Solutions Manual for A Student's Guide to Analytical Mechanics

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Solutions to Exercises Chapter 1

Exercise 1.1

Okay, suppose the equation of motion for a falling body were actually

$$\frac{d^3x}{dt^3} = -D. \tag{1.1}$$

You would solve this in the normal way. Integrate once with respect to time to get

$$\frac{d^2x}{dt^2} = -Dt + a_0,$$

where a_0 is a constant of integration, the initial acceleration in this case. Then integrate again to get

$$\frac{dx}{dt} = -\frac{1}{2}Dt^2 + a_0 t + v_0,$$

where v_0 is the initial velocity. A final integration gives

$$x = -\frac{1}{6}Dt^3 + \frac{1}{2}a_0 t^2 + v_0 t + x_0.$$

(a) "Released from rest" means no initial acceleration or velocity in this case. Then the unfortunate mass starts at height x_0 above the ground and hits the ground ($x = 0$) at a time given by

$$0 = -\frac{1}{6}Dt^3 + x_0,$$

or at time $t = (6x_0/D)^{1/3}$. By contrast, in actual gravity, the same reasoning gives a time of fall $t = (2x_0/g)^{1/2}$, where g is the gravitational acceleration. Thus the time of fall would have a different power-law scaling with distance (cube root) in the weird, artificial world we propose than in the observed world.

(b) The solution $x(t)$ for our artificial equation of motion is a cubic polynomial, you know that this can have the shape described. Here's an example where we posit a crazy world where the equation of motion is (1.1) and in which $D = 1$ m/s^3. A mass is launched from a height $x_0 = 56$ m (about the height of the Leaning Tower of Pisa) with initial velocity $v_0 = -10$ m/s and initial acceleration (although how would you even do this?) $a_0 = 5$ m/s^2. The height of this mass as a function of time is shown in the figure. Under these circumstances, the mass would fall for a little while, then rise before falling again. If the world were really described by an equation of motion like (1.1), you think someone would have noticed!

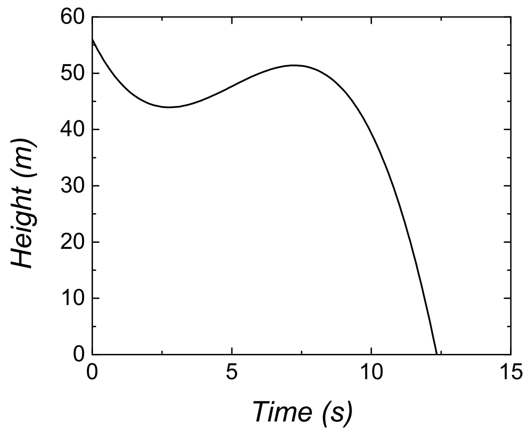

Figure 1.1 Height of a mass launched from a great height, versus time, in an unphysical world.

Exercise 1.2

"Obvious" is of course a subjective term, but the tension nevertheless does vary during the pendulum's swing. The tension is trying to hold the mass on its circular path, against the forces of gravity and centrifugal force. Let's consider a special case, where the pendulum swings through a semicircular arc, where its angle ϕ measured from the vertical goes from $-90°$ to $+90°$, where $\phi = 0°$ represents the mass at its lowest point of swing. ϕ is defined in Figure (2.1) of the book.

When the pendulum is lowest, $\phi = 0$, it is also moving the fastest. The tension has to pull against not only the full weight of the mass, but also against the centrifugal force $ml\dot{\phi}^2$, where l is the pendulum's length. Vice versa, when the pendulum swings to $\phi = 90°$ and comes to rest before starting its descent, there is no centrifugal force for an object at rest; and moreover the tension does not pull against gravity, which is perpendicular to the string at this point. Thus the tension is maximal when $\phi = 0°$ and minimal (for this particular example) when $\phi = \pm 90°$. The general formula for the tension is given as a formula in Eqn. (2.6) of Chapter 2.

Exercise 1.3

The force as written is the force on mass 1 due to mass 2, $\mathbf{F}_{1,2}$ (this is worth verifying!). In terms of the coordinates given, we have explicitly

$$\mathbf{F}(\mathbf{r}_1, \mathbf{r}_2) = Gm_1m_2\frac{\mathbf{r}}{r^3} = \mathbf{F}_{1,2} = -\mathbf{F}_{2,1}.$$

Then, given the prescription

$$\begin{aligned}\mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \\ \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1,\end{aligned}$$

the equation of motion of the center of mass coordinate is

$$\begin{aligned}\ddot{\mathbf{R}} &= \frac{m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2}{m_1 + m_2} \\ &= \frac{\mathbf{F}_{1,2} + \mathbf{F}_{2,1}}{m_1 + m_2} = 0.\end{aligned}$$

Thus the center of mass coordinate does not care about the forces at all. Vice versa, the equation of motion for the relative coordinate is

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 \\ &= \frac{\mathbf{F}_{2,1}}{m_2} - \frac{\mathbf{F}_{1,2}}{m_1} \\ &= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{2,1} = \frac{m_1 + m_2}{m_1m_2}\mathbf{F}_{2,1}.\end{aligned}$$

So, defining the relative force $\mathbf{F} = \mathbf{F}_{2,1}$ for notational convenience, the equation of motion is

$$\mu\ddot{\mathbf{r}} = \mathbf{F},$$

where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass.

Exercise 1.4

The minus sign on dx in Eqn. (1.5) is there because you are integrating from high up to down low, that is, the steps you take in height are negative. The integral is then

$$\begin{aligned}
t_p &= \int_h^0 \frac{-dx}{\sqrt{2g(h-x)+v_0^2}} \\
&= \frac{1}{\sqrt{2g}} \int_0^h dx (h + v_0^2/(2g) - x)^{-1/2} \\
&= \frac{1}{\sqrt{2g}} (-2)(h + v_0^2/(2g) - x)^{1/2}\Big|_0^h \\
&= -\frac{2}{\sqrt{2g}} \left[\sqrt{\frac{v_0^2}{2g}} - \sqrt{h + \frac{v_0^2}{2g}} \right] \\
&= -\frac{|v_0|}{g} + \sqrt{\frac{2h}{g} + \frac{v_0^2}{g^2}}. \qquad (1.2)
\end{aligned}$$

If instead you compute the full solution, it looks like this:

$$x(t) = -\frac{1}{2} g t^2 + v_0 t + h.$$

So when does the balloon hit the ground? When $x(t) = 0$, where t solves the quadratic equation

$$t^2 - \frac{2v_0}{g} t - \frac{2h}{g} = 0.$$

Let's assume the balloon was thrown with downward initial velocity, $v_0 = -|v_0|$, so that we don't have to mess around with finding the time to reach its apex, then the time to fall from the apex to the ground. In this case the quadratic equation gives the solutions

$$t = -\frac{|v_0|}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2h}{g}}.$$

You can't use the one with the minus sign, which would give $t < 0$, and the balloon would hit the ground before you threw it. The positive sign gives the same answer as (1.2).

Exercise 1.5

Conservation of energy for the spring says

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2.$$

You can solve this for the velocity,

$$\frac{dx}{dt} = \dot{x} = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2}.$$

You integrate this easily, since it is a first order, separable equation:

$$\begin{aligned} t = \int_0^t dt' &= \int_0^x dx' \frac{dx'}{\sqrt{2E/m - (k/m)x'^2}} \\ &= \sqrt{\frac{m}{k}} \int_0^x \frac{dx'}{\sqrt{2E/k - x'^2}} \\ &= \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{x}{\sqrt{2E/k}}\right). \end{aligned}$$

This has solved the problem inside-out. You don't want t as a function of x, but rather x as a function of t. This gives you

$$x(t) = \sqrt{\frac{2E}{k}} \sin\left(\sqrt{\frac{k}{m}}t\right).$$

You will recognize this as sinusoidal motion with angular frequency $\omega = \sqrt{k/m}$.

Notice that we have here neglected a constant of integration, thereby asserting that this is the particular solution to the motion where $x = 0$ when $t = 0$. A nonzero constant of integration can make this more general.

2

Solutions to Exercises Chapter 2

Exercise 2.1

I can't pretend to know what language you're using to solve these differential equations of motion, but it seems likely that many students have access to Good Old Mathematica. So, I'll describe the numerical solutions using this. For starters, let's see how to get Mathematica to solve a simpler differential equation, for the mass on a spring:

$$\ddot{x} = -(k/m)x.$$

This requires the Mathematica function `NDSolve`, which means, "solve differential equations numerically."

You would first define the values of the constants, let's say

```
k=1;
m=1;
```

Then the differential equation is defined in Mathematica syntax as

```
x''[t]==-(k/m)*x[t]
```

Note the square brackets enclosing the argument of the function. Note the use of a double equal sign. The differential equation must also have initial conditions, on both the initial position and the initial velocity. Let's say we release the mass from rest $\dot{x}(0) = 0$, from a position $x(0) = 0.5$ m. These conditions would be written

```
x[0]==0.5,x'[0]==0
```

These are also written with double equal signs. the whole set of things specifying the equation and initial conditions is then enclosed in curly brackets:

```
{x''[t]==-(k/m)*x[t],x[0]==0.5,x'[0]==0}
```

For the numerical solution you need also to specify the range of the dependent variable, time in this case. This is given as three items collected in curly brackets: the variable name, its initial value, and its final value. If we were calculating the motion between zero and ten seconds, say, then we would have

```
{t,0,10}
```

The command `NDSolve` requires three arguments, separated by commas: the differential equations and initial conditions; the dependent variable or variables to be solved for; and the range of the independent variable. These arguments are set inside of square brackets. The whole thing is set equal to some variable, here called "`springsolution`," that represents the solution. The relevant command is

```
springsolution=NDSolve[{x''[t]==-(k/m)*x[t],x[0]==0.5,x'[0]==0},x,{t,0,10}];
```

When you type this in, nothing happens, but the solution is stored, as a quantity ready to by interpolated and plotted, in `springsolution`. To plot this solution, you can use the `Plot` command. This requires as an argument the variable you're plotting, `x[t]`, along with the range of `t` to be plotted. In addition, to make sure that you're plotting the solution that you just calculated above, you refer to the x variable using the syntax

```
x[t]/.springsolution
```

the gist of which is, "plot `x[t]` using the solution from `springsolution`." Therefore the useful plot command here is

```
Plot[x[t]/.springsolution,{t,0,10}],AxesLabel->{"t (s)" ,"x
(m)"}]
```

This includes Mathematica's way of labeling the axes; no plot is any use without axes! The text in the axis labels has to be in quotes, because that's just how Mathematica rolls. The output of this plot is shown below. There are lots of further ways to manipulate the figure, of course, which you can see in documentation for Mathematica.

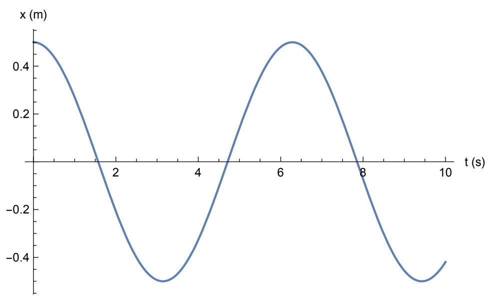

Figure 2.1 Motion of a particular mass on a particular spring, sponsored by Mathematica.

(a) Now to the business at hand. Let's consider as an example a pendulum of length $l = 1$ m, swinging in standard gravity with $g = 9.8$ m/s^2. Then let's drop this pendulum from rest at an initial angle $\theta_0 = 0.5$ radians, so the initial y is $y_0 = l \sin\theta_0$. The commands that generate the interpolating function are

```
g=9.8;
l=1;
ph0=0.5;
y0=l*Sin[ph0];
cartesian=NDSolve[{Sqrt[l^2-y[t]^2]*y''[t]
+(y[t]/Sqrt[l^2-y[t]^2])*(y'[t]^2+y[t]*y''[t]+y[t]^2*y'[t]^2/(l^2-y[t]^2))==-g*y[t],
y[0]==y0,y'[0]==0},y,{t,0,10}];
```

Notice here that "`Sqrt`" is Mathematica's way of declaring a square root; and that squaring a quantity is described by "`^2`." Functions such as `Sqrt` and `Sin` have their arguments in square brackets. The last three lines in the above list are all part of the same command, but are split here so that they fit on the page.

The result is plotted using the command

```
Plot[y[t]/.cartesian,{t,0,10},AxesLabel->{``t (sec)'' ,``y (m)''}]
```

which results in the figure below.

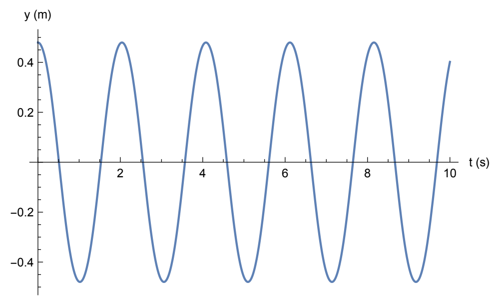

Figure 2.2 Motion of the pendulum as described in the text.

You would get the same result if you solved the problem in polar coordinates of course, using

```
g=9.8;
l=1;
phi0=0.5;
polar=NDSolve[{phi''[t]==-(g/l)*Sin[phi[t]],phi[0]==phi0,phi'[0]==0},phi,{t,0,10}];
Plot[l*Sin[phi[t]]/.polar,{t,0,10},AxesLabel->{``t (sec)'',``phi
(m)''}]
```

Interestingly, if you start with initial conditions $\theta_0 > \pi/2$, the the Cartesian version doesn't work.

(b) We have

$$\begin{aligned} y &= l \sin\phi \\ \dot{y} &= l\dot{\phi}\cos\phi \\ \ddot{y} &= l\ddot{\phi}\cos\phi - l\dot{\phi}^2 \sin\phi \\ l^2 - y^2 &= l^2 \cos^2\phi. \end{aligned}$$

The first term on the left of Equation (2.3) in the book,using the positive square root of $\cos^2\phi$, is then

$$\begin{aligned} (l^2 - y^2)^{1/2}\ddot{y} &= l\cos\phi\left[l\ddot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi\right] \\ &= l^2\ddot{\phi}\cos^2\phi - l^2\dot{\phi}^2\sin\phi\cos\phi. \end{aligned}$$

The piece in the big square brackets is

$$\begin{aligned} \dot{y}^2 + y\ddot{y} + \frac{y^2\dot{y}^2}{l^2 - y^2} &= l^2\dot{\phi}^2\cos^2\phi + l\sin\phi\left(l\ddot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi\right) + \frac{l^2\sin^2\phi l^2\dot{\phi}^2\cos^2\phi}{l^2\cos^2\phi} \\ &= l^2\ddot{\phi}\sin\phi\cos\phi + l^2\dot{\phi}^2\cos^2\phi, \end{aligned}$$

thanks to some fast-thinking cancellations. The second term on the left of (2.3) in the book is then

$$\begin{aligned} \frac{y}{(l^2 - y^2)^{1/2}}\left[\dot{y}^2 + y\ddot{y} + \frac{y^2\dot{y}^2}{l^2 - y^2}\right] &= \frac{l\sin\phi}{l\cos\phi}\left[l^2\ddot{\phi}\sin\phi\cos\phi + l^2\dot{\phi}^2\cos^2\phi\right] \\ &= l^2\ddot{\phi}\sin^2\phi + l^2\dot{\phi}^2\sin\phi\cos\phi. \end{aligned}$$

Adding everything together and setting this sum equal to $-gy$ gives

$$l^2\ddot{\phi} = -gl\sin\phi,$$

which is the desired (and much simpler) equation.

Exercise 2.2

Overall, $\dot{\phi}$ would have some mean value given by 2π divided by the period of rotation. The pendulum doesn't go at constant angular velocity of course, going faster (larger $\dot{\phi}$ at the bottom of its swing, and slower (smaller $\dot{\phi}$) at the top of its swing. The instantaneous value of $\dot{\phi}$ can never be negative in this rotational motion, however.

Exercise 2.3

This can be done the same way as in Exercise 2.1, but plotting a different outcome. Here is an example for a pendulum of length $l = 1$ m, mass $m = 0.1$ kg, released from rest at $\phi_0 = \pi/2$.

```
g=9.8;
l=1;
phi0=Pi/2;
polar=NDSolve[{phi''[t]==-(g/l)*Sin[phi[t]],phi[0]==phi0,phi'[0]==0},phi,{t,0,3}];
Plot[phi[t]]/.polar,{t,0,3},AxesLabel->{''t (sec)'',''phi (rad)''}]
Plot[(m*l*phi'[t]^2+m*g*Cos[phi[t]])/.polar,{t,0,3},AxesLabel->{''t
(sec)'',''tau (kg-m/s^2)''}]
```

These plots are shown in the figure below, for a time just longer than one period. In this case, the tension is a maximum when $\phi = 0$ and the pendulum is swinging fastest, and the tension vanishes when $\phi = \pi/2$, as anticipated in Exercise 1.2. If you mess around with the initial condition ϕ_0, you should find that 1) for $\phi_0 < \pi/2$, the tension is always positive, since the rod must always pull against gravity and possible centrifugal force; and 2) for $\phi_0 > \pi/2$, as shown in the examples in the book, the tension must sometimes become negative, to keep the mass from falling in toward the pivot point.

