Solutions to the problems in

Elastic Wave Propagation and Generation in Seismology

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CHAPTER 1

•Problem 1. An example is given by the reflection of axes referred to at the end of §1.3.

•Problem 2. Counterclockwise rotation. Fig. 9.11 shows this type of rotation for $\alpha = 45°$. Let the angle between x_1 and x'_1 be any α. Then, using the definition (1.3.1):

$$a_{11} = \mathbf{e}'_1.\mathbf{e}_1 = \cos\alpha$$

$$a_{22} = \mathbf{e}'_2.\mathbf{e}_2 = 1$$

$$a_{33} = \mathbf{e}'_3.\mathbf{e}_3 = \cos\alpha$$

$$a_{13} = \mathbf{e}'_1.\mathbf{e}_3 = \cos(\pi/2 - \alpha) = \sin\alpha$$

$$a_{31} = \mathbf{e}'_3.\mathbf{e}_1 = \cos(\pi/2 + \alpha) = -\sin\alpha$$

The other a_{ij} are zero because $\mathbf{e}_2$ and $\mathbf{e}'_2$ are perpendicular to the other unit vectors. Then

$$\mathbf{A} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

For the clockwise rotation, $a_{13} = \cos(\pi/2 + \alpha)$ and $a_{31} = \cos(\pi/2 - \alpha)$ and

$$\mathbf{A} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

•Problem 3.

(a) Multiply (1.3.2) scalarly with $\mathbf{e}_k$:

$$\mathbf{e}'_i.\mathbf{e}_k = \sum_{j=1}^{3} a_{ij}\mathbf{e}_j.\mathbf{e}_k = \sum_{j=1}^{3} a_{ij}\delta_{jk} = a_{ik}$$

Because of (1.2.7) the only nonzero term is the one shown above. This result is similar to (1.3.1) with j replaced by k.

(b) Start with

$$\mathbf{e}_j = a_{ij}\mathbf{e}'_i$$

To verify that this expression is correct multiply scalarly with $\mathbf{e}'_k$:

$$\mathbf{e}'_k.\mathbf{e}_j = a_{ij}\mathbf{e}'_k\mathbf{e}'_i = a_{ij}\delta_{ik} = a_{kj}$$

Then

$$\mathbf{v} = v_j\mathbf{e}_j = v_j a_{ij}\mathbf{e}'_i = v'_i\mathbf{e}'_i; \qquad v'_i = a_{ij}v_j$$

•Problem 4.

$$\mathbf{u} = (0.30, 0.50)$$

For Fig. 1.2a, add λ to the components of $\mathbf{u}$:

$$\mathbf{v} = (u_1 + \lambda, u_2 + \lambda) = (v_1, v_2) = (0.55, 0.75)$$

The rotation is counterclockwise and the corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} \cos 40° & \sin 40° \\ -\sin 40° & \cos 40° \end{pmatrix}$$

Apply $\mathbf{A}$ to $\mathbf{u}$. This gives $\mathbf{u}' = (u'_1, u'_2)$:

$$u'_1 = a_{11}u_1 + a_{12}u_2 = 0.55$$
$$u'_2 = a_{21}u_1 + a_{22}u_2 = 0.19$$

Add λ to the components of $\mathbf{u}'$:

$$\mathbf{v}' = (v'_1, v'_2) = (u'_1 + \lambda, u_2 + \lambda') = (0.80, 0.44)$$

Inverse rotation matrix:

$$\mathbf{A}^{-1} = \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

Apply $\mathbf{A}^{-1}$ to $\mathbf{v}'$. The result are the components of $\mathbf{v}'$ in the unprimed coordinate system. Let c_1 and c_2 indicate these components. Then

$$c_1 = a_{11}b_1 + a_{21}b_2 = 0.33 \neq v_1$$
$$c_2 = a_{12}b_1 + a_{22}b_2 = 0.85 \neq v_2$$

For Fig. 1.2b, λ is a factor for the vector components

$$\mathbf{v} = (\lambda v_1, \lambda v_2) = (0.39, 0.65)$$

Then, proceeding as before

$$\mathbf{u}' = (0.55, 0.19)$$
$$\mathbf{v}' = (0.72, 0.25)$$
$$(c_1, c_2) = (v_1, v_2)$$

•Problem 5.

(a)

$$\nabla.(\mathbf{a} \times \mathbf{b}) = (\epsilon_{ijk}a_jb_k)_{,i} = \epsilon_{ijk}a_{j,i}b_k + \epsilon_{ijk}a_jb_{k,i} = b_k\epsilon_{kij}a_{j,i} - a_j\epsilon_{jik}b_{k,i} = \mathbf{b}.(\nabla\times\mathbf{a}) - \mathbf{a}.(\nabla\times\mathbf{b})$$

(b)

$$\nabla.(f\mathbf{a}) = (f\mathbf{a})_{i,i} = f_{,i}a_i + fa_{i,i} = (\nabla f).\mathbf{a} + f(\nabla.\mathbf{a})$$

(c)

$$(\nabla \times (f\mathbf{a}))_{,i} = \epsilon_{ijk}(f\mathbf{a})_{k,j} = \epsilon_{ijk}f_{,j}a_k + \epsilon_{ijk}fa_{k,j} = (\nabla f) \times \mathbf{a} + f(\nabla \times \mathbf{a})$$

(d)

$$(\nabla \times \mathbf{r})_i = \epsilon_{ijk} x_{k,j} = \epsilon_{ijk}\delta_{kj} = \epsilon_{ikk} = 0$$

(e)

$$(\mathbf{a}.\nabla\mathbf{r})_j = ((\mathbf{a}.\nabla)\mathbf{r})_j = a_i x_{j,i} = a_i\delta_{ij} = a_j$$

(f)

$$(\nabla|\mathbf{r}|)_i = (\sqrt{x_j x_j})_{,i} = \frac{1}{2|\mathbf{r}|}(x_j x_j)_{,i} = \frac{1}{|\mathbf{r}|}x_j x_{j,i} = \frac{1}{|\mathbf{r}|}x_j\delta_{ji} = \frac{x_i}{|\mathbf{r}|} = \left(\frac{\mathbf{r}}{|\mathbf{r}|}\right)_i$$

Note: $(x_j x_j)_{,i} = x_j x_{j,i} + x_{j,i} x_j = 2x_j x_{j,i} = 2x_j\delta_{ji} = 2x_i$.

•Problem 6. $|\mathbf{v}|^2 = v_i v_i = a_{ji} v'_j a_{ki} v'_k = \delta_{jk} v'_j v'_k = v'_j v'_j = |\mathbf{v}'|^2$

•Problem 7. Let $\mathbf{n}'$ be equal to (1, 0, 0), (0,1,0), (0, 0, 1). Then (1.4.8) gives

$$(\tau'_{l1} - a_{li}a_{1j}\tau_{ij}) = 0$$

$$(\tau'_{l2} - a_{li}a_{2j}\tau_{ij}) = 0$$

$$(\tau'_{l3} - a_{li}a_{3j}\tau_{ij}) = 0$$

respectively. The three equations can be written as in (1.4.9).

•Problem 8. The x_i are the components of a vector ($\mathbf{r}$).

•Problem 9.

$$(\nabla(\nabla.\mathbf{u}) - \nabla\times\nabla\times\mathbf{u})_i = (u_{j,j})_{,i} - \epsilon_{ijk}(\nabla\times\mathbf{u})_{k,j} = u_{j,ji} - \epsilon_{ijk}\epsilon_{klm}u_{m,lj} =$$

$$= u_{j,ji} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, u_{m,lj} = u_{j,ji} - u_{j,ij} + u_{i,jj} = (\nabla^2\mathbf{u})_i$$

In the expression to the left of the last equals sign the first two terms are equal to each other.

•Problem 10. Let i be equal to 1. Writing in full we have:

$$\epsilon_{1jk}\alpha_{jk} = \epsilon_{123}\alpha_{23} + \epsilon_{132}\alpha_{32} = \alpha_{23} - \alpha_{32} = 0$$

Therefore, $\alpha_{23} = \alpha_{32}$. Similar results are obtained when $i = 2$ and $i = 3$.

•Problem 11.

(a) Let

$$|B| = \epsilon_{ijk} b_{i1} b_{j2} b_{k3}$$

(column expansion) and let C be the matrix obtained by interchanging the first and second columns of B. Then

$$|C| = \epsilon_{ijk} b_{i2} b_{j1} b_{k3} = \epsilon_{jik} b_{j2} b_{i1} b_{k3} = -\epsilon_{ijk} b_{i1} b_{j2} b_{k3} = -|B|$$

The expression to the right of the second equals sign is obtained by interchanging i and j. A similar argument shows that this result applies when any other pair of columns or rows is interchanged.

(b) Let the first two columns of B be equal to each other; i.e., $b_{i1} = b_{i2}$. Then

$$|B| = \epsilon_{ijk} b_{i1} b_{j2} b_{k3} = \epsilon_{ijk} b_{i2} b_{j1} b_{k3} = \epsilon_{ijk} b_{j1} b_{i2} b_{k3} = -\epsilon_{jik} b_{j1} b_{i2} b_{k3} = -|B|$$

Therefore, $B = 0$. The expression to the right of the third equals sign is obtained from that on the left by a change in the order of the factors. A similar argument shows that this result applies to any other pair of columns or rows.

(c) Let

$$d_1 = \epsilon_{lmn}|B|$$

$$d_2 = \epsilon_{ijk} b_{il} b_{jm} b_{kn}$$

$$d_3 = \epsilon_{ijk} b_{li} b_{mj} b_{nk}$$

Unless l, m and n are all different, $d_2 = d_3 = 0$ because of (b) and d_1 is also equal to zero. If $(l, m, n) = (1, 2, 3)$, $d_2 = d_3$ from the definition of determinant and $d_1 = |B|$. If (l, m, n) is an even permutation of $(1, 2, 3)$, d_1 and d_2 have an even number of permutations and $d_2 = d_3 = |B| = d_1$. If (l, m, n) is an odd permutation, then $d_2 = d_3 = -|B| = d_1$.

•Problem 12. Apply the result of Problem 1.11(c) with $B = A$, where A is a rotation matrix, use $|A| = 1$, and contract with a_{pn}:

$$\epsilon_{lmn} a_{pn} = \epsilon_{ijk} a_{il} a_{jm} a_{kn} a_{pn} = \epsilon_{ijk} a_{il} a_{jm} \delta_{kp} = \epsilon_{ijp} a_{il} a_{jm}$$

Here p is a free index. Changing it to k gives

$$\epsilon_{lmn} a_{kn} = \epsilon_{ijk} a_{il} a_{jm}$$

•Problem 13. Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Show that $w'_r = a_{rn} w_n$.

$$w'_r = \epsilon_{rst} u'_s v'_t = \epsilon_{rst} a_{sp} u_p a_{tq} v_q = \epsilon_{rst} a_{sp} a_{tq} u_p v_q = \epsilon_{str} a_{sp} a_{tq} u_p v_q$$

$$= \epsilon_{pqn} a_{rn} u_p v_q = a_{rn} \epsilon_{npq} u_p v_q = a_{rn} w_n$$

where the result of Problem 1.12 was used.

•Problem 14.

(a) Start with

$$t_{ij} v_j = \lambda v_i$$

Assume λ is complex. Then,

$$t_{ij} v_j^* = \lambda^* v_i^*$$

where the * indicates complex conjugation. Contract the first equation with v_i^* and the second with v_i

$$t_{ij} v_j v_i^* = \lambda v_i v_i^*$$

$$t_{ij} v_j^* v_i = \lambda^* v_i^* v_i$$

The fourth equation can be rewritten as follows

$$t_{ij} v_j^* v_i = t_{ji} v_j^* v_i = t_{ij} v_i^* v_j = \lambda^* v_i^* v_i$$

The symmetry of t_{ij} was used. Subtract the third equation from the last one

$$0 = (\lambda^* - \lambda) v_i^* v_i$$

Then, assuming $v_i^* v_i \neq 0$

$$\lambda^* = \lambda$$

and λ is real.

(b) Start with

$$t_{ij} u_j = \lambda u_i$$

$$t_{ij} v_j = \mu v_i$$

Contract the first equation with v_i and the second with u_i

$$t_{ij} u_j v_i = \lambda u_i v_i$$

$$t_{ij} v_j u_i = \mu v_i u_i$$

The fourth equation can be rewritten as follows

$$t_{ji} v_i u_j = t_{ij} v_i u_j = \mu v_i u_i$$

Subtract the third equation from the last one:

$$0 = (\mu - \lambda) v_i u_i$$

Therefore, if $\lambda \neq \mu$

$$v_i u_i = 0$$

If $\lambda = \mu$ we cannot say anything about $v_i u_i$.

•Problem 16.

