Appendix 11

Solutions of Some Exercises

A11.1 Chapter 1

Exercise 1.4

In a dust universe with curvature and with a cosmological constant the Friedmann equation can be written in the form

$$\dot{a}^2 = a^2\left[-K + \frac{C}{a} + \frac{1}{3}\Lambda a^2\right] \equiv G(a). \tag{A11.1}$$

Here

$$C = \frac{8\pi G}{3}\rho_m a^3 = \Omega_m H_0^2 a_0^3 = \begin{cases} \frac{\Omega_m}{H_0|\Omega_k|^{3/2}} = \frac{2q_0}{H_0|1-2q_0|^{3/2}} & \text{if } \Omega_k \neq 0 \\ \Omega_m H_0^2 & \text{if } \Omega_k = 0 \text{ and } a_0 = 1. \end{cases} \tag{A11.2}$$

If the curvature is negative and $\Lambda > 0$, G is strictly positive and we find an expanding solution for all times. At late times, curvature becomes negligible and the universe expands like $a \propto 1/|t| \propto \exp(\sqrt{\Lambda/3}\tau)$. If $\Lambda < 0$ the square bracket is decreasing and G has a zero, $G(a_c) = 0$. At this point expansion turns into contraction and the universe recollapes.

The case $K = 0$ can be solved explicitly, leading to

$$a^3(\tau) = \begin{cases} \frac{3C}{2\Lambda}\big(\cosh(\sqrt{3\Lambda}t) - 1\big) & \Lambda > 0, a_{\min} = \big(3C/2\Lambda\big)^{1/3} \\ \frac{-3C}{2\Lambda}\big(1 - \cos\big(\sqrt{-3\Lambda}t\big)\big) & \Lambda < 0. \end{cases} \tag{A11.3}$$

The qualitative behavior is like for $K < 0$.

The case $K > 0$ is most interesting. The function G can be written as $G(a) = aP(a)$, where P is a third-order polynomial that has one or three real roots. In the dashed region of Fig. A11.1, P has one real root, but for a negative value of a. Hence the universe expands forever. In the upper left region, with a high cosmological constant, the scale factor has a minimum. Such a universe has no big bang but comes out of a previous contracting phase. It is called a bouncing solution. For a value of $\Omega_m > 0.01$ one finds a maximum redshift $z_{\max} < 4$ for a bouncing universe. Hence they cannot explain cosmological data like quasars and galaxies at a redshift of 6 or even the CMB. Solutions below the dashed region

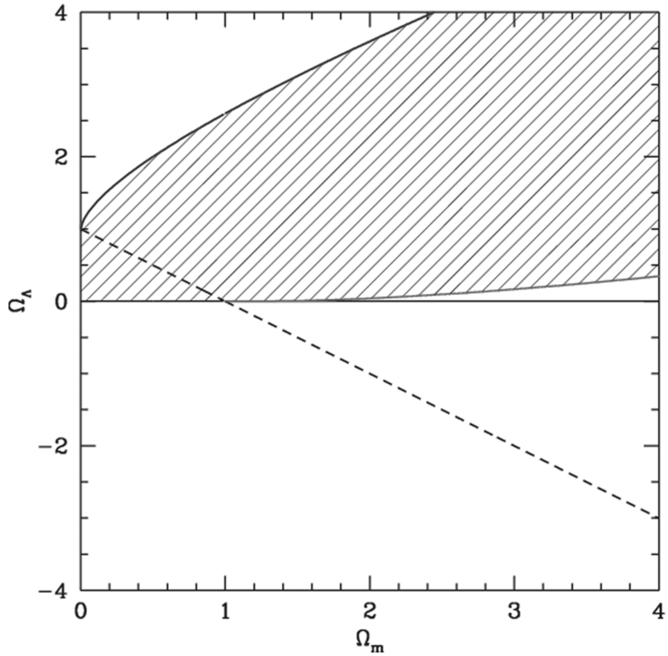

Fig. A11.1 The kinematics of a universe with matter density parameter Ω_m and cosmological constant parameter Ω_Λ. The universes with parameters above the dashed line are positively curved, those below negatively. The universes with values $(\Omega_m, \Omega_\Lambda)$ in the dashed region emerge from a big bang and expand forever. Those below emerge from a big bang and recollapse into a big crunch, and those above emerge from a collapsing universe; they have no big bang in the past.

emerge from a big bang but recollapse eventually, when either the negative cosmological constant or the positive curvature term render $G(a_{\max}) = 0$.

A11.2 Chapter 2

Exercise 2.1

We want to show that

$$L_X g = a^2 \left[-2\left(\frac{\dot a}{a} T + \dot T\right) dt^2 + 2(\dot L_i - T_{,i}) dt\, dx^i + \left(2\frac{\dot a}{a} T \gamma_{ij} + L_{i|j} + L_{j|i}\right) dx^i\, dx^j \right], \tag{A11.4}$$

for $X = T\partial_t + L^i \partial_i$ and $g = a^2(t)[-dt^2 + \gamma_{ij}\, dx^i\, dx^j] = a^2(t) S_{\mu\nu}\, dx^\mu\, dx^\nu$.

We use $L_X a^2 = 2\dot a a T$ and $L_X(a^2 S) = L_X(a^2)S + a^2 L_X S$. Furthermore, we show in the text that follows that for an arbitrary metric S, we have

$$(L_X S)_{\mu\nu} = X_{\mu;\nu} + X_{\nu;\mu}, \tag{A11.5}$$

where here ; denotes the covariant derivative w.r.t. the metric S. For our metric S all Christoffel symbols involving a "0" vanish, so that $X_{\nu;0} = X_{\nu,0}$ and $X_{0;\nu} = X_{0,\nu}$. Furthermore $X_{i;j} = X_{i|j}$, where $_|$ denotes the covariant derivative w.r.t. the three-dimensional metric γ. With this we obtain

$$L_X g = 2\frac{\dot{a}}{a} T a^2 S + a^2 \left(-2\dot{T}\, dt^2 - 2(T_{,i} - \dot{L}_i)\, dt\, dx^i + (L_{i|j} + L_{j|i})\, dx^i\, dx^j\right), \tag{A11.6}$$

which agrees with Eq. (A11.4). It remains to show Eq. (A11.5). For this we use the general expression (A2.20). For a doubly covariant tensor field this gives

$$\begin{aligned}(L_X S)_{\alpha\beta} &= X^\mu S_{\alpha\beta,\mu} + X^\mu{}_{,\alpha} S_{\mu\beta} + X^\mu{}_{,\beta} S_{\mu\alpha} \\ &= X_\nu \left(S^{\mu\nu} S_{\alpha\beta,\mu} + S^{\mu\nu}{}_{,\alpha} S_{\mu\beta} + S^{\mu\nu}{}_{,\beta} S_{\mu\alpha}\right) + X_{\alpha,\beta} + X_{\beta,\alpha}.\end{aligned}$$