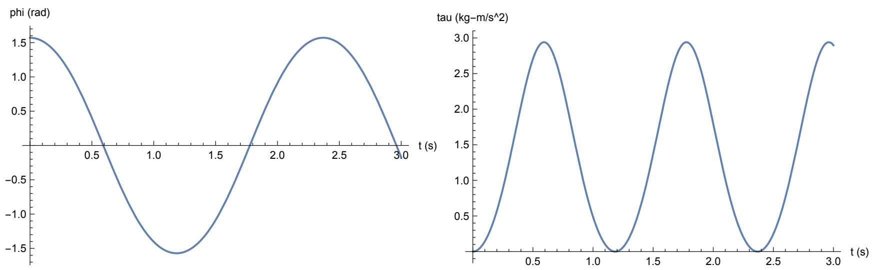

Figure 2.3 (a) The swing of the pendulum as described in the text. (b) the tension in the string over this same tim e.

Exercise 2.4

We start with the equations of motion

$$\frac{d\phi}{dt} = \frac{p_\phi}{ml^2}$$
$$\frac{dp_\phi}{dt} = -mgl\sin\phi,$$

with the momentum related to the generalized velocity $\dot{\phi}$ by

$$p_\phi = ml^2\frac{d\phi}{dt}.$$

For a set of two coupled first order differential equations, there is a standard trick for eliminating one of the quantities to get a second order equation in the other. Suppose we are looking for an equation for ϕ, we could take another derivative of ϕ to get

$$\begin{aligned}\frac{d^2\phi}{dt^2} &= \frac{1}{ml^2}\frac{dp_\phi}{dt} \\ &= \frac{1}{ml^2}(-mgl\sin\phi) = -\frac{g}{l}\sin\phi,\end{aligned}$$

which is the desired equation of motion.

Interestingly, this does not go the other way. Suppose you try to write an equation for the momentum, by writing

$$\begin{aligned}\frac{d^2p_\phi}{dt^2} &= -mgl\cos\phi\dot{\phi} \\ &= -mgl\cos\phi\left(\frac{p_\phi}{ml^2}\right) = -\frac{g}{l}\cos\phi p_\phi.\end{aligned}$$

So to find the time-dependent momentum you are still obliged to know the coordinate ϕ. This is a theme we develop throughout the book: Momentum depends in a *simple and predictable way* on velocity (they are simply proportional in this example), and can be eliminated straightforwardly. On the other hand, forces depend on coordinates in a different way for each force, and this dependence has to be carried along.

The numerical solution, based on what we had above, can be described by the Mathematica statement

```
hamiltonian=NDSolve[{p'[t]==-m*g*l*Sin[phi[t]],phi'[t]==p[t]/(m*l*l),
p[0]==p0,phi[0]==phi0},{phi,p},{t,tmin,tmax}]
```

for some suitable values of `g, l, m, phi0, p0, tmin, tmax`.

Here we will give one example where $g = 9.8$ m/s^2, $l = 1$ m, the pendulum is released from rest at an initial angle $\phi = 1$ radian, and the time interval goes from −4 s to +4s. We will try two masses, $m = 0.5$ kg, and $m = 2$ kg. By running the code you can verify that the period of motion is the same for both masses. To plot the phase space trajectory, you need another Mathematica function,

```
ParametricPlot[{phi[t],p[t]}/.hamiltonian,{t,tmin,tmax},AxesLabel->{''phi
(rad)'',''p (kg-m^2/s^2)''}]
```

The resulting phase space plots are shown in the figure on the next page. Here I have taken the liberty of putting them both on the same plot. Both pendulums swing through the same angular range, but the more massive one (orange) experiences greater momenta, although it travels at the same velocity as the lighter one.

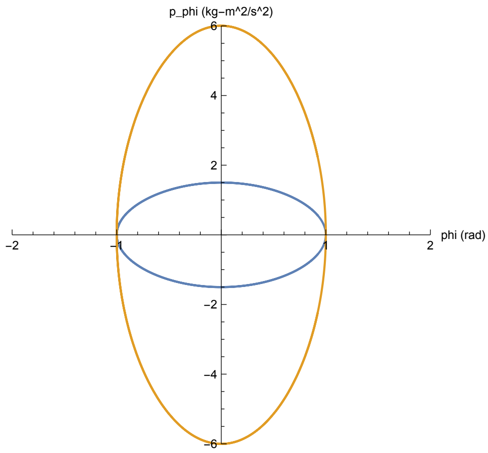

Figure 2.4 Phase space plots of two pendulums, identical except that one has mass 0.5 kg (blue), while the other has mass 2 kg (orange), hence greater momenta.

Exercise 2.5

This is kind of a neat mathematical trick that's sometimes useful in evaluating physically relevant integrals.In this case the integral is

$$A = 2\int_{-\phi_0}^{\phi_0} d\phi \sqrt{2ml^2\left(E - \frac{1}{2}mgl\phi^2\right)}.$$

First you notice that the limits of integration are determined by the values of ϕ where the thing inside the square root vanishes, otherwise the square root (which represents a momentum) would be complex number. The limits of the classically allowed region are given by

$$\phi_0^2 = \frac{2E}{mgl}.$$

Now, we factor as much as we can out of the integral to get

$$A = 2\sqrt{2ml^2E}\int_{-\phi_0}^{\phi_0} d\phi \sqrt{1 - \frac{\phi^2}{(2E/mgl)}}$$

Next, define a new coordinate $x = \phi/\phi_0$, then the integral becomes

$$A = 2\sqrt{2ml^2E}\sqrt{\frac{2E}{mgl}}\int_{-1}^{1} dx\sqrt{1 - x^2}$$

$$= 4\sqrt{\frac{l}{g}}E\int_{-1}^{1} dx\,\sqrt{1-x^2}.$$

In this way, all the dependences on the physical, dimensionful quantities are given explicitly. The integral itself is just a dimensionless quantity to be evaluated once and you're done with it. In this case, even that's easy. The integral represents the area of a semicircle of radius 1, and therefore has the value $\pi/2$. From this the Action integral is $A = 2\pi\sqrt{l/g}E$.

Exercise 2.6

Well, it says "convince yourself," so I don't know what I can do for you. Still, consider this. Position and momenta are constantly changing, so the configuration of the pendulum, represented as a point on the phase space curve, must be moving around. It's just a question of which way it goes.

Suppose the pendulum is moving according to the closed ovaly curve and at some time it has positive angle $\phi > 0$ and positive momentum $p_\phi > 0$, that is, the phase space point lies in the first quadrant of the diagram. Well, then the pendulum will continue moving to larger ϕ until it gets to the rightmost part of the oval. To do this, it must have gone clockwise. You can make similar arguments regardless of the signs of ϕ and p_ϕ.

For the wiggly curve shown, the momentum is always positive and the motion is always toward larger ϕ, to the right on the curve. There could, however, be a similar curve that lies entirely at negative p_ϕ, representing the pendulum whirling in the other direction. In this case the motion along the phase space trajectory would go to the left.

Exercise 2.7

The units of action are areas in phase space, that is, units of p times the units of q. But in terms of units, if $p = \partial L/\partial\dot{q}$ and L has units of energy, then the action has units (*energy / (units of q / time)* $\times$ *units of q* = *energy* $\times$ *time*.

3
Solutions to Exercises Chapter 3

Exercise 3.1

The constraint here is that the length of the string does not change. If you pull the free end down a distance l, what happens to the rest? That length l of string comes out of the two segments that support pulley 2. Each of these segments will shorten by $l/2$, with $P2$ turning to make this possible.

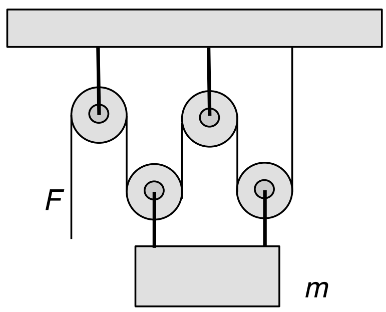

Figure 3.1 More pulleys, giving more mechanical advantage.

Therefore, if you were to apply a downward force F on the free end, the virtual work done in pulling a virtual displacement δl down is positive, since the force and displacement are in the same direction. At the same time, lifting the mass m a virtual distance $\delta l/2$ makes a negative virtual work against gravity.

The net virtual work must be zero,

$$\delta l F + (-\delta l/2)mg = 0$$

so that $F = mg/2$.

The figure shows an option, not necessarily the best one, for a set of four pulleys. Two are anchored to the ceiling, two to the mass to be hoisted. By the same argument, a virtual displacement δl of the free end is distributed four ways among the rope segments between the pulleys. The force you need to exert is therefore $F = mg/4$.

Extra Credit. Try to work out the Spanish Barton, shown in the figure below. Here pulley P_1 is attached to the ceiling, and the string that runs over it supports pulleys P_2 and P_3. The string you pull on runs over pulley P_2, then under pulley P_3, then is attached to the ceiling. The mass is supported from pulley P_3 as shown. For massless, frictionless, ideal pulleys, show that you can support the mass m by exerting a force $mg/4$, as above, but this time you need only three pulleys.

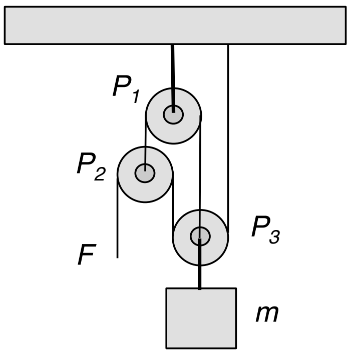

Figure 3.2 The Spanish Barton.

Exercise 3.2

Let's take as a coordinate system the one shown in Figure 3.2 of the text: $\hat{x}$ and $\hat{y}$ for horizontal and vertical with respect to the ground; and $\hat{s}$ and $\hat{n}$ for along and normal to the plane. The little mass m that slides on the inclined plane is what makes this complicated, because its motion depends on the motion of the inclined plane, hence on both the coordinates x and s.

In more detail: if the plane accelerates, it accelerates only in the x direction. But the little mass m is then moving along in a non-inertial coordinate frame. The safest bet is to write the acceleration of the mass in the inertial coordinates (x, y):

$$\begin{aligned} a_x &= \ddot{x} + \cos\alpha\ddot{s} \\ a_y &= -\sin\alpha\ddot{s}, \end{aligned}$$

while the inclined plane, able to move only in the x direction, has acceleration $\ddot{x}$.

To employ d'Alembert's principle, we take as the applied forces the weights of the two objects. The virtual work of the inclined plane, which can have only virtual displacements in x, is

$$\begin{aligned} (\mathbf{F}^{\mathrm{a}}_{\mathrm{plane}} - m_{\mathrm{plane}}\mathbf{a}_{\mathrm{plane}}) \cdot \delta x\hat{x} &= (-Mg\hat{y} - M\ddot{x}) \cdot \delta x\hat{x} \\ &= -M\ddot{x}\delta x. \end{aligned}$$

The virtual work of the mass on the plane, which can be impacted by virtual displacements in either x or s, is

$$\begin{aligned} (\mathbf{F}^{\mathrm{a}}_{\mathrm{mass}} - m_{\mathrm{mass}}\mathbf{a}_{\mathrm{mass}}) \cdot (\delta x\hat{x} + \delta s\hat{s}) &= \Big(- mg\hat{y} - m(\ddot{x} + \cos\alpha\ddot{s})\hat{x} + m\sin\alpha\ddot{s}\hat{y}\Big) \cdot (\delta x\hat{x} + \delta s\hat{s}) \\ &= \Big(- m(\ddot{x} + \cos\alpha\ddot{s})\Big)\delta x \\ &\quad + \Big(mg\sin\alpha - m(\ddot{x} + \cos\alpha\ddot{s})\cos\alpha - m\sin^2\alpha\ddot{s}\Big)\delta s, \end{aligned}$$

where we have used the geometrical facts $\hat{y} \cdot \hat{x} = 0$, $\hat{y} \cdot \hat{s} = -\sin\alpha$, and $\hat{x} \cdot \hat{s} = \cos\alpha$.

According to d'Alembert, the sum of the two virtual work terms must add to zero:

$$\Big(- M\ddot{x} - m(\ddot{x} + \cos\alpha\ddot{s})\Big)\delta x + \Big(mg\sin\alpha - m(\ddot{x} + \cos\alpha)\cos\alpha - m\sin^2\alpha\ddot{s}\Big)\delta s = 0$$

Now, the virtual displacements δx and δs are independent degrees of freedom, so we can set the coefficients of these two separately to zero. This gives

$$\begin{aligned} (M + m)\ddot{x} + m\cos\alpha\ddot{s} &= 0 \\ \cos\alpha\ddot{x} + \ddot{s} &= g\sin\alpha. \end{aligned}$$

This is a system of equations that can be solved for the accelerations:

$$\ddot{s} = \frac{g\sin\alpha}{1 - \frac{m}{M+m}\cos^2\alpha}$$

$$\ddot{x} = -\frac{mg\sin\alpha\cos\alpha}{(M+m) - m\cos^2\alpha}.$$

Does this make sense? For a very massive inclined plane, $M \gg m$, we have $\ddot{s} = g\sin\alpha$ like normal, while $\ddot{x} = 0$, the plane isn't going anywhere. Also, if $\alpha = 0$, the plane is horizontal and both objects remain stationary. If $\alpha = \pi/2$, then the plane does not move, but the mass falls with acceleration g.

You should come back later, after the next chapter, and try this again using Lagrange's equation. It will be a lot simpler, and you should get the same answer!

Exercise 3.3

Define downward to be the positive x direction, and positive δx to be a downward virtual displacement of mass m. A positive δx would turn the wheel a virtual angle $\delta\theta$, which would shift mass m' by the virtual displacement $\delta x' = -R'\delta\theta = -(R'/R)\delta x$. That is, m' goes up when m goes down.

The principle of d'Alembert applied to this circumstance yields

$$(mg - ma)\delta x + (m'g - m'a')\delta x' = 0,$$

or

$$(mg - ma)\delta x + (m'g - m'a')\left(-\frac{R'}{R}\right)\delta x = 0.$$

Further, the accelerations are related by the fact that the angular acceleration of the wheels is

$$\alpha = \frac{a}{R} = -\frac{a'}{R'}.$$

Then from d'Alembert we extract

$$mg - ma - m'g\frac{R'}{R} + m'a\frac{R'^2}{R^2} = 0.$$

Solving for a, we get

$$a = gR\frac{mR - m'R'}{mR^2 + m'R'^2}.$$

The other acceleration is

$$a' = -\frac{R'}{R}a = -gR'\frac{mR - m'R'}{mR^2 + m'R'^2}.$$

Exercise 3.4 (I am indebted to Claudio Mazzoleni for finding a major mstake in the original solution.)

(a) For this rigid configuration, both masses are descried by the same swing angle ϕ. Virtual displacements are all of the form $\delta\phi\hat{\phi}$, but this must be adapted to the problem. d'Alembert's principle is

$$\sum_{i=1}^{2}(\mathbf{F}_i^{\mathrm{a}} - m_i\mathbf{a}_i)\cdot\delta\mathbf{x}_i = 0.$$

We know that the virtual displacements are related, since the masses swing through the same virtual angle $\delta\phi$. This angle corresponds to a different virtual displacement in length for the two masses, however, since they are different distances from the pivot:

$$\delta\mathbf{x}_i = l_i\delta\phi\hat{\phi}.$$

Evaluating gravitational forces, and accelerations in polar coordinates, as in the chapter, d'Alembert's principle becomes

$$\left[(-m_1 g\sin\phi - m_1 l_1\ddot{\phi})l_1 + (-m_2 g\sin\phi - m_2 l_2\ddot{\phi})l_2\right]\delta\phi = 0.$$

This is true for any virtual displacement $\delta\phi$, so the term in square brackets must be zero. Solving for acceleration,

$$\begin{aligned}\ddot{\phi} &= -\frac{m_1 l_1 + m_2 l_2}{m_1 l_1^2 + m_2 l_2^2} g\sin\phi\\ &= -\frac{g}{L}\sin\phi,\end{aligned}$$

where the effective length is given by

$$L = \frac{m_1 l_1^2 + m_2 l_2^2}{m_1 l_1 + m_2 l_2}.$$

This expression has vaguely familiar elements to it. If you recall that the momentum of inertia about the pivot is

$$I = m_1 l_1^2 + m_2 l_2^2$$

and that the center of mass is a distance

$$l_{CM} = \frac{m_1 l_1 + m_2 l_2}{m_1 + m_2}$$

from the pivot, then this pendulum of mass $m = m_1 + m_2$ has the effective length

$$L = \frac{I}{ml_{CM}}.$$

This is a result that is generically true for a *physical pendulum*, one whose mass is not concentrated in a point, but that is characterized by values of m, l_{CM}, and I.

Now, suppose $l_1 < l_2$, as shown. Then we have

$$\frac{L}{l_1} = \frac{m_1 + m_2(l_2/l_1)^2}{m_1 + m_2(l_2/l_1)} > 1,$$
$$\frac{L}{l_2} = \frac{m_1(l_1/l_2)^2 + m_2}{m_1(l_1/l_2) + m_2} < 1,$$

so L is intermediate between l_1 and l_2. Thus the angular frequency of the compound pendulum, $\sqrt{g/L}$, is also intermediate between the frequencies of the two uncoupled pendulums.