(a) Start with (1.4.10) and contract i and j:

$$t'_{ii} = a_{ik} a_{il} t_{kl} = \delta_{kl} t_{kl} = t_{ll}$$

(b) Start with (1.4.80)

$$v_k t_{km} = \lambda v_m$$

and write the tensor and vector using (1.3.9) and (1.4.12):

$$a_{ik} v'_i a_{nk} a_{lm} t'_{nl} = \delta_{in} v'_i a_{lm} t'_{nl} = a_{lm} v'_n t'_{nl} = \lambda a_{lm} v'_l$$

Therefore

$$a_{lm}(v'_n t'_{nl} - \lambda v'_l) = 0$$

Contracting with a_{km} gives

$$v'_n t'_{nk} = \lambda v'_k$$

•Problem 17. Introduce (1.4.107) in (1.4.113) and use (1.4.65) and $\delta_{kk} = 3$:

$$w_i = \frac{1}{2}\epsilon_{ijk} W_{jk} = \frac{1}{2}\epsilon_{ijk}\epsilon_{jkl} w_l = -\frac{1}{2}\epsilon_{jik}\epsilon_{jkl} w_l = -\frac{1}{2}\left(\delta_{ik}\delta_{kl} - \delta_{il}\delta_{kk}\right) w_l =$$

$$-\frac{1}{2}\left(\delta_{il} - 3\delta_{il}\right) w_l = w_i$$

•Problem 18. We must show that $w'_i = a_{ip}w_p$. Start with

$$w'_i = \frac{1}{2}\epsilon_{ijk}W'_{jk} = \frac{1}{2}\epsilon_{ijk}a_{jm}a_{kn}W_{mn} = \frac{1}{2}\epsilon_{jki}a_{jm}a_{kn}W_{mn} = \frac{1}{2}\epsilon_{mnp}a_{ip}W_{mn} =$$

$$\frac{1}{2}a_{ip}\epsilon_{mnp}W_{mn} = a_{ip}w_p$$

where the result of Problem 1.12 was used.

•Problem 19.
(a)

$$\begin{vmatrix} -\lambda & w_3 & -w_2 \\ -w_3 & -\lambda & w_1 \\ w_2 & -w_1 & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2 + w_1^2 + w_2^2 + w_3^2\right) = -\lambda\left(\lambda^2 + |\mathbf{w}|^2\right) = 0$$

Then

$$\lambda_1 = 0; \qquad \lambda_2 = i|\mathbf{w}|; \qquad \lambda_3 = -i|\mathbf{w}|$$

(b)

$$W_{ij}w_j = \epsilon_{ijk}w_k w_j = 0$$

so that λ_1 is an eigenvalue. Here we used the fact that $w_k w_j$ is symmetric in k and j and (1.4.60).

•Problem 20. Use (1.4.113) with $W_{ij} = a_i b_j - b_i a_j$ and (1.4.56):

$$w_i = \frac{1}{2}\epsilon_{ijk}\left(a_j b_k - b_j a_k\right) = \frac{1}{2}\epsilon_{ijk}a_j b_k - \frac{1}{2}\epsilon_{ikj}b_k a_j = \epsilon_{ijk}a_j b_k = (\mathbf{a}\times\mathbf{b})_i$$

•Problem 21.

$$A \approx \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix}$$

The condition for the approximation is

$$\frac{1-\cos\alpha}{\cos\alpha} \le 0.01 \quad \Rightarrow \quad \frac{1}{1.01} \le \cos\alpha \quad \Rightarrow \quad \alpha \le 8^\circ$$

Then

$$100\times\frac{|\cos 8^\circ - 1|}{\cos 8^\circ} = 0.98$$

and

$$100\times\frac{|\sin 8^\circ - \alpha_r|}{\sin 8^\circ} = 0.33; \qquad \alpha_r = \frac{8\pi}{180}$$

•Problem 22. Use the definition (1.6.31):

$$\mathcal{T}_c\cdot\mathbf{v} = t_{ij}\mathbf{e}_j\mathbf{e}_i.\mathbf{v} = t_{ij}\mathbf{e}_j(\mathbf{e}_i.\mathbf{v}) = t_{ij}v_i\mathbf{e}_j$$

On the other hand:

$$\mathbf{v}.\mathcal{T} = \mathbf{v}.t_{ij}\mathbf{e}_i\mathbf{e}_j = t_{ij}(\mathbf{v}.\mathbf{e}_i)\mathbf{e}_j = t_{ij}v_i\mathbf{e}_j = \mathcal{T}_c\cdot\mathbf{v}$$

CHAPTER 2

•Problem 1. Use

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Then

$$2\varepsilon_{23,23} = \left(u_{2,3} + u_{3,2}\right)_{,23} = u_{2,323} + u_{3,223} = \left(u_{2,2}\right)_{3,3} + \left(u_{3,3}\right)_{2,2} = \left(\varepsilon_{22}\right)_{,33} + \left(\varepsilon_{33}\right)_{,22}$$

This proves (2.4.4).

For (2.4.5) write the individual terms separately:

$$\varepsilon_{33,12} = u_{3,312}$$

$$-\varepsilon_{12,33} = -\frac{1}{2}\left(u_{1,233} + u_{2,133}\right)$$

$$\varepsilon_{23,13} = \frac{1}{2}\left(u_{2,313} + u_{3,213}\right)$$

$$\varepsilon_{31,23} = \frac{1}{2}\left(u_{3,123} + u_{1,323}\right)$$

Adding the last three equations and using $u_{i,jkl} = u_{i,kjl} = u_{i,ljk} = ...$, gives the first equation.

•Problem 2. Using (2.3.15)

$$x_{i,1} = u_{i,1} + \delta_{i1}$$

Then

$$d\mathbf{r}^{(1)} = (1 + u_{1,1}, u_{2,1}, u_{3,1})$$

and

$$ds_1^2 = d\mathbf{r}^{(1)}.d\mathbf{r}^{(1)} = (1 + u_{1,1})^2 + u_{2,1}^2 + u_{3,1}^2 = 1 + 2u_{1,1} + u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2 \approx 1 + 2u_{1,1}$$

Then, using $(1+a)^n \approx 1 + na$ for $a << 1$, we get

$$ds_1 \approx (1 + 2u_{1,1})^{1/2} \approx 1 + u_{1,1}$$

•Problem 3.

$$w_i = -\frac{1}{2}(\nabla\times\mathbf{u})_i = -\frac{1}{2}\epsilon_{ijk}u_{k,j} = -\frac{1}{2}\delta_{i3}\epsilon_{321}u_{1,2} = \frac{1}{2}u_{1,2}\delta_{i3} = \frac{1}{2}\alpha\delta_{i3}$$

and

$$\mathbf{w} = \frac{1}{2}\alpha(0,0,1) = \frac{1}{2}\alpha\mathbf{e}_3$$

•Problem 4. Write an equation similar to (1.4.90):

$$\begin{vmatrix} -\lambda & \alpha & 0 \\ \alpha & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda(-\lambda^2 + \alpha^2) = 0$$

Solving for λ gives $\lambda_1 = 0$, $\lambda_2 = \alpha$, $\lambda_3 = -\alpha$. Therefore, in the rotated system the tensor has the expression given in (2.7.24). To determine the eigenvectors solve equations similar to (1.4.92) for these three eigenvalues. The corresponding unit eigenvectors are $(0, 0, \pm 1)$, $(1/\sqrt{2})(1, 1, 0)$, and $(1/\sqrt{2})(1, -1, 0)$.

•Problem 5. Let m be mass. Then

$$\frac{\rho - \rho_o}{\rho_o} = \frac{1}{(m/V_o)}\left(\frac{m}{V} - \frac{m}{V_o}\right) = \frac{V_o - V}{V} = \frac{V_o - (V_o + dV)}{V_o + dV} = -\frac{dV}{V_o + dV}$$

$$= -\frac{dV/V_o}{1 + dV/V_o} = -\frac{\varepsilon_{ii}}{1 + \varepsilon_{ii}} \approx -\varepsilon_{ii}(1 - \varepsilon_{ii}) \approx -\varepsilon_{ii} = -\nabla.\mathbf{u}$$

Here we used (2.4.22) and $(1 + x)^{-1} \approx 1 - x$ for small x.

•Problem 6. Let $\mathbf{b} = (1/2)\nabla \times \mathbf{u}$. Then

$$\mathcal{I}\times\mathbf{b} = \mathbf{e}_p\mathbf{e}_p \times \mathbf{b} = \mathbf{e}_p(\mathbf{e}_p \times \mathbf{b})$$

and

$$(\mathcal{I}\times\mathbf{b})_{ij} = \delta_{pi}\epsilon_{jkl}\delta_{pk}b_l = \delta_{ki}\epsilon_{jkl}b_l = \epsilon_{jil}b_l = -\epsilon_{ijl}b_l$$

Now use $b_l = (1/2)\epsilon_{lmn}u_{n,m}$. Then

$$\left(\mathcal{I}\times\left(\frac{1}{2}\nabla\times\mathbf{u}\right)\right)_{ij} = -\frac{1}{2}\epsilon_{ijl}\epsilon_{lmn}u_{n,m} = -\frac{1}{2}\epsilon_{lij}\epsilon_{lmn}u_{n,m} = \frac{1}{2}(u_{i,j} - u_{j,i}) = \omega_{ij} = (\Omega)_{ij}$$

Here (1.4.65) and (2.5.3) were used.

•Problem 7. If $\mathbf{v}$ is an eigenvector of T and λ is the corresponding eigenvalue, then

$$\mathsf{T}\mathbf{x} = \lambda\mathbf{x}$$

and

$$\mathbf{x}^T\mathsf{T}\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} = \lambda|\mathbf{x}|^2$$

Because the left-hand side and $|\mathbf{x}|^2$ are positive, $\lambda > 0$.

•Problem 8.

(a) Start with an expression similar to (2.5.1) using the Eulerian description:

$$du_i = u_i(\mathbf{r} + d\mathbf{r}) - u_i(\mathbf{r}) = \frac{\partial u_i}{\partial x_j}dx_j = u_{i,j}dx_j = (\mathbf{u}\nabla)_{ij}dx_j = dx_j(\nabla\mathbf{u})_{ji}$$

Write this expression in vector form:

$$d\mathbf{u} = \mathbf{u}\nabla.d\mathbf{r} = d\mathbf{r}.\nabla\mathbf{u}$$

(see (1.6.32)). Now use (2.5.4):

$$d\mathbf{R} = d\mathbf{r} - d\mathbf{u} = d\mathbf{r} - d\mathbf{r}.\nabla\mathbf{u} = d\mathbf{r}.(\mathcal{I} - \nabla\mathbf{u})$$

(b)

$$D^2 = [d\mathbf{r}.(\mathcal{I} - \nabla\mathbf{u})].[d\mathbf{r}.(\mathcal{I} - \nabla\mathbf{u})] = [d\mathbf{r}.(\mathcal{I} - \nabla\mathbf{u})].[(\mathcal{I} - \mathbf{u}\nabla).d\mathbf{r}] =$$

$$d\mathbf{r}.[(\mathcal{I} - \nabla\mathbf{u}).(\mathcal{I} - \mathbf{u}\nabla)].d\mathbf{r} = d\mathbf{r}.(\mathcal{I} - \nabla\mathbf{u} - \mathbf{u}\nabla + \nabla\mathbf{u}.\mathbf{u}\nabla).d\mathbf{r} \approx d\mathbf{r}.(\mathcal{I} - 2\mathcal{E}).d\mathbf{r}$$

Here (1.6.32) and (2.6.3) were used.