For the last equals sign we simply inserted $X^\mu = X_\nu S^{\nu\mu}$. We now take the derivative of the identity $S^{\nu\mu} S_{\mu\beta} = \delta^\nu_\beta$ w.r.t. α. This yields $S^{\mu\nu}{}_{,\alpha} S_{\mu\beta} = -S^{\mu\nu} S_{\mu\beta,\alpha}$. Correspondingly $S^{\mu\nu}{}_{,\beta} S_{\mu\alpha} = -S^{\mu\nu} S_{\mu\alpha,\beta}$. Inserting this above and using the definition

$$X_{\alpha;\beta} = X_{\alpha,\beta} - \Gamma^\mu_{\alpha\beta} X_\mu \quad \text{with} \quad \Gamma^\beta_{\mu\nu} = \frac{1}{2} S^{\beta\alpha}\left(S_{\mu\alpha,\nu} + S_{\nu\alpha,\mu} - S_{\mu\nu,\alpha}\right),$$

we obtain (A11.5).

Exercise 2.3

In synchronous gauge ($A = B = 0$) we have

$$\Psi = -k^{-1}(\mathcal{H}\sigma + \dot{\sigma}) \qquad \text{and} \tag{A11.7}$$

$$V = v - \sigma. \tag{A11.8}$$

For a pure dust universe Eq. (2.119) reduces to

$$\dot{V} + \mathcal{H}V = k\Psi. \tag{A11.9}$$

Inserting the expressions above this yields

$$\dot{v} + \mathcal{H}v = 0, \tag{A11.10}$$

which only has a decaying solution, $v \propto 1/a$, that is, the only possible nondecaying solution is $v = 0$.

Exercise 2.4

We consider a perturbed FL universe containing two noninteracting fluids with energy densities ρ_α and pressure P_α. The total energy density and pressure are $\rho = \rho_1 + \rho_2$ and $P = P_1 + P_2$. We first note that for both components the intrinsic entropy perturbation is given by

$$\Gamma_\alpha = \pi_L^{(\alpha)} - \frac{c_\alpha^2}{w_\alpha}\delta_\alpha = \frac{\delta P_\alpha}{P_\alpha} - c_\alpha^2 \frac{\delta\rho_\alpha}{P_\alpha} \tag{A11.11}$$

and the total sound speed is

$$c_s^2 = \frac{\dot P_1 + \dot P_2}{\dot\rho} = \frac{c_1^2\dot\rho_1 + c_2^2\dot\rho_2}{\dot\rho} = \frac{(1+w_1)c_1^2\rho_1 + (1+w_2)c_2^2\rho_2}{(1+w)\rho}. \qquad \text{(A11.12)}$$

For the second equality sign we have used that both components are separately conserved. Defining now $R = \rho_2/\rho$, so that $\rho_1/\rho = 1 - R$, we can also write

$$(1+w)c_s^2 = (1+w_1)c_1^2(1-R) + c_2^2(1+w_2)R. \qquad \text{(A11.13)}$$

Let us first assume $\Gamma_\alpha = 0$, so that $\delta P_\alpha = c_\alpha^2\delta\rho_\alpha$. The total entropy perturbation is then given by $\Gamma = \Gamma_{\rm rel}$ with

$$P\Gamma_{\rm rel} = c_1^2\delta\rho_1 + c_2^2\delta\rho_2 - c_s^2(\delta\rho_1 + \delta\rho_2) = \left(c_1^2 - c_s^2\right)\delta\rho_1 + \left(c_2^2 - c_s^2\right)\delta\rho_2. \qquad \text{(A11.14)}$$

To express $\Gamma_{\rm rel}$ in terms of gauge-invariant variables we now use

$$\delta\rho_\alpha = [D_g^{(\alpha)} + (1+w_\alpha)(3H_L + H_T)]\rho_\alpha.$$

Inserting this in Eq. (A11.14) yields

$$\begin{aligned} w\Gamma_{\rm rel} = {}& \left(c_1^2 - c_s^2\right)(1-R)D_g^{(1)} + \left(c_2^2 - c_s^2\right)RD_g^{(2)} + (3H_L + H_T) \\ & \times \left[\left(c_1^2 - c_s^2\right)(1-R)(1+w_1) + \left(c_2^2 - c_s^2\right)R(1+w_2)\right]. \end{aligned} \qquad \text{(A11.15)}$$

Using Eq. (A11.13) and

$$1 + w = \frac{\rho + P}{\rho} = \frac{\rho_1 + P_1 + \rho_2 + P_2}{\rho} = (1+w_1)(1-R) + (1+w_2)R,$$

we find that the square bracket above vanishes and $\Gamma_{\rm rel}$ is gauge invariant, as it should be. In fact, with the relation (A11.13)

$$[\quad] = c_1^2(1-R)(1+w_1) + c_2^2R(1+w_2) - c_s^2(1+w) = 0.$$

Multiplying Eq. (A11.15) with $1 + w$ and using Eq. (A11.13) to replace c_s^2 finally leads to

$$w(1+w)\Gamma_{\rm rel} = R(1-R)\left(c_1^2 - c_2^2\right)\left[(1+w_2)D_g^{(1)} - (1+w_1)D_g^{(2)}\right]. \qquad \text{(A11.16)}$$

From this equation we already conclude that $\Gamma_{\rm rel}$ vanishes if both sound speeds are equal, $c_1^2 = c_2^2$, or if one of the two components is largely subdominant, $R \simeq 0$ or $R \simeq 1$. If neither of these conditions is fulfilled, perturbations are adiabatic if

$$(1+w_2)D_g^{(1)} = (1+w_1)D_g^{(2)} \quad \text{(adiabaticity)}. \qquad \text{(A11.17)}$$