(b) Now suppose the masses are described by separate angles ϕ_1 and ϕ_2, but constrained by $\phi_1 = \phi_2$, so that any virtual displacements satisfy $\delta\phi_1 = \delta\phi_2$. We can introduce a Lagrange multiplier τ, so called because it needs to be a torque in this case (units: Newton-meters, or Joules). We add the condition

$$\tau(\delta\phi_1 - \delta\phi_2) = 0$$

to the usual d'Almbert principle, to get

$$\Big[(-m_1 g \sin\phi_1 - m_1 l_1 \ddot{\phi}_1) l_1 + \tau\Big]\delta\phi_1 + \Big[(-m_2 g \sin\phi_2 - m_2 l_2 \ddot{\phi}_2) l_2 - \tau\Big]\delta\phi_2 = 0.$$

Now the two virtual displacements can be regarded as varying individually, leading to the separate equations

$$-m_1 g l_1 \sin\phi_1 - m_1 l_1^2 \ddot{\phi}_1 + \tau = 0$$
$$-m_2 g l_2 \sin\phi_2 - m_2 l_2^2 \ddot{\phi}_2 - \tau = 0.$$

If you add these two equations together and recognize that $\phi_1 = \phi_2 = \phi$, then you return to the equation of motion $\ddot{\phi} = -(g/L)\sin\phi$ described above. However, if you subtract the second equation from the first, you get

$$-(m_1 l_1 - m_2 l_2) g \sin\phi - (m_1 l_1^2 - m_2 l_2^2)\ddot{\phi} + 2\tau = 0.$$

For small amplitude of swing, $\sin\phi \approx \phi$ and $\ddot{\phi} \approx -(g/L)\phi$, in which case

$$2\tau = (m_1 l_1 - m_2 l_2) g\phi + (m_1 l_1^2 - m_2 l_2^2)\left(-\frac{g}{L}\right)\phi$$

Substituting for L and doing some algebra, we arrive at the expression for the torque of constraint,

$$\tau = \frac{m_1 m_2 l_1 l_2 (l_2 - l_1)}{m_1 l_1^2 + m_2 l_2^2} g\phi.$$

Notice that when the lengths are the same, $l_1 = l_2$, the torque of constraint is zero. Effectively, this becomes two pendulums of the same length, which would swing in unison anyway, without being constrained to do so,.

4
Solutions to Exercises Chapter 4

Exercise 4.1

Given the coordinete definitions

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta,$$

calculating time derivatives and kinetic energies runs on autopilot. Start with

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 &= \Big(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi\Big)^2 \\ &\quad + \Big(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi\Big)^2 \\ &= \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\phi}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta,\end{aligned}$$

which simplifies due to trigonometry and a lot of cancellation of cross terms. Next calculate the z part:

$$\begin{aligned}\dot{z}^2 &= \Big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\Big)^2 \\ &= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\end{aligned}$$

Adding these together gets us to the simple expression

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta).$$

This you can stick into Lagrange's equations and you're ready to go.

As for the acceleration, I'm not sure you really want to have this much fun, but here we go anyway. The trick is of course that the unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ all depend explicitly on time. As we did in Chapter 2, the best way to deal with this is to express these in terms of the time-independent Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$. This can be done by means of a confusing vector diagram (confusing to me, anyway!) or more analytically. Starting from $\hat{r} = \mathbf{r}/r$, we get

$$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}.$$

Meanwhile, the unit vector in ϕ is still tracking rotation around the z axis and is the same as it was in polar coordinates

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}.$$

Finally, $\hat{\theta}$ is orthogonal to the other two, and is in fact given by

$$\hat{\theta} = \hat{\phi} \times \hat{r} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}.$$

(Quick check: if $\theta = \pi/2$, $\phi = 0$, then $\hat{r} = \hat{x}$, $\hat{\phi} = \hat{y}$, and we should get $\hat{\theta} = -\hat{z}$, which we do, so the sign is right.)

Now we can take derivatives like there's no tomorrow:

$$\begin{aligned}\dot{\hat{r}} &= (\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\hat{x} + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\hat{y} - \dot{\theta}\sin\theta\hat{z} \\ &= \dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\hat{\phi} \\ \dot{\hat{\theta}} &= (-\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\cos\theta\sin\phi)\hat{x} + (-\dot{\theta}\sin\theta\sin\phi + \dot{\phi}\cos\theta\cos\phi)\hat{y} - \dot{\theta}\cos\theta\hat{z} \\ &= -\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\hat{\phi}\end{aligned}$$

In these two expressions, you can just read off the unit vectors, which are written above. For the $\hat{\phi}$ derivative,

$$\dot{\hat{\phi}} = -\dot{\phi}\cos\phi\hat{x} - \dot{\phi}\sin\phi\hat{y},$$

it's straightforward to find the projection of this vector on the axes,

$$\begin{aligned}\dot{\hat{\phi}} \cdot \hat{r} &= -\dot{\phi}\sin\theta \\ \dot{\hat{\phi}} \cdot \hat{\theta} &= -\dot{\phi}\cos\theta \\ \dot{\hat{\phi}} \cdot \hat{\phi} &= 0,\end{aligned}$$

so that

$$\dot{\hat{\phi}} = -\dot{\phi}\sin\theta\hat{r} - \dot{\phi}\cos\theta\hat{\theta}.$$

That's all preliminary. Now we are ready to calculate, first, the velocity in spherical coordinates:

$$\begin{aligned}\dot{\mathbf{r}} &= \dot{r}\hat{r} + r\dot{\hat{r}} \\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}.\end{aligned}$$

And then, at last, the acceleration:

$$\ddot{\mathbf{r}} = (\ddot{r}\hat{r} + \dot{r}\dot{\hat{r}}) + (\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}) + (\dot{r}\dot{\phi}\sin\theta\hat{\phi} + r\ddot{\phi}\sin\theta\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\phi\hat{\phi} + r\dot{\phi}\sin\theta\dot{\hat{\phi}}).$$

From here, you substitute the expressions for the time derivatives of the unit vectors, and after a whole lot of fun algebra you get

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta)\hat{\theta}$$
$$+(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\phi)\hat{\phi}.$$

Exercise 4.2

(a) If σ takes a fixed value, then $\tau = x/\sigma$ is uniquely determined by x. We substitute into the equation for y and get

$$y = \frac{1}{2}\left[\left(\frac{x}{\sigma}\right)^2 - \sigma^2\right], \quad \text{fixed } \sigma.$$

This describes a parabola in the x-y plane, which is concave upward and has a minimum at $(x, y) = (0, -\sigma^2/2)$. Likewise, the relation for fixed τ is

$$y = \frac{1}{2}\left[\tau^2 - \left(\frac{x}{\tau}\right)^2\right], \quad \text{fixed } \tau,$$

describing a parabola that is concave downward and has maximum at $(x, y) = (0, \tau^2/2)$. Several of these parabolas are shown in the figure below.

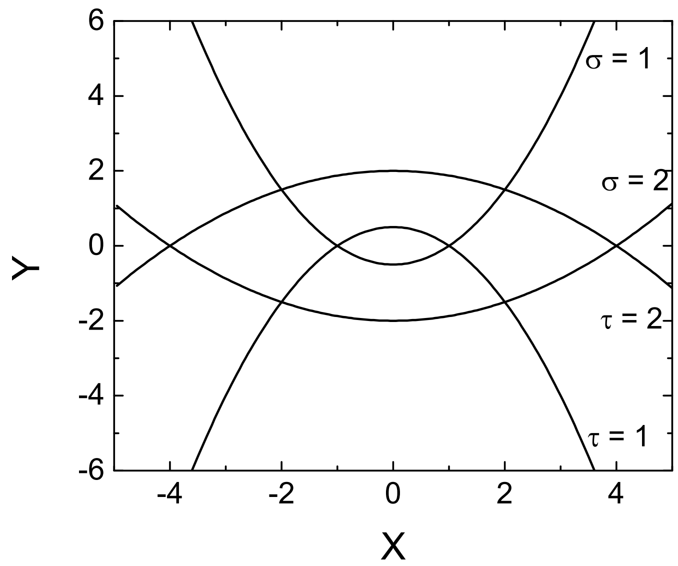

Figure 4.1 Selected surfaces of constant τ or σ.

(b)

$$\dot{x}^2 + \dot{y}^2 = \left(\dot{\sigma}\tau + \sigma\dot{\tau}\right)^2 + \left(\tau\dot{\tau} - \sigma\dot{\sigma}\right)^2$$
$$= \left(\sigma^2 + \tau^2\right)\left(\dot{\sigma}^2 + \dot{\tau}^2\right),$$

so the free-particle Lagrangian is

$$L = \frac{1}{2}m\left(\sigma^2 + \tau^2\right)\left(\dot{\sigma}^2 + \dot{\tau}^2\right).$$

Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \frac{d}{dt}\left(m(\sigma^2 + \tau^2)\dot{\sigma})\right)$$
$$= m\left(\sigma^2 + \tau^2\right)\ddot{\sigma} + 2m\sigma\dot{\sigma}^2 + 2m\tau\dot{\tau}\dot{\sigma}$$
$$\frac{\partial L}{\partial \sigma} = m\sigma\dot{\sigma}^2 + m\sigma\dot{\tau}^2,$$

so Lagranges's equation for σ is

$$\left(\sigma^2 + \tau^2\right)\ddot{\sigma} + \sigma\left(\dot{\sigma}^2 - \dot{\tau}^2\right) + 2\tau\dot{\sigma}\dot{\tau} = 0.$$

Similarly,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tau}}\right) = m\left(\sigma^2 + \tau^2\right)\ddot{\tau} + 2m\sigma\dot{\sigma}\dot{\tau} + 2m\tau\dot{\tau}^2$$
$$\frac{\partial L}{\partial \tau} = m\tau\dot{\sigma}^2 + m\tau\dot{\tau}^2,$$

so that Lagrange's equation for τ is

$$\left(\sigma^2 + \tau^2\right)\ddot{\tau} + \tau\left(\dot{\tau}^2 - \dot{\sigma}^2\right) + 2\sigma\dot{\sigma}\dot{\tau} = 0.$$

(c) The first thing is, let's get the initial conditions written in terms of σ and τ. From $y(0) = 0$ we get $\sigma_0 = \tau_0$, and then from $x(0) = x_0$ we must have

$$\sigma_0 = \tau_0 = \sqrt{x_0}.$$

Given this, $\dot{x}(0) = \dot{\sigma}_0\tau_0 + \sigma_0\dot{\tau}_0 = 0$ implies $\dot{\sigma}_0 = -\dot{\tau}_0$. Finally, $\dot{y}(0) = \sqrt{x_0}\dot{\tau}_0 - \sqrt{x_0}\dot{\sigma}_0 = v_0$ implies that

$$\dot{\tau}_0 = \frac{v_0}{2\sqrt{x_0}}, \quad \dot{\sigma}_0 = -\frac{v_0}{2\sqrt{x_0}}.$$

Using the Mathematica commands as described in the solutions for chapter 2, I show below an example of the motion in (σ, τ) coordinates. In this case I have $x_0 = 1$ m, $\dot{y}_0 = 1$ m/s. Notice that in this case where $x = x_0$ is fixed and $y = v_0 t$, we can write

$$y = \frac{1}{2}\left[\tau^2 - \frac{x_0^2}{\tau^2}\right] = v_0 t.$$

This actually affords the analytic solution

$$\tau = \sqrt{v_0 t + \sqrt{v_0^2 t^2 + x_0^2}}$$

for the curve shown.

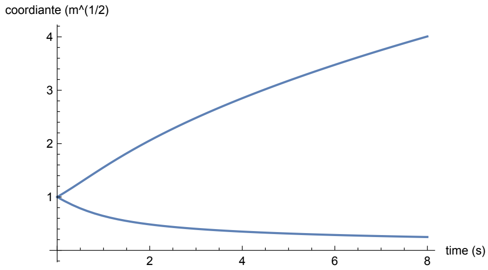

Figure 4.2 Motion of a particular free particle, as seen by τ (upper curve) and σ (lower curve).

Exercise 4.3

I'm thinking of a parabolic wire that opens upward, so the motion will be constrained to some region of space, and the bead will slide back and forth like a weird pendulum. In parabolic coordinates the parabola of constraint will correspond to some constant value of σ, which we hold constant for this exercise. Given that $\dot{\sigma} = 0$ and that the gravitational potential energy $V = mgy$, a suitable Lagrangian in τ is

$$L = \frac{1}{2}\left(\sigma^2 + \tau^2\right)\dot{\tau}^2 - \frac{mg}{2}\left[\tau^2 - \sigma^2\right].$$

Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tau}}\right) = m\left(\sigma^2 + \tau^2\right)\ddot{\tau} + 2m\tau\dot{\tau}^2$$

$$\frac{\partial L}{\partial \tau} = m\tau\dot{\tau}^2 - mg\tau$$

Thus the equation of motion is

$$\left(\sigma^2 + \tau^2\right)\ddot{\tau} + \tau\dot{\tau}^2 = -g\tau,$$

where σ serves as a parameter that describes the shape of the parabolic wire.

Exercise 4.4

The coordinates of the bead are time-dependent:

$$\begin{aligned} x &= R\sin\theta\cos(\omega t) \\ y &= R\sin\theta\sin(\omega t) \\ z &= R\cos\theta. \end{aligned}$$

So,

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= \Big(R\dot{\theta}\cos\theta\cos(\omega t) - R\omega\sin\theta\sin(\omega t)\Big)^2 + \Big(R\dot{\theta}\cos\theta\sin(\omega t) + R\omega\sin\theta\cos\omega t\Big)^2 \\ &= R^2\dot{\theta}^2\cos^2\theta + R^2\omega^2\sin^2\theta, \end{aligned}$$

and

$$\dot{z}^2 = \Big(-R\dot{\theta}\sin\theta\Big)^2 = R^2\dot{\theta}^2\sin^2\theta.$$

Meanwhile, the potential energy is

$$V = mgz = mgR\cos\theta,$$

so the Lagrangian is

$$L - \frac{1}{2}\Big(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta\Big) - mg\cos\theta.$$

with equation of motion

$$mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta.$$

Exercis 4.5

What is meant is spherical coordinates, in terms of which the kinetic energy is worked out above. The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V(r),$$

for whatever spherically symmetric potential you want to put in there. The equations of motion come from applying the formulas, and are, in this case

$$\begin{aligned} r^2\ddot{\phi}\sin^2\theta + 2r\dot{r}\dot{\phi}\sin^2\theta + 2r^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta &= 0 \\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} &= r^2\dot{\phi}^2\sin\theta\cos\theta \\ m\ddot{r} &= mr^2\dot{\theta}^2 + mr\dot{\phi}^2\sin^2\theta - \frac{\partial V}{\partial r}. \end{aligned}$$

It's not completely clear that this is a win. The three coordinates are still pretty interrelated in these equations, although it is nice that the potential energy only appears in one of them, I guess. Later in the book we will see that the parts of kinetic energy that involve the angular coordinates θ and ϕ deal with the angular momentum, which is pretty standard. The only thing that makes different problems different is the functional form of $V(r)$.

Exercise 4.6

This is one of these great problems where you could start off thinking, "Well, if *this* thing moves *this* way, then this *other* thing will move *that* way..." and go down a rabbit hole of confusion. Luckily, Langrange's way is much easier. You don't need to know what all the velocities of the masses are; you just need to know what form they take in an inertial frame.

First, note that there are two degrees of freedom here, the horizontal motion of block 1 and the vertical motion of block 3. Block 2 is then constrained by how block 3 drags it, plus how the pulley moves along with block 1. Moreover, the horizontal velocity of 3 is the same as that of 1.

So, given the coordinate x_1 of block 1, its velocity in the inertial frame of the ground is simply

$$\mathbf{v}_1 = \dot{x}_1 \hat{x}.$$

The height of block 3 is an inertial coordinate, since it is not attached to anything that can accelerate in y. Let y_3 be the height of block 3, measured so that $y_3 = 0$ when the block is at the height of the pulley. Then the velocity of this block is

$$\mathbf{v}_3 = \dot{x}_1 \hat{x} + \dot{y}_3 \hat{y}.$$

Finally, there's block 2, which is pulled by the falling block 3. Let η_2 be the position of block 2 with respect to the end of block 1, so that its velocity in the inertial frame is $\dot{x}_2 = \dot{x}_1 + \dot{\eta}_2$. The constraint is that the length of the string, l, is constant, so that $\eta_2 = l - y_3$. (As 3 falls and y_3 becomes more negative, η_2 becomes more positive, pulling to the right.) The velocity of block 2 is then entirely given by the other velocities:

$$\mathbf{v}_2 = (\dot{x}_1 - \dot{y}_3)\hat{x}.$$

That's all the thiking we have to do. Now Lagrange takes over. The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\Big(\dot{x}_1 - \dot{y}_3\Big)^2 + \frac{1}{2}m_3\Big(\dot{x}_1^2 + \dot{y}_3^2\Big) - m_3 g y_3 \\ &= \frac{1}{2}\Big(m_1 + m_2 + m_3\Big)\dot{x}_1^2 + \frac{1}{2}\Big(m_2 + m_3\Big)\dot{y}_3^2 - m_2\dot{x}_1\dot{y}_3 - m_3 g y_3. \end{aligned}$$

To save writing, we'll use the notation $M = m_1 + m_2 + m_3$ for the total mass of all three blocks. Lagrange's equation for x_1 is

$$M\ddot{x}_1 - m_2\ddot{y}_3 = 0,$$

which is a relation between the two accelerations. Lagrange's equation for y_3, which has the gravitational force in it, is

$$\Big(m_2 + m_3\Big)\ddot{y}_3 - m_2\ddot{x}_1 = -m_3 g.$$

Eliminating $\ddot{y}_3$ from this equation and solving for $\ddot{x}_1$ gives the result:

$$\ddot{x}_1 = -\frac{m_2 m_3}{(m_2 + m_3)M - m_2^2}\,g.$$

So in the end, it goes *that* way, i.e., the big block accelerates to the left in the figure.