(c) Using (2.6.4) the factor in the last equation in (b) becomes

$$d\mathbf{r}'.[(1 - 2\epsilon_1)\mathbf{e}'_1\mathbf{e}'_1 + (1 - 2\epsilon_2)\mathbf{e}'_2\mathbf{e}'_2 + (1 - \epsilon_3)\mathbf{e}'_3\mathbf{e}'_3].d\mathbf{r}'$$

where $d\mathbf{r}' = dx_i\mathbf{e}'_i$. This gives three terms of the form

$$dx'_i\mathbf{e}'_i.(1 - 2\epsilon_J)\mathbf{e}'_J\mathbf{e}'_J.dx'_k\mathbf{e}'_k = (1 - 2\epsilon_J)dx'_idx'_k(\mathbf{e}'_i.\mathbf{e}'_J)(\mathbf{e}'_J.\mathbf{e}'_k) = (1 - 2\epsilon_J)(dx'_J)^2; \qquad J = 1, 2, 3$$

Therefore,

$$D^2 = (1 - 2\epsilon_1)(dx'_1)^2 + (1 - 2\epsilon_2)(dx'_2)^2 + (1 - 2\epsilon_3)(dx'_3)^2$$

(d) The volume of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

is $(4/3)\pi abc$. Therefore, the volume of the ellipsoid in (c) is given by

$$V = \frac{4}{3}\pi D^3[(1 - 2\epsilon_1)(1 - 2\epsilon_2)(1 - 2\epsilon_3)]^{-1/2} \approx V_o(1 + \epsilon_1)(1 + \epsilon_1)(1 + \epsilon_1)$$

$$\approx V_o(1 + \epsilon_1 + \epsilon_2 + \epsilon_3) = V_o(1 + \nabla.\mathbf{u})$$

For the first approximation use $(1 - 2d)^{-1/2} \approx 1 + d$ for small d. For the second approximation second order terms were neglected. In the last step the result of Problem 1.16a was used. Finally,

$$\frac{V - V_o}{V_o} = \nabla\mathbf{u}$$

CHAPTER 3

•Problem 1. Start with

$$\frac{D}{Dt}(fg) = \frac{\partial}{\partial t}(fg) + \frac{\partial}{\partial x_k}(fg)\frac{\partial x_k}{\partial t} = g\frac{\partial f}{\partial t} + f\frac{\partial g}{\partial t} + \left(g\frac{\partial f}{\partial x_k} + f\frac{\partial g}{\partial x_k}\right)\frac{\partial x_k}{\partial t}$$

$$= g\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_k}\frac{\partial x_k}{\partial t}\right) + f\left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x_k}\frac{\partial x_k}{\partial t}\right) = g\frac{Df}{Dt} + f\frac{Dg}{Dt}$$

•Problem 2. Because J is a function of $\mathbf{R}$ and t, $D/Dt = \partial/\partial t$. Start with $J = \epsilon_{ijk}x_{1,i}x_{2,j}x_{3,k}$. Then

$$\frac{DJ}{Dt} = \epsilon_{ijk}\left[\left(\frac{\partial}{\partial t}x_{1,i}\right)x_{2,j}x_{3,k} + \left(\frac{\partial}{\partial t}x_{2,j}\right)x_{1,i}x_{3,k} + \left(\frac{\partial}{\partial t}x_{3,k}\right)x_{1,i}x_{2,j}\right]$$

Consider the first term. The time derivative gives:

$$\frac{\partial}{\partial t}x_{1,i} = \frac{\partial}{\partial t}\frac{\partial x_1}{\partial X_i} = \frac{\partial}{\partial X_i}\frac{\partial x_1}{\partial t} = \frac{\partial v_1}{\partial X_i}$$

Use $v_i = v_i(x_1, x_2, x_3, t)$; $x_j = (X_1, X_2, X_3)$. Then

$$\frac{\partial v_1}{\partial X_i} = \frac{\partial v_1}{\partial x_1}\frac{\partial x_1}{\partial X_i} + \frac{\partial v_1}{\partial x_2}\frac{\partial x_2}{\partial X_i} + \frac{\partial v_1}{\partial x_3}\frac{\partial x_3}{\partial X_i} = \frac{\partial v_1}{\partial x_l}\frac{\partial x_l}{\partial X_i} = \frac{\partial v_1}{\partial x_l}x_{l,i}$$

Then the first term becomes

$$\epsilon_{ijk}\left(\frac{\partial}{\partial t}x_{1,i}\right)x_{2,j}x_{3,k} = \epsilon_{ijk}\frac{\partial v_1}{\partial x_l}x_{l,i}x_{2,j}x_{3,k}$$

If $l = 1$, the right-hand side of the expression above gives

$$\frac{\partial v_1}{\partial x_1}\epsilon_{ijk}x_{1,i}x_{2,j}x_{3,k} = \frac{\partial v_1}{\partial x_1}\,J$$

If $l = 2$ we get

$$\frac{\partial v_1}{\partial x_2}\epsilon_{ijk}x_{2,i}x_{2,j}x_{3,k} = 0$$

Here we used the fact that $x_{2,i}x_{2,j}$ is symmetric in i and j (1.4.60). Alternatively, we are in the case of Problem 1.11(b). If $l = 3$ we also get zero.

Similarly, the second and third terms of the first equation give

$$\frac{\partial v_2}{\partial x_2}\,J \qquad \text{and} \qquad \frac{\partial v_3}{\partial x_3}J$$

Finally, adding the three terms together gives:

$$\frac{DJ}{Dt} = J\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) = J(\nabla.\mathbf{v}) = Jv_{k,k}$$

•Problem 3. Given

$$\int_{V_o} \rho_o dV_o = \int_V \rho dV$$

change the variables of integration x_i to X_i in the second integral. This changes the integration volume to V_o. When changing variables the Jacobian (2.2.3) is needed. This gives

$$\int_{V_o} \rho_o dV_o = \int_{V_o} \rho J dV_o$$

Then

$$\int_{V_o} (\rho J - \rho_o) dV_o = 0$$

and

$$\rho J = \rho_o$$

provided that the integrand is continuous (see the comments that follow (3.5.3)).

•Problem 4.

(a) From Problem 3.3, because ρ_o depends on $\mathbf{R}$ only

$$\frac{D}{Dt}(\rho J) = 0$$

Now use the results of Problems 3.1 and 3.2:

$$\frac{D}{Dt}(\rho J) = J\frac{D\rho}{Dt} + \rho\frac{DJ}{Dt} = J\left(\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u}\right) = 0$$

and

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} \ = 0$$

because $J \neq 0$ (p. 42).

(b) Use the result in (a) and (3.2.4,6)

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} \ = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x_k}\frac{\partial x_k}{\partial t} + \rho v_{k,k} = \frac{\partial\rho}{\partial t} + \rho_{,k}v_k + \rho v_{k,k} = \frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = \frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v})$$

•Problem 5. Convert the integral over V into an integral over V_o (the volume before deformation) and note that V_o is fixed in time, so that differentiation and integration commute. Then use the result of Problem 3.1 and the first equation in Problem 3.4:

$$\frac{d}{dt}\int_V \rho\phi dV = \int_{V_o} \frac{D}{Dt}(J\rho\phi)dV_o = \int_{V_o}\left[\phi\frac{D}{Dt}(J\rho) + J\rho\frac{D\phi}{Dt}\right]dV_o = \int_{V_o} J\rho\frac{D\phi}{Dt}dV_o = \int_V \rho\frac{D\phi}{Dt}dV$$

•Problem 6. Using (3.3.2) and (3.3.3) we obtain

$$\int_S \mathbf{T}dS + \int_V \rho\left(\mathbf{f} - \frac{D\mathbf{v}}{Dt}\right)dV = 0$$

The surface S is made of three surfaces; S^+, S^- and δS. Let r and h be the radius and thickness of the disk. The volume of the disk is $\pi r^2 h$ and the area of δS is $2\pi r h$. We will use the following result

$$\int_V f(x_1, x_2, x_3) dV \le \max\{|f|\} V$$

where f is any function defined within V. A similar result applies if the integration domain is a surface. Apply this to the jth components of the surface and volume integrals in the first equation

$$\int_{\delta S} T_j dS \le T_{max} \int_{\delta S} dS = T_{max} 2\pi r h$$

$$\int_V \rho \left(f_j - \frac{Dv_j}{Dt} \right) dV \le I_{max} \int_V dV = I_{max} \pi r^2 h$$

where T_{max} and I_{max} are the maximum values of the absolute values of the integrands. When h goes to zero, the two integrals go to zero.

•Problem 7.

(a). Equation of the plane through points A, B, C:

$$a x_1 + b x_2 + c x_3 = d$$

Let

$$\mathbf{p} = (a, b, c); \qquad \mathbf{n} = (n_1, n_2, n_3) = \frac{1}{|\mathbf{p}|}(a, b, c)$$

where $\mathbf{p}$ and $\mathbf{n}$ are a vector and a unit vector both normal to the plane. Also, if $\mathbf{x} = (x_1, x_2, x_3)$ is a vector from the origin to any point on the plane and $\mathbf{p}$ is the vector from the origin to the point on the plane for which $\mathbf{p}$ is perpendicular to the plane, then the equation of the plane is

$$\mathbf{p}.\mathbf{x} = |\mathbf{p}| h = d$$

where h is the distance from the origin to te plane (see figure 3.3, where the point P is the origin, and Problem 19, page 21, of Spiegel (1959)). From this equation we have

$$|\mathbf{p}| = \frac{d}{h}; \qquad \mathbf{n} = \frac{h}{d}(a, b, c)$$

Now refer to Fig. 3.3. The coordinates of the points A, B, and C are

$$\left(\frac{d}{a}, 0, 0\right) = \left(\frac{h}{n_1}, 0, 0\right)$$

$$\left(0, \frac{d}{b}, 0\right) = \left(0, \frac{h}{n_2}, 0\right)$$

$$\left(0, 0, \frac{d}{c}\right) = \left(0, 0, \frac{h}{n_2}\right)$$

respectively. The area dS_n of the triangle ABC is one-half of the absolute value of the vector product of the vectors $\overrightarrow{BA}$ and $\overrightarrow{CA}$

$$dS_n = \frac{1}{2}\left|\overrightarrow{B-A}\times\overrightarrow{C-A}\right| = \frac{1}{2}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -h/n_1 & h/n_2 & 0 \\ -h/n_1 & 0 & h/n_3 \end{vmatrix} = \frac{1}{2}h^2\left|\left(\frac{1}{n_2n_3}, \frac{1}{n_1n_3}, \frac{1}{n_1n_2}\right)\right|$$

$$= \frac{1}{2}\frac{h^2}{n_1n_2n_3}|\mathbf{n}| = \frac{1}{2}\frac{h^2}{n_1n_2n_3}$$

and the volume of the tetrahedron is given by

$$V = \frac{1}{2}\int_0^h dS_n(h')dh' = \frac{1}{2}\frac{1}{n_1n_2n_3}\int_0^h h'^2 dh' = \frac{1}{6}\frac{h^3}{n_1n_2n_3} = \frac{1}{3}hdS_n$$

where $dS_n(h')$ is the area of the triangle parallel to ABC a distance h' from the origin.

(b) As in Problem 3.6, and using the results in (a) and (c), we have

$$\int_S T_j dS \le T_{max}\int_S dS = T_{max}(dS_n + dS_1 + dS_2 + dS_3) \propto T_{max}dS_n$$

because $|n_i| \le 1$

$$\int_V \rho\left(f_j - \frac{Dv_j}{Dt}\right)dV \le I_{max}\int_V dV = \frac{1}{3}I_{max}hdS_n$$

As h goes to zero, the volume and the surface of the tetrahedron go to zero, but the volume vanishes faster (because of the factor h).

(c) As done in (a), for dS_3 we have

$$dS_3 = \frac{1}{2}\left|\overrightarrow{A-P}\times\overrightarrow{B-P}\right| = \frac{1}{2}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ h/n_1 & 0 & 0 \\ 0 & h/n_2 & 0 \end{vmatrix} = \frac{1}{2}\frac{h^2}{n_1n_2}|\mathbf{e}_3| = n_3dS_n$$

Similarly

$$dS_1 = n_1dS_n; \qquad dS_2 = n_2dS_n$$

•Problem 8.

(a) Equation of the plane and normal unit vector:

$$x_1 + 3x_2 + 3x_3 = 3$$

$$\mathbf{n} = \frac{1}{\sqrt{19}}(1,3,3)$$

Stress vector. Use (3.4.15):

$$\mathbf{T} = \frac{1}{\sqrt{19}}\begin{pmatrix}1 & 3 & 3\end{pmatrix}\begin{pmatrix}1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1\end{pmatrix} = \frac{1}{\sqrt{19}}(1,0,0)$$

(b) Normal and shearing stress vectors. Use (3.8.2) and (3.8.4):

$$\mathbf{T}^N = (\mathbf{T}.\mathbf{n})\mathbf{n} = \frac{1}{19}\mathbf{n}$$

$$\mathbf{T}^S = \frac{1}{19\sqrt{19}}(18, -3, -3)$$

Furthermore,

$$\mathbf{T}^S.\mathbf{n} = 0$$

so that $\mathbf{T}^S$ is in the plane determined in (a).

(c) Solving the eigenvalue problem gives $\tau_1 = 1$, $\tau_1 = 0$ and $\tau_1 = -2$. Using (3.10.14), (3.10.16) and (3.10.17) we obtain

$$C_1 : \quad (\tau_n + 1)^2 + \tau_s^2 \geq 1$$

$$C_2 : \quad (\tau_n + 1/2)^2 + \tau_s^2 \leq (3/2)^2$$

$$C_3 : \quad (\tau_n - 1/2)^2 + \tau_s^2 \geq (1/2)^2$$

CHAPTER 4

•Problem 1. Use an argument similar to that used in connection with (1.4.68). Start with

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}$$

$$\tau'_{ij} = c'_{ijkl}\varepsilon'_{kl}$$

and write τ'_{ij} and ε'_{kl} in terms of the unprimed variables,

$$a_{im}a_{jn}\tau_{mn} = c'_{ijkl}a_{kp}a_{lq}\varepsilon_{pq}$$

Use the first equation to rewrite τ_{mn} and move the expression on the right-hand side to the left. This gives

$$\left(a_{im}a_{jn}c_{mnpq} - c'_{ijkl}a_{kp}a_{lq}\right)\varepsilon_{pq} = 0$$

which means that

$$c'_{ijkl}a_{kp}a_{lq} = a_{im}a_{jn}c_{mnpq}$$

Contract with a_{rp} and a_{sq}

$$c'_{ijkl}a_{kp}a_{rp}a_{lq}a_{sq} = c'_{ijkl}\delta_{kr}\delta_{ls} = c'_{ijrs} = a_{im}a_{jn}a_{rp}a_{sq}c_{mnpq}$$

The last equality is the transformation law for a fourth-order tensor.