To determine Γ when $\Gamma_\alpha \neq 0$ we simply note that in this case $\delta P_\alpha = P_\alpha\Gamma_\alpha + c_\alpha^2\delta\rho_\alpha$ so that

$$P\Gamma = P_1\Gamma_1 + P_2\Gamma_2 + P\Gamma_{\rm rel}.$$

Inserting our result for $\Gamma_{\rm rel}$ we find

$$\Gamma = \frac{w_1}{w}(1-R)\Gamma_1 + \frac{w_2}{w}R\Gamma_2 + \Gamma_{\rm rel}. \qquad \text{(A11.18)}$$

We now want to derive an evolution equation for $\Gamma_{\rm rel}$ in the case where $\Gamma_\alpha = 0$ and $w_\alpha = c_\alpha^2 =$ constant for both components. We use the conservation equation (2.115), which in this case reduces to

$$\dot{D}_g^{(\alpha)} = -k(1 + w_\alpha)V_\alpha. \tag{A11.19}$$

Defining

$$f = \frac{R(1 - R)}{w(1 + w)}\left(c_1^2 - c_2^2\right),$$

the derivative of $\Gamma_{\rm rel}$ can be written as

$$\dot{\Gamma}_{\rm rel} = \frac{\dot{f}}{f}\Gamma_{\rm rel} + kf(1 + w_1)(1 + w_2)[V_2 - V_1]. \tag{A11.20}$$

This shows that even if perturbations of a two-component fluid are initially adiabatic, they develop a relative entropy perturbation if $V_1 \neq V_2$. This is already clear from the adiabaticity condition (A11.17), which cannot be maintained if $V_1 \neq V_2$ due to the time evolution of $D_g^{(\alpha)}$ given in Eq. (A11.19). Especially on sub-Hubble scales, where V_1 and V_2 evolve differently (we consider the nontrivial case $c_1 \neq c_2$), adiabaticity between different components cannot be maintained. When talking about adiabatic perturbations, we therefore always refer to super-Hubble scales.

A11.3 Chapter 3

Exercise 3.1

We want to show that only exponential potentials allow for power law inflation, $a \propto t^q$ with some constant q, and we want to express q in terms of the parameters of the potential. We assume a spatially flat FL universe, $K = 0$.

For a spatially flat FL universe, the Friedmann equation and energy–momentum conservation (or the first and second Friedmann equations) imply

$$\dot{\mathcal{H}} = -\frac{1 + 3w}{2}\mathcal{H}^2.$$

Now if $a \propto t^q$ we have $\mathcal{H} = q/t$ and $\dot{\mathcal{H}} = -q/t^2$. Inserting this above gives

$$q = \frac{2}{1 + 3w} \quad \text{hence} \quad w = \frac{2 - q}{3q} = \text{constant}.$$

From this we also conclude that inflation, that is, $w < -1/3$, is obtained if and only if $q < 0$. Hence for an *expanding* and inflating universe with an expansion law of the form $a \propto (t/t_0)^q$ we have to choose t and t_0 negative in order for a to increase with t. That is, conformal time is negative during inflation.

Furthermore, integrating $d\tau = a\,dt \propto t^q\,dt$ yields $\tau \propto t^{q+1}$; hence

$$a \propto \tau^p \quad \text{with} \quad p = \frac{q}{q + 1} = \frac{2}{3 + 3w}.$$

Since

$$w = P/\rho = a^2 P/(a^2\rho) = \frac{\frac{1}{2}\dot{\varphi}^2 - a^2 W}{\frac{1}{2}\dot{\varphi}^2 + a^2 W} = \text{constant},$$

and

$$a^2\rho = \frac{1}{2}\dot{\varphi}^2 + a^2 W = 3M_P^2\mathcal{H}^2 \propto 1/t^2$$

it follows that both $\frac{1}{2}\dot{\varphi}^2$ and a^2W are proportional to $1/t^2$. More precisely,

$$\dot{\varphi} = \sqrt{a^2(\rho + P)} = \sqrt{3(1+w)}M_P\mathcal{H} = \sqrt{3(1+w)}q\frac{M_P}{t} \tag{A11.21}$$

$$\varphi = M_P\sqrt{2q(1+q)}\log(t/t_*), \tag{A11.22}$$

where t_* is an integration constant. But also $W = (\rho - P)/2$ is a power law in t. This is possible only if $W \propto \exp(-\alpha\varphi/M_P) \propto t^{-\alpha\sqrt{2q(1+q)}}$. To determine α we use that $a^2W \propto 1/t^2$; hence

$$t^{2q-\alpha\sqrt{2q(1+q)}} \propto t^{-2}, \tag{A11.23}$$

which implies

$$\alpha^2 = \frac{2(1+q)}{q} \qquad \text{or} \qquad q(\alpha) = \frac{2}{\alpha^2 - 2}. \tag{A11.24}$$

Inserting this into the expressions for w and p we find

$$w(\alpha) = \frac{\alpha^2 - 3}{3}, \qquad p(\alpha) = \frac{2}{\alpha^2}. \tag{A11.25}$$

The universe is inflating when $w < -1/3$; hence $\alpha^2 < 2$. De Sitter inflation is obtained in the limit $\alpha \to 0$.

A11.4 Chapter 4

Exercise 4.4

We start with Eq. (4.137), which yields

$$\frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell^{(V)}P_\ell(\mathbf{n}\cdot\mathbf{n}')$$
$$= \sum_\ell \frac{(2\ell+1)^2}{(2\pi)^3}\int d^3k\, M_\ell^{(V)}(k)P_\ell(\mu)P_\ell(\mu')(\mathbf{n}\cdot\mathbf{n}' - \mu\mu').$$

For the last factor we made use of Eq. (4.138). Before we continue we now show Eq. (4.194). The addition theorem of spherical harmonics yields

$$\int d\Omega_{\hat{\mathbf{k}}}\, P_\ell(\mu) P_{\ell'}(\mu') = \frac{(4\pi)^2}{(2\ell+1)(2\ell'+1)} \sum_{mm'} \int d\Omega_{\hat{\mathbf{k}}}\, Y_{\ell m}(\hat{\mathbf{k}})\, Y^*_{\ell m}(\mathbf{n}) Y^*_{\ell' m'}(\hat{\mathbf{k}}) Y_{\ell' m'}(\mathbf{n}').$$

Using the orthogonality of spherical harmonics, this implies

$$\int d\Omega_{\hat{\mathbf{k}}}\, P_\ell(\mu) P_{\ell'}(\mu') = \delta_{\ell\ell'} \frac{(4\pi)^2}{(2\ell+1)^2} \sum_m Y^*_{\ell m}(\mathbf{n}) Y_{\ell m}(\mathbf{n}') = \frac{4\pi}{2\ell+1} P_\ell(\mathbf{n}\cdot\mathbf{n}').$$