Exercise 4.7

(a) The equation for the trajectory is

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2.$$

The initial condition for this trajectory is standard: $x(t = 0) = 0$ implies $x_0 = 0$. But the Hamilton's principle point of view requires also a terminal condition. If $x(t_1) = x_1$, then

$$x_1 = v_0 t_1 + \frac{1}{2}at_1^2,$$

or

$$v_0 = \frac{x_1}{t_1} - \frac{1}{2}at_1.$$

(b) Having specified the initial and terminal conditions, we must determine the value of the acceleration a that can meet these conditions under the actual action of gravity. The Lagrangian for this problem is

$$L = \frac{1}{2}m(v_0 + at)^2 - mg\left(v_0 t + \frac{1}{2}at^2\right),$$

which for fixed x_1, t_1 is still a function of a (and time). After a great deal of elementary but tedious algebra, one finds the action

$$S = \int_0^{t_1} dt L(t) = \frac{mt_1^3}{24}a^2 + \frac{mgt_1^3}{12}a + \frac{mx_1^2}{2t_1} - \frac{mgx_1t_1}{2}.$$

The physically correct path is the one for which a minimizes S, which is easily seen to be $a = -g$. Thus the trajectory $x(t) = v_0 t - (1/2)dt^2$ is the correct one, that exhibits acceleration $-g$.

Suppose you went to the extra term and proposed a trajectory

$$x(t) = v_0 t + \frac{1}{2}at^2 + \frac{1}{6}jt^3.$$

Then by the same reasoning you would have

$$v_0 = \frac{x_1}{t_1} - \frac{1}{2}at_1 - \frac{1}{6}jt_1^2.$$

$$L = \frac{1}{2}m\left(v_0 + at + \frac{1}{2}jt^2\right)^2 - mg\left(v_0 t + \frac{1}{2}at^2 + \frac{1}{6}jt^3\right),$$

Then S is an even more complicated function of a and now j, too (I recommend using Mathematica to get the algebra right). It is still a quadratic function of a and j, though, whose extremum you find easily by setting

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial j} = 0,$$

which yields $a = -g$ and $j = 0$. You can't fool Hamilton's principle: it knows there is no t^3 term in the trajectory of an object falling under gravity.

5
Solutions to Exercises Chapter 5

Exercise 5.1

Given

$$\mathbf{S} = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2$$
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

we have

$$\dot{\mathbf{r}}_1 = \frac{\dot{\mathbf{S}} + \beta\dot{\mathbf{r}}}{\alpha + \beta}$$
$$\dot{\mathbf{r}}_2 = \frac{\dot{\mathbf{S}} - \alpha\dot{\mathbf{r}}}{\alpha + \beta}$$

so that the Lagrangian is

$$L = \frac{1}{2}\left[\frac{m_1 + m_2}{(\alpha+\beta)^2}\right]\dot{\mathbf{S}}^2 + \frac{1}{2}\left[\frac{m_1\beta^2 + m_2\alpha^2}{(\alpha+\beta)^2}\right]\dot{\mathbf{r}} + \frac{1}{2}\left[\frac{2m_1\beta}{(\alpha+\beta)^2} - \frac{2m_2\alpha}{(\alpha+\beta)^2}\right]\dot{\mathbf{S}}\cdot\dot{\mathbf{r}} - V(\mathbf{r}).$$

The conjugate momentum to $\mathbf{S}$ is

$$\frac{\partial L}{\partial\dot{\mathbf{S}}} = \frac{m_1 + m_2}{(\alpha+\beta)^2}\dot{\mathbf{S}} + \left[\frac{m_1\beta}{(\alpha+\beta)^2} - \frac{m_2\alpha}{(\alpha+\beta)^2}\right]\dot{\mathbf{r}}.$$

The "simplest form" of this would contain no admixture of $\dot{\mathbf{r}}$, and one way to achieve this is to have $\alpha = m_1, \beta = m_2$. And then we get $\mathbf{S} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2$ as usual. Well, usually you would also divide by M to give $\mathbf{S}$ the units of length.

Exercise 5.2

The Lagrangian was worked out in Exercise 4.2:

$$L = \frac{m}{2}\left(\sigma^2 + \tau^2\right)\left(\dot{\sigma}^2 + \dot{\tau}^2\right),$$

leading to the momenta

$$p_\sigma = \frac{\partial L}{\partial \dot\sigma} = m\left(\sigma^2 + \tau^2\right)\dot\sigma$$
$$p_\tau = \frac{\partial L}{\partial \dot\tau} = m\left(\sigma^2 + \tau^2\right)\dot\tau.$$

Becasue L has nonzero partial derivatives with respect to σ and τ, in general these momenta depend on the fictitious forces $\partial L/\partial\sigma$, $\partial L/\partial\tau$ and are therefore not conserved.

Now, let

$$\sigma = \rho\cos\alpha$$
$$\tau = \rho\sin\alpha.$$

Then by an easy calculation

$$L = \frac{m}{2}\rho^2\left(\dot\rho^2 + \rho^2\dot\alpha^2\right) = \frac{m}{2}\rho^2\dot\rho^2 + \frac{m}{2}\rho^4\dot\alpha^2.$$

This Lagrangian is independent of α, so the momentum

$$p_\alpha = \frac{\partial L}{\partial\dot\alpha} = m\rho^4\dot\alpha \equiv C$$

is a constant of the motion.

Then the "*ma*" portion of Lagrange's equation for ρ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\rho}\right) = m\rho^2\ddot\rho + 2m\rho\dot\rho^2,$$

while the generalized force is

$$\frac{\partial L}{\partial\rho} = m\rho\dot\rho^2 + 2m\rho^3\dot\alpha^2$$
$$= m\rho\dot\rho^2 + \frac{2C^2}{m\rho^5}.$$

Thus the equation of motion for the *free particle* in this weird coordinate ρ is

$$m\rho^2\ddot\rho + m\dot\rho\rho^2 = \frac{2C^2}{m\rho^5}.$$

Exercise 5.3

(a) The momentum of each object is still its mass times its velocity. The momentum of the cart is $M\dot x$. The mass on the pendulum, as seen in the inertial frame of the ground, has velocity $d(x + l\sin\phi)/dt$, and so its momentum is $m\dot x + l\dot\phi\cos\phi$. So the total horizontal momentum of the pendulum on the cart is

$(M+m)\dot{x}+ml\dot{\phi}\cos\phi$, even though in the text we got it from the slick Lagrangian method.

(b) Using the ingredients given in the main text, we have from conservation of p_x, and setting this conserved value equal to zero,

$$(M+m)\dot{x}+ml\dot{\phi}\cos\phi = 0,$$

or

$$\dot{x} = -\frac{ml}{M+m}\dot{\phi}\cos\phi.$$

Substituting this into the expression for p_ϕ we have

$$\begin{aligned} p_\phi &= ml^2\dot{\phi} + ml\dot{x}\cos\phi \\ &= ml^2\dot{\phi} - \frac{m^2l^2}{M+m}\dot{\phi}\cos^2\phi. \end{aligned}$$

Thus the "*ma*" part of Lagrange's equation is

$$\frac{dp_\phi}{dt} = ml^2\ddot{\phi} - \frac{m^2l^2}{M+m}\ddot{\phi}\cos^2\phi + \frac{m^2l^2}{M+m}\dot{\phi}^2\sin\phi.$$

Making the same substitution for $\dot{x}$, the generalized force is

$$\begin{aligned} \frac{\partial L}{\partial\phi} &= -ml\dot{x}\dot{\phi}\sin\phi - mgl\sin\phi \\ &= \frac{m^2l^2}{M+m}\dot{\phi}^2\sin\phi\cos\phi - mgl\sin\phi. \end{aligned}$$

Setting these equal and re-arranging gives the result.

Exercise 5.4

Using the methods outlined in the solutions to Chapter 2, we set up and integrate the equations of motion. For concreteness, I will assume here the same pendulum that was discussed in Chapter 5. It has length $l = 1$ m, mass $m = 1$ kg, and is sitting on a cart of mass $M = 2$ kg.

As a start, we imagine releasing the mass from rest from an initial angle $\phi_0 = 2$ rad, just as in the example of Chapter 2. In this case the cart is also initially at rest. The resulting motion of ϕ versus time is shown in Figure 5.1. Here the red curve describes the pendulum on the cart, while the blue curve is the same pendulum, released in the same way, but anchored to the ground. So, as described before, the period of the pendulum is reduced when it sits on the cart.

What about if it goes all the way around instead of back-and forth? Here is an example, like the one in Chapter 2, where the initial angle is $\phi_0 = 2$ rad, and

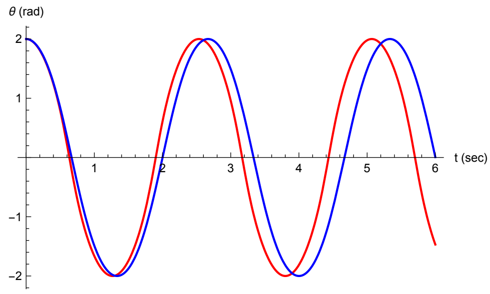

Figure 5.1 The pendulum swinging on a cart (red) and on solid ground (blue).

the initial generalized velocity is $\dot{\phi}_0 = 3.5$ rad/s. The initial velocity of the cart is irrelevant for what we're asking about here (check this!). Nevertheless, here I set it equal to $= ml\dot{\phi}_0 \cos\phi_0/(M+m)$, so that the conserved momentum p_x is zero.

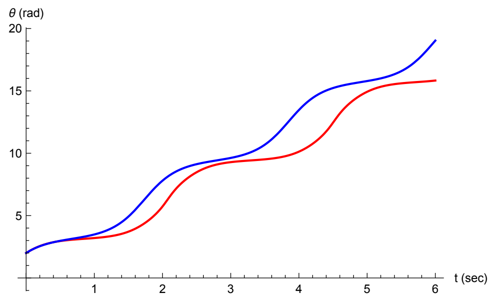

Figure 5.2 The pendulum going round and round on a cart (red) and on solid ground (blue).

This result is shown in Figure 5.2. The result is the opposite of the back-and-forth case: when the pendulum goes round and round on the cart (red, lower curve), it actually lags behind the motion it would have had on solid ground (blue, upper curve). As you can see, the difference occurs mostly at times like ≈ 1 s, where $\phi \approx \pi$ places the pendulum at the top of its trajectory.

Exercise 5.5

We can use the same coordinate system we did in Chapter 2, with $\hat{x}$ pointing downward. The difference is that the point of support is accelerating. This adds a height $-at^2/2$ to the pendulum, where upward acceleration has $a > 0$ and the rise corresponds to more negative values of x in our coordinate system. Thus the coordinates in an inertial frame are

$$x = l\cos\phi - \frac{1}{2}at^2, \quad y = l\sin\phi,$$

and the velocities are

$$\dot{x} = -l\dot{\phi}\sin\phi - at, \quad \dot{y} = l\dot{\phi}\cos\phi.$$

The rest is amazingly straightforward; this is what Lagrange is for, after all. Given the potential energy $V = -mgx$, the Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + malt\dot{\phi}\sin\phi + \frac{1}{2}ma^2t^2 + mgl\cos\phi - \frac{1}{2}mgat^2.$$

The momentum conjugate to ϕ is

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ml^2\dot{\phi} + malt\sin\phi,$$

with time derivative

$$\frac{dp_\phi}{dt} = ml^2\ddot{\phi} + mal\sin\phi + malt\dot{\phi}\cos\phi.$$

This must be set equal to the generalized force,

$$\frac{\partial L}{\partial\phi} = -mgl\sin\phi + malt\dot{\phi}\cos\phi.$$

Note that the last term in each of these expressions cancels (no malt today!) and after some algebra we arrive at the equation of motion

$$\ddot{\phi} = -\frac{g+a}{l}\sin\phi.$$

Why, this is awesome. It is the equation of motion as if the acceleration due to gravity were $g + a$ rather than just a. From the pendulum's point of view, the acceleration could come from either real gravity near the surface of the Earth, or from the uniformly accelerating elevator, or both.

Exercise 5.6

The angle at which the platform is rotated at time t is given generically by $\Phi(t)$. In the example of the chapter, the rotation rate was constant and Φ was equal to Ωt. But now we allow $\Phi(t)$ to be an arbitrary function of time. Working out the Lagrangian, it is similar to what we had before,

$$L = \frac{1}{2}m\left(\dot{\eta}^2 + \dot{\xi}^2\right) + m\dot{\Phi}\left(\eta\dot{\xi} - \xi\dot{\eta}\right) + \frac{1}{2}m\dot{\Phi}^2\left(\eta^2 + \xi^2\right),$$

basically replacing Ω by $\dot{\Phi}$.

Let's start this time with the generalized forces, which look pretty much the same as before:

$$Q_\eta = \frac{\partial L}{\partial \eta} = m\dot{\Phi}\dot{\xi} + m\dot{\Phi}^2\eta$$
$$Q_\xi = \frac{\partial L}{\partial \xi} = -m\dot{\Phi}\dot{\eta} + m\dot{\Phi}^2\xi.$$

Th real difference comes from the additional time dependence of the momenta. These momenta are

$$p_\eta = \frac{\partial L}{\partial \dot{\eta}} = m\dot{\eta} - m\dot{\Phi}\xi$$
$$p_\xi = \frac{\partial L}{\partial \dot{\xi}} = m\dot{\xi} + m\dot{\Phi}\eta.$$

The time derivatives of these are

$$\frac{dp_\eta}{dt} = m\ddot{\eta} - m\dot{\Phi}\dot{\xi} - m\ddot{\Phi}\xi$$
$$\frac{p_\xi}{dt} = m\ddot{\xi} + m\dot{\Phi}\eta + m\ddot{\Phi}\eta.$$

Lagrange's equations of motion, written to emphasize the fictitious forces, are

$$m\ddot{\eta} = 2m\dot{\Phi}\dot{\xi} + m\dot{\Phi}^2\eta + m\ddot{\Phi}\xi$$
$$m\ddot{\xi} = -2m\dot{\Phi}\dot{\eta} + m\dot{\Phi}^2\xi - m\ddot{\Phi}\eta.$$

On the right-hand side of these equations, you can recognize the expressions for the Coriolis and centrifugal forces, although they might vary in time now that $\dot{\Phi}$ need not be constant. The new force, proportional to the second derivative of Φ, can be written as

$$\mathbf{F}_{\text{Euler}} = m\ddot{\Phi}\mathbf{r} \times \hat{z},$$

and is called the *Euler force* (Calkin, p. 57). The delightfully-named website "revolvy.com" describes it thus: "The Euler force will be felt by a person riding a merry-go-round. As the ride starts, the Euler force will be the apparent force pushing the person to the back of the horse, and as the ride comes to a stop, it will be the apparent force pushing the person towards the front of the horse. A person on a horse close to the perimeter of the merry-go-round will perceive a greater apparent force than a person on a horse closer to the axis of rotation."

Exercise 5.7

This is a poorly worded question. If z is the axis of rotation, then z is the same in both the rotating and inertial frames, and a term $m\dot{z}^2/2$ is added to the Lagrangian. Big deal.

A better question is to do what is proposed, and to limit motion to the surface of the planet, at radius R. Then in the rotating coordinate system

$$(\eta, \xi, z) = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$$

we are interested in the Coriolis and centrifugal forces in terms on θ and ϕ, the co-latitude and longitude of the planet.[1] In these coordinates the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{\eta}^2 + \dot{\xi}^2 + \dot{z}^2\right) + m\Omega\left(\eta\dot{\xi} - \xi\dot{\eta}\right) + \frac{1}{2}m\Omega^2\left(\eta^2 + \xi^2\right) \\ &= \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\dot{\phi}^2\sin^2\theta\right) + mR^2\Omega\dot{\phi}\sin^2\theta + \frac{1}{2}mR^2\Omega^2\sin^2\theta. \end{aligned}$$

The momenta are

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial\dot{\theta}} = mR^2\dot{\theta} \\ p_\phi &= \frac{\partial L}{\partial\dot{\phi}} = mR^2\left(\dot{\phi} + \Omega\right)\sin\theta. \end{aligned}$$

Note that the Lagrangian does not depend on ϕ, whereby p_ϕ is a conserved quantity, the component of angular momentum along z as seen in the inertial frame. This is very similar to the case seen in the rotating platform; note that $mR^2\sin^2\theta$ is the moment of inertia of the mass about the axis at colatitude θ.

Meanwhile, the equation of motion for θ is

$$mR^2\ddot{\theta} = mR^2\left(\dot{\phi} + \Omega\right)^2\sin\theta\cos\theta.$$

In terms of the constant angular momentum p_ϕ, this is a self-contained equatoin for θ:

$$mR^2\ddot{\theta} = \frac{p_\phi^2}{mR^2\sin\theta}\cos\theta.$$

Exercise 5.8

We have already worked out the kinetic energy. Including the potential due to the spring leads to the Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + mr^2\Omega\dot{\phi} + \frac{1}{2}mr^2\Omega^2 - \frac{1}{2}kr^2.$$

Of course there is still a conserved quantity: because L is independent of ϕ, the conjugate angular momentum $p_\phi = mr^2\left(\dot{\phi} + \Omega\right) \equiv \mathcal{L}$ is still conserved.

[1] θ is measured from the north pole down toward the south, but latitude is measured from the equator either up north or down south.