•Problem 2. Use $c_{ijkl} = c_{jikl}$. The coefficients of λ and μ have the symmetry of c_{ijkl}. Therefore only the third term in (4.6.1) must be considered

$$\nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = \nu(\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ik})$$

Use (1.4.65) and $\epsilon_{mji} = -\epsilon_{mij}$. This gives

$$2\nu\epsilon_{mij}\epsilon_{mkl} = 0; \qquad m \neq i, j$$

When $i \neq j$ and $k \neq l$, this gives $\nu = 0$. When $i = j$ or $k = l$, the factor in parentheses in the third term of (4.6.1) is identically equal to zero, and (4.6.2) is satisfied.

•Problem 3. Start with

$$\varepsilon_{ij}x_j = \nu x_i$$

and use (4.6.8). This gives

$$\varepsilon_{ij}x_j = a(\tau_{ij} - b\delta_{ij})x_j = a\tau_{ij}x_j - abx_i = \nu x_i$$

where a and b represent obvious scalar factors. Then

$$\tau_{ij}x_j = \frac{1}{a}(\nu + b)x_i$$

Therefore, ε_{ij} and τ_{ij} have the same principal directions.

•Problem 4. From (4.6.13) we obtain

$$\lambda + \mu = \frac{\lambda}{2\sigma}$$

which can be rewritten as

$$\mu = \frac{\lambda(1-2\sigma)}{2\sigma}$$

Rewrite (4.6.12) as

$$\lambda + \mu = \frac{\mu}{Y}(3\lambda + 2\mu)$$

Replace $\lambda + \mu$ and μ using the first two expressions

$$\frac{\lambda}{2\sigma} = \frac{\lambda}{Y}\frac{1-2\sigma}{2\sigma}\left(3\lambda + \lambda\frac{1-2\sigma}{\sigma}\right)$$

Simple manipulations give

$$\lambda = \frac{Y\sigma}{(\sigma+1)(1-2\sigma)}$$

so that

$$\frac{\lambda(1-2\sigma)}{2\sigma} = \frac{Y}{2(\sigma+1)} = \mu$$

•Problem 5. Divide the numerator and denominator on the right of (4.6.12) and (4.6.13) by λ

$$Y = \frac{\mu(3 + 2\mu/\lambda)}{1 + \mu/\lambda}$$

$$\sigma = \frac{1}{2(1 + \mu/\lambda)}$$

and let λ go to ∞. This gives $Y = 3\mu$ and $\sigma = 1/2$.

•Problem 6. Replace (4.6.14) and (4.6.15) in (4.6.20).

$$k = \frac{Y\sigma}{(1+\sigma)(1-2\sigma)} + \frac{Y}{3(1+\sigma)} = \frac{Y}{3(1-2\sigma)}$$

•Problem 7. From (4.5.13) and (4.6.3)

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

Then

$$W = \frac{1}{2}\tau_{ij}\varepsilon_{ij} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{1}{2}\lambda\left(\varepsilon_{kk}\right)^2 + \mu\left(\varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{21}^2 + \varepsilon_{22}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2 + \varepsilon_{32}^2 + \varepsilon_{33}^2\right)$$

$$= \frac{1}{2}\lambda\left(\varepsilon_{kk}\right)^2 + \mu\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\varepsilon_{12}^2 + 2\varepsilon_{23}^2 + 2\varepsilon_{13}^2\right)$$

The symmetry of ε_{ij} was used.

•Problem 8. Using (4.7.3)

$$\varepsilon_{ij}\varepsilon_{ij} = (\bar{\varepsilon}_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij})(\bar{\varepsilon}_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}) = \bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij} + \frac{2}{3}\bar{\varepsilon}_{ij}\varepsilon_{kk}\delta_{ij} + \frac{1}{9}\left(\varepsilon_{kk}\right)^2\delta_{ij}\delta_{ij} =$$

$$\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij} + \frac{2}{3}\bar{\varepsilon}_{ii}\varepsilon_{kk} + \frac{1}{9}(\varepsilon_{kk})^2\delta_{ii} = \bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij} + \frac{1}{3}(\varepsilon_{kk})^2$$

Recall that $\bar{\varepsilon}_{ij}$ has zero trace and $\delta_{ii} = 3$. Now introduce this expression in the first equality of (4.7.1) and use (4.6.20)

$$W = \frac{1}{2}\lambda(\varepsilon_{kk})^2 + \mu\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij} + \frac{1}{3}\mu(\varepsilon_{kk})^2 = \frac{1}{2}k(\varepsilon_{kk})^2 + \mu\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij}$$

•Problem 9.

$$\nabla\cdot[\nabla(\nabla\cdot\mathbf{u})] = [\nabla(\nabla\cdot\mathbf{u})]_{i,i} = (\nabla.\mathbf{u})_{,ii} = \nabla^2(\nabla\cdot\mathbf{u})$$

$$\nabla\cdot(\nabla\times\nabla\times\mathbf{u}) = [\nabla\times(\nabla\times\mathbf{u})]_{i,i} = (\epsilon_{ijk}(\nabla\times\mathbf{u})_{k,j})_{,i} = [\epsilon_{ijk}\epsilon_{klm}u_{m,lj}]_{,i} = [u_{j,ij} - u_{i,jj}]_{,i}$$

$$= u_{j,iji} - u_{i,jji} = u_{i,jij} - u_{i,jji} = 0$$

Here (1.4.65) was used and because i and j are dummy indices, $u_{j,iji} = u_{i,jij}$.

$$\{\nabla\times[\nabla(\nabla\cdot\mathbf{u})]\}_i = \epsilon_{ijk}(u_{l,l})_{,kj} = 0$$

because $(u_{l,l})_{,kj}$ is symmetric in k and j.

$$[\nabla\times(\nabla\times\nabla\times\mathbf{u})]_i = \epsilon_{ijk}(u_{l,klj} - u_{k,llj}) = \epsilon_{ijk}(u_{l,l})_{,kj} - (\epsilon_{ijk}u_{k,j})_{,ll} = -[(\nabla\times\mathbf{u})_i]_{,ll} = -[\nabla^2(\nabla\times\mathbf{u})]_i$$

•Problem 10. From (4.8.5)

$$\mu = \rho\beta^2$$

$$\lambda = \rho\alpha^2 - 2\mu = \rho(\alpha^2 - 2\beta^2)$$

Introduce these expressions in (4.6.13)

$$\sigma = \frac{1}{2}\frac{\alpha^2 - 2\beta^2}{\alpha^2 - \beta^2}$$

•Problem 11.

$$\nabla.\mathbf{u} = u_{3,3}$$

$$\nabla(\nabla.\mathbf{u}) = u_{3,33}\mathbf{e}_3$$

$$\nabla\times\mathbf{u} = (u_{3,2}, -u_{3,1}, 0) = \mathbf{0}$$

$$\frac{\partial^2\mathbf{u}}{\partial t^2} = \frac{\partial^2 u_3}{\partial t^2}\mathbf{e}_3$$

Because all the relevant nonzero terms correspond to vector components in the x_3 direction, when introduced in (4.8.4) they satisfy

$$\alpha^2\frac{\partial^2 u_3}{\partial x_3^2} = \frac{\partial^2 u_3}{\partial t^2}$$

•Problem 12.

$$\nabla.\mathbf{u} = u_{2,2} = 0$$

$$\nabla \times \mathbf{u} = -u_{2,3}\mathbf{e}_1$$

$$\nabla \times \nabla \times \mathbf{u} = -u_{2,33}\mathbf{e}_2$$

.

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\partial^2 u_2}{\partial t^2}\mathbf{e}_2$$

Introducing these expressions in (4.8.4) shows that

$$\beta^2 \frac{\partial^2 u_2}{\partial x_3^2} = \frac{\partial^2 u_2}{\partial t^2}$$

CHAPTER 5

•Problem 1. Application of the boundary conditions gives

$$h(-x/c) + g(x/c) = F(x) \tag{1}$$

$$h'(-x/c) + g'(x/c) = G(x) \tag{2}$$

Integrate (2)

$$-c\, h(-x/c) + c\, g(x/c) = \int_0^x G(s)ds + a \tag{3}$$

Rewrite (3) as

$$-h(-x/c) + g(x/c) = \frac{1}{c}\int_0^x G(s)ds + k; \qquad\qquad k = \frac{a}{c} \tag{4}$$

Solve (1) and (4) for $h(-x/c)$ and $g(x/c)$

$$g(x/c) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(s)ds + \frac{k}{2} \tag{5}$$

$$h(-x/c) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(s)ds - \frac{k}{2} \tag{6}$$

Replace x by $x + ct$ in (5) and by $x - ct$ in (6)

$$g(t + x/c) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(s)ds + \frac{k}{2} \tag{7}$$

$$h(t - x/c) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(s)ds - \frac{k}{2} \tag{8}$$

Adding (7) and (8) gives $\psi(x,t)$

$$\psi(x,t) = \frac{1}{2}[F(x+ct) - F(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds \tag{9}$$

•Problem 2. The Laplacian in spherical coordinates has the following expression

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Here θ and ϕ are the angles shown in Fig. 9.3. Do not confuse the angle ϕ with the function ϕ used in §5.5. When $u = u(r)$, the Laplacian reduces to the first term

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$$

•Problem 3.

$$\nabla\cdot\ \mathbf{M} = M_{i,i} = \epsilon_{ijk}\phi_{,ji}a_k = 0$$

on account of the symmetry of $\phi_{,ji}$.

•Problem 4.

$$\psi_{,p} = -\mathrm{i}(\mathbf{k}.\mathbf{r})_{,p}\psi = -\mathrm{i}(k_j r_j)_{,p}\psi = -\mathrm{i}k_j r_{j,p}\psi = -\mathrm{i}k_j\delta_{jp}\psi = -\mathrm{i}k_p\psi$$

•Problem 5.

$$\mathbf{N}_p = \frac{1}{k}\epsilon_{pjq}M_{q,j} = -\mathrm{i}\frac{1}{k}\epsilon_{pjq}[(\mathbf{k}\times\mathbf{a})_q\psi]_{,j} = -\mathrm{i}\frac{1}{k}\epsilon_{pjq}(\mathbf{k}\times\mathbf{a})_q\psi_{,j}$$

$$= -\mathrm{i}\frac{1}{k}\epsilon_{pjq}(\mathbf{k}\times\mathbf{a})_q\,(-\mathrm{i}k_j\psi) = -\frac{1}{k}\epsilon_{pjq}(\mathbf{k}\times\mathbf{a})_q k_j\psi = \frac{1}{k}\epsilon_{pqj}(\mathbf{k}\times\mathbf{a})_q k_j\psi$$

•Problem 6. The spatial derivatives are the same as those in §5.6. The only difference is the replacement of $-\ddot{\psi}/c^2$ by $k_c^2\psi$ in equations such as (5.6.9), (5.6.14) and (5.6.22).

•Problem 7. From (5.8.60)

$$u_1 = c_1\cos(\omega t - k_1x_1)$$

$$u_3 = -c_3\sin(\omega t - k_1x_1)$$

Then

$$\frac{u_1^2}{c_1^2} + \frac{u_3^2}{c_3^2} = \cos^2(\omega t - k_1x_1) + \sin^2(\omega t - k_1x_1) = 1$$

•Problem 8. The real parts of the displacements given by (5.8.53)-(5.8.55) are

$$\mathbf{u}_P = A(l,0,n)\cos\left[\omega\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\alpha}\right)\right]$$

$$\mathbf{u}_{SV} = B\,(-n,0,l)\cos\left[\omega\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\beta}\right)\right]$$

$$\mathbf{u}_{SH} = C(0,1,0)\cos\left[\omega\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\beta}\right)\right]$$

In all cases,

$$\mathbf{p}.\mathbf{r} = lx_1 + nx_3$$

To get τ_{ij} for the displacements given above use (5.9.5). For the P waves we have

$$\lambda u_{k,k} = \lambda Al(-\sin[...])(-l\omega/\alpha) + \lambda An(-\sin[...])(-n\omega/\alpha) = \lambda\frac{A}{\alpha}\omega\sin[...]$$

because $l^2 + n^2 = |\mathbf{p}|^2 = 1$. The ellipsis represent the argument of the cosine in the expression for $\mathbf{u}_P$.

Let

$$a_{ij} = \mu\,(u_{i,j} + u_{j,i})$$

Then

$$a_{11} = 2\mu\frac{A}{\alpha}\omega l^2\sin[...]$$

$$a_{33} = 2\mu\frac{A\omega}{\alpha}\omega n^2\sin[...]$$

$$a_{13} = a_{31} = 2\mu ln\frac{A\omega}{\alpha}\sin[...]$$

All the other terms are zero because either $u_2 = 0$ or u_1 and u_3 do not depend on x_2.

