Chapter 5

changes on pp 126 - 128. Changes in red.

Where derivatives of the parameters can be neglected, the matrix $\mathcal{L}_E$ is diagonalizable into P- and S-wave components. Define a matrix of partial derivatives $\mathbf{\Pi}$ as

$$\mathbf{\Pi} \equiv \begin{pmatrix} i\,\alpha/\omega & 0 & 0 & 0 \\ 0 & i\,\beta/\omega & 0 & 0 \\ 0 & 0 & i\,\beta/\omega & 0 \\ 0 & 0 & 0 & i\,\beta/\omega \end{pmatrix} \begin{pmatrix} \partial_x & \partial_y & \partial_z \\ 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} = \frac{i}{\omega} \begin{pmatrix} \alpha\,\partial_x & \alpha\,\partial_y & \alpha\,\partial_z \\ 0 & -\beta\,\partial_z & \beta\,\partial_y \\ \beta\,\partial_z & 0 & -\beta\,\partial_x \\ -\beta\,\partial_y & \beta\,\partial_x & 0 \end{pmatrix}. \tag{5.22}$$

Acting on a displacement vector $\boldsymbol{u}$, the first row of $\mathbf{\Pi}$ is proportional to the divergence of $\boldsymbol{u}$, and the remaining rows the curl of $\boldsymbol{u}$ (see Appendix B):

$$\mathbf{\Psi} \equiv \begin{pmatrix} \psi \\ \boldsymbol{A} \end{pmatrix} = \mathbf{\Pi}\,\boldsymbol{u} = \frac{i}{\omega} \begin{pmatrix} \alpha\,\partial_x\,u_x + \alpha\,\partial_y\,u_y + \alpha\,\partial_z\,u_z \\ \beta\,\partial_y\,u_z - \beta\,\partial_z\,u_y \\ \beta\,\partial_z\,u_x - \beta\,\partial_x\,u_z \\ \beta\,\partial_x\,u_y - \beta\,\partial_y\,u_x \end{pmatrix} = \frac{i}{\omega} \begin{pmatrix} \alpha\,\boldsymbol{\nabla}\cdot\boldsymbol{u} \\ \beta\,(\boldsymbol{\nabla}\times\boldsymbol{u})_x \\ \beta\,(\boldsymbol{\nabla}\times\boldsymbol{u})_y \\ \beta\,(\boldsymbol{\nabla}\times\boldsymbol{u})_z \end{pmatrix}. \tag{5.23}$$

The transformation operator Π has been normalized as suggested by Yanglei Zou of MOSRP. With this normalization, the P and S wave functions ψ and A retain the dimension of displacement, and plane-wave reflections assume the familiar Zoeppritz form as presented in Aki and Richards (1980). Though $\mathbf{\Pi}$ appears to map a 3-dimensional vector into a 4-dimensional space, the space is restricted by the constraint $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})=\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$ (see equation (B-34)). Thus, there are only three independent dimensions present.

If $\mathbf{\Pi}$ is postmultiplied by its transpose, the result is a four-by-four matrix:

$$\mathbf{\Pi}\,\mathbf{\Pi}^T \equiv \frac{-1}{\omega^2} \begin{pmatrix} \alpha^2\,\nabla^2 & 0 & 0 & 0 \\ 0 & \beta^2\,\nabla^2 - \beta^2\,\partial_x\partial_x & -\beta^2\,\partial_x\partial_y & -\beta^2\,\partial_x\partial_z \\ 0 & -\beta^2\,\partial_x\partial_y & \beta^2\,\nabla^2 - \beta^2\,\partial_y\partial_y & -\beta^2\,\partial_y\partial_z \\ 0 & -\beta^2\,\partial_x\partial_z & -\beta^2\,\partial_y\partial_z & \beta^2\,\nabla^2 - \beta^2\,\partial_z\partial_z \end{pmatrix}. \tag{5.24}$$