The radial equation of motion is

$$\begin{aligned}\frac{dp_r}{dt} = m\ddot{r} = \frac{\partial L}{\partial r} &= mr\left(\dot{\phi} + \Omega\right)^2 - kr \\ &= \frac{\mathcal{L}^2}{mr^3} - kr.\end{aligned}$$

This is just the equation of motion for a mass subject to both the spring force and the centrifugal force. A circular orbit results when the two forces balance, i.e., when

$$r = \left(\frac{\mathcal{L}^2}{mk}\right)^{1/4}.$$

Exercise 5.9

There's no harm here in including an electric potential, and making the treatment more complete. If there were *only* an electric potential ϕ, then it generates a conservative force $-q\nabla\phi$ on a particle of charge q, and the usual Lagrangian business applies. More generally, in the presence of a vector potential that can vary in time and space, we recall the electromagnetic fields generated:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

The claim is that a suitable Lagrangian for a charged particle of charge q and mass m is given in terms of these potentials as

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A},$$

where the last term is the one suggested in the chapter. The generalized momentum conjugate to the particle's coordinate is

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{q}{c}\mathbf{A},$$

where the notation suggests that you take the derivative separately with respect to x, y, z and collect these into the vector $\mathbf{p}$.

Now the momentum depends on the vector potential, which can vary with the coordinate of the particle, which is of course in general moving from place to place. The time derivative of this momentum is then

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= m\ddot{\mathbf{r}} + \frac{q}{c}\left(\dot{x}\frac{\partial \mathbf{A}}{\partial x} + \dot{y}\frac{\partial \mathbf{A}}{\partial y} + \dot{z}\frac{\partial \mathbf{A}}{\partial z}\right) + \frac{q}{c}\frac{\partial \mathbf{A}}{\partial t} \\ &= m\ddot{\mathbf{r}} + \frac{q}{c}\left(\dot{\mathbf{r}} \cdot \nabla\right)\mathbf{A} + \frac{q}{c}\frac{\partial \mathbf{A}}{\partial t}.\end{aligned}$$

The generalized force is given by the gradient of the Lagrangian,

$$\nabla L = -q\nabla\phi + \frac{q}{c}\nabla\left(\dot{\mathbf{r}}\cdot\mathbf{A}\right).$$

Putting these together, we get Lagrange's equation

$$m\ddot{\mathbf{r}} = q\left(-\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) + \frac{q}{c}\left[\nabla\left(\dot{\mathbf{r}}\cdot\mathbf{A}\right) - \left(\dot{\mathbf{r}}\cdot\nabla\right)\mathbf{A}\right].$$

Getting there! Now we have to recall an old vector identity that you probably saw once or twice and then forgot about:

$$\dot{\mathbf{r}}\times\left(\nabla\times\mathbf{A}\right) = \nabla\left(\dot{\mathbf{r}}\cdot\mathbf{A}\right) - \left(\dot{\mathbf{r}}\cdot\nabla\right)\mathbf{A}$$

(irritating, but doable by writing specific components out). Substituting this, we get the equation of motion

$$m\ddot{\mathbf{r}} = q\left(-\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) + \frac{q}{c}\dot{\mathbf{r}}\times\left(\nabla\times\mathbf{A}\right),$$

and using the electromagnetic facts above this is

$$m\ddot{\mathbf{r}} = q\mathbf{E} + \frac{q}{c}\dot{\mathbf{r}}\times\mathbf{B}.$$

The right hand side has the electrostatic force, plus the usual Lorentz force.

Exercise 5.10

Whoops, this was pretty much done at the end of Sec. 5.6. Were you paying attention?

6
Solutions to Exercises Chapter 6

Exercise 6.1

Note that the Lagrangian is already a function of positions q_a and velocities v_a. So this whole issue of derivatives at fixed momenta versus derivatives at fixed velocities, which we labored over so intensely in the chapter, has no place here. All we're really doing in this exercise is to reduce the f second-order equations of motion to $2f$ first-order equations of motion.

Half of these equations of motion are easy, they are just

$$\frac{dq_a}{dt} = v_a,$$

perfectly acceptable as v_a is one of the quantities we carry around. The equations for the v_a come from Lagrange's equations. The force part, $\partial L/\partial q_a$, is unchanged. But the dynamical part is more complicated. L is a function of all the coordinates and all the velocities, all of which vary in time, and maybe an explicit function of time as well. We get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_a}\right) = \sum_b \left[\frac{\partial^2 L}{\partial q_b \partial v_a}\frac{dq_b}{dt} + \frac{\partial^2 L}{\partial v_b \partial v_a}\frac{dv_b}{dt}\right] + \frac{\partial^2 L}{\partial t \partial v_a}.$$

Thus the other half of the equations of motion would read

$$\sum_b \left[\frac{\partial^2 L}{\partial q_b \partial v_a} v_b + \frac{\partial^2 L}{\partial v_b \partial v_a}\frac{dv_b}{dt}\right] + \frac{\partial^2 L}{\partial t \partial v_a} = \frac{\partial L}{\partial q_a}.$$

These equations confound together all the velocities and their derivatives. They are somewhat more complicated and unwieldy that Hamilton's equations, which cleanly separate out the derivatives of their objects of study, namely, the momenta.

Well, but maybe we can make a little headway using matrices and such, as we did in the chapter. Let's try this in the simplest case where the Lagrangian is written

$$L = \frac{1}{2}\dot{q}^T A\dot{q} - V = \frac{1}{2}\sum_a v_a A_{ab} v_a - V$$

for some inertia matrix A and regarding velocities $v_a = \dot{q}_a$ as variables. Components of the generalized forces are as always

$$\frac{\partial L}{\partial q_c} = -\frac{\partial V}{\partial q_c} + \frac{1}{2}\sum_a v_a \frac{\partial A_{ab}}{\partial q_c} v_b$$

The kinetic part of Lagrange's equations are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial v_c}\right) &= \frac{d}{dt}\left(\sum_b A_{cb} v_b\right) \\
&= \sum_b A_{cb}\frac{dv_b}{dt} + \sum_{ab}\frac{\partial A_{cb}}{\partial q_a}\frac{dq_a}{dt} v_b \\
&= \sum_b A_{cb}\frac{dv_b}{dt} + \sum_{ab} v_a \frac{\partial A_{cb}}{\partial q_a} v_b,
\end{aligned}$$

where, instead of treating the matrix product Av as its own thing, we go ahead and take the time derivative of this product, preserving the velocities v.

After some rearrangement, Lagrange's equations are then

$$\sum_b A_{cb}\dot{v}_b = -\frac{\partial V}{\partial q_c} + \frac{1}{2}\sum_a v_a \frac{\partial A_{ab}}{\partial q_c} v_b - \sum_{ab} v_a \frac{\partial A_{cb}}{\partial q_a} v_b$$

Now, this is maybe not as clean as Hamilton's equations, but it does have a kind of "$ma = F$" vibe to it, if you look. On the left side we have the product of the inertia matrix (playing the role of "m") times the accelerations, although this is a matrix product in general. On the right hand side we have, in order: the applied forces; the usual fictitious forces; and a new kind of force that is downright delusional.

Let's see what it looks like in a simple case, motion in a plane in polar coordinates, just like in the Chapter. In this case the inertia matrix is

$$A = \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}.$$

Then the fictitious forces have components

$$\begin{aligned}
F_{\mathbf{f},r} &= \frac{1}{2}\begin{pmatrix} v_r & v_\phi \end{pmatrix}\frac{\partial}{\partial r}\begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}\begin{pmatrix} v_r \\ v\phi \end{pmatrix} = mrv_\phi^2 \\
F_{\mathbf{f},\phi} &= \frac{1}{2}\begin{pmatrix} v_r & v_\phi \end{pmatrix}\frac{\partial}{\partial \phi}\begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}\begin{pmatrix} v_r \\ v\phi \end{pmatrix} = 0.
\end{aligned}$$

The delusional forces are a little more awkward, but in this case A is diagonal, $b = c$, and we can calculate components

$$F_{\mathbf{d},r} = -\sum_a v_a \frac{\partial A_{rr}}{\partial q_a} v_r = v_r(0)v_r - v_\phi(0)v_r = 0$$
$$F_{\mathbf{d},\phi} = \sum_a v_a \frac{\partial A_{\phi\phi}}{\partial q_a} v_\phi = -v_r(2mr)v_\phi - v_r(0)v_\phi = -2mrv_r v_\phi.$$

The equations of motion for the velocities are

$$m\dot{v}_r = -\frac{\partial V}{\partial r} + mrv_\phi^2$$
$$mr^2\dot{v}_\phi = -\frac{\partial V}{\partial \phi} - 2mrv_r v_\phi,$$

which must of course be complemented by

$$\dot{r} = v_r$$
$$\dot{\phi} = v_\phi$$

to make a solvable system of equations.

Now, I'm not an expert in this form of the equations of motion (I don't think anybody is, honestly), but let's see what we've got. First, using these equations and by direct substitution of $p_r = mv_r$, $p_\phi = mr^2 v_\phi$, you can recover the usual Hamilton's equations.[1] The equations in terms of velocities are less convenient, however, in that the velocities are coupled between one equation and the other.

More to the point, the whole idea of constants of the motion gets hidden. Suppose that $\partial V/\partial\phi = 0$, then angular momentum should be conserved. do the equations tell us this? Sort of. In this case the equation for v_ϕ is

$$mr^2\dot{v}_\phi + 2mrv_r v_\phi = 0.$$

Using $v_r = \dot{r}$, you could then recognize this as reading

$$\frac{d(mr^2 v_\phi)}{dt} = 0.$$

There's your constant of the motion, but you kind of had to go digging for it, whereas Hamilton's equations hand it to you on a silver platter.

Exercise 6.2

The coordinates in the inertial frame read

$$x = r\cos\Omega t \quad y = r\sin\Omega t.$$

[1] Exercises within exercises!

We've seen approximately a million times how to construct the Lagrangian for something like this:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\Omega^2\right).$$

r is the only degree of freedom. Its conjugate momentum is $p_r = \partial L/\partial\dot{r} = m\dot{r}$. Lagrange's equations are

$$\frac{dp_r}{dt} = m\ddot{r} = \frac{\partial L}{\partial r} = mr\Omega^2$$

or $\ddot{r} = \Omega^2 r$. The general solution to this is $r(t) = Ae^{\Omega t} + Be^{-\Omega t}$; a particular solution with $\dot{r}(0) = 0$, is $r(t) = r_0 \cosh \Omega t$.

On the other hand, the Hamiltonian is

$$\begin{aligned} H &= p_r\dot{r} - L \\ &= \frac{p_r^2}{2m} - \frac{1}{2}m\Omega^2 r^2. \end{aligned}$$

Hamilton's equations are

$$\begin{aligned} \frac{dp_r}{dt} &= -\frac{\partial H}{\partial r} = m\Omega^2 r \\ \frac{dr}{dt} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}. \end{aligned}$$

These equations are fine and give the right answer. Ironically, the easiest way to see this it to kind of undo the transformation, and to replace this pair of first-order equations with a single, second-order equation. Thus

$$\frac{d^2r}{dt^2} = \frac{1}{m}\frac{dp_r}{dt} = \frac{1}{m}m\Omega^2 r,$$

and this is Lagrange's equation again. If the solution is $r = r_0 \cosh \Omega t$, then the momentum will be $mr_0\Omega \sinh \Omega t$.

This is a little bit silly, but the point is you can certainly *formulate* equations of motion for this problem using Hamilton's procedure. How you *solve* the equations is up to you.

Exercise 6.3

You don't really need Lagrangian (let alone Hamiltonian) mechanics for this, but what the heck. The potential energy of the mass M is $V = Mg(r - r_0)$ for some reference length r_0: if the radius of the whirling string increases, it raises M and increases the kinetic energy. Thus the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - Mg(r - r_0).$$

In the usual way, the equation of motion for ϕ is $d(mr^2\dot{\phi})/dt = 0$ and identifies the angular momentum $p_\phi = mr^2\dot{\phi} = \mathcal{L}$ as a fixed quantity. Then Lagrange's equation for r is

$$m\ddot{r} = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 - Mgr = \frac{\mathcal{L}^2}{mr^3} - Mg.$$

A stable orbit would have $\ddot{r} = 0$, and hence for angular momentum $\mathcal{L}$ the radius of this orbit would be

$$r = \left(\frac{\mathcal{L}^2}{mMg}\right)^{1/3}.$$

Okay, you could use Hamiltonians too. The Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{\mathcal{L}^2}{2mr^2} + Mg(r - r_0).$$

The equation of motion for the momentum is

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{\mathcal{L}^2}{mr^3} - Mg.$$

Zeroing the change in momentum, $dp_r/dt = 0$, results in the same condition as above.

Exercise 6.4

(i) Give me two matrices A and B. For either one, say A, the transpose A^T is defined in terms of matrix elements by

$$A^T_{ij} = A_{ji}.$$

By the rules of matrix multiplication, matrix elements of the product AB are

$$(AB)_{ij} = \sum_k A_{ik}B_{kj}.$$

Then the matrix elements of the transpose of AB are

$$(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk}B_{ki} = \sum_k B_{ki}A_{jk} = \sum_k B^T_{ik}A^T_{kj} = (B^TA^T)_{ij}.$$

So, $(AB)^T = B^TA^T$.

Exercise 6.5

Let's suppose for the moment that you didn't have the genius required to know that you should use the Legendre transformation to go from Lagrangians to Hamiltonians. What would you do instead? As always, you identify the momentum and try to write its equation of motion in terms of T_p.

The chapter begins this for us. The kinetic energy in terms of velocities is written in matrix form as

$$T_{\dot{q}} = \frac{1}{2}\dot{q}^T A\dot{q} + \dot{q}^T b + c.$$

This gives momentum

$$p = A\dot{q} + b,$$

which is inverted to give velocity as a function of momentum:

$$\dot{q} = A^{-1}(p - b).$$

Our general strategy is as it was in the chapter. We will first find the equation of motion for a component of the generalized momentum from Lagrange's equation, then translate this expression into something that depends on momentum:

$$\begin{aligned}
\frac{dp_a}{dt} = \frac{\partial(T_{\dot{q}} - V)}{\partial q_a} &= \frac{1}{2}\dot{q}^T \frac{\partial A}{\partial q_a}\dot{q} + \dot{q}^T \frac{\partial b}{\partial q_a} + \frac{\partial c}{\partial q_a} - \frac{\partial V}{\partial q_a} \\
&= \frac{1}{2}(p - b)^T (A^{-1})^T \frac{\partial A}{\partial q_a} A^{-1}(p - b) + (p - b)^T (A^{-1})^T \frac{\partial b}{\partial q_a} + \frac{\partial c}{\partial q_a} - \frac{\partial V}{\partial q_a} \\
&= -\frac{1}{2}(p - b)^T \frac{\partial A^{-1}}{\partial q_a}(p - b) + (p - b)^T A^{-1} \frac{\partial b}{\partial q_a} + \frac{\partial c}{\partial q_a} - \frac{\partial V}{\partial q_a}. \qquad (6.1)
\end{aligned}$$

where we have used the result $\partial A^{-1}/\partial q_a = -A^{-1}(\partial A \partial q_a)A^{-1}$ again.

Next, we directly construct the kinetic energy as a function of momenta:

$$T_p = T_{2,p} + T_{1,p} + T_{0,p},$$

where

$$\begin{aligned}
T_{2,p} &= \frac{1}{2}\left((p - b)^T (A^{-1})^T\right) A \left(A^{-1}(p - b)\right) = \frac{1}{2}(p - b)^T A^{-1}(p - b) \\
T_{1,p} &= (p - b)^T A^{-1} b \\
T_{0,p} &= c.
\end{aligned}$$

Based on our experience with non-moving coordinate systems, we know that the main thing that has to happen here is a change in sign of the $T_{2,p}$ component, so as to get an appropriate minus sign in the generalized force, as in the first term of dp_a/dt above. An easy way to do this (HERE IS THE TRICK) is to write $T_{1,p}$ as

$$\begin{aligned}
T_1(p) &= (p - b)^T A^{-1} b \\
&= -(p - b)^T A^{-1}(p - b) + (p - b)A^{-1} p.
\end{aligned}$$

Then the Quantity Formerly Known As The Lagrangian can be written

$$L_p = -H + (p-b)^T A^{-1} p,$$

where we have carefully, and with forethought, introduced a symbol

$$H = \frac{1}{2}(p-b)^T A^{-1}(p-b) - c + V.$$

What is it good for? Look at the following derivative

$$-\frac{\partial H}{\partial q_a} = -\frac{1}{2}(p-b)^T \frac{\partial A^{-1}}{\partial q_a}(p-b) + \frac{1}{2}\frac{\partial b^T}{\partial q_a} A^{-1}(p-b) + \frac{1}{2}(p-b)^T A^{-1}\frac{\partial b}{\partial q_a} + \frac{\partial c}{\partial q_a} - \frac{\partial V}{\partial q_a}.$$

The second and third terms in this are transposes of each other, but they are also numbers. And by golly, a number (a 1×1 matrix) is its own transpose. The result:

$$-\frac{\partial H}{\partial q_a} = -\frac{1}{2}(p-b)^T \frac{\partial A^{-1}}{\partial q_a}(p-b) + (p-b)^T A^{-1}\frac{\partial b}{\partial q_a} + \frac{\partial c}{\partial q_a} - \frac{\partial V}{\partial q_a},$$

which is exactly the generalized force applied and set equal to dp_a/dt, see above. In summary, the portion H of the transformed Lagrangian gives us the equation of motion

$$\frac{dp_a}{dt} = -\frac{\partial H}{\partial q_a}.$$

There is another necessary equation of motion, the one for q_a. We're kind of conditioned to think of this as coming from the derivative of H with respect to momentum. Is it?

$$\begin{aligned}
\frac{\partial H}{\partial p_a} &= \frac{\partial}{\partial p_a}\left[\frac{1}{2}(p-b)^T A^{-1}(p-b)\right] \\
&= \frac{1}{2}A^{-1}(p-b) + \frac{1}{2}(p-b)^T A^{-1} \\
&= A^{-1}(p-b) = \dot{q}_a = \frac{dq_a}{dt}.
\end{aligned}$$

These are, of course, Hamilton's equations.