For the SV waves

$$\lambda u_{k,k} = \lambda nl\omega\frac{B}{\beta}\sin[...] - \lambda ln\omega\frac{B}{\beta}\sin[...] = 0$$

(as expected)

$$a_{11} = -2\mu\frac{B}{\beta}\omega nl\sin[...]$$

$$a_{33} = 2\mu\frac{B\omega}{\beta}\omega nl\sin[...]$$

$$a_{13} = a_{31} = \mu(l^2 - n^2)\frac{B\omega}{\beta}\sin[...]$$

All the other terms are zero.
For the SH waves (nonzero terms)

$$a_{12} = a_{21} = \mu\frac{C}{\beta}\omega l\sin[...]$$

$$a_{23} = a_{32} = \mu\frac{C}{\beta}\omega n\sin[...]$$

•Problem 9. To obtain the expressions (5.9.9)-(5.9.11) we must perform the matrix operations indicated in (5.9.3). The expressions for τ_{ij} are given in (5.9.6)-(5.9.8). The velocity vectors are obtained by taking the derivatives with respect to t of the displacement vectors given in Problem 5.8, which gives

$$\dot{\mathbf{u}}_P = -A\omega(l, 0, n)^T sin[...]$$

$$\dot{\mathbf{u}}_{SV} = -A\omega(-n, 0, l)^T\sin[...]$$

$$\dot{\mathbf{u}}_{SH} = -A\omega(0, 1, 0)^T\sin[...]$$

The transposition is required for the matrix multiplications. For the P waves we need the following product

$$\begin{pmatrix} (\lambda + 2\mu l^2) & 0 & 2\mu ln \\ 0 & \lambda & 0 \\ 2\mu ln & 0 & (\lambda + 2\mu n^2) \end{pmatrix}\begin{pmatrix} l \\ 0 \\ n \end{pmatrix} = \begin{pmatrix} (\lambda + 2\mu l^2)l + 2\mu ln^2 \\ 0 \\ 2\mu l^2 n + (\lambda + 2\mu n^2)n \end{pmatrix}$$

$$= (\lambda + 2\mu)\begin{pmatrix} l \\ 0 \\ n \end{pmatrix} = (\lambda + 2\mu)\mathbf{p}$$

When all the factors are included we obtain

$$\mathbf{E}_P = \frac{A^2\omega^2}{\alpha}(\lambda + 2\mu)\mathbf{p}\sin^2[...] = \rho\alpha\omega^2 A^2\mathbf{p}\sin^2\left[\omega\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\alpha}\right)\right]$$

where (4.8.5a) was used.

For the SV waves we have

$$\begin{pmatrix} -2nl & 0 & (l^2-n^2) \\ 0 & 0 & 0 \\ (l^2-n^2) & 0 & 2nl \end{pmatrix} \begin{pmatrix} -n \\ 0 \\ l \end{pmatrix} = \mathbf{p}$$

and

$$\mathbf{E}_{SV} = \mu \frac{B^2\omega^2}{\beta}\mathbf{p}\sin[...] = \rho\beta\omega^2 B^2 \mathbf{p}\sin^2\left[\omega\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\beta}\right)\right]$$

where (4.8.5b) was used.

For the SH waves we have

$$\begin{pmatrix} 0 & l & 0 \\ l & 0 & n \\ 0 & n & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{p}$$

and

$$\mathbf{E}_{SH} = \mu \frac{C^2\omega^2}{\beta}\mathbf{p}\sin[...] = \rho\beta\omega^2 C^2 \mathbf{p}\sin^2\left[\omega\left(t - \frac{\mathbf{p}\cdot\mathbf{r}}{\beta}\right)\right]$$

•Problem 10. It is straightforward to derive (5.9.12)-(5.9.14) because $\mathbf{a}_3.\mathbf{p} = n$ and $\mathbf{p}.\mathbf{r} = lx_1$ when $x_3 = 0$.

•Problem 11. In general, if $T = 2\pi/\omega$

$$I = \frac{1}{T}\int_0^T \sin^2 \underbrace{(\omega t - b)}_{u} dt = \frac{1}{T\omega}\int_{-b}^{\omega T - b} \sin^2 u du = \frac{1}{2\omega T}\left(u - \frac{1}{2}\sin 2u\right)\Big|_{-b}^{\omega T - b} = \frac{1}{2}$$

Applying this result to (5.9.12)-(5.9.14) immediately gives (5.9.15)-(5.9.17).

CHAPTER 6

•Problem 1. From (4.6.3), the general expression for τ_{3i} is

$$\tau_{3i} = \lambda\delta_{3i}u_{k,k} + \mu(u_{i,3} + u_{3,i})$$

(1) *P waves:*

$$\tau_{31} = \mu(u_{1,3} + u_{3,1})$$

For the incident waves (see (6.2.4))

$$u_{1,3} = \mathrm{i}A\sin e\frac{\omega\cos e}{\alpha}\exp[...] = u_{3,i}$$

where the ellipsis indicate the argument in the exponential in the expression for $\mathbf{u}_P$. Then

$$\tau_{31} = 2\mu u_{1,3}$$

Similar arguments show that this expression applies to the reflected and transmitted waves.

$\tau_{32} = 0$ because $u_2 = 0$ and u_3 has no dependence on x_2.

$$\tau_{33} = \lambda(u_{1,1} + u_{3,3}) + 2\mu u_{3,3} = \lambda u_{1,1} + (\lambda + 2\mu)u_{3,3}$$

(2) *SV waves:*

$$\tau_{31} = \mu(u_{1,3} + u_{3,1})$$

(the two partial derivatives are not equal)

$\tau_{32} = 0$ because $u_2 = 0$ and u_3 has no dependence on x_2.

$$\tau_{33} = 2\mu u_{3,3}$$

because the divergence of $\mathbf{u}_{SV}$ is zero.

(3) *SH waves:* τ_{31} and τ_{33} are equal to zero because $u_1 = u_3 = 0$ and u_2 does not depend on x_2.

$$\tau_{32} = \mu(u_{2,3} + u_{3,2}) = \mu u_{2,3}$$

•Problem 2. The factor $b_j - b_k$ is a wave number, so that

$$\int_{-\infty}^{\infty} e^{\mathrm{i}(b_j - b_k)x_1}dx_1 = \mathcal{F}\{1\} = 2\pi\delta(b_j - b_k)$$

(see (A.62)) Interchanging the order of the integration and the summation in the left-hand side of (6.5.12) and using the result above gives the right-hand side. Now use the fact that $\delta(b_j - b_k)$ is different from zero if $b_j = b_k$ and equal to zero if $b_j \neq b_k$ (see the comments following (A.14). Then, if $k = 1$, the right-hand side of (6.5.12) gives

$$2\pi\sum_{j=0}^{3} a_j\delta(b_j - b_1) = 2\pi a_1\delta(0) = 0$$

so that $a_1 = 0$. Similar results are obtained with $k = 2, 3$.

•Problem 3.

(a) The third term in (6.5.10) is equal to zero when $f_1 = 0, \pi/2$, which means that a_3 in (6.5.11) is equal to zero. Applying an argument similar to that used in the previous problem we conclude that $b_1 = b_2$ or $e = e_1$. Now use (6.5.14) with $a_3 = 0$

$$\frac{A}{\alpha}(\lambda + 2\mu\cos^2 e) = -\frac{A_1}{\alpha}(\lambda + 2\mu\cos^2 e)$$

so that $A_1 = -A$. This result agrees with (6.5.20) for $f = 0$ (recall that f_1 was renamed f). For $f_1 = \pi/2$ see (c).

(b) Use the results above and $f_1 = 0$. After canceling the common factor of $\exp(\mathrm{i}\omega t)$, from (6.5.9) we get

$$2\frac{A}{\alpha}\sin 2e\exp(-\mathrm{i}\omega x_1 \sin e/\alpha) + \frac{B_1}{\beta}\exp(0) = 0$$

This implies that the argument of the exponential in the first term must be zero, which in turn means $e = 0$. But then the first term must be equal to zero, which means that $B_1 = 0$. This result agrees with (6.5.21) for $e = f = 0$. The result $e = f = 0$ agrees with Snell's law.

(c) From (a) we know that $A_1 = -A$ and $e_1 = e$. Use these results and $f_1 = \pi/2$ with (6.5.9)

$$2\frac{A}{\alpha}\sin 2e\exp(-\mathrm{i}\omega x_1 \sin e/\alpha) - \frac{B_1}{\beta}\exp(-\mathrm{i}\omega x_i/\beta) = 0$$

which implies

$$\frac{\sin e}{\alpha} = \frac{1}{\beta}$$

or

$$\sin e = \frac{\alpha}{\beta}$$

However, because $\alpha > \beta$, this equation cannot be satisfied. This also agrees with Snell's law, which shows that f_1 cannot be equal to $\pi/2$.

•Problem 4. The denominator of (6.5.21) is always positive because $e \le \pi/2$. The numerator is positive or zero as long as $f \le \pi/4$, in which case $B_1/A \le 0$. Let us investigate whether f can be larger than $\pi/4$. The largest value of f is attained for $e = \pi/2$, in which case, from Snell's law

$$\sin f = \frac{\beta}{\alpha}$$

Then, if $f > \pi/4$

$$\frac{\beta}{\alpha} > \frac{\sqrt{2}}{2}$$

or

$$\alpha^2 < 2\beta^2$$

However, from Problem 4.10 we see that this condition corresponds to a negative Poisson's ratio. Therefore, as long as $\sigma > 0$, $f < \pi/4$ and $B_1/A \le 0$.

•Problem 5. Start with (6.5.8) and use (6.5.15).

Aside from the exponential factor, the component in the $\mathbf{a}_1$ direction is given by

$$A\left(\sin e + \sin e\frac{A_1}{A} - \cos f\frac{B_1}{A}\right)$$

Use (6.5.20) and (6.5.21) and operate. Aside from a factor of A, the numerator of the resulting expression is

$$\sin e\left[\sin 2e\ \sin 2f + \left(\frac{\alpha}{\beta}\right)^2\cos^2 2f\right] + \sin e\left[\sin 2e\ \sin 2f - \left(\frac{\alpha}{\beta}\right)^2\cos^2 2f\right]$$

$$+\cos f\ \left(2\frac{\alpha}{\beta}\sin 2e\ \cos 2f\right) = \sin 2e\left[2\sin e\sin 2f + 2\frac{\alpha}{\beta}\cos f\cos 2f\right]$$

Rewrite the second term within the brackets

$$2\frac{\alpha}{\beta}\cos f\cos 2f = 2\frac{\alpha}{\beta}\cos f(1-2\sin^2 f) =$$

$$2\frac{\alpha}{\beta}\cos f - 2\frac{\alpha}{\beta}\sin f\ \sin 2f = 2\frac{\alpha}{\beta}\cos f - 2\sin e\sin 2f$$

(Snell's law was used). Introducing this result in the previous expression gives

$$2\frac{\alpha}{\beta}\cos f\ \sin 2e = 2\frac{\alpha}{\beta}\cos f 2\frac{\alpha}{\beta}\sin f\cos e = 2\left(\frac{\alpha}{\beta}\right)^2\sin 2f\ \cos e$$

(Snell's law was used).

Aside from the exponential factor, the component in the $\mathbf{a}_3$ direction is given by

$$A\left(-\cos e + \cos e\frac{A_1}{A} + \sin f\frac{B_1}{A}\right)$$

Use (6.5.20) and (6.5.21) and operate. Aside from a factor of A, the numerator of the resulting expression is

$$-2\left(\frac{\alpha}{\beta}\right)^2\cos e\cos^2 f - 2\frac{\alpha}{\beta}\sin f\sin 2e\cos 2f = -2\left(\frac{\alpha}{\beta}\right)^2\cos e(\cos^2 2f + 2\sin^2 f\cos 2f)$$

$$= -2\left(\frac{\alpha}{\beta}\right)^2\cos e\cos 2f(\cos 2f + 2\sin^2 f) = -2\left(\frac{\alpha}{\beta}\right)^2\cos e\cos 2f$$

(Snell's law was used).

•Problem 6. Start with (6.5.27), divide by $\alpha\cos e$ and use Snell's law written as $\beta/\alpha = \sin f/\sin e$. This immediately gives (6.5.28).

•Problem 7. Equations (5.9.9)-(5.9.11) give the energy flux in the direction of propagation per unit area. These equations are general and apply to the P and SV displacements included in (6.5.8). Multiplication by the cross-sectional areas of the beams and averaging as in Problem 5.10 gives (6.5.29)-(6.5.31).

•Problem 8. After multiplying by β^2/α^2 and using Snell's law ($\beta/\alpha = \sin f/\sin e$) the numerator of (6.5.36) can be written as

$$2\sin e\cos e\sin 2f\left(\frac{\beta}{\alpha}\right)^2 - \cos^2 f = 2\cos e\sin 2f\ \frac{\beta}{\alpha}\sin f - \cos^2 2f = 0$$

for a value of f to be determined. In addition

$$\frac{\beta}{\alpha}\cos e = \frac{\beta}{\alpha}(1-\sin^2 e)^{1/2} = \left(\frac{\beta^2}{\alpha^2} - \sin^2 f\right)^{1/2}$$

Combining these two equations gives

$$2\sin f\sin 2f\left(\frac{\beta^2}{\alpha^2} - \sin^2 f\right)^{1/2} - \cos^2 2f = 0$$

This equation can be solved doing a forward search with a computer. The angles are 30° and 34.26° for $\alpha/\beta = \sqrt{3}$.