Operating on a 4-vector $(\psi,\ \boldsymbol{A})$, we have

$$\mathbf{\Pi}\,\mathbf{\Pi}^T \begin{pmatrix} \psi \\ \boldsymbol{A} \end{pmatrix} = -\frac{\nabla^2}{\omega^2} \begin{pmatrix} \alpha^2\,\psi \\ \beta^2\,\boldsymbol{A} \end{pmatrix} + \frac{1}{\omega^2} \begin{pmatrix} 0 \\ \beta^2\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) \end{pmatrix}. \tag{5.25}$$

For the subspace of vectors where $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$,

$$\boldsymbol{\Pi}\,\boldsymbol{\Pi}^{\mathrm{T}}\begin{pmatrix}\psi\\ \boldsymbol{A}\end{pmatrix}=\mathcal{J}\begin{pmatrix}\psi\\ \boldsymbol{A}\end{pmatrix}, \tag{5.26}$$

where

$$\mathcal{J}=-\frac{\nabla^2}{\omega^2}\begin{pmatrix}\alpha^2 & 0 & 0 & 0\\ 0 & \beta^2 & 0 & 0\\ 0 & 0 & \beta^2 & 0\\ 0 & 0 & 0 & \beta^2\end{pmatrix}. \tag{5.27}$$

The matrix $\mathcal{J}$ can be formally inverted to obtain:

$$\mathcal{J}^{-1}=-\nabla^{-2}\omega^2\begin{pmatrix}\alpha^{-2} & 0 & 0 & 0\\ 0 & \beta^{-2} & 0 & 0\\ 0 & 0 & \beta^{-2} & 0\\ 0 & 0 & 0 & \beta^{-2}\end{pmatrix}. \tag{5.28}$$

Provided derivatives of λ, μ, and ρ are neglectable, the elastic wave operator $\mathcal{L}_{\mathrm{E}}$ can be diagonalized with the operator Π:

$$\mathcal{L}_{\mathrm{D}}=\boldsymbol{\Pi}\,\mathcal{L}_{\mathrm{E}}\,\boldsymbol{\Pi}^{\mathrm{T}}\,\mathcal{J}^{-1}=\begin{pmatrix}\mathcal{L}_{\mathrm{P}} & 0 & 0 & 0\\ 0 & \mathcal{L}_{\mathrm{S}} & 0 & 0\\ 0 & 0 & \mathcal{L}_{\mathrm{S}} & 0\\ 0 & 0 & 0 & \mathcal{L}_{\mathrm{S}}\end{pmatrix}, \tag{5.29}$$

valid, as for $\boldsymbol{\Pi}\,\boldsymbol{\Pi}^{T}$, in the subspace $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$, and where $\mathcal{L}_{\mathrm{P}}$ and $\mathcal{L}_{\mathrm{S}}$ are P- and S-wave operators

$$\mathcal{L}_{\mathrm{P}}(\boldsymbol{x},\,t)=\rho\,\alpha^2\left(\nabla^2+\frac{\omega^2}{\alpha^2}\right), \tag{5.30}$$

$$\mathcal{L}_{\mathrm{S}}(\boldsymbol{x},\,t)=\rho\,\beta^2\left(\nabla^2+\frac{\omega^2}{\beta^2}\right), \tag{5.31}$$

and where α and β are P- and S-wave velocities, related to the original parameters as

$$\alpha=\sqrt{\frac{\lambda+2\,\mu}{\rho}}\,, \tag{5.32}$$

and

$$\beta=\sqrt{\frac{\mu}{\rho}}\,. \tag{5.33}$$

We note that the matrix $\mathcal{J}=\mathbf{\Pi}\,\mathbf{\Pi}^{T}$, acting on a solution $\mathbf{\Psi}$ to the wave equation $\mathcal{L}_D\mathbf{\Psi}=0$, yields the unit matrix.

Note that the original equation (5.33) appearing at the top of page 128 has been replaced with the new equation (5.33) above, and the text “The expression (5.29) can be formally ....” above the old equation (5.33) has been deleted. Equation (5.34) on p 128 and the text immediately above it are unchanged.