For the last equals sign we have again applied the addition theorem. With the help of the recursion relation

$$\mu P_\ell(\mu) = \frac{\ell+1}{2\ell+1} P_{\ell+1}(\mu) + \frac{\ell}{2\ell+1} P_{\ell-1}(\mu),$$

we can now perform the angular integration,

$$\begin{aligned}
&\int d^3k\, M^{(V)}_\ell(k) P_\ell(\mu) P_\ell(\mu')(\mathbf{n}\cdot\mathbf{n}' - \mu\mu') \\
&= 4\pi \int dk\, k^2 M^{(V)}_\ell(k) \left[\frac{1}{2\ell+1} (\mathbf{n}\cdot\mathbf{n}') P_\ell(\mathbf{n}\cdot\mathbf{n}') \right. \\
&\quad \left. - \frac{(\ell+1)^2}{(2\ell+1)^2(2\ell+3)} P_{\ell+1}(\mathbf{n}\cdot\mathbf{n}') - \frac{\ell^2}{(2\ell+1)^2(2\ell-1)} P_{\ell-1}(\mathbf{n}\cdot\mathbf{n}') \right] \\
&= \frac{4\pi}{(2\ell+1)^2} \int dk\, k^2 M^{(V)}_\ell(k) \left[\frac{(\ell+1)(\ell+2)}{2\ell+3} P_{\ell+1}(\mathbf{n}\cdot\mathbf{n}') + \frac{\ell(\ell-1)}{2\ell-1} P_{\ell-1}(\mathbf{n}\cdot\mathbf{n}') \right].
\end{aligned}$$

Identifying the coefficient of P_ℓ finally results in

$$C_\ell = \frac{2\ell(\ell+1)}{\pi(2\ell+1)^2} \int dk\, k^2 \left[M^{(V)}_{\ell+1}(k) + M^{(V)}_{\ell-1}(k) \right]. \tag{A11.26}$$

A11.5 Chapter 5

Exercise 5.3

We consider the following parameterization of a 2D tensor field:

$$T_{ab} = \alpha\delta_{ab} + \gamma\epsilon_{ab} + \left(\partial_a\partial_b - \frac{1}{2}\delta_{ab}\Delta\right)\varepsilon + \frac{1}{2}\left(\epsilon_{ac}\partial^c\partial_b + \epsilon_{bc}\partial^c\partial_a\right)\beta. \tag{A11.27}$$

We want to show that every tensor field can be written in this form. Clearly, there are as many parameters on the right-hand side as there are components of T_{ab}, so this may work as a general parameterization. Note that in flat space raising and lower indices is done with δ_{ab}, so it does not change anything.

(1) We first determine the parameters α to β from T_{ab}. A straightforward calculation yields

$$\alpha = \frac{1}{2}\text{trace } T = \frac{1}{2}\left(T_{11} + T_{22}\right) \quad \text{(A11.28)}$$

$$\gamma = \frac{1}{2}\epsilon^{ab}T_{ab} = \frac{1}{2}\left(T_{12} - T_{21}\right) \quad \text{(A11.29)}$$

$$\varepsilon = 2\Delta^{-2}\left(\partial^a\partial^b T_{ab}\right) - \Delta^{-1}\left(T_{11} + T_{22}\right) \quad \text{(A11.30)}$$

$$\beta = 2\Delta^{-2}\left(\epsilon_{ac}\partial^c\partial_b T^{ab}\right) - \Delta^{-1}\left(T_{12} - T_{21}\right). \quad \text{(A11.31)}$$

These equations have unique solutions $\alpha,\ \gamma,\ \varepsilon,\ \beta$ [we assume that our functions decay at infinity, e.g., that they are in $L^2(\mathbb{R}^2)$]. Inversely we obtain

$$T_{11} = \alpha + \frac{1}{2}\left(\partial_1^2 - \partial_2^2\right)\varepsilon + \partial_1\partial_2\beta \quad \text{(A11.32)}$$

$$T_{12} = \gamma + \partial_1\partial_2\varepsilon + \frac{1}{2}\left(\partial_2^2 - \partial_1^2\right)\beta \quad \text{(A11.33)}$$

$$T_{22} = \alpha - \frac{1}{2}\left(\partial_1^2 - \partial_2^2\right)\varepsilon - \partial_1\partial_2\beta \quad \text{(A11.34)}$$

$$T_{21} = -\gamma + \partial_1\partial_2\varepsilon + \frac{1}{2}\left(\partial_2^2 - \partial_1^2\right)\beta. \quad \text{(A11.35)}$$

Inserting the expressions for $\alpha,\ \gamma,\ \varepsilon,\ \beta$ shows that our identities are consistent.

(2) As ϵ_{ab} changes sign under parity and δ_{ab} as well as $\partial_a\partial_b$ do not, we find that for T_{ab} to be a normal 2-tensor that does not change sign under parity, we must request that α and ε are even under parity while γ and β change sign under parity. In other words, α and ε are scalars while γ and β are pseudo-scalars.

(3) The polarization from Thomson scattering is a symmetric and traceless tensor; hence $\alpha = \gamma = 0$ and it is of the form

$$\mathcal{P}_{ab} = \left(\partial_a\partial_b - \frac{1}{2}\delta_{ab}\Delta\right)\varepsilon + \frac{1}{2}\left(\epsilon_{ac}\partial^c\partial_b + \epsilon_{bc}\partial^c\partial_a\right)\beta. \quad \text{(A11.36)}$$

Using Eqs. (5.24) we have

$$\mathcal{E} = \partial_a\partial_b\mathcal{P}_{ab} - \epsilon_{cd}\epsilon_{ab}\partial_c\partial_a\mathcal{P}_{bd} = 2\partial_a\partial_b\mathcal{P}_{ab} = \Delta^2\varepsilon. \quad \text{(A11.37)}$$

For the second equality we used that in two dimensions $\epsilon_{cd}\epsilon_{ab} = \delta_{ca}\delta_{db} - \delta_{da}\delta_{cb}$ and that P_{ab} is traceless. Using also the fact that P_{ab} is symmetric, we find, inserting (5.25) for $\mathcal{B}$,

$$\mathcal{B} = -2\epsilon_{bc}\partial_a\partial_b\mathcal{P}_{ac} = \Delta^2\beta. \quad \text{(A11.38)}$$

Therefore, the decomposition (A11.36) is entirely equivalent to the decomposition of the polarization into $\mathcal{E}$ and $\mathcal{B}$ modes.

