Exercise: concepts from chapter 3

Reading: Fundamentals of Structural Geology, Ch 3

1) The natural representation of a curve, $\mathbf{c} = \mathbf{c}(s)$, satisfies the condition $|d\mathbf{c}/d\mathbf{s}| = 1$, where *s* is the natural parameter for the curve.

- a) Describe in words and a sketch what this condition means.
- b) Demonstrate that the following vector function (3.6) is the natural representation of the circular helix (Fig. 1) by showing that it satisfies the condition $|d\mathbf{c}/d\mathbf{s}| = 1$.

$$\mathbf{c}(s) = a \cos\left[\left(a^{2} + b^{2}\right)^{-1/2} s\right] \mathbf{e}_{x} + a \sin\left[\left(a^{2} + b^{2}\right)^{-1/2} s\right] \mathbf{e}_{y} + b\left(a^{2} + b^{2}\right)^{-1/2} s \mathbf{e}_{z}$$
(1)
$$a > 0, \quad -\infty < b < +\infty$$

c) Use (1) and MATLAB to plot a 3D image of the circular helix ($a = 1, b = 1/2\pi$). An example is shown in Figure 1. Describe and label *a* and *b*.



Figure 1. Circular helix with unit tangent (red), principal normal (green), and binormal (blue) vectors at selected locations.

2) An arbitrary representation of a curve, $\mathbf{c} = \mathbf{c}(t)$, satisfies the condition $|\mathbf{d}\mathbf{c}/\mathbf{d}t| = \mathbf{d}s/\mathbf{d}t$, where *t* is the arbitrary parameter and *s* is the natural parameter for the curve.

a) Demonstrate that the following vector function (3.2) is an arbitrary representation of the circular helix by showing that it satisfies this condition.

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$$\mathbf{c}(t) = (a\cos t)\mathbf{e}_x + (a\sin t)\mathbf{e}_y + bt\mathbf{e}_z, \ a > 0, \ -\infty < b < +\infty$$
(2)

- b) Show how this condition and the chain rule are used to derive the equation (3.8) for the unit tangent vector for an arbitrary representation of a curve and then use this equation to derive the unit tangent vector for the circular helix (3.11). In the process show how *t* and *s* are related.
- c) Using your result from part b) for $\mathbf{t}(t)$ write the equation for the unit tangent vector, $\mathbf{t}(s)$, as a function of the natural parameter. Use this equation and MATLAB to plot a 3D image of a set of unit tangent vectors on the circular helix ($a = 1, b = 1/2\pi$) as in Figure 1.

3) The curvature vector, scalar curvature, and radius of curvature are three closely related quantities (Fig. 3.10) that help to describe a curved line in three-dimensional space.

- a) Derive equations for the curvature vector, $\mathbf{k}(s)$, the scalar curvature, $\kappa(s)$, and the radius of curvature, $\rho(s)$, for the natural representation of the circular helix (1).
- b) Show how these equations reduce to the special case of a circle.
- c) Derive an equation for the unit principal normal vector, $\mathbf{n}(s)$, for the circular helix as given in (1).
- d) Use MATLAB to plot a 3D image (Fig. 1) of a set of unit principal normal vectors on the circular helix ($a = 1, b = 1/2\pi$). Describe the orientation of these vectors with respect to the circular helix itself, and the Cartesian coordinates.
- e) Derive an equation for the unit binormal vector, $\mathbf{b}(s)$, for the circular helix (1). This is the third member of the moving trihedron.
- f) Use MATLAB to plot a 3D image (Fig. 1) of a set of unit binormal vectors on the circular helix ($a = 1, b = 1/2\pi$).

4) If $\mathbf{c} = \mathbf{c}(t)$ is the arbitrary parametric representation of a curve, then a general definition of the scalar curvature is given in (3.26) as:

$$\kappa(t) = \left| \frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} \times \frac{\mathrm{d}^2 \mathbf{c}}{\mathrm{d}t^2} \right| / \left| \frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} \right|^3 \tag{3}$$

- a) Show how this relationship may be specialized to plane curves lying in the (*x*, *y*)-plane where $\mathbf{c}(t) = c_x \mathbf{e}_x + c_y \mathbf{e}_y$ and the components are arbitrary functions of *t*.
- b) Further specialize this relationship for the plane curve lying in the (x, y)-plane where the parameter is taken as x instead of t, so one may write $c_x = x$ and $c_y = f(x)$ such that $\mathbf{c}(x) = x\mathbf{e}_x + f(x)\mathbf{e}_y$ and the normal curvature is:

$$\kappa(x) = \left| \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \right| / \left[1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x} \right)^2 \right]^{3/2} \tag{4}$$

c) Evaluate the error introduced in the often-used approximation $\kappa(x) \sim |d^2 f/dx^2|$ by plotting the following ratio as a function of the slope, df/dx, in MATLAB:

$$\frac{\kappa(exact) - \kappa(approx)}{\kappa(exact)}$$
(5)

Develop a criterion to limit errors to less than 10% in practical applications.

