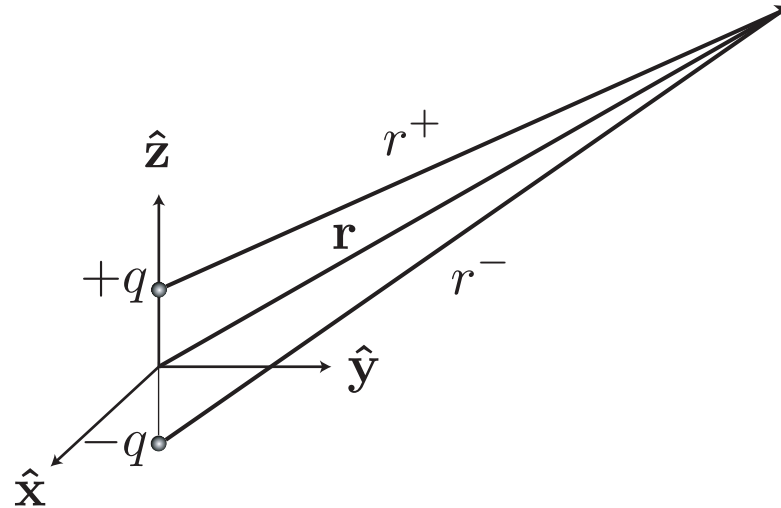
EECS 117

Lecture 8: Electrostatic Potential & Gauss Law

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Electric Dipole



- An electric dipole is two equal and opposite charges q separated by a distance d
- Consider the potential due to an electric dipole at points far removed from the dipole $r \gg d$

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon r^+} + \frac{-q}{4\pi\epsilon r^-} = \frac{q}{4\pi\epsilon} \left(\frac{1}{r^+} - \frac{1}{r^-} \right)$$

Law of Cosines

- Vector summation is fundamentally related to a triangle. Note that we can think of adding two vectors \mathbf{r} and $\hat{\mathbf{z}}d/2$ as forming $\mathbf{r}^+ = \mathbf{r} + \hat{\mathbf{z}}d/2$
- The length of a vector is given by $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$
- We can therefore express r^+ in terms of r and d

$$r^+ = \sqrt{\mathbf{r}^+ \cdot \mathbf{r}^+} = \sqrt{\left(\mathbf{r} + \frac{d}{2}\hat{\mathbf{z}}\right) \cdot \left(\mathbf{r} + \frac{d}{2}\hat{\mathbf{z}}\right)}$$

- Or simplifying a bit and recalling that $r \gg d$

$$r^+ = r \sqrt{1 + \left(\frac{d}{2r}\right)^2 + \frac{d}{r} \cos \theta} \approx r \sqrt{1 + \frac{d}{r} \cos \theta}$$

Back to Potential Calculation

- So we have

$$\frac{1}{r^+} = \frac{1}{r} \frac{1}{\sqrt{1 + \frac{d}{r} \cos \theta}} \approx \frac{1}{r} \left(1 - \frac{d}{2r} \cos \theta \right)$$

and

$$\frac{1}{r^+} \approx \frac{1}{r} \left(1 + \frac{d}{2r} \cos \theta \right)$$

- So that the potential is given by

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon} \left(\frac{1}{r^+} - \frac{1}{r^-} \right) = \frac{-qd \cos \theta}{4\pi\epsilon r^2}$$

- Unlike an isolated point charge, the potential drops like $1/r^2$ rather than $1/r$

Electric Field of a Dipole

- The potential calculation was easy since we didn't have to deal with vectors
- The electric field is simply $\mathbf{E} = -\nabla\phi$
- Since our answer is in spherical coordinates, but there is no φ variation due to symmetry, we have

$$\nabla\phi = \hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \hat{\mathbf{r}}\frac{2qd\cos\theta}{4\pi\epsilon r^3} + \hat{\theta}\frac{1}{r}\frac{qd}{4\pi\epsilon r^2}\sin\theta$$

- Thus the electric field is given by

$$\mathbf{E} = -\frac{qd}{4\pi\epsilon r^3} \left(\hat{\mathbf{r}}2\cos\theta + \hat{\theta}\sin\theta \right)$$

Electric Flux Density

- It shall be convenient to define a new vector D in terms of E

$$\mathbf{D} = \epsilon \mathbf{E}$$

- Note that the units are simply C/m^2 .
- We call this the “flux” density because the amount of it’s flux crossing any sphere surrounding a point source is the same

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^\pi \frac{q}{4\pi r^2} r^2 \sin \theta d\theta d\varphi = \frac{q}{4\pi} 2\pi 2 = q$$

Gauss' Theorem

- Because of the $1/r^2$ dependence of the field, the integral is a constant
- Gauss' law proves that for *any* surface (not just a sphere), the result is identical

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = q$$

- Furthermore by superposition, the result applies to any distribution of charge

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho dV = q_{\text{inside}}$$

Gauss' Law “Proof”

- First of all, does it makes sense? For instance, if we consider a region with no net charge, then the flux density crossing the surface is zero. This means that the flux lines entering the surface from the left equal the flux lines leaving the surface on the right
- We can prove that the flux crossing an infinitesimal surface of any shape is the same as the flux crossing a radial cone
- Notice that if the surface is tilted relative to the radial surface by an angle θ , it's cross-sectional area is larger by a factor of $1/\cos \theta$
- The flux is therefore a constant

$$d\Psi = \mathbf{D} \cdot d\mathbf{S} = D_r dS \cos \theta = D_r \frac{dS'}{\cos \theta} \cos \theta = D_r dS$$

Application of Gauss' Law

- For any problem with symmetry, it's easy to calculate the fields directly using Gauss' Law
- Consider a long (infinite) charged wire. If the charge density is $\lambda \text{ C/m}$, then by symmetry the field is radial
- Applying Gauss law to a small concentric cylinder surrounding the wire

$$\oint \mathbf{D} \cdot d\mathbf{S} = D_r 2\pi r d\ell$$

- Since the charge inside the cylinder is simply $\lambda d\ell$

$$D_r 2\pi r d\ell = \lambda d\ell$$

$$D_r = \frac{\lambda}{2\pi r}$$

Spherical Cloud of Charge

- Another easy problem is a cloud of charge Q . Since the charge density is uniform $\rho = Q/V$, and $V = \frac{4}{3}\pi a^3$

$$\oint \mathbf{D} \cdot d\mathbf{S} = \begin{cases} \rho \frac{4}{3}\pi r^3 = Q \left(\frac{r}{a}\right)^3 & r < a \\ Q & r > a \end{cases}$$

- But $\oint \mathbf{D} \cdot d\mathbf{S} = 4\pi r^2 D_r$

$$D_r = \begin{cases} \frac{Qr}{4\pi a^3} & r < a \\ \frac{Q}{4\pi r^2} & r > a \end{cases}$$

Charge Sheet

- Consider an infinite plane that is charged uniformly with surface charge density ρ_s
- By symmetry, the flux through the sides of a centered cylinder intersecting with the plane is zero and equal at the top and bottom
- The flux crossing the top, for instance, is simply DdS , where D only can have a \hat{y} component by symmetry
- The total flux is thus $2DdS$. Applying Gauss' Law

$$2DdS = \rho_s dS$$

- The electric field is therefore

$$\mathbf{E} = \hat{y} \begin{cases} \frac{\rho_s}{2\epsilon_0} & y > 0 \\ \frac{-\rho_s}{2\epsilon_0} & y < 0 \end{cases}$$