It looks like we found the piece we need, the Hamiltonian H, residing in a part of the properly transformed Lagrangian. What of the leftover piece? We have

$$L = -H + (p-b)^T A^{-1} p = -H + \dot{q}^T p,$$

the same relation that defines the Legendre transformation.

Exercise 6.6

As related in the chapter, the Lagrangian for this situation is

$$L = \frac{1}{2}m(\dot{\eta}^2 + \dot{\xi}^2) + m\Omega(\eta\dot{\xi} - \xi\dot{\eta}) + \frac{1}{2}m\Omega^2(\eta^2 + \xi^2),$$

with momenta

$$p_\eta = \frac{\partial L}{\partial \dot{\eta}} = m\dot{\eta} - m\Omega\xi$$
$$p_\xi = \frac{\partial L}{\partial \dot{\xi}} = m\dot{\xi} + m\Omega\eta.$$

Therefore, the velocities in terms of momenta are

$$\dot{\eta} = \frac{p_\eta}{m} + \Omega\xi$$
$$\dot{\xi} = \frac{p_\xi}{m} - \Omega\eta.$$

So after some irritating algebra you find the Hamiltonian

$$H = \frac{p_\eta^2}{2m} + \frac{p_\xi^2}{2m} + \Omega(\xi p_\eta - \eta p_\xi).$$

The equations of motion for the momenta then stand alone, apart from those for the coordinates:

$$\frac{dp_\eta}{dt} = -\frac{\partial H}{\partial \eta} = \Omega p_\xi$$
$$\frac{dp_\xi}{dt} = -\frac{\partial H}{\partial \xi} = -\Omega p_\eta.$$

This leads to things like $\ddot{p}_\eta = -\Omega^2 p_\eta$, $\ddot{p}_\xi = -\Omega^2 p_\xi$, whose solutions are clearly just sines and cosines. The momenta evolve simply in time.

Let's give the mass initial conditions like $\eta(0) = \eta_0$, $\dot{\eta}(0) = 0$, $\xi(0) = 0$, $\dot{\xi}(0) = 0$. Then the initial conditions for the momenta are $p_\eta(0) = 0$, $p_\xi(0) = m\Omega\eta_0$, and

$$p_\eta = m\Omega\eta_0 \sin\Omega t$$
$$p_\xi = m\Omega\eta_0 \cos\Omega t.$$

Then the coordinates evolve in time as

$$\dot{\eta} = \frac{p_\eta}{m} + \Omega\xi = \Omega\eta_0 \sin\Omega t + \Omega\xi$$
$$\dot{\xi} = \frac{p_\xi}{m} - \Omega\eta = \Omega\eta_0 \cos\Omega t - \Omega\eta.$$

These, too, can be separated by taking another derivative:

$$\ddot{\eta} = -\Omega^2\eta + 2\Omega^2\eta_0 \cos\Omega t$$
$$\ddot{\xi} = -\Omega^2\xi - 2\Omega^2\eta_0 \sin\Omega t,$$

at which point you may be wondering why we bothered with all this Hamiltonian stuff.

Anyway, equations like these are solved by standard methods, like you would find in the classic text by Boas.[2] Consider η and ξ as the real and imaginary parts of a complex variable $z = \eta + i\xi$, in terms of which both the equations are collected together into

$$\ddot{z} + \Omega^2 z = 2A\exp(-i\Omega t),$$

where $A = \Omega^2\eta_0$ is real-valued. If the right-hand side of this equation were zero, the equation would be what is called *homogeneous*, and you would know the sine and cosine (or complex exponential) solutions. But the term on the right, the driving, or source, term, complicates things. The theory of differential equations tells us that the general solution to the equation for z is

$$z(t) = a\exp(i\Omega t) + b\exp(-i\Omega t) + z_p(t),$$

where z_p is *any* solution to the equation that is independent of (i.e., cannot be written as a linear combination of) the homogeneous solutions $\exp(\pm i\Omega t)$. The constants a and b serve to set the initial conditions. You can freely add or subtract the homogeneous solutions as much as you want, and still satisfy the differential equation.

The function z_p is called a *particular solution* to the equation. We only need one; where do we get it? For details, I refer the reader to the book by Boas, or equivalent. The gist of it is, you rewrite the differential equation

$$\ddot{z}_p + \Omega^2 z_p = 2A\exp(-i\Omega t),$$

as

$$\left(\frac{d}{dt} + i\Omega\right)\left(\frac{d}{dt} - i\Omega\right)z_p = 2A\exp(-i\Omega t).$$

Then we introduce an auxiliary function u so that

$$\left(\frac{d}{dt} + i\Omega\right)u = 2A\exp(-i\Omega t)$$
$$\left(\frac{d}{dt} - i\Omega\right)z_p = u.$$

[2] M. L. Boas, *Mathematical Methods in the Physical Sciences*, 3rd ed. Wiley 2006, see Chapter 8.

This is a pair of first-order equations, which can be solved sequentially, first for $u(t)$, then for $z_p(t)$, by using the technique of integrating factors.

I won't go into the details here, but suffice it to say that a perfectly good particular solution is given by

$$z_p(t) = \frac{iA}{\Omega} t \exp(-i\Omega t) = i\eta_0 \Omega t \exp(-i\Omega t),$$

as you can verify by direct substitution. The form of the solutions is then

$$\begin{aligned} \eta(t) &= \Re(z(t)) \\ \xi(t) &= \Im(z(t)), \end{aligned}$$

where

$$z(t) = z(t) = a \exp(i\Omega t) + b \exp(-i\Omega t) + i\eta_0 \Omega t \exp(-i\Omega t).$$

This is enough to get the idea. As time goes by, the coordinates in the rotating frame, in addition to going round and round, also expand linearly in time away from the center of rotation.

Exercise 6.7

The Lagrangian is worked out in previous chapters:

$$L = \frac{1}{2} m \left(\sigma^2 + \tau^2\right)\left(\dot{\sigma}^2 + \dot{\tau}^2\right).$$

The momenta are

$$\begin{aligned} p_\sigma &= m\left(\sigma^2 + \tau^2\right)\dot{\sigma} \\ p_\tau &= m\left(\sigma^2 + \tau^2\right)\dot{\tau}. \end{aligned}$$

The Lagrangian translated into momenta is

$$\begin{aligned} L_p &= \frac{1}{2} m \left(\sigma^2 + \tau^2\right)\left[\left(\frac{p_\sigma^2}{m(\sigma^2 + \tau^2)}\right)^2 + \left(\frac{p_\tau^2}{m(\sigma^2 + \tau^2)}\right)^2\right] \\ &= \frac{1}{2m(\sigma^2 + \tau^2)}\left(p_\sigma^2 + p_\tau^2\right). \end{aligned}$$

Meanwhile,

$$\dot{q}^T p = \frac{p_\sigma}{m(\sigma^2 + \tau^2)} p_\sigma + \frac{p_\tau}{m(\sigma^2 + \tau^2)} p_\tau.$$

The difference is the Hamiltonian,

$$H = \frac{1}{2m} \frac{p_\sigma^2 + p_\tau^2}{\sigma^2 + \tau^2}.$$

Exercise 6.8

Let's focus on the x axis, since that's where the action is. There are three energies to consider: the kinetic energy

$$T = \frac{1}{2}m\dot{x}^2,$$

the energy of the stretched spring,

$$V_s = \frac{1}{2}kx^2,$$

and the energy of the charge in the electric field,

$$V_e = -\mathbf{x} \cdot \mathbf{F} = e\mathcal{E}_0 x \cos \Omega t.$$

Together these make the Lagrangian,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - e\mathcal{E}_0 x \cos \Omega t.$$

There's nothing especially tricky in going from velocity to momentum $p = m\dot{x}$ here. The Hamiltonian is

$$\begin{aligned} H &= \dot{x}p - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + V_s + V_e \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2 + e\mathcal{E}_0 x \cos \Omega t, \end{aligned}$$

which you better believe is the total energy. It is not, however, a conserved quantity, for

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -e\mathcal{E}_0 \Omega x \sin \Omega t,$$

which varies with time. The field adds energy to the electron when the electron is already moving in the direction of acceleration, and loses it when the reverse is true.

Exercise 6.9

I'll do you one better. We've already seen that the general Lagrangian of a charged particle in a magnetic field is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}.$$

Thus, for example, the x component of the generalized momentum is

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{q}{c}A_x,$$

and similarly for y and z. The rest is starting to look pretty familiar. We have

$$\begin{aligned}
\dot{\mathbf{r}} \cdot \mathbf{p} &= \frac{p^2}{m} - \frac{q}{mc}\mathbf{p} \cdot \mathbf{A} \\
\frac{1}{2}m\dot{\mathbf{r}}^2 &= \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 \\
&= \frac{p^2}{2m} - \frac{q}{mc}\mathbf{p} \cdot \mathbf{A} + \frac{q^2}{2mc^2}A^2 \\
\frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A} &= \frac{q}{mc}\mathbf{p} \cdot \mathbf{A} - \frac{q^2}{mc^2}A^2.
\end{aligned}$$

From this we construct the Hamiltonian in the usual way

$$\begin{aligned}
H &= \frac{p^2}{2m} - \frac{q}{mc}\mathbf{p} \cdot \mathbf{A} + \frac{q^2}{2mc^2}A^2 \\
&= \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2.
\end{aligned}$$

You recognize this as just $(1/2)m\dot{\mathbf{r}}^2$, but the rules say, we have to write it in terms of momentum.

In the case at hand, with $\mathbf{B} = B\hat{z}$, and the usual choice of $\mathbf{A} = (B/2)(-y\hat{x} + x\hat{y})$, we get

$$\frac{q}{c}\mathbf{A} = \frac{qB}{2c}(-y\hat{x} + x\hat{y}) = \frac{m\omega_c}{2}(-y\hat{x} + x\hat{y}),$$

in terms of the cyclotron frequency ω_c. **Note:** On p. 104 of the text, the cyclotron frequency is wrong. The correct definition is $\omega_c = qB/mc$. This specific Hamiltonian is then

$$H = \frac{1}{2m}\left(p_x + \frac{m\omega_c}{2}y\right)^2 + \frac{1}{2m}\left(p_y - \frac{m\omega_c}{2}x\right)^2.$$

To find the circles, it is of course useful to go into polar coordinates. The Lagrangian is given in Chapter 5,

$$\begin{aligned}
L &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{qB}{2c}r2\dot{\phi} \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}m\omega_c r^2\dot{\phi},
\end{aligned}$$

given in terms of the cyclotron frequency $\omega_c = qB/mc$. The conjugate momenta are

$$\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + \frac{1}{2}m\omega_c r^2.
\end{aligned}$$

Then after the usual business,

$$\dot{r}p_r + \dot{\phi}p_\phi = \frac{p_r^2}{m} + \frac{p_\phi^2}{mr^2} - \frac{1}{2}\omega_c p_\phi$$
$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{1}{2}\omega_c p_\phi + \frac{1}{8}m\omega_c^2 r^2$$
$$\frac{1}{2}m\omega_c r^2\dot{\phi} = \frac{1}{2}\omega_c p_\phi - \frac{1}{4}m\omega_c^2 r^2,$$

from which fragments the Hamiltonian is constructed,

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{1}{2}\omega_c p_\phi + \frac{1}{8}m\omega_c^2 r^2.$$

We move on to Hamilton's equations. An easy one is

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0,$$

meaning that the momentum p_ϕ is conserved; no surprises there. What is its constant value? Well, for a circular orbit we require both r and $\dot{r}$ to be constant:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} = 0,$$
$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{1}{4}m\omega_c^2 r = 0.$$

This last equation identifies the constant angular momentum as

$$p_\phi = \pm\frac{1}{2}m\omega_c r^2.$$

There are two choices here, and we are looking for the negative one. This is because the equation of motion for ϕ gives

$$\begin{aligned}\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} - \frac{1}{2}\omega_c \\ &= \frac{1}{mr^2}\left(-\frac{1}{2}m\omega_c r^2\right) - \frac{1}{2}\omega_c \\ &= -\omega_c.\end{aligned}$$

So ϕ runs around the circle with angular frequency $-\omega_c$. We could have alternatively taken $p_\phi = +m\omega_c r^2/2$, which would give $\dot{\phi} = 0$. So, yes, a charged particle could be motionless in a magnetic field, but this is less interesting.

7
Solutions to Exercises Chapter 7

Exercise 7.1

(a) First let's recall what the equations of motion are. If

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

then

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\ \dot{p} &= -\frac{\partial H}{\partial x} = -m\omega^2 x. \end{aligned}$$

By directly substituting everything in sight, these equations can be recast as equations for the new variables:

$$\begin{aligned} \dot{x}' &= \omega p_x' \\ \dot{p}_x' &= -\omega x'. \end{aligned}$$

This looks nice, and brings a kind of symmetry to the proceedings. But look: If we had made this substitution directly into the Hamiltonian, we'd get a new function

$$H' = \frac{1}{2}p_x'^2 + \frac{1}{2}x'^2.$$

Do Hamilton's equations, applied to this function, get the right equations of motion for x', p_x'? Not likely! There is no ω even. The equations, if we interpret H' as a Hamiltonian, are

$$\begin{aligned} \dot{x}' &= \frac{\partial H'}{\partial p_x'} = p_x' \\ \dot{p}_x' &= -\frac{\partial H'}{\partial x'} = -x'. \end{aligned}$$

(b) We still obviously have $\dot{\bar{p}} = -\partial\bar{H}/\partial\bar{q} = 0$, so that's not the problem. If all the Hamiltonian business works out, we'd expect $\dot{\bar{q}} = \partial\bar{H}/\partial\bar{p} = \omega$. Is it? Let's find out:

$$\begin{aligned}\dot{\bar{q}} &= \frac{1}{1+(p/q)^2}\left[\frac{1}{q}\dot{p} - \frac{p}{q^2}\dot{q}\right]\\ &= \frac{1}{1+(p/q)^2}\left[\frac{1}{q}(-\omega q) - \frac{p}{q^q}\omega p\right]\\ &= -\omega.\end{aligned}$$

So, no, in this formulation this equation has the opposite sign.

Exercise 7.2

(a) Are you kidding me? If

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2,$$

then

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

and Hamilton's equations of motion are

$$\begin{aligned}\dot{x} &= \frac{p}{m}\\ \dot{p} &= -m\omega^2 x,\end{aligned}$$

with solutions

$$\begin{aligned}x(t) &= x_0 \cos\omega t\\ p(t) &= -m\omega x_0 \sin\omega t.\end{aligned}$$

(b) Suddenly it's not so funny. The Lagrangian

$$\bar{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + m\omega x\dot{x}$$

gives us a different conjugate momentum

$$\bar{p} = \frac{\partial\bar{L}}{\partial\dot{x}} = m\dot{x} + m\omega x$$

and a new Hamiltonian

$$\bar{H} = \dot{x}\bar{p} - \bar{L}_p = \frac{\bar{p}^2}{2m} - \omega x\bar{p} + m\omega^2 x^2.$$

This looks like (and is) a confusing mess, but it should still get the job done. Hamilton's equations are now

$$\dot{x} = \frac{\partial \bar{H}}{\partial \bar{p}} = \frac{\bar{p}}{m} - \omega x$$
$$\dot{\bar{p}} = -\frac{\partial \bar{H}}{\partial x} = \omega \bar{p} - 2\omega^2 x.$$

Well, and why not? There are different equations for different mathematical quantities.

We can solve these equations in the usual, anticlimactic way, by taking another time derivative. Using Hamilton's equations, we get

$$\ddot{x} = \frac{\dot{\bar{p}}}{m} - \omega \dot{x}$$
$$= -\omega^2 x,$$

This is the usual equation of motion for x, with solution (for example)

$$X(t) = x_0 \cos \omega t.$$

The corresponding momentum is

$$\bar{p} = m\dot{x} + m\omega x = m\omega x_0 \left(\cos \omega t - \sin \omega t \right).$$

(c) A phase space orbit for this Hamiltonian is shown in Figure 7.1. In this example, $m = 1$, $\omega = 2\pi/1\mathrm{sec}$, and $x_0 = 0.1\mathrm{m}$. This orbit is still elliptical, but is tilted at some weird angle.

Exercise 7.3

You have to choose the signs knowing where you're going with this. First, recall the basic dependence of the type-1 generating function $\Lambda_1(q, \bar{q})$:

$$d\Lambda_1 = pdq - \bar{p}d\bar{q} + (\bar{H} - H).$$

To get $\Lambda_3(p, \bar{q})$, we need to subtract away the part that goes as pdq. So define

$$\Lambda_3 = \Lambda_1 - pq.$$

Then

$$d\Lambda_3 = pdq - \bar{p}d\bar{q} + (\bar{H} - H) - pdq - qdp$$
$$= -\bar{p}d\bar{q} - qdp + (\bar{H} - H)dt$$
$$= \frac{\partial \Lambda_3}{\partial \bar{q}} d\bar{q} + \frac{\partial \Lambda_3}{\partial p} dp + \frac{\partial \Lambda_3}{\partial t} dt.$$

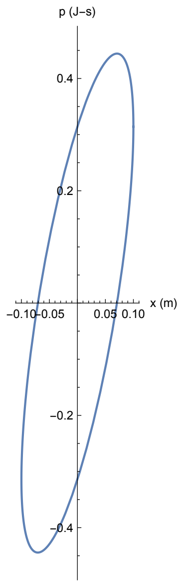

Figure 7.1 Typical phase space trajectory for the weird Hamiltonian $\bar{H}$ discussed in the text.