5) If $\mathbf{c} = \mathbf{c}(t)$ is the arbitrary parametric representation of a curve, then a general definition of the scalar torsion is given is (3.50) as:

$$\tau(t) = \left[\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}^{2}\mathbf{c}}{\mathrm{d}t^{2}} \times \frac{\mathrm{d}^{3}\mathbf{c}}{\mathrm{d}t^{3}}\right)\right] / \left|\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} \times \frac{\mathrm{d}^{2}\mathbf{c}}{\mathrm{d}t^{2}}\right|^{2}$$
(6)

a) Derive an expression for the scalar torsion for the parametric representation of the circular helix of radius *a* and pitch *b* as given by:

$$\mathbf{c}(t) = (a\cos t)\mathbf{e}_x + (a\sin t)\mathbf{e}_y + bt\mathbf{e}_z, \ a > 0, \ -\infty < b < +\infty$$
(7)

b) Describe the scalar torsion for circular helix ($a = 1, b = 1/2\pi$) using the MATLAB plot prepared for question 3) and the change in orientation of the moving trihedron with respect to the natural parameter *s*.

6) The two intrinsic scalar properties of continuous curves that determine their shape in three dimensions are the curvature (3.20) and the torsion (3.48). For the circular helix these properties are constants given by:

$$\kappa = a / \left(a^2 + b^2\right), \quad \tau = b / \left(a^2 + b^2\right) \tag{8}$$

- a) Use MATLAB to plot two 3D graphs, one for the scalar curvature and the other for the torsion. Use a range for the radius, *a*, of $0 \le a \le 3$ and for the pitch, *b* of $-1.5 \le b \le +1.5$.
- b) Study the graphs constructed in part a) and describe the interesting features. For example, describe how the curvature varies with the radius for constant pitch, including a pitch of zero. Also describe how the torsion varies with the radius for constant pitch, including a pitch of +1and -1.

7) The tangent plane at a point on a surface is illustrated in Figure 3.19. For geological surfaces, such as the top of a sedimentary formation, the orientation of the tangent plane at a point on the surface is measured using strike and dip, or dip direction and dip. Given a general parametric representation of a surface s(u, v), where u and v are scalar quantities that are the parameters of the surface, the tangent plane to the surface is defined in (3.63) as:

$$\mathbf{P} = \mathbf{s} + h \frac{\partial \mathbf{s}}{\partial u} + k \frac{\partial \mathbf{s}}{\partial v}, \quad -\infty \le h, k \le +\infty$$
(9)

As an example consider the parametric representation of the circular paraboloid given using the two parameters, *u* and *v*:

$$\mathbf{s}(u,v) = u\mathbf{e}_x + v\mathbf{e}_y - (u^2 + v^2)\mathbf{e}_z$$
(10)

This is an idealization of a geologic dome, which is a common trap for hydrocarbons.

- a) Use MATLAB to plot the circular paraboloid over the following ranges of the two parameters: $-1 \le u \le +1$, $-1 \le v \le +1$. Use the function surf to create a color-filled wire-frame of the surface.
- b) Derive the general equations for the tangent plane, **P**, to this surface. In doing so define the tangent vectors for the *u* and *v*-parameter curves on this surface.
- c) Evaluate your equation for the tangent plane at the point $u_0 = -0.25$, $v_0 = -0.25$. Plot a small patch of the tangent plane on the circular paraboloid and print these combined images. Move the tangent plane patch around on the paraboloid and

make sure by visual inspection that it remains tangent to the surface at the point you have specified.

8) Given a general parametric representation of a surface s(u, v), where u and v are the parameters, the unit normal vector to the surface is defined in (3.73) as:

$$\mathbf{N} = \left(\frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v}\right) / \left|\frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v}\right|$$
(11)

- a) Derive the equation for the unit normal vector, **N**, for the circular paraboloid (dome) defined in equation (10).
- b) At each intersection of the *u* and *v*-parameter curves on the circular paraboloid plot an arrow representing the unit normal vector.
- c) Suppose the parameters (u, v) are associated with the geographic coordinates (Easting, Northing). Determine the plunge direction and plunge (α_n, ϕ_n) for the normal line at the point $u_0 = -0.25$, $v_0 = -0.25$. Use this information to find the strike and dip (α_s, ϕ_s) of surface at that point.
- 9) The coefficients of the first fundamental form are used to calculate arc lengths of curves. The coefficients of the first fundamental form are defined in (3.85) as:

$$E = \frac{\partial \mathbf{s}}{\partial u} \cdot \frac{\partial \mathbf{s}}{\partial u}, \quad F = \frac{\partial \mathbf{s}}{\partial u} \cdot \frac{\partial \mathbf{s}}{\partial v}, \quad G = \frac{\partial \mathbf{s}}{\partial v} \cdot \frac{\partial \mathbf{s}}{\partial v}$$
(12)