A11.6 Chapter 6

Exercise 6.1

Because of statistical homogeneity, a 3-point function depends only on the differences $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$,

$$\langle X(\mathbf{x}_1)X(\mathbf{x}_3)X(\mathbf{x}_3)\rangle = \xi_3(\mathbf{r}_{12}, \mathbf{r}_{32}). \tag{A11.39}$$

Here we use that $\mathbf{r}_{13} = \mathbf{r}_{12} - \mathbf{r}_{32}$ is not an independent variable. Fourier transforming this expression we obtain

$$\begin{aligned}
&\int d^3x_1 d^3x_2 d^3x_3 e^{i(\mathbf{k}_1\mathbf{x}_1+\mathbf{k}_2\mathbf{x}_2+\mathbf{k}_3\mathbf{x}_3)}\langle X(\mathbf{x}_1)X(\mathbf{x}_3)X(\mathbf{x}_3)\rangle \\
&= \int d^3r_{12} d^3x_2 d^3r_{32} e^{i(\mathbf{k}_1\mathbf{r}_{12}+\mathbf{k}_3\mathbf{r}_{32}+\mathbf{x}_2(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3))}\xi_3(\mathbf{r}_{12}, \mathbf{r}_{32}) \\
&= (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\int d^3r_{12} d^3r_{32} e^{i(\mathbf{k}_1\mathbf{r}_{12}+\mathbf{k}_3\mathbf{r}_{32})}\xi_3(\mathbf{r}_{12}, \mathbf{r}_{32}) \\
&= (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B(\mathbf{k}_1, \mathbf{k}_3).
\end{aligned} \tag{A11.40}$$

Since the first line of this equation as well as the Dirac delta are symmetric in $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, this is also true for $B(\mathbf{k}_1, \mathbf{k}_3)$. Here we have suppressed the variable $\mathbf{k}_2 = -(\mathbf{k}_1 + \mathbf{k}_3)$. We now want to show that B depends only on the moduli $k_i = |\mathbf{k}_i|$. For this we use that the cosine of the angle between $\mathbf{k}_1$ and $\mathbf{k}_3$ is given by

$$\mu \equiv \frac{\mathbf{k}_1 \cdot \mathbf{k}_3}{k_1 k_3} = \frac{k_2^2 - k_1^2 - k_3^2}{2k_1 k_3}. \tag{A11.41}$$

Therefore if we can show that B depends only on k_1, k_3, and μ we are done. For this, without loss of generality, we choose the z-direction as the direction of $\mathbf{k}_1$ and denote by $\mu_{ij} = \cos\theta_{ij}$, where θ_{ij} is the polar angle of $\mathbf{r}_{ij}$. We also use that

$$\mathbf{k}_3 = k_3(\mu\mathbf{e}_z + \sqrt{1-\mu^2}\mathbf{e}_\perp), \tag{A11.42}$$

and, again without loss of generality, we identify the direction $\mathbf{e}_\perp$ that is normal to $\mathbf{e}_z$ with the x-direction, so that $\mathbf{r}_{ij}\mathbf{e}_\perp = r_{ij}\cos\varphi_{ij}$. With these choices of coordinate directions we have

$$\mathbf{k}_1\mathbf{r}_{12} = \mu_{12} r_{12} k_1 \text{ and } \quad \mathbf{k}_3\mathbf{r}_{32} = \left(\mu\mu_{32} + \sqrt{1-\mu^2}\sqrt{1-\mu_{32}^2}\right) r_{32} k_3. \tag{A11.43}$$

Note also that due to statistical isotropy apart from r_{12} and r_{32}, ξ_3 depends only on the cosine of the angle between $\mathbf{r}_{12}$ and $\mathbf{r}_{32}$, which is given by

$$\nu = \mu_{12}\mu_{32} + \sqrt{(1-\mu_{12}^2)(1-\mu_{32}^2)}\cos(\varphi_{12} - \varphi_{32}). \tag{A11.44}$$

Using spherical coordinates we can transform $\varphi_{12} \to \varphi_{12} - \varphi_{32} \equiv \varphi$. With this the integral (A11.40) becomes

$$B(k_1, k_3, \mu) = \int r_{12}^2 dr_{12} d\mu_{12} d\varphi r_{32}^2 dr_{32} d\mu_{32} d\varphi_{32} \xi_3(r_{12}, r_{32}, \nu(\mu_{12}, \mu_{32}, \varphi)) \times \exp\left[i\left(r_{12}k_1\mu_{12} + r_{32}k_3\left(\mu\mu_{32} + \sqrt{1-\mu^2}\sqrt{1-\mu_{32}^2}\cos\varphi_{32}\right)\right)\right]. \tag{A11.45}$$

Finally, one can perform the integration over φ_{32}, which yields

$$B(k_1, k_3, \mu) = 2\pi \int r_{12}^2 dr_{12} d\mu_{12} d\varphi r_{32}^2 dr_{32} d\mu_{32} \xi_3(r_{12}, r_{32}, \nu(\mu_{12}, \mu_{32}, \varphi)) \times J_0\left(r_{32}k_3\sqrt{1-\mu^2}\sqrt{1-\mu_{32}^2}\right) \exp\left[i\left(r_{12}k_1\mu_{12} + r_{32}k_3\mu\mu_{32}\right)\right]. \tag{A11.46}$$

Here J_0 denotes the Bessel function of order 0 and μ can be written as a function of the k_i via Eq. (A11.41). In principle one could also convert the integral over φ or the one over μ_{32} into an integral over ν but with awkward boundary conditions and with a not very illuminating result. In Eq. (A11.46) it is no longer evident that B is symmetric under the exchange of the k_i. But we know that this must be true because of the original expression given on the first line of Eq. (A11.40).