This function of $(\bar{q}, p)$ therefore generates the other quantities via

$$q = -\frac{\partial \Lambda_3}{\partial p}$$
$$\bar{p} = -\frac{\partial \Lambda_3}{\partial \bar{q}}$$
$$\bar{H} - H = \frac{\partial \Lambda_3}{\partial t}.$$

What's left? Λ_1 depends on coordinates q and $\bar{q}$. Λ_2 and Λ_3 swapped out one or the other of these for the conjugate momentum. Looks like Λ_4 will be a function of both momenta. We can start with Λ_1, then take the coordinates away via

$$\Lambda_4 = \Lambda_1 - pq + \bar{p}\bar{q},$$

so

$$\begin{aligned} d\Lambda_4 &= pdq - \bar{p}d\bar{q} + (\bar{H} - H)dt - pdq - qdp + \bar{p}d\bar{q} + \bar{q}d\bar{p} \\ &= -qdp + \bar{q}d\bar{p} + (\bar{H} - H)dt \\ &= \frac{\partial \Lambda_4}{\partial p}dp + \frac{\partial \Lambda_4}{\partial \bar{p}}d\bar{p} + \frac{\partial \Lambda_4}{\partial t}dt, \end{aligned}$$

giving the transformations

$$q = -\frac{\partial \Lambda_4}{\partial p}$$
$$\bar{q} = \frac{\partial \Lambda_4}{\partial \bar{p}}$$
$$\bar{H} - H = \frac{\partial \Lambda_4}{\partial t}.$$

Exercise 7.4

As established above, the Hamiltonian for this charged particle is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 .$$

Now, the Hamilton-Jacobi equation requires replacing the components of momenta with the relevant gradients with respect to coordinates. In polar coordinates, this means

$$\mathbf{p} \to \frac{\partial W}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial W}{\partial \phi}\hat{\phi}.$$

So if the vector potential is written $\mathbf{A} = (1/2)Br\phi$, the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial W}{\partial \phi} - \frac{qB}{2c}r\right)^2\right] = E.$$

Try separating this using $W = W_r(r) + W_\phi(\phi)$. You get

$$\frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2 + \left(\frac{1}{r}\frac{dW_\phi}{d\phi} - \frac{qB}{2c}r\right)^2\right] = E.$$

This can be re-arranged to isolate the ϕ part:

$$\frac{dW_\phi}{d\phi} = \frac{qB}{2c}r^2 + r\sqrt{2mE - \left(\frac{dW_r}{dr}\right)^2}$$

The left side depends on on ϕ, the right side only on r. So they must be equal to the same constant, α_ϕ. Using this constant we can solve for dW_r/dr:

$$\frac{dW_r}{dr} = \sqrt{2mE - \left(\frac{\alpha_\phi}{r} - \frac{qB}{2c}r\right)^2}$$

The generating function is therefore

$$W = \int dr\sqrt{2mE - \left(\frac{\alpha_\phi}{r} - \frac{qB}{2c}r\right)^2} + \alpha_\phi\phi.$$

Using this generating function, the conjugate momenta are

$$p_\phi = \frac{\partial W}{\partial \phi} = \alpha_\phi,$$
$$p_r = \frac{\partial W}{\partial r} = \sqrt{2mE - \left(\frac{\alpha_\phi}{r} - \frac{qB}{2c}r\right)^2}.$$

Thus α_ϕ is just the conserved angular momentum as you might have guessed. This momentum is defined just as if the particle were moving in an effective radial potential

$$V(r) = \frac{1}{2m}\left(\frac{\alpha_\phi}{r} - \frac{qB}{2c}r\right)^2.$$

Exercise 7.5

The free particle Hamiltonian is

$$H = \frac{p^2}{2m}.$$

The corresponding Hamilton-Jacobi equation for S is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t} = 0.$$

This turns out to be separable as a product, $S(x,t) = X(x)T(t)$. Try it. You get

$$\frac{1}{2m}\left(\frac{\partial X}{\partial x}\right)^2 T^2 + \frac{\partial T}{\partial t}X = 0.$$

Dividing by XT^2, you get two terms which are independent,

$$\frac{1}{2mX}\left(\frac{\partial X}{\partial x}\right)^2 + \frac{1}{T^2}\frac{\partial T}{\partial t} = 0.$$

Set the first term of this equal to a constant α and the second to $-\alpha$. Then

$$X^{-1/2}\frac{dX}{dx} = \sqrt{2m\alpha},$$
$$2X^{1/2} = \sqrt{2m\alpha}(x - x_0),$$
$$X = \frac{m\alpha}{2}(x - x_0)^2$$

and

$$T^{-2}\frac{dT}{dt} = -\alpha$$
$$-T^{-1} = -\alpha(t - t_0)$$

$$T = \frac{1}{\alpha(t - t_0)}.$$

Then the generating function is

$$S = XT = \frac{m(x - x_0)^2}{2(t - t_0)}.$$

Exercise 7.6

This is tricky. I will try to motivate how the search for a generating function S goes, but frankly (spoiler alert) I know how it comes out, too. We are set on solving the partial differential Hamilton-Jacobi (H-J) equation

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - mAxt + \frac{\partial S}{\partial t} = 0.$$

We would love to do this by separation of variables, which was a basic ingredient in all of our differential equations classes. A provisional S might be

$$S^{\text{prov}}(x, t) = f(x) + g(t),$$

or maybe a product $f(x)g(t)$. But you can quickly substitute these in and see that they are no help, largely because of that irritating xt term, and those darn kids. But there is a gimmick to get this term out of the way. Suppose you add in another thing, $h(x, t)$, chosen so that its time derivative cancels the $-mAxt$. That is you would want

$$\frac{\partial h}{\partial t} = mAxt, \quad \text{or}$$
$$h = \frac{1}{2}mAxt^2.$$

Then your provisional S becomes

$$S^{\text{prov}}(x, t) = f(x) + g(t) + \frac{1}{2}mAxt^2.$$

Substituting this into the H-J equation you get

$$\frac{1}{2m}\left(\frac{df}{dx} + \frac{1}{2}mAt^2\right) - mAxt + \left(\frac{dg}{dt} + mAxt\right) = 0,$$
$$\frac{1}{2m}\left(\frac{df}{dx} + \frac{1}{2}mAt^2\right) + \frac{dg}{dt} = 0.$$

Now look at this. The *only* x dependence here is in df/dx, and we put that in there ourselves. There's no point in making extra complications for ourselves, so let's let $f = 0$. Then your provisional S function is

$$S^{\text{prov}} = g(t) + \frac{1}{2}mAxt^2.$$

But wait! There is one thing more. By the rules of Hamilton-Jacobi theory, the momentum is given by

$$p = \frac{\partial S^{\text{prov}}}{\partial x} = \frac{1}{2}mAt^2.$$

This would require the momentum to be zero at time $t = 0$. We can make this more general by adding in a term $p_0 x$, where p_0 is the initial momentum. Thus our final, working generating function is

$$S(x,t) = g(t) + \frac{1}{2}mAxt^2 + p_0 x.$$

Is there anything else, Lieutenant Columbo? I hope not.

Let's take this generating function out for a spin. Substituting into the H-J equation we get

$$\frac{1}{2m}\left(\frac{1}{2}mAt^2 + p_0\right)^2 + \frac{dg}{dt} = 0.$$

This is an ordinary differential equation, with solution

$$g(t) = -\frac{p_0^2}{2m}t - \frac{p_0 A}{6}t^3 - \frac{mA^2}{40}t^5.$$

So there it is, but $g(t)$ is not *that* critical to solving the problem. The main thing is, the momentum is now given for free as a function of time,

$$p = \frac{\partial S}{\partial x} = \frac{1}{2}mAt^2 + p_0.$$

Finally, we can refer to Hamilton's equation for x,

$$\begin{aligned}\dot{x} = \frac{\partial H}{\partial p} &= \frac{p}{m}\\ &= \frac{1}{2}At^2 + \frac{p_0}{m}.\end{aligned}$$

This problem is now reduced to quadratures - our favorite thing. The solution for x is

$$x(t) = \frac{1}{6}At^3 + \frac{p_0}{m}t + x_0.$$

At this point I can't help but laugh, because what is that thing? You know, Hamilton's equation for the momentum is

$$\dot{p} = -\frac{\partial H}{\partial x} = mAt,$$

whose solution is just $mAt^2/2 + p_0$, as we got the long way above.

8

Solutions to Exercises Chapter 8

Exercise 8.1

The action is the integral of momentum over a full period of the motion. This is twice the integral from the left turning point $-x_t$ to the right turning point x_t, or four times the integral from zero to x_t:

$$J = 4\int_0^{x_t} dx\,\sqrt{2m(E - V_0\tan^2(x/a))},$$

where the turning point is given by $x_t = a\tan^{-1}\sqrt{E/V_0}$. Asserting that this integral can be done is different from saying that it's easy. This trick I learned from the solutions manual to Calkin's great book. First write

$$\begin{aligned}\sqrt{E - V_0\tan^2(x/a)} &= \frac{E - V_0\tan^2(x/a)}{\sqrt{E - V_0\tan^2(x/a)}}\\ &= \frac{E\cos^2(x/a) - V_0(1-\cos^2(x/a))}{\cos^2(x/2)\sqrt{E - V_0\tan^2(x/a)}}\\ &= \frac{(E+V_0)\cos(x/a)}{\cos(x/a)\sqrt{E - V_0\tan^2(x/a)}} - \frac{V_0}{\cos^2(x/a)\sqrt{E - V_0\tan^2(x/a)}}.\end{aligned}$$

In the first term here, write

$$\begin{aligned}\cos(x/a)\sqrt{E - V_0\tan^2(x/a)} &= \sqrt{E(1-\sin^2(x/a)) - V_0\sin^2(x/a)}\\ &= \sqrt{E - (E+V_0)\sin^2(x/a)}.\end{aligned}$$

The action integral takes the much more manageable form

$$J = 4\sqrt{2m}\left[\sqrt{E+V_0}\int_0^{x_t} dx\frac{\cos(x/a)}{\sqrt{E/(E+V_0) - \sin^2(x/a)}} - \sqrt{V_0}\int_0^{x_t} dx\frac{1/\cos^2(x/a)}{\sqrt{E/V_0 - \tan^2(x/a)}}.\right]$$

Using $\sin\theta = \sqrt{E/(E+V_0)}\sin(x/a)$, the first integral becomes

$$\int_0^{\theta_t} \frac{a\sqrt{E/(E+V_0)}\cos\theta d\theta}{\sqrt{E/(E+V_0)(1-\sin^2\theta)}} = a\int_0^{\pi/2} d\theta = \frac{\pi a}{2},$$

and using $\tan(x/a) = \sqrt{E/V_0}\sin\theta$ the second integral becomes

$$\int_0^{\theta_t} \frac{a\sqrt{E/V_0}\cos\theta d\theta}{\sqrt{(E/V_0)(1-\sin^2\theta)}} = a\int_0^{\pi/2} d\theta = \frac{\pi a}{2}.$$

Thus the action is

$$J = 4\sqrt{2m}\left(\frac{\pi a}{2}\right)\left[\sqrt{E+V_0} - \sqrt{V_0}\right].$$

The energy is given in terms of the action as

$$E = \left(\frac{J}{2\pi\sqrt{2m}a} + \sqrt{V_0}\right)^2 - V_0,$$

which is refreshingly *not* linear in the energy, as it would be for the harmonic oscillator.

The frequency of this motion is

$$\nu = \frac{dE}{dJ} = \frac{J}{4\pi^2 ma^2} + \frac{\sqrt{V_0}}{\pi\sqrt{2m}a}.$$

Now, look at this. At low enough energy, everything is supposed to be a harmonic oscillator. Well, if $E \ll V_0$, we have $\sqrt{E+V_0} - \sqrt{V_0} \approx (E/2)\sqrt{V_0}$ and the relation between energy and action is

$$E \approx \sqrt{\frac{V_0}{2m}}\frac{1}{\pi a}J,$$

with angular frequency

$$\omega = 2\pi\frac{dE}{dJ} \approx 2\sqrt{\frac{V_0}{2m}}\frac{1}{a}.$$

For small excursion a away from zero, the potential is cast as a harmonic oscillator with effective angular frequency ω_{eff}, where

$$V \approx V_0\left(\frac{x}{a}\right)^2 = \frac{1}{2}m\omega_{eff}^2 x^2,$$

which also gives $\omega_{eff} = 2\sqrt{V_0/2m}/a$.

Exercise 8.2

Don't get carried away here. We still want $d\omega/dt$ to be small, so we can make Taylor series expansions where necessary. Also, any given period is still pretty much oscillating at frequency ω_0, so we will feel free to use this when averaging over a period. Thus if $\omega = \omega_0 + \omega' t$, where $\omega' = d\omega/dt$, we have

$$\sin(\omega_0 t + \omega' t^2) \approx \sin\omega_0 t + \omega' t^2 \cos\omega_0 t$$
$$\cos(\omega_0 t + \omega' t^2) \approx \cos\omega_0 t - \omega' t^2 \sin\omega_0 t$$

Here $\omega' t^2$ is regarded as small, for the period of frequency ω_0 that starts at time $t = 0$.

As described in the chapter, the angle is given by

$$\begin{aligned}\phi &= \sqrt{\frac{2E}{mgl}}\sin(\omega_0 t + \omega' t^2)\\ &\approx \sqrt{\frac{2E}{mgl}}\left[\sin\omega_0 t + \omega' t^2 \cos\omega_0 t\right],\end{aligned}$$

where we ignore the phase δ that is irrelevant here. The velocity of this angle is, expanded to order ω',

$$\begin{aligned}\dot\phi &= \sqrt{\frac{2E}{mgl}}(\omega_0 + 2\omega' t)\cos(\omega_t + \omega' t)\\ &\approx \sqrt{\frac{2E}{mgl}}(\omega_0 + 2\omega' t)\left[\cos\omega_0 t - \omega' t^2 \sin\omega_0 t\right]\\ &\approx \sqrt{\frac{2E}{mgl}}\left[\omega_0\cos\omega_0 t + 2\omega' t\cos\omega_0 t - \omega_0\omega' t^2 \sin\omega_0 t\right]\end{aligned}$$

In constructing the rate of change of energy with length,

$$\frac{dE}{dl} = \frac{1}{2}mg\phi^2 - ml\dot\phi^2,$$

we will expand out to terms linear in ω'. So we get

$$\begin{aligned}\frac{1}{2}mg\phi^2 &\approx \frac{1}{2}mg\frac{2E}{mgl}\left(\sin\omega_0 t + \omega' t^2 \cos\omega_0 t\right)^2\\ &\approx \frac{E}{l}\left(\sin^2\omega_0 t + 2\omega' t^2 \sin\omega_0 t\cos\omega_0 t\right).\end{aligned}$$

$$\begin{aligned}-ml\dot\phi^2 &\approx -ml\frac{2E}{mgl}\left(\omega_0\cos\omega_0 t + 2\omega' t\cos\omega_0 t - \omega_0\omega' t^2 \sin\omega_0 t\right)^2\\ &\approx -\frac{2E}{l}\left(\cos^2\omega_0 t + 4\frac{\omega'}{\omega_0}t\cos^2\omega_0 t - 2\omega' t^2 \sin\omega_o t\cos\omega_0 t\right).\end{aligned}$$

This has used $\omega_0^2 = g/l$.

Now we average everything over one period $T = 2\pi/\omega_0$. The averages of $\sin^2$ and $\cos^2$ are equal to $1/2$, as always. We also use the averages

$$\frac{1}{T}\int_0^T dt\, t^2 \sin\omega_0 t \cos\omega_0 t = -\frac{\pi}{2\omega_0}$$
$$\frac{1}{T}\int_0^T dt\, t \cos^2\omega_0 t = \frac{\pi}{2\omega_0}. \tag{8.1}$$

The result is

$$\begin{aligned}\frac{dE}{dl} &= -\frac{1}{2}\frac{E}{l} - \frac{7\pi E}{l}\frac{\omega'}{\omega_0^2} \\ &= -\frac{1}{2}\frac{E}{l}\left[1 + 14\pi\frac{1}{\omega_0^2}\frac{d\omega}{dt}\right].\end{aligned}$$

The second term in brackets is the correction for finite (and constant) rate of change $d\omega/dt$ of the frequency. If this is supposed to be less than some tolerance δ (like 1 percent), then the fractional change of the frequency during a single period T should be

$$T\frac{1}{\omega_0}\frac{d\omega}{dt} = \frac{2\pi}{\omega_0}\frac{1}{\omega_0}\frac{d\omega}{dt} \leq 7\delta.$$

Exercise 8.3

Here follows the Mathematica code that produced the figure. See the solutions to Chapter 2 for more information on how to set up these differential equations.