•Problem 9.

(a) Hilbert transform of $\cos at$. Assume that $a > 0$. Use the method described after (B.14). Start with the Fourier transform

$$\mathcal{F}\{\cos at\} = \pi[\delta(\omega - a) + \delta(\omega + a)]$$

(see Problem A.3). Change the phase as indicated in (B.13). The first and second deltas are nonzero for $\omega > 0$ and $\omega < 0$, respectively. Then

$$\pi[\mathrm{i}\delta(\omega - a) - \mathrm{i}\delta(\omega + a)] = \mathrm{i}\pi[\delta(\omega - a) - \delta(\omega + a)] = -\mathcal{F}\{\sin at\}$$

(see Problem A.3). Therefore

$$\mathcal{H}\{\cos at\} = -\sin at$$

(a) Hilbert transform of $\sin at$. Proceed as in (a). The Fourier transform is given in Problem A.3. Change the phase

$$\mathrm{i}\pi[-\mathrm{i}\delta(\omega + a) - \mathrm{i}\delta(\omega - a)] = \pi[\delta(\omega + a) + \delta(\omega - a)] = \mathcal{F}\{\cos at\}$$

Therefore

$$\mathcal{H}\{\sin at\} = \cos at$$

(a) Hilbert transform of δ. The Fourier transform is equal to 1 (see A.60). Change the phase and use (A.75)

$$\mathrm{i}\ \mathrm{sgn}\omega = \mathcal{F}\left\{\frac{-1}{\pi t}\right\}$$

Therefore

$$\mathcal{H}\{\delta\} = -\frac{1}{\pi t}$$

•Problem 10. For the reflection coefficient start with the expression to the left of the last equality in (6.6.11), divide the numerator and denominator by $\beta\beta'$ and use $\mu = \rho\beta^2$ and an equivalent relation for μ'. For the transmission coefficient start with (6.6.9) and divide the numerator and denominator by β. This gives

$$\frac{C_1}{C} = \frac{(\mu/\beta)\cos f - (\mu'/\beta')\cos f'}{(\mu/\beta)\cos f + (\mu'/\beta')\cos f'} = \frac{\rho\beta\cos f - \rho'\beta'\cos f'}{\rho\beta\cos f + \rho'\beta'\cos f'}$$

$$\frac{C'}{C} = \frac{2(\mu/\beta)\cos f}{(\mu/\beta)\cos f + (\mu'/\beta')\cos f')} = \frac{2\rho\beta\cos f}{\rho\beta\cos f + \rho'\beta'\cos f'}$$

To compute the impedance for each of the three waves use $\tau_{32}/\dot{u}_2$ (see page 163) with τ_{32} given by (6.4.8). For the incident wave from the first term in (6.6.1) we have

$$\tau_{32} = \mu u_{2,3} = \mathrm{i}\mu C\omega\frac{\cos f}{\beta}\exp[...]$$

$$\dot{u}_2 = \mathrm{i}C\omega\exp[...]$$

(the ellipsis indicate the argument in the exponential) and

$$\frac{\tau_{32}}{\dot{u}_2} = \frac{\mu}{\beta}\cos f = \rho\beta\cos f$$

For the reflected and transmitted waves use the second term in (6.6.1) and (6.6.2). This gives $-\rho\beta\cos f$ and $\rho'\beta'\cos f'$.

•Problem 11. For $\mathbf{u}$, start with (6.6.1) written as

$$\mathbf{u} = \mathbf{a}_2 C\exp[\mathrm{i}\omega(t - x_1/c)]\left\{\exp(\mathrm{i}\omega x_3\cos f/\beta) + \frac{C_1}{C}\exp(-\mathrm{i}\omega x_3\cos f/\beta)\right\}$$

where $c = \beta/\sin f$. Use (6.6.21) for C_1/C (assume $\omega > 0$)

$$\mathbf{u} = \mathbf{a}_2 C\exp[\mathrm{i}\omega(t - x_1/c)]\{\exp(\mathrm{i}\omega x_3\cos f/\beta) - \exp[-\mathrm{i}(\omega x_3\cos f/\beta) - 2\mathrm{i}\chi]\}$$

Now consider the factor in braces after it has been multiplied and divided by $\exp\mathrm{i}\chi$

$$\exp(-\mathrm{i}\chi)\{\exp[\mathrm{i}(\omega x_3\cos f/\beta + \chi)] - \exp[-\mathrm{i}(\omega x_3\cos f/\beta + \chi)]\}$$
$$= 2\mathrm{i}\exp(-\mathrm{i}\chi)\sin\left(\frac{\omega x_3\cos f}{\beta} + \chi\right) = 2\exp[\mathrm{i}(\pi/2 - \chi)]\sin\left(\frac{\omega x_3\cos f}{\beta} + \chi\right)$$

For $\mathbf{u}'$, multiply and divide (6.6.2) by C and use (6.6.22).

$$\mathbf{u}' = 2\mathbf{a}_2 C\sin\chi\ \exp(\mathrm{i}\omega x_3\cos f'/\beta')\exp[\mathrm{i}\omega(t - x_1/c)]\exp[\mathrm{i}(\pi/2 - \chi)]$$

and use Snell's law to modify the exponent involving $\cos f'$

$$\mathrm{i}\frac{\cos f'}{\beta'} = \left(\frac{\sin^2 f'}{\beta'^2} - \frac{1}{\beta'^2}\right)^{1/2} = \beta^{-1}\left(\sin^2 f - \frac{\beta^2}{\beta'^2}\right)^{1/2}$$

•Problem 12. We need the real part of the displacement $\mathbf{u}'$ given in Problem 6.11. Using $\exp(\mathrm{i}\pi/2) = \mathrm{i}$ we have

$$\mathcal{R}\{\exp[\mathrm{i}\omega(t - x_1/c) + \mathrm{i}(\pi/2 - \chi)]\} = \mathcal{R}\{\mathrm{i}\ \exp[\mathrm{i}\omega(t - x_1/c) - \chi)]\} = -\sin[\omega(t - x_1/c) - \chi)]$$

Therefore

$$\mathcal{R}\{\mathbf{u}'\} = -2\mathbf{a}_2 C \sin\chi \sin[\omega(t - x_1/c) - \chi)] \exp[\omega x_3 (\sin^2 f - \beta^2/\beta'^2)^{1/2}/\beta]$$

To get energy use (5.9.4),(5.9.3b) and (4.6.3)

$$\mathcal{P}_{SH} = \mathbf{a}_3.\mathbf{E}_{SH} = -\delta_{3j}\tau_{ij}\dot{u}'_i = -\tau_{i3}\dot{u}'_i = -\mu' u'_{2,3}\dot{u}'_2$$

(recall that $u'_1 = u'_3 = 0$). Using this expression with $\mathcal{R}\{\mathbf{u}'\}$ and setting $x_3 = 0$ gives

$$\mathcal{P}_{SH} = 2\sin^2\chi \; \frac{\mu'}{\beta} C^2\omega^2 \left(\sin^2 f' - \frac{\beta}{\beta^2}\right)^{1/2} \sin 2[\omega(t - x_1/c) - \chi)]$$

Note that $u'_{2,3}$ and $\dot{u}'_2$ contribute a factor of $4\sin[...]\cos[...] = 2\sin 2[...]$.
Using $T = \pi/\omega$

$$\int_o^T \sin 2[\omega(t - x_1/c) - \chi)]dt = 0$$

and $\mathcal{P}_{SH} = 0$.

•Problem 13. For normal incidence (6.6.56) reduces to

$$\left(\frac{A_1}{A}\right)^2 + \frac{\rho'}{\rho}\frac{\sin 2e'}{\sin 2e}\left(\frac{A'}{A}\right)^2 = 1$$

We need the following result

$$\frac{\sin 2e'}{\sin 2e} = \frac{\sin e'}{\sin e}\frac{\cos e'}{\cos e} = \frac{\alpha'}{\alpha}\frac{\cos e'}{\cos e}$$

where Snell's law was used. When e goes to zero, e' goes to zero, their cosines go to one and

$$\lim_{e\to 0}\frac{\sin 2e'}{\sin 2e} = \frac{\alpha'}{\alpha}$$

Using this result in the energy equation together with (6.6.50) and (6.6.51) gives

$$\left(\frac{\rho'\alpha' - \rho\alpha}{\rho'\alpha' + \rho\alpha}\right)^2 + \frac{4\rho'\alpha'\rho\alpha}{(\rho'\alpha' + \rho\alpha)^2} = 1$$

•Problem 14. Because $\mu = 0$ for an inviscid fluid, this problem is similar to that discussed in §6.5.1.

CHAPTER 7

•Problem 1. Multiply (7.3.11) by $\mathrm{i}\tan K$, add the result to (7.3.10) and use (7.3.19)

$$A(1+\mathrm{i}\tan K)\mathrm{e}^{-\mathrm{i}K} - B(1-\mathrm{i}\tan K)\mathrm{e}^{\mathrm{i}K} = 0$$

Using

$$1 \pm \mathrm{i}\tan K = \frac{2\mathrm{e}^{\pm\mathrm{i}K}}{\mathrm{e}^{\mathrm{i}K}+\mathrm{e}^{-\mathrm{i}K}}$$

the previous expression gives

$$\frac{2}{\mathrm{e}^{\mathrm{i}K}+\mathrm{e}^{-\mathrm{i}K}}(A-B) = 0$$

so that $A = B$.

•Problem 2. Start with (6.9.16a). For grazing incidence $f = \pi/2$, which means that $c = \beta$ (see (6.9.2)). Use (6.9.5a).Then

$$\theta = Hk\eta' = H\frac{\omega}{\beta}\eta' = H\omega\sqrt{\frac{1}{\beta'^2}-\frac{1}{\beta^2}} = m\pi$$

(we are assuming $\beta > \beta'$) and

$$\omega = \frac{m\pi}{H\sqrt{\frac{1}{\beta'^2}-\frac{1}{\beta^2}}} = \frac{m\pi\beta'}{H\sqrt{1-\frac{\beta'^2}{\beta^2}}}$$

This equation is (7.3.32) with $m = N-1$ and $\beta_1 = \beta'$.

•Problem 3. Start with (4.6.3)

$$\tau_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j}+u_{j,i})$$

If λ and μ depend on position

$$\begin{aligned}\tau_{ij,j} &= \lambda_{,j}\delta_{ij}u_{k,k} + \lambda\delta_{ij}u_{k,kj} + \mu_{,j}(u_{i,j}+u_{j,i}) + \mu(u_{i,jj}+u_{j,ij})\\ &= \lambda_{,i}u_{k,k} + \lambda u_{k,ki} + \mu_{,j}(u_{i,j}+u_{j,i}) + \mu(u_{i,jj}+u_{k,ki})\\ &= \mu u_{i,jj} + (\lambda+\mu)u_{k,ki} + \lambda_{,i}u_{k,k} + \mu_{,j}(u_{j,i}+u_{i,j})\\ &= \mu(\nabla^2\mathbf{u})_i + (\lambda+\mu)(\nabla(\nabla.\mathbf{u}))_i + (\nabla\lambda)_i\nabla.\mathbf{u} + [(\nabla\mu).(\nabla\mathbf{u}+\mathbf{u}\nabla)]_i\end{aligned}$$

If λ and μ depend on z only,

$$\lambda_{,i} = \frac{\mathrm{d}\lambda}{\mathrm{d}z}\delta_{i3}; \qquad \nabla\lambda = \frac{\mathrm{d}\lambda}{\mathrm{d}z}\mathbf{a}_z$$

(the equation on the right gives the third term in (7.3.46)) and

$$\mu_{,i} = \frac{\mathrm{d}\mu}{\mathrm{d}z}\delta_{i3}; \qquad \nabla\mu = \frac{\mathrm{d}\mu}{\mathrm{d}z}\mathbf{a}_z$$

Then the fourth term of $\tau_{ij,j}$ includes the factor

$$[\mathbf{a}_z.(\nabla\mathbf{u}+\mathbf{u}\nabla)]_j = \delta_{i3}(\nabla\mathbf{u}+\mathbf{u}\nabla)_{ij} = u_{j,3}+u_{3,j}$$

(any subindex can be used in the left-hand side). Now we show that this result is equal to the following factor in (7.3.46)

$$\left(2\frac{\partial\mathbf{u}}{\partial z}+\mathbf{a}_z\times\nabla\times\mathbf{u}\right)_j = 2u_{j,3}+\epsilon_{jik}\delta_{i3}\epsilon_{kmn}u_{n,m} = 2u_{j,3}+\epsilon_{kj3}\epsilon_{kmn}u_{n,m} = u_{j,3}+u_{3,j}$$

where (1.4.65) was used. The factor $\mathrm{d}\mu/\mathrm{d}z$ is common to both terms. The last term in (7.3.46) comes from the corresponding term in (4.2.6).