Exercise 6.2

The coefficient $\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle$ is obtained from the 3-point function by integrating with the corresponding spherical harmonics,

$$\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle = \int \xi_3(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_2) Y^*_{\ell_1 m_1}(\mathbf{n}_1) Y^*_{\ell_2 m_2}(\mathbf{n}_2) Y^*_{\ell_3 m_3}(\mathbf{n}_3) d\Omega_1 d\Omega_2 d\Omega_3. \tag{A11.47}$$

On the other hand, we have expression (6.38) for ξ_3. Using the addition theorem of spherical harmonics,

$$P_L(\mu_{ij}) = \frac{4\pi}{2L+1} \sum_M Y_{LM}(\mathbf{n}_i) Y^*_{LM}(\mathbf{n}_j), \tag{A11.48}$$

Eq. (6.38) leads to three integrals of the following form:

$$\int Y^*_{\ell_i m_i}(\mathbf{n}_i) Y_{L_i M_i}(\mathbf{n}_i) Y^*_{L_{[i-1]} M_{[i-1]}}(\mathbf{n}_i), \tag{A11.49}$$

where $[i-1] = i-1$ for $i = 2, 3$ and $[i-1] = 3$ for $i = 1$. Using the triple integrals of spherical harmonics given in Appendix 4, Section A4.2.3, we find

$$\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3}\rangle = (4\pi)^{3/2} \sum_{L_i, M_i} \prod_{i=1}^{3} \sqrt{2\ell_i + 1} \begin{pmatrix} \ell_i & L_i & L_{[i-1]} \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \ell_i & L_i & L_{[i-1]} \\ m_i & M_i & M_{[i-1]} \end{pmatrix} b^{(2)}_{L_1 L_2 L_3}. \tag{A11.50}$$

The factors $(-1)^{M_i}$ multiply together to 1 since $M_1 + M_2 + M_3 = 0$, as is easy to check. Now together with (A4.61) Eq. (6.40) implies

$$\begin{aligned} &\sqrt{\frac{\prod_{i=1}^{3}(2\ell_i + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} b_{\ell_1 \ell_2 \ell_3} \\ &= \sum_{m_1 m_2 m_3} \langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3}\rangle \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad \text{(A11.51)} \\ &= (4\pi)^{3/2} \sum_{m_1 m_2 m_3;\, L_i M_i} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \prod_{i=1}^{3} \sqrt{2\ell_i + 1} \\ &\quad \times \begin{pmatrix} \ell_i & L_i & L_{[i-1]} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_i & L_i & L_{[i-1]} \\ m_i & M_i & M_{[i-1]} \end{pmatrix} b^{(2)}_{L_1 L_2 L_3}. \qquad \text{(A11.52)} \end{aligned}$$

Deviding by the prefactor we find

$$b_{\ell_1 \ell_2 \ell_3} = \sum_{L_i} Q^{L_1 L_2 L_3}_{\ell_1 \ell_2 \ell_3} b^{(2)}_{L_1 L_2 L_3}, \quad \text{where} \tag{A11.53}$$

$$Q^{L_1 L_2 L_3}_{\ell_1 \ell_2 \ell_3} = (4\pi)^2 \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1} \sum_{m_i; M_i} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \prod_{i=1}^{3} \begin{pmatrix} \ell_i & L_i & L_{[i-1]} \\ 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} \ell_i & L_i & L_{[i-1]} \\ m_i & M_i & M_{[i-1]} \end{pmatrix}. \tag{A11.54}$$

Here the sums over all m_i and M_i are always understood as sums from $-\ell_1$ to ℓ_i and $-L_i$ to L_i respectively.

Exercise 6.7

Let us first show that

$$V_1(\nu) = \int_{\partial K(\nu)} ds = \frac{1}{4} \int_{\mathbb{S}^2} d\Omega \delta(u(\mathbf{n}) - \nu) \sqrt{(\nabla u)^2}. \tag{A11.55}$$

To show this let us consider a small neighborhood of a given point $\mathbf{n}_0$ on the curve $u(\mathbf{n}) = \nu$. In this neighborhood we may parameterize this curve by some function $\mathbf{n}(t)$. (We assume that ν is not a local maximum; otherwise the curve $u(\mathbf{n}) = \nu$ shrinks to a point.) The length of a part of our curve is then given by the integral of $\sqrt{\dot{\mathbf{n}}^2}dt$. Choosing local coordinates on the sphere along the curve and orthogonal to it we find for the small part of the curve that we parameterize as $\mathbf{n}(t)$

$$L = \int \sqrt{(\dot{\mathbf{n}})^2}dt = \int \delta(u(\mathbf{n}) - \nu)\sqrt{(\dot{\mathbf{n}})^2}d\Omega. \tag{A11.56}$$

By construction $u(\mathbf{n}(t)) = \nu$ and therefore

$$\frac{du(\mathbf{n}(t))}{dt} = \nabla u(\mathbf{n}(t)) \cdot \dot{\mathbf{n}} = 0. \tag{A11.57}$$

Since we are in two dimensions, this implies that

$$\dot{n}_i = \alpha\epsilon_{ij}\nabla_j u(\mathbf{n}(t)), \text{ or equivalently } \epsilon_{ki}\dot{n}_i = -\alpha\nabla_k u(\mathbf{n}(t)). \tag{A11.58}$$

The proportionality factor depends on our parameterization and we can choose it to be unity. Equation (A11.58) then implies that $(\dot{\mathbf{n}})^2 = (\nabla u)^2$, which leads to (A11.55).

We now also want to show that

$$V_2(\nu) = \int_{\partial K(\nu)} \kappa(s)ds = \frac{1}{2\pi}\int_{\mathbb{S}^2} d\Omega\delta(u(\mathbf{n}) - \nu)\frac{\sum_{ij=1}^{2}(-1)^{j+i+1}\nabla_i u\nabla_j u\nabla_i\nabla_j u}{(\nabla u)^2}. \tag{A11.59}$$

Using $ds = \delta(u(\mathbf{n}) - \nu)\sqrt{(\nabla u)^2}d\Omega$ we simply need to show that on the curve $u(\mathbf{n}) = \nu$ the geodesic curvature is given by

$$\kappa(\mathbf{n}) = \frac{\sum_{ij=1}^{2}(-1)^{j+i+1}\nabla_i u\nabla_j u\nabla_i\nabla_j u}{(\nabla u)^{3/2}}. \tag{A11.60}$$

To show this we now derive $u(\mathbf{n}(t)) = \nu$ a second time, leading to

$$\nabla_i\nabla_j u(\mathbf{n}(t))\dot{n}_i\dot{n}_j + \nabla_j u(\mathbf{n}(t))\ddot{n}_j = 0, \tag{A11.61}$$

$$\nabla_i\nabla_j u(\mathbf{n}(t))\dot{n}_i\dot{n}_j = -\dot{n}_i\ddot{n}_j\epsilon_{ij}. \tag{A11.62}$$

For the second equality we made use of Eq. (A11.58) (with $\alpha = 1$). Now the general expression for the geodesic curvature of an arbitrary line can be found in a generic geometry book; it is

$$\kappa(\mathbf{n}(t)) = \frac{\dot{n}_i\ddot{n}_j\epsilon_{ij}}{(\dot{\mathbf{n}})^{3/2}}. \tag{A11.63}$$

Inserting $\dot{\mathbf{n}}$ and $\dot{n}_i\ddot{n}_j\epsilon_{ij}$ from Eqs. (A11.58) and (A11.62) we find Eq. (A11.60).