The setup:

```
g=9.8;
l=1;
phi0=0.01;
phip0=0;
omega=Sqrt[g/l];
T=2*Pi/omega:
Numper=30;
```

Define a length that varies linearly in time, reducing to ϵ of its original length in a time $Nper \times T$. (Note the underscore after the t, this is part of Mathematica syntax.)

```
epsilon=0.6;
```

```
lp=(epsilon-1)/(Numper*T);
length[t_]:=l*(1-t*(1-epsilon)/(Numper*T));
```

Next set up the differential equation

```
polar=NDSolve[{phi''[t]+2*lp*phi'[t]/length[t]==-(g/length[t]*phi[t],phi[0]==phi0,
phi'[0]==phip0},phi,{t,0,Numper*T}];
```

Then plot it. The command

```
Plot[phi[t]/.ploar,{t,0,Numper*T}]
```

gives the figure in the main text. Of more interest is to plot the total energy,

```
Plot[(1/2)*length[t]^2*phi[t]^2+(1/2)*g*length[t]*phi[t]^2/.polar,{t,0,Numper*T},
AxesLabel->{''t (s)'',''E (J)''}]
```

This gives the figure below. There is an overall rise in the total energy, plus a lot of small wiggles, which are the things we average over.

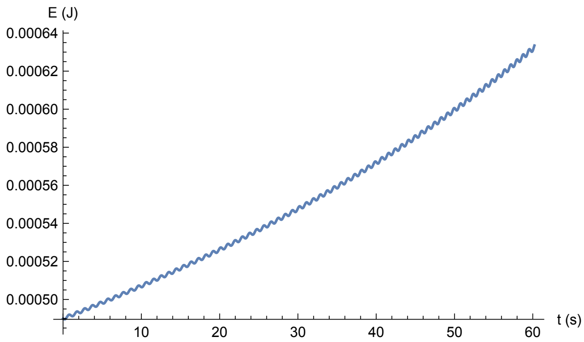

Figure 8.1 Up up and away! The energy of the pendulum as it is gradually shortened.

Exercise 8.4

(Note the following is stolen almost entirely from Goldstein, who then goes on to make a full discussion of perturbation theory.)

Starting from the Hamiltonian

$$\bar{H} = \frac{\omega}{2\pi}J + \frac{\dot{\omega}}{\omega}\frac{J}{4\pi}\sin 4\pi\beta,$$

Hamilton's equations in the action-angle coordinates are

$$\dot{J} = -\frac{\partial \bar{H}}{\partial \beta} = -\frac{\dot{\omega}}{\omega}J\cos 4\pi\beta$$

$$\dot{\beta} = \frac{\partial \bar{H}}{\partial J} = \frac{\omega}{2\pi} + \frac{\dot{\omega}}{4\pi\omega}\sin 4\pi\beta.$$

We will consider the small quantity $\epsilon \equiv \dot{\omega}/\omega$ to be both small and approximately constant in time over a period. We write the frequency therefore as $\omega = \omega_0 + \epsilon t$. When $\epsilon = 0$, the zero-order solution is $\beta_0 = \omega_0/2\pi$. Substituting this on the right-hand side of the equation for β, we have a next order equation

$$\dot{\beta}_1 = \frac{\omega_0}{2\pi} + \frac{\omega_0}{2\pi}\epsilon t + \frac{\epsilon}{4\pi}\sin 2\omega_0 t.$$

This you can just integrate:

$$\beta_1(t) = \frac{\omega_0}{2\pi}t + \frac{\omega_0\epsilon}{4\pi}t^2 + \frac{\epsilon}{4\pi}\frac{1}{2\omega_0}(1 - \cos 2\omega_0 t),$$

where the 1 in front of the cosine term is a constant of integration that ensures $\beta_1(0) = 0$.

The equation of motion for the action is then

$$\begin{aligned}\dot{J} &= -\epsilon J \cos 4\pi\beta_1\\ &= -\epsilon J \cos\left[2\omega_0 t + \omega_0\epsilon t^2 + \frac{\epsilon}{2\omega_0}(1 - \cos 2\omega_0 t)\right].\end{aligned}$$

Treating the ϵ parts as some small quantity δ, we expand the cosine as

$$\cos(2\omega_0 t + \delta) = \cos 2\omega_0 t \cos\delta - \sin 2\omega_0 t \sin\delta \approx \cos 2\omega_0 t - \delta \sin 2\omega_0 t$$

to get

$$\frac{\dot{J}}{J} = -\epsilon\cos 2\omega_0 t + \omega_0\epsilon^2 t^2 \sin 2\omega_0 t + \frac{\epsilon^2}{2\omega_0}\sin 2\omega_0 t(1 - \cos 2\omega_0 t).$$

Now it's safe to average this over a period $T = 2\pi/\omega_0$. The sines and cosines average to zero, but the remaining average is

$$\frac{1}{T}\int_0^T dt\, t^2 \sin 2\omega_0 t = -\frac{T^2}{4\pi}. \tag{8.2}$$

Then, if we take $\dot{J}/J$ approximately constant during this period, we can get the fractional change in action over a period of the motion:

$$\begin{aligned}\frac{\Delta J}{J} &\approx T\left(\omega_0\epsilon^2\left(-\frac{T^2}{4\pi}\right)\right)\\ &= -\frac{(\epsilon\tau)^2}{2}.\end{aligned}$$

Therefore the action does change, but only to second order in the rate of change of the pendulum's frequency.

Exercise 8.5

In case you do not have access to the paper of Tufillaro *et al*, here is what they do. If the length r changes, then the kinetic energy of m changes in the usual way, but also mass M goes up or down with velocity $\dot{r}$. Thus the kinetic energy is

$$T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right).$$

Likewise the potential energy is

$$V = gr(M - m\cos\phi),$$

since as one mass goes up, the other goes down. So the Lagrangian is

$$L = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + gr(m\cos\phi - M).$$

This gives the conjugate momenta

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi} \\ p_r &= \frac{\partial L}{\partial\dot{r}} = (M+m)\dot{r}. \end{aligned}$$

Lagrange's equations read

$$\begin{aligned} &\frac{d}{dt}(mr^2\dot{\phi}) = -mgr\sin\phi \\ &(M+m)\ddot{r} = mr\dot{\phi}^2 + g(m\cos\theta - M). \end{aligned}$$

The first of these is written to emphasize the usual role of angular momentum of mass m, governed by the torque applied by gravity.

The transition to the Hamiltonian is straightforward here, since the coordinate system is not explicitly moving. We have

$$H = T + V = \frac{p_r^2}{M+m} + \frac{p_\phi^2}{2mr^2} + gr(M - m\cos\phi).$$

The real fun comes when Tufillaro describes another coordinate system to make progress:

$$\begin{aligned} r &= \frac{1}{2}\left(\xi^2 + \eta^2\right) \\ \phi &= 2\tan^{-1}\left[(\xi^2 - \eta^2)/2\xi\eta\right], \end{aligned}$$

but we do not take this up here.

9
Solutions to Exercises Chapter 9

Exercise 9.1

The trajectory equation is

$$\frac{dz}{dx} = \sqrt{\frac{E_z - mgz}{E_x}},$$

or

$$(E_z - mgz)^{-1/2}dz = E_x^{-1/2}dx.$$

Integrating both sides,

$$-\frac{2}{mg}(E_z - mgz)^{1/2} = \frac{x}{\sqrt{E_x}} + C.$$

For a trajectory that starts at $x = 0$, $z = 0$, the constant of integration must be $C = -2\sqrt{E_z}/mg$. The solution is then

$$\sqrt{E_z - mgz} = -\frac{mg}{2\sqrt{E_x}}x + \sqrt{E_z}.$$

Solving for z, we get

$$z = -\frac{mg}{4E_x}x^2 + \sqrt{\frac{E_z}{E_x}}x :$$

a parabola!

Exercise 9.2

The orbit equation is

$$\frac{d\phi}{dr} = \frac{1}{r^2}\frac{\sqrt{C}}{\sqrt{-A + 2B/r - C/r^2}},$$

where as a shorthand I have written

$$\begin{aligned} A &= -2mE \\ B &= GMm^2 \\ C &= L_z^2. \end{aligned}$$

The variables separate in the first-order differential equation sense, whereby a solution is

$$\phi - \phi_0 = \int dr \frac{1}{r^2} \frac{\sqrt{C}}{\sqrt{-A + 2B/r - C/r^2}}$$

This is clearly too many $1/r$'s, so we substitute $u = 1/r$ to get

$$\phi - \phi_0 = -\int du \frac{\sqrt{C}}{\sqrt{-A + 2Bu - Cu^2}}.$$

Next you can complete the square in the denominator,

$$-A + 2Bu - Cu^2 = -C\left[\left(u - \frac{B}{C}\right)^2 - \left(\frac{B}{C}\right)^2 + \frac{A}{C}\right].$$

The integral becomes

$$\phi - \phi_0 = -\int du \frac{1}{\sqrt{[B^2/C^2 - A/C] - (u - B/C)^2}}.$$

This integral you can do by introducing a new angle α via

$$u - \frac{B}{C} = \sqrt{\frac{B^2}{C^2} - \frac{A}{C}} \cos\alpha.$$

This angle is, amazingly, directly related to the rotation angle ϕ. After substituting,

$$\phi - \phi_0 = \int \frac{\sin\alpha d\alpha}{\sqrt{1 - \cos^2\alpha}} = \alpha.$$

Substituing this in, we have

$$\frac{1}{r} - \frac{B}{C} = \sqrt{\frac{B^2}{C^2} - \frac{A}{C}} \cos(\phi - \phi_0).$$

And so the radius is

$$\begin{aligned} r &= \frac{C}{B + \sqrt{B^2 - AC}\cos(\phi - \phi_0)} \\ &= \frac{L_z^2}{GMm^2 + \sqrt{G^2M^2m^4 + 2mL_z^2E}\cos(\phi - \phi_0)}. \end{aligned}$$

Note this has a + sign rather than the − sign given in the text. This is an ambiguity in the initial angle, ϕ_0 versus $\phi_0 - \pi$. For concreteness, let's have r take its maximum value when $\phi = 0$, which would give us

$$r = \frac{L_z^2}{GMm^2 - \sqrt{G^2M^2m^4 + 2mL_z^2E}\cos\phi}.$$

Let's see if we can identify this as the equation of an ellipse. What do we know about ellipses? Well, given two points in a plane, $\mathbf{r}_1$ and $\mathbf{r}_2$, you look for all the points $\mathbf{r}$ for which the sum of the distance from $\mathbf{r}$ to these two fixed points is a constant:

$$|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2| = d.$$

For our particular ellipse, let's take the two points to be $(0, 0)$ and $(0, f)$ (these are the *foci* of the ellipse). A point (x, y) is on the ellipse if

$$x^2 + y^2 + (x - f)^2 + y^2 = d^2,$$

or

$$\begin{aligned}(x^2 + y^2) - fx &= \frac{1}{2}(d^2 - f^2)\\ r - fr\cos\phi &= \frac{1}{2}(d^2 - f^2),\end{aligned}$$

Thus

$$r = \frac{(d^2 - f^2)/2}{1 - f\cos\phi};$$

apart from redefining the constants, this is clearly the form of r versus ϕ for the comet's orbit.

Exercise 9.3

In matrices, the relation between the old coordinates and the new is

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

A 2×2 matrix is easily inverted, so by matrices we also have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (\delta\eta_1 - \beta\eta_2)/\Delta \\ (-\gamma\eta_1 + \alpha\eta_2)/\Delta \end{pmatrix},$$

where $\Delta = \alpha\delta - \beta\gamma$.

The kinetic energy in the new coordinates is then

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \\ &= \frac{m}{2\Delta^2}\left[(\delta^2+\gamma^2)\dot{\eta}_1^2 + (\beta^2+\alpha^2)\dot{\eta}_2^2 - 2(\alpha\gamma+\beta\delta)\dot{\eta}_1\dot{\eta}_2\right]. \end{aligned}$$

To have "clean" coordinates that avoid cross terms, we should set $(\alpha\gamma+\beta\delta) = 0$, or

$$\alpha\gamma = -\beta\delta.$$

The potential energy in these coordinates is

$$\begin{aligned} V &= \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k'(x_1-x_2)^2 \\ &= \frac{k}{2\Delta^2}\left[(\delta^2+\gamma^2)\eta_1^2 + (\beta^2+\alpha^2)\eta_2^2 - 2(\alpha\gamma+\beta\delta)\eta_1\eta_2\right] \\ &\quad + \frac{k'}{2\Delta^2}\left[(\delta+\gamma)^2\eta_1^2 + (-\beta-\alpha)^2\eta_2^2 + 2(\delta+\gamma)(-\beta-\alpha)\eta_1\eta_2\right]. \end{aligned}$$

To get rid of the cross terms here, we require $(\alpha\gamma+\beta\delta) = 0$ (which we already knew), and also

$$(\delta+\gamma)(-\beta-\alpha) = 0.$$

So one of these factors has to be zero. Pick one: let's say $\delta = -\gamma$. Then, because $\alpha\gamma = -\beta\delta$, we must have $\alpha = \beta$. This defines all the Greek letters apart from some overall scaling. If we take $\alpha = \gamma = 1$ arbitrarily, then the new coordinates become

$$\begin{aligned} \eta_1 &= x_1 + x_2 \\ \eta_2 &= x_1 - x_2. \end{aligned}$$

Here are the center of mass and relative coordinates apart from whatever other scaling you might find convenient.

Exercise 9.4

The coordinates of the atoms in the molecule can be, from left to right, x_1, x_2, x_3. They are referred to some fixed origin, but this will almost not matter, since the potential energy depends on the relative coordinates. The Lagrangian is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k(x_1-x_2)^2 - \frac{1}{2}k(x_2-x_3)^2.$$

The conjugate momenta are just simple masses times velocities, so Lagrange's equations are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + kx_2 \\ M\ddot{x}_2 &= kx_1 - 2kx_2 + kx_3 \\ m\ddot{x}_3 &= kx_2 - kx_3. \end{aligned}$$

To find normal modes, we assert that each coordinate is capable of moving at the same (yet unknown) frequency, $x_j(t) = x_j(0)\exp(i\omega t)$. Substituting this into the equations of motion and re-arranging, we get

$$\begin{pmatrix} -k + m\omega^2 & k & 0 \\ k & -2k + M\omega^2 & k \\ 0 & k & -k + m\omega^2 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By the rules of linear algebra, this is only possible if the determinant of the 3×3 matrix is equal to zero. That is to say,

$$(-k + m\omega^2)\left[(-2k + M\omega^2)(-k + m\omega^2) - k^2\right] - k\left[k(-k + m\omega^2) - 0\right] = 0,$$

which simplifies to

$$\omega^2(m\omega^2 - k)(Mm\omega^2 - k(M + 2m)) = 0.$$

The three roots are determined by the three ways this product can be zero. An obvious one is

$$\omega^2_{\text{antisym}} = \sqrt{\frac{k}{m}},$$

which is the frequency of a simple mass m on a spring k. This corresponds to the case where the two end masses oscillate back and forth relative to the central mass, which stays put. It is the antisymmetric stretch mode.

A second mode is

$$\omega^2_{\text{sym}} = \sqrt{\frac{k(M + 2m)}{Mm}} = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)},$$

the ansitymmetric mode. In this case, both the end masses move left and right in synchronized motion, while the central mass moves opposite these to preserve the center of mass. The final mode is

$$\omega_{\text{cm}} = 0.$$

This is the motion of the center of mass itself, which has no restoring force. The molecule will just drift off forever and not come back, so the "period" of its oscillation is $1/(2\pi 0) = \infty$.

Exercise 9.5

Based on the Lagrangian in the chapter, Lagrange's equations of motion are

$$2\ddot{\phi}_1 + \cos(\phi_1 - \phi_2)\ddot{\phi}_2 + \sin(\phi_1 - \phi_2)\dot{\phi}_2^2 = -2\frac{g}{l}\sin\phi_1$$
$$\ddot{\phi}_2 + \cos(\phi_1 - \phi_2)\ddot{\phi}_1 - \sin(\phi_1 - \phi_2)\dot{\phi}_1^2 = -\frac{g}{l}\sin\phi_2.$$

For small amplitude swings, we ignore the terms proportional to $\dot{\phi}_i^2$ and set the cosines equal to 1. This gives us

$$2\ddot{\phi}_1 + \ddot{\phi}_2 + 2\omega_0^2\phi_1 = 0$$
$$\ddot{\phi}_1 + \ddot{\phi}_2 + \omega_0^2\phi_2 = 0,$$

in terms of the frequency of a single pendulum $\omega_0 = \sqrt{g/l}$.

Again, we assert that the angles can swing at the same frequency ω, if only we knew what that frequency is. We write $\phi_j(t) = \phi_j(0)\exp(i\omega t)$ and insert these into the equations of motion to get

$$\begin{pmatrix} -2\omega^2 + 2\omega_0^2 & -\omega^2 \\ -\omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The frequencies are determined by setting the determinant of this matrix equal to zero, which gives

$$\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0.$$

This has the roots

$$\omega^2 = (2 \pm \sqrt{2})\omega_0.$$

The modes themselves are determined by the relative amplitudes $\phi_1(0)$ and $\phi_2(0)$. These are related by either of the linear equations suggested by the matrix equation above, for example

$$(-2\omega^2 + 2\omega_0^2)\phi_1(0) - \omega^2\phi_2(0) = 0.$$

The natural relative amplitudes in a given mode have the ratio

$$\frac{\phi_2(0)}{\phi_1(0)} = \frac{-2\omega^2 + 2\omega_0^2}{\omega^2}$$
$$= \frac{\mp\sqrt{2}}{2 \pm \sqrt{2}}.$$

Thus in the lower frequency mode $\omega^2 = (2 - \sqrt{2})\omega_0^2$, the angles swing in phase, although with different amplitudes. For the higher frequency mode $\omega^2 = (2 + \sqrt{2})\omega_0^2$, the swings are out of phase.