•Problem 4. Aside from a factor of $\mathrm{e}^{-\gamma_\alpha kz}\mathrm{e}^{\mathrm{i}k(ct-x)}$ we have

$$\tau_{33} = \lambda u_{1,1}+(\lambda+2\mu)u_{3,3} = -\lambda A\mathrm{i}k + A(-\mathrm{i}\gamma_\alpha)(-\gamma_\alpha)k = A\mathrm{i}k\left[-\lambda+(\lambda+2\mu)\gamma_\alpha^2\right]$$

$$= A\mathrm{i}k\left[-\lambda+(\lambda+2\mu)\left(1-\frac{c^2}{\alpha^2}\right)\right] = A\mathrm{i}k\left(2\mu-\mu\frac{\alpha^2}{\beta^2}\frac{c^2}{\alpha^2}\right) = A\mathrm{i}k\mu\left(2-\frac{c^2}{\beta^2}\right)$$

where (7.2.14) and $\lambda+2\mu=\mu\alpha^2/\beta^2$ were used (see (4.8.5)). When $z=0$, $\mathrm{e}^{-\gamma_\alpha kz}=1$.

•Problem 5. For the displacement given by (7.4.35) we have

$$(\nabla\times\mathbf{u})_1 = \epsilon_{123}u_{3,2}+\epsilon_{132}u_{2,3} = 0$$

(because u_3 does not depend on x_2 and $u_2=0$)

$$(\nabla\times\mathbf{u})_2 = \epsilon_{213}u_{3,1}+\epsilon_{231}u_{1,3} = (\mathrm{i}kW-\mathrm{i}U')\mathrm{e}^{\mathrm{i}k(ct-x)}\mathbf{a}_2 \equiv a(x,z)\mathbf{a}_2$$

$$(\nabla\times\mathbf{u})_3 = \epsilon_{312}u_{2,1}+\epsilon_{321}u_{1,2} = 0$$

Using (7.4.37)

$$\nabla(\nabla\cdot\mathbf{u}) = (-\mathrm{i}k(-kU+W'),0,-kU'+W'')\mathrm{e}^{\mathrm{i}k(ct-x)}$$

Use

$$\nabla\times\nabla\times\mathbf{u} = \nabla\times(0,a,0)$$

Then

$$(\nabla\times\nabla\times\mathbf{u})_1 = \epsilon_{132}a_{,3}$$

$$(\nabla\times\nabla\times\mathbf{u})_2 = 0$$

$$(\nabla\times\nabla\times\mathbf{u})_3 = \epsilon_{312}a_{,1}$$

$$\mathbf{a}_z\times\nabla\times\mathbf{u} = a\mathbf{a}_3\times\mathbf{a}_2 = -a\mathbf{a}_1 = (-\mathrm{i}kW+\mathrm{i}U',0,0)\mathrm{e}^{ik(ct-x)}$$

NOTE: There is a typo in (7.4.42); the parenthesis after U' should not be there.

•Problem 6. After introducing (7.4.37)-(7.4.42) in (7.3.46), combining the terms with $\nabla(\nabla\cdot\mathbf{u})$, canceling the exponential factor, and rearranging we get
Horizontal component:

$$(\lambda+2\mu)(-k^2U+kW')+\mu(-kW'+U'')+\mu'(2U'+kW-U')+\omega^2\rho U$$

$$= U\left[\rho\omega^2 - (\lambda+2\mu)k^2\right] + \lambda kW' + \mu kW' + \mu U'' + \mu' U' + \mu' kW$$

$$U\left[\rho\omega^2 - (\lambda+2\mu)k^2\right] + \lambda kW' + \frac{d}{dz}\left[\mu(U'+kW)\right] = 0$$

In the last step the following was used

$$\mu kW' + \mu U'' + \mu' U' + \mu' kW = \frac{\mathrm{d}}{\mathrm{d}z}(\mu U') + \frac{\mathrm{d}}{\mathrm{d}z}(k\mu W) = \frac{\mathrm{d}}{\mathrm{d}z}(\mu U' + k\mu W)$$

Vertical component:

$$= (\lambda+2\mu)(-kU' + W'') - \mu(k^2 W - kU') + \lambda'(-kU + W') + 2\mu' W' + \omega^2 W\rho$$

$$= W(\rho\omega^2 - k^2\mu) - \mu kU' + (\lambda+2\mu)W'' - \lambda kU' - \lambda' kU + \lambda' W' + 2\mu' W'$$

$$= W(\rho\omega^2 - k^2\mu) - \mu kU' + \frac{d}{dz}\left[(\lambda+2\mu)W' - k\lambda U\right] = 0$$

In the last step the following was used

$$(\lambda+2\mu)W'' + (\lambda'+2\mu')W' - k(\lambda U' + \lambda' U) = \frac{\mathrm{d}}{\mathrm{d}z}\left[(\lambda+2\mu)W' - k\lambda U\right]$$

•Problem 7. For (7.5.7) and (7.5.8), start with (7.5.1) and (7.5.2) with $z = 0$ and equate the corresponding x and z components.

For (7.5.9), add (7.4.7) and (7.4.8), add (7.5.4) and (7.5.6), equate the corresponding results, cancel a common factor of k and use $1 + \gamma_\beta^2 = 2 - c^2/\beta^2$.

For (7.5.10), add (7.4.10) and (7.4.11), add (7.5.3) and (7.5.5), and equate the corresponding results.

•Problem 8. Form (7.5.7)-(7.5.10), $F(c)$ is given by

$$F(c) = \begin{vmatrix} 1 & \mathrm{i}\gamma_\beta & -1 & \mathrm{i}\gamma_{\beta'} \\ -\mathrm{i}\gamma_\alpha & 1 & -\mathrm{i}\gamma_{\alpha'} & -1 \\ -2\mu\gamma_\alpha & -\mathrm{i}\mu(1+\gamma_\beta^2) & -2\mu'\gamma_{\alpha'} & \mathrm{i}\mu'(1+\gamma_{\beta'}^2) \\ \mathrm{i}\mu(1+\gamma_\beta^2) & -2\mu\gamma_\beta & -\mathrm{i}\mu'(1+\gamma_{\beta'}^2) & -2\mu'\gamma_{\beta'} \end{vmatrix}$$

Multiply the second and fourth rows by i. This multiplies the determinant by i^2

$$F(c) = -\begin{vmatrix} 1 & \mathrm{i}\gamma_\beta & -1 & \mathrm{i}\gamma_{\beta'} \\ \gamma_\alpha & \mathrm{i} & \gamma_{\alpha'} & -\mathrm{i} \\ -2\mu\gamma_\alpha & -\mathrm{i}\mu(1+\gamma_\beta^2) & -2\mu'\gamma_{\alpha'} & \mathrm{i}\mu'(1+\gamma_{\beta'}^2) \\ -\mu(1+\gamma_\beta^2) & -2\mathrm{i}\mu\gamma_\beta & \mu'(1+\gamma_{\beta'}^2) & -2\mathrm{i}\mu'\gamma_{\beta'} \end{vmatrix}$$

Multiply the second and fourth columns by $-\mathrm{i}$. This multiplies the determinant by i^2

$$F(c) = \begin{vmatrix} 1 & \gamma_\beta & -1 & \gamma_{\beta'} \\ \gamma_\alpha & 1 & \gamma_{\alpha'} & -1 \\ -2\mu\gamma_\alpha & -\mu(1+\gamma_\beta^2) & -2\mu'\gamma_{\alpha'} & \mu'(1+\gamma_{\beta'}^2) \\ -\mu(1+\gamma_\beta^2) & -2\mu\gamma_\beta & \mu'(1+\gamma_{\beta'}^2) & -2\mu'\gamma_{\beta'} \end{vmatrix}$$

Dividing the third and fourth rows by μ multiplies the determinant by $1/\mu^2$, but because we will set it to zero this factor can be ignored, although it should have been included in (7.5.11).

•Problem 9.

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$$

$$v^2 \frac{\partial^2 f}{\partial x^2} = -v^2 k^2 f$$

Introduce these two expressions in (7.6.1), use (7.6.3), change signs and cancel a common factor of f. This gives (7.6.4).

•Problem 10. Let

$$\mathcal{F}\{g(ax)\} = \int_{-\infty}^{\infty} g(ax) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x$$

assume that $a > 0$ and introduce the change of variables $ax = u$, so that $\mathrm{d}x = \mathrm{d}u/a$. Then

$$\mathcal{F}\{g(ax)\} = \frac{1}{a} \int_{-\infty}^{\infty} g(u) \mathrm{e}^{\mathrm{i}(k/a)u} \mathrm{d}u = \frac{1}{a} G\left(\frac{k}{a}\right) \equiv \frac{1}{|a|} G\left(\frac{k}{a}\right)$$

where $G(k) = \mathcal{F}\{g(x)\}$.

If $a < 0$ introduce the same change of variable but use $a = -|a|$, so that $ax = -|a|x = u$ and $\mathrm{d}x = -\mathrm{d}u/|a|$. Then

$$\mathcal{F}\{g(ax)\} = -\frac{1}{|a|} \int_{\infty}^{-\infty} g(u) \mathrm{e}^{\mathrm{i}(k/a)u} \mathrm{d}u = \frac{1}{|a|} \int_{-\infty}^{\infty} g(u) \mathrm{e}^{\mathrm{i}(k/a)u} \mathrm{d}u = \frac{1}{|a|} G\left(\frac{k}{a}\right)$$

Fourier transform of a Gaussian function

$$\mathcal{F}\{\mathrm{e}^{-at^2}\} = \frac{\pi^2}{a^2} \mathrm{e}^{\omega^2/4a}$$

(e.g., Papoulis, 1962).

•Problem 11. Let

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\frac{1}{2} t \omega_o'' \, (k-k_o)^2)} \mathrm{d}k$$

Let $k - k_o = u$, so that $\mathrm{d}k = \mathrm{d}u$, and $a = t\omega_o''/2$. Then

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}au^2} \mathrm{d}u = \frac{1}{\pi} \int_0^{\infty} \cos au^2 \; \mathrm{d}u + \mathrm{i}\frac{1}{\pi} \int_0^{\infty} \sin au^2 \; \mathrm{d}u$$

If $\omega_o'' < 0$, $a < 0$. Let $a = -|a|$. Then

$$\cos au^2 = \cos(-|a|)u^2 = \cos |a| u^2$$

$$\sin au^2 = \sin(-|a|)u^2 = -\sin |a| u^2$$

Therefore, for any a

$$I = \int_0^{\infty} \cos |a| u^2 \; \mathrm{d}u + \mathrm{i} \; \mathrm{sgn} \; a \int_0^{\infty} \sin |a| u^2 \; \mathrm{d}u = \frac{1}{2\pi} \sqrt{\frac{\pi}{|a|}} \left(\cos \frac{\pi}{4} + \mathrm{i} \; \mathrm{sgn} \; a \sin \frac{\pi}{4} \right)$$

$$= \frac{1}{\sqrt{4\pi|a|}} e^{i(\pi/4)\text{sgn } |a|} = \frac{1}{\sqrt{2\pi t|\omega_o|}} e^{i(\pi/4)\text{sgn } \omega_o''}$$

•Problem 12. In (7.6.17) the phase should be $\omega(k)t+\psi(k)-kx$. Then (7.6.28), (7.6.29) and (7.6.30) become

$$\phi(k) = \omega(k) + \frac{1}{t}\psi(k) - k\frac{x}{t}$$

$$\phi'(k_o) = \omega'(k_o) + \frac{1}{t}\psi'(k_o) - \frac{x}{t} = 0$$

$$\phi(k) \approx \phi(k_o) + \frac{1}{2}\phi''(k_o)(k-k_o)^2$$

where

$$\phi''(k_o) = \omega''(k_o) + \frac{1}{t}\psi''(k_o)$$

In this case $\phi''(k_o)$ replaces the $\omega''(k_o)$ in (7.6.32) with ϕ_o given by

$$\phi_o = \phi(k_o) = \omega(k_o) + \frac{1}{t}\psi(k_o) - k_o\frac{x}{t}$$

•Problem 13. Start with

$$\tan\eta_1 kH = \frac{\mu_2\gamma_2}{\mu_1\eta_1} \tag{1}$$

Take logarithms on both sides

$$\ln\tan\eta_1 kH = \ln\mu_2\gamma_2 - \ln\mu_1\eta_1 \tag{2}$$

Take derivative with respect to k

$$\frac{1}{\tan\eta_1 kH}\frac{d}{dk}\tan\eta_1 kH = \frac{1}{\mu_2\gamma_2}\frac{d}{dk}(\mu_2\gamma_2) - \frac{1}{\mu_1\eta_1}\frac{d}{dk}(\mu_1\eta_1) \tag{3}$$

Operate

$$\frac{\sec^2\eta_1 kH}{\tan\eta_1 kH}\frac{d}{dk}(\eta_1 kH) = \frac{1}{\gamma_2}\frac{d\gamma_2}{dk} - \frac{1}{\gamma_1}\frac{d\eta_1}{dk} \tag{4}$$