A11.7 Chapter 7

Exercise 7.1

We consider a mass M positioned at $\mathbf{x} = 0$ with gravitational potential $\Psi = GM/r$. To first order in Ψ the corresponding metric is given by

$$ds^2 = -(1 + 2\Psi)\,dt^2 + (1 - 2\Psi)\,d\mathbf{x}^2.$$

We want to determine the deflection of a photon in this metric. Angles are invariant under conformal transformations of the geometry. We may therefore calculate the deflection in the conformally related metric $d\tilde{s}^2 = (1 + 2\Psi)\,ds^2$. To first order in Ψ we have

$$d\tilde{s}^2 = -(1 + 4\Psi)\,dt^2 + d\mathbf{x}^2.$$

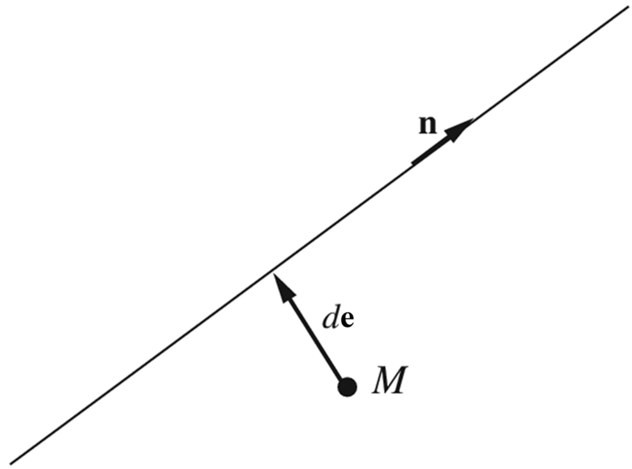

Fig. A11.2 A photon passing the mass M in direction $\mathbf{n}$ with impact parameter d.

We consider a photon along the unperturbed path $\mathbf{x}(s) = d\mathbf{e} + s\mathbf{n}$. The spatial unit vector $\mathbf{n}$ is the direction of motion of the photon and $\mathbf{e}$ is a spatial unit vector normal to $\mathbf{n}$. Hence d is the impact parameter, that is, the closest distance of the photon from the mass M at $\mathbf{x} = 0$; see Fig. A11.2. The unperturbed photon velocity is given by $(n^\mu) = (1, \mathbf{n})$. Since Ψ is spherically symmetric, angular momentum is conserved and also the perturbed motion will be in the plane $(\mathbf{e}, \mathbf{n})$. We define the perturbed velocity by

$$(n^\mu + \delta n^\mu) = (1 + \delta n^0, \mathbf{n} + \delta\mathbf{n}).$$

As it lies in the plane $(\mathbf{e}, \mathbf{n})$, the spatial part of δn^μ is of the form $\delta\mathbf{n} = \varphi\mathbf{e} + \alpha\mathbf{n}$, where φ is the deflection angle and α is related to the gravitational redshift. The Christoffel symbols are of first order in Ψ, so that the first-order equation of motion for the photon trajectory gives

$$\delta\dot{n}^\mu + \tilde{\Gamma}^\mu_{00} + 2\tilde{\Gamma}^\mu_{0j}n^j + \tilde{\Gamma}^\mu_{ij}n^i n^j = 0.$$

For the metric $d\tilde{s}^2$ the only nonvanishing Christoffel symbols are

$$\tilde{\Gamma}^0_{0i} = \tilde{\Gamma}^0_{i0} = \tilde{\Gamma}^i_{00} = 2\partial_i\Psi.$$

For the deflection angle we therefore obtain

$$\dot{\varphi} = (\delta\dot{\mathbf{n}} \cdot \mathbf{e}) = -2\mathbf{e} \cdot \nabla\Psi = 2MG\frac{d}{(d^2 + s^2)^{3/2}}.$$

Integrating this from $s = -\infty$ to $s = \infty$ yields

$$\varphi = \frac{4MG}{d}. \qquad \text{(A11.64)}$$

A11.8 Chapter 8

Exercise 8.1

We want to show the following theorem:

Theorem: Let $\xi(\mathbf{r})$ be a correlation function that depends on the orientation of $\mathbf{r}$ only via its scalar product with one fixed given direction $\mathbf{n}$ (e.g., the line of sight). Denoting the corresponding direction cosine by μ and expanding ξ in Legendre polynomials, we have

$$\xi(\mathbf{r}) = \sum_n \xi_n(r)L_n(\mu), \qquad \mu = \hat{\mathbf{r}} \cdot \mathbf{n}. \tag{A11.65}$$

In this situation the Fourier transform of ξ, the power spectrum, is of the form

$$P(\mathbf{k}) = \sum_n p_n(k)L_n(\nu), \quad \nu = \hat{\mathbf{k}} \cdot \mathbf{n} \quad \text{where} \tag{A11.66}$$

$$p_n(k) = 4\pi i^n \int_0^\infty dr r^2 j_n(kr)\xi_n(r), \quad \text{and} \tag{A11.67}$$

$$\xi_n(r) = \frac{(-i)^n}{2\pi^2} \int_0^\infty dk k^2 j_n(kr) p_n(k). \tag{A11.68}$$

Proof The Fourier transform of ξ is defined as

$$P(\mathbf{k}) = \int d^3r e^{i\mathbf{r}\cdot\mathbf{k}} \xi(\mathbf{r}). \tag{A11.69}$$

We use that

$$e^{i\mathbf{r}\cdot\mathbf{k}} = \sum_\ell i^\ell (2\ell+1) j_\ell(kr) L_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}),$$

where L_ℓ is the Legendre polynomial of degree ℓ. Hence

$$L_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{k}}) Y^*_{\ell m}(\hat{\mathbf{r}}) \;=\; \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{r}}) Y^*_{\ell m}(\hat{\mathbf{k}});$$