The arc length of a curve $\mathbf{c}[u(t), v(t)]$ on a surface, $\mathbf{s}(u,v)$ is defined in (3.89) as:

$$s = \int_{a}^{b} \left[E\left(\frac{du}{dt}\right)^{2} + 2F\left(\frac{du}{dt}\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^{2} \right]^{1/2} dt$$
(13)

As an example consider the parametric representation of the elliptic paraboloid:

$$\mathbf{s}(u,v) = u\mathbf{e}_{x} + v\mathbf{e}_{y} + (u^{2} + v^{2})\mathbf{e}_{z}$$
(14)

- a) Consider the *u*-parameter curve $\mathbf{c}(u, 0.7)$ and calculate the arc length of this parabola from u = -1m to u = +1m.
- b) Use MATLAB to plot the elliptic paraboloid (14) and the parabolic curve c(u, 0.7).

10) The coefficients of the first fundamental form may be used to calculate surface area (Fig. 3.26) given the parametric representation of a surface. The general equation for the surface area in terms of the parametric representation $\mathbf{s}(u,v)$ and the coefficients of the first fundamental form is given in (3.95) as:

$$A = \iint \left(EG - F^2 \right)^{1/2} \mathrm{d}u \mathrm{d}v \tag{15}$$

Consider the following representation of the sphere of radius *a*:

$$\mathbf{s}(\theta,\phi) = (a\cos\theta\sin\phi)\mathbf{e}_x + (a\sin\theta\sin\phi)\mathbf{e}_y + (a\cos\phi)\mathbf{e}_z$$

0(16)

- a) Derive an equation for the surface area of the sector of the sphere within the range $0 \le \theta \le \pi/4$ and $0 < \phi \le \pi/6$ and evaluate this for a = 1m.
- b) Plot this sector of the sphere in 3D using MATLAB.

11) The general equation for the unit normal vector to a surface with the parametric representation s(u,v) is given in (3.73) as:

$$\mathbf{N} = \left(\frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v}\right) / \left|\frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v}\right|$$
(17)

As an example consider the surface with the parametric representation:

$$\mathbf{s}(u,v) = u\mathbf{e}_x + v\mathbf{e}_y + (u^2 - v^2)\mathbf{e}_z$$
(18)

- a) Use MATLAB to plot a 3D illustration of this surface as a wire frame or other suitable graph and describe the shape. This is the hyperbolic paraboloid (Fig. 3.29c).
- b) Derive an equation for the unit normal vector, N(u, v), of the hyperbolic paraboloid given in (18). Compare your equation to (3.75) which is the unit normal vector of the elliptic paraboloid and describe the differences.
- c) Use MATLAB to plot the unit normal vectors on the hyperbolic paraboloid.

12) The first and second fundamental forms are used to characterize the local shape of surfaces (Fig. 3.29) including familiar shapes of geological surfaces (Fig. 3.32). As an example consider the hyperbolic paraboloid which is better known in the geological context as a saddle structure:

$$\mathbf{s}(u,v) = u\mathbf{e}_x + v\mathbf{e}_y + (u^2 - v^2)\mathbf{e}_z$$
(19)

- a) Derive equations for the coefficients [E(u, v), F(u, v), G(u, v)] of the first fundamental form for the hyperbolic paraboloid and write down the equation for the first fundamental form, *I*.
- b) Derive equations for the coefficients [L(u, v), M(u, v), N(u, v)] of the second fundamental form for the hyperbolic paraboloid and write down the equation for the second fundamental form, *II*.
- c) Use the coefficients of the second fundamental form to identify whether the point on the hyperbolic paraboloid at the origin (x = y = z = 0) is elliptic, hyperbolic, parabolic or planar. What can you conclude about the shape at an arbitrary point on the hyperbolic paraboloid?
- d) The normal curvature, κ_n , for a surface is given in (3.120) by the ratio of the two fundamental forms, $\kappa_n = II/I$. Derive an equation for the normal curvature of the hyperbolic paraboloid in terms of the parameters (u, v) and their differentials (du, dv). Use the equation for κ_n to determine the normal curvature at the origin (x = y = z = 0) for this surface.
- e) Plot the distribution of normal curvature at the origin as a function of orientation using MATLAB. Hint: consider the circular path around the origin defined by $du^2 + dv^2 = 1$, and plot κ_n versus the angle measured from the *u*-axis. Identify the values of the principal normal curvatures and the principal directions at the origin.