Using (1)

$$\frac{\sec^2\eta_1 kH}{\tan\eta_1 kH} = \frac{1+\tan^2\eta_1 kH}{\tan\eta_1 kH} = \frac{{\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2}{\mu_1\eta_1\mu_2\gamma_2} \tag{5}$$

From (4) and (5)

$$\frac{{\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2}{\mu_1\eta_1\mu_2\gamma_2}\left(\eta_1 H + kH\frac{d\eta_1}{dk}\right) = \frac{1}{\gamma_2}\frac{d\gamma_2}{dk} - \frac{1}{\gamma_1}\frac{d\eta_1}{dk} \tag{6}$$

From (7.3.5)

$$\frac{d\eta_1}{dk} = \frac{1}{\eta_1{\beta_1}^2}c\frac{dc}{dk}; \qquad \frac{d\gamma_2}{dk} = -\frac{1}{\gamma_2{\beta_2}^2}c\frac{dc}{dk} \tag{7}$$

Introduce (7) in (6)

$$\frac{{\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2}{\mu_1\mu_2\gamma_2}H = -\frac{1}{{\gamma_2}^2{\beta_2}^2}c\frac{dc}{dk} - \frac{1}{{\eta_1}^2{\beta_1}^2}c\frac{dc}{dk} - c\frac{dc}{dk}\left(kH\frac{{\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2}{\mu_1\eta_1\mu_2\gamma_2}\frac{1}{\eta_1{\beta_1}^2}\right)$$

$$= -c\frac{\mathrm{d}c}{\mathrm{d}k}\left[\frac{\mu_1{\eta_1}^2{\beta_1}^2\mu_2 + \mu_1\mu_2{\beta_2}^2{\gamma_2}^2 + kH({\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2)\gamma_2{\beta_2}^2}{\mu_1{\eta_1}^2{\beta_1}^2\mu_2{\beta_2}^2{\gamma_2}^2}\right] \tag{8}$$

Solve for $\mathrm{d}c/\mathrm{d}k$

$$\frac{\mathrm{d}c}{\mathrm{d}k} = -\frac{1}{c}\frac{B_1}{B_2} \tag{9}$$

where

$$B_1 = (\mu_1^2\eta_1^2 + \mu_2^2\eta_2^2)\eta_1^2\beta_1^2\beta_2^2\gamma_2 H \tag{10}$$

and

$$B_2 = \mu_1\mu_2\left(\eta_1^2\beta_1^2 + \beta_2^2\gamma_2^2\right) + kH(\mu_1^2\eta_1^2 + \mu_2^2\eta_2^2)\gamma_2\beta_2^2 \tag{11}$$

Use this expression with (7.6.15)

$$U = c + k\frac{\mathrm{d}c}{\mathrm{d}k} = c - \frac{k}{c}\frac{B_1}{B_2} = \frac{1}{c}\left(\frac{c^2 B_2 - kB_1}{B_2}\right) \tag{12}$$

$$c^2 B_2 = \underbrace{c^2\mu_1\mu_2(\eta_1^2\beta_1^2 + \beta_2^2\gamma_2^2)}_{I} + \underbrace{c^2 kH(\mu_1^2\eta_1^2 + \mu_2^2\eta_2^2)\gamma_2\beta_2^2}_{II} \tag{13}$$

$$II - kB_1 = kH\gamma_2({\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2){\beta_2}^2(c^2 - {\eta_1}^2{\beta_1}^2) = kH\gamma_2({\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2){\beta_2}^2{\beta_1}^2 \tag{14}$$

The following was used

$${\beta_1}^2 = c^2 - {\eta_1}^2{\beta_1}^2 \tag{15}$$

(square both sides of (7.3.5a)). Extracting a common factor of I/c^2 (see (13)), from (12) we obtain

$$U = \frac{\beta_1^2}{c}\left(\frac{c^2/\beta_1^2 + \Omega}{1+\Omega}\right) \tag{16}$$

where

$$\Omega = kH\gamma_2\frac{({\mu_1}^2{\eta_1}^2 + {\mu_2}^2{\gamma_2}^2){\beta_2}^2}{\mu_1\mu_2(\eta_1^2\beta_1^2 + \beta_2^2\gamma_2^2)} \tag{17}$$

From (15) and by squaring both sides of (7.3.5b) we get

$$\eta_1^2\beta_1^2 = c^2 - {\beta_1}^2; \qquad \beta_2^2\gamma_2^2 = {\beta_2}^2 - c^2 \tag{18}$$

so that

$$\eta_1^2\beta_1^2 + \beta_2^2\gamma_2^2 = {\beta_2}^2 - {\beta_1}^2 \tag{19}$$

Then

$$\Omega = kH\gamma_2\left[\frac{\mu_1{\eta_1}^2{\beta_2}^2}{\mu_2({\beta_2}^2 - {\beta_1}^2)} + \frac{\mu_2{\gamma_2}^2{\beta_2}^2}{\mu_1({\beta_2}^2 - {\beta_1}^2)}\right] \tag{20}$$

Now use

$$\frac{\mu_1}{\mu_2} = \frac{\rho_1{\beta_1}^2}{\rho_2{\beta_2}^2} \tag{21}$$

(see (4.8.5b)) and (18) in (20). This gives

$$\Omega = kH\gamma_2 \left[\frac{\rho_1}{\rho_2}\left(\frac{c^2-\beta_1^2}{\beta_2^2-\beta_1^2}\right) + \frac{\mu_2}{\mu_1}\left(\frac{\beta_2^2-c^2}{\beta_2^2-\beta_1^2}\right)\right] \tag{22}$$

From (16), if c goes to β_1, U goes to β_1. If c goes to β_2, γ_2 and Ω go to zero and U goes to β_2.

•Problem 14. Equation (7.6.50) is obtained by straight substitution. When $k_o > 0$ the phase in (7.6.32) can be written as $\phi = |\omega_o|t - |k_o|x + \pi/4$. When $k_o < 0$, $k_o = -|k_o|$, $\omega_o = -|\omega_o|$ (see (7.6.47)) and sgn $\omega_o'' = -1$ (see (7.6.48). In this case the phase is $-\phi$. The two contributions give $2\cos\phi$. Then, from (7.6.32)

$$f(x,t) = \frac{1}{\sqrt{2\pi t|\omega_o''|}}\cos\left(\omega_o t - k_o x + \frac{\pi}{4}\right)$$

The argument of the cosine in (7.6.51) comes from (7.6.50). The factor in front of the cosine comes from two times the coefficient in (7.6.32) and (7.6.49)

$$\frac{2}{\sqrt{2\pi t}}\frac{1}{\sqrt{|\omega_o''|}} = \frac{2}{\sqrt{2\pi t}}\frac{\sqrt{vt^3a}}{(v^2t^2-x^2)^{3/4}} = \sqrt{\frac{2va}{\pi}}\frac{t}{(v^2t^2-x^2)^{3/4}}$$

•Problem 15. After the change of variables, $\mathrm{d}u = \mathrm{sgn}\ c\ \mathrm{d}s/|c|^{1/3}$. If $c > 0$, sgn $c = 1$ and

$$\int_{-\infty}^{\infty} ...\ \mathrm{d}u = \frac{1}{|c|^{1/3}}\int_{-\infty}^{\infty} ...\ \mathrm{d}s$$

where the ellipsis represent the integrand. If $c < 0$, sgn $c = -1$ and

$$\int_{-\infty}^{\infty} ...\ \mathrm{d}u = -\frac{1}{|c|^{1/3}}\int_{\infty}^{-\infty} ...\ ds = \frac{1}{|c|^{1/3}}\int_{-\infty}^{\infty} ...\ \mathrm{d}s$$

Therefore, after the change in variable the integral in (7.6.58) has the same expression for c positive or negative. Then the exponent in the integrand of (7.6.58) can be written as

$$ub + c\frac{u^3}{3} = zs + \frac{s^3}{3}; \qquad z = \frac{b\ \mathrm{sgn}\ c}{|c|^{1/3}}$$

Note that $|c|^3\mathrm{sgn}\ c = c^3$ for $c > 0$ or $c < 0$. Therefore, (7.6.58) becomes

$$I = \frac{1}{2\pi}\frac{1}{|c|^{1/3}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(zs+s^3/3)}\mathrm{d}s = \frac{1}{|c|^{1/3}}\mathrm{Ai}(z)$$

CHAPTER 8

•Problem 1. Label the six terms on the right sides of (8.3.4) and (8.3.6)as follows

$$(\mathrm{c}(U_k f)_{,l})_{,j} = \mathrm{c}_{,j}(U_k f)_{,l} + \mathrm{c}(U_k f)_{,lj} = \underbrace{\mathrm{c}_{,j} U_{k,l} f}_{I} + \underbrace{\mathrm{c}_{,j} U_k f_{,l}}_{II}$$

$$+ \underbrace{\mathrm{c} U_{k,lj} f}_{III} + \underbrace{\mathrm{c} U_{k,l} f_{,j}}_{IV} + \underbrace{\mathrm{c} U_{k,j} f_{,l}}_{V} + \underbrace{\mathrm{c} U_k f_{,lj}}_{VI}$$

$$f_{,lj} = \underbrace{\ddot{f} T_{,l} T_{,j}}_{VII} - \underbrace{\dot{f} T_{,lj}}_{VIII}$$

The coefficient of $\ddot{f}$ in (8.3.8) comes from VI, VII and (3.8.7). The coefficient of $\dot{f}$ from II, IV, V, VI, $VIII$ and (8.3.5). The coefficient of f from I and III.

•Problem 2. Interchange j and l in the expression for Γ_{ik} and use the symmetry of c_{ijkl} (see (4.5.11))

$$\Gamma_{ik} = \frac{1}{\rho} c_{ijkl} T_{,l} T_{,j} = \frac{1}{\rho} c_{ilkj} T_{,j} T_{,l} = \frac{1}{\rho} c_{kjil} T_{,l} T_{,j} = \Gamma_{ki}$$

(the obvious equality $T_{,j} T_{,l} = T_{,l} T_{,j}$ was used).

•Problem 3. The determinant is

$$D = \begin{vmatrix} B + C\mathcal{T}_{11} & C\mathcal{T}_{12} & C\mathcal{T}_{13} \\ C\mathcal{T}_{12} & B + C\mathcal{T}_{22} & C\mathcal{T}_{23} \\ C\mathcal{T}_{13} & C\mathcal{T}_{23} & B + C\mathcal{T}_{33} \end{vmatrix}$$

where $\mathcal{T}_{ij} = T_{,i} T_{,j}$. Expanding the determinant gives

$$D = (B+C\mathcal{T}_{11})(B+C\mathcal{T}_{22})(B+C\mathcal{T}_{33}) + 2C^3 \mathcal{T}_{12}\mathcal{T}_{23}\mathcal{T}_{13} - C^2[\mathcal{T}_{13}^2(B+C\mathcal{T}_{22}) + \mathcal{T}_{12}^2(B+C\mathcal{T}_{33}) + \mathcal{T}_{23}^2(B+C\mathcal{T}_{11})]$$

After operating D becomes

$$D = B^3 + a_1 B^2 C + a_2 B C^2 + a_3 C^3$$

The coefficients a_k depend on $\mathcal{T}_{ij}$, which can written in full using $\mathcal{T}_{IJ}\mathcal{T}_{JL}\mathcal{T}_{IL} = T_{,1}^2 T_{,2}^2 T_{,3}^2$ (no summation over capital indices) and similar relations. After that is done it is found that $a_2 = a_3 = 0$. The following derivation, however, is shorter and shows the expressions for the a_k. The starting point is (1.4.55)

$$D = \epsilon_{ijk} d_{i1} d_{j2} d_{k3} = \epsilon_{ijk}(B\delta_{i1} + C\mathcal{T}_{i1})(B\delta_{j2} + C\mathcal{T}_{j2})(B\delta_{k3} + C\mathcal{T}_{k3}) =$$

$$\epsilon_{123} B^3 + B^2 C(\epsilon_{1j3}\mathcal{T}_{j2} + \epsilon_{i23}\mathcal{T}_{i1}) + \epsilon_{12k}\mathcal{T}_{k3} + BC^2(\epsilon_{ij3}\mathcal{T}_{i1}\mathcal{T}_{j2} + \epsilon_{1jk}\mathcal{T}_{j2}\mathcal{T}_{k3} + \epsilon_{i2k}\mathcal{T}_{i1}\mathcal{T}_{k3}) + C^3 \epsilon_{ijk}\mathcal{T}_{i1}\mathcal{T}_{j2}\mathcal{T}_{k3}$$

The coefficient of B^3 is one. To determine the coefficient of $B^2 C$ recall that ϵ_{ijk} is zero when there are repeated indices. Then the coefficient becomes

$$\epsilon_{123}(\mathcal{T}_{11} + \mathcal{T}_{22} + \mathcal{T}_{33}) = T_{,1}T_{,1} + T_{,2}T_{,2} + T_{,3}T_{,3} = T_{,1}^2 + T_{,2}^2 + T_{,3}^2 = |\nabla T|^2$$

The coefficient of BC^2 is the sum of three similar terms. The first involves ϵ_{ij3}, which