$Y_{\ell m}$ are the spherical harmonics. Inserting these identities in (A11.69) using the ansatz (A11.65) for the correlation function, we obtain

$$P(\mathbf{k}) = \sum_{\ell m} \sum_{nm'} \frac{(4\pi)^2 i^\ell}{2\ell+1} \int d^3r \xi_n(r) j_\ell(kr) Y_{\ell m}(\hat{\mathbf{k}}) Y^*_{\ell m}(\hat{\mathbf{r}}) Y_{nm'}(\hat{\mathbf{r}}) Y^*_{nm'}(\mathbf{n}). \tag{A11.70}$$

Using the orthogonality relation of spherical harmonics, the integration over directions gives

$$P(\mathbf{k}) = 4\pi \sum_n i^n \int_0^\infty dr r^2 \xi_n(r) j_n(kr) L_n(\nu). \tag{A11.71}$$

Identification of the expansion coefficients yields (A11.67). Equation (A11.68) is obtained in the same way using the inverse Fourier transform,

$$\xi(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k e^{-i\mathbf{k}\cdot\mathbf{r}} P(\mathbf{k}).$$

□

Clearly, if $\xi(\mathbf{r}) = \langle\Delta(\mathbf{x})\Delta(\mathbf{x}+\mathbf{r})\rangle$ is independent of $\mathbf{x}$ (Δ is statistically homogeneous), ξ does not depend on the sign of $\mathbf{r}$ and in the sum above only ξ_n with even n's can contribute so that $P(\mathbf{k})$ is real.

A11.9 Chapter 9

Exercise 9.9.2

We parameterize the initial conditions by

$$C_{ij} = \langle X_i(\mathbf{k})X_j^*(\mathbf{k}')\rangle = A_{ij}(k/H_0)^{n_{ij}}\delta(\mathbf{k}-\mathbf{k}').$$

Clearly, for C_{ij} to be positive semidefinite for all values of k, the matrix $A_{ij} = C_{ij}(k = H_0)$ has to be positive semidefinite. Let us now consider $i \neq j$ with $A_{ij} \neq 0$. If neither $n_{ii} \leq n_{ij} \leq n_{jj}$ nor $n_{jj} \leq n_{ij} \leq n_{ii}$ is true, n_{ij} is either the largest or the smallest of these three spectral indices. Let us first assume it to be the smallest. To show that C_{ij} is not positive semidefinite, we have to find a vector V so that $C_{mn}V^mV^n < 0$. If $A_{ij} > 0$, we choose $V^i = -V^j = 1$, and if $A_{ij} < 0$, we choose $V^i = V^j = 1$, so that $A_{ij}V^iV^j = -|A_{ij}|$ (no sum!). Since n_{ij} is smaller than n_{ii} and n_{jj} we can choose k to be sufficiently small so that $|A_{ij}|(k/H_0)^{n_{ij}} \gg |A_{ii}|(k/H_0)^{n_{ii}}$ and $|A_{ij}|(k/H_0)^{n_{ij}} \gg |A_{jj}|(k/H_0)^{n_{jj}}$. Setting all other components of V to 0 we obtain for such values of k

$$\sum_{mn} V^mV^nC_{mn}(k) = -|A_{ij}|(k/H_0)^{n_{ij}} + A_{ii}(k/H_0)^{n_{ii}} + A_{jj}(k/H_0)^{n_{jj}} < 0.$$

If n_{ij} is larger than n_{ii} and n_{jj} we just have to choose k sufficiently large.

A11.10 Chapter 10

Exercise 10.3

We want to compute the integral

$$J_{BE}(x_c) = \int_0^1 \frac{dx}{x}\frac{e^x \exp[-2x_c/x]}{e^x - 1} \tag{A11.72}$$

for small values of x_c; more precisely, $0 < x_c \ll 1$. We want to show that

$$J_{BE}(x_c) = \frac{1}{2x_c} - \frac{1}{2}\log(x_c) + \text{higher order}, \tag{A11.73}$$

where "higher order" denotes terms of order unity and terms that vanish for $x_c \to 0$. To compute the integral (A11.72) we use the series expansion

$$\frac{te^t}{e^t - 1} = \sum_{m=0} B_m \frac{t^m}{m!} \tag{A11.74}$$

Here B_m are the Bernoulli numbers (see Abramowitz and Stegun, 1970), given by[1]

$$B_0 = 1, \quad B_1 = 1/2, \quad B_2 = 1/6, \quad B_3 = 0, \quad B_4 = -1/30, \tag{A11.75}$$

$$B_{2n+1} = 0, \quad B_{2n} = -(-1)^n \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n), \quad \text{for } n > 1. \tag{A11.76}$$

Here ζ denotes the Riemann zeta-function. With this

$$J_{BE}(x_c) = \sum_{m=0} B_m \frac{I_m(x_c)}{m!} \qquad \text{where} \tag{A11.77}$$

$$I_m(x_c) = \int_0^1 dx x^{m-2} \exp[-2x_c/x] \tag{A11.78}$$

With the variable transform $y = 1/x$ we obtain

$$I_m(x_c) = \int_1^\infty dy y^{-m} \exp[-2x_c y] = E_m(2x_c), \tag{A11.79}$$

where E_m denotes the well-known exponential integral function of order m. E_0 is elementary and yields the first part of our result, $I_0 = e^{-2x_c}/(2x_c) \simeq 1/(2x_c)$. The exponential integral of order 1 has the asymptotic behavior $E_1(2x_c) = \mathrm{Ei}(2x_c) \simeq -\log(2x_c) - \gamma + \mathcal{O}(x_c)$, where $\gamma \simeq 0.577$ is the Euler–Mascheroni constant. Then, as $E'_m(2x_c) = -E_{m-1}(2x_c)$ it follows that the exponential integral of order $m \geq 1$ behaves as $E_m(2x_c) \simeq (2x_c)^{m-1} \log(2x_c)$+const. for small $x_c \ll 1$. This proves Eq. (A11.73).

As a final remark let me mention that such integrals are often estimated using a saddle point approximation. While the behavior $J_{BC} \propto x_c^{-1}$ is recovered by this method also here, the prefactor is wrong. One can actually show that in this case the saddle point approximation obtains corrections that scale like x_c^{-1} at every order and is therefore useless.

[1] One often finds $B_1 = -1/2$. This depends on the definition of B_n as $B_n = b_n(0)$ or $B_n = b_n(1)$, where $b_n(x)$ are the Bernoulli polynomials; see Abramowitz and Stegun (1970). Here we use the second identification, which gives $B_1 = 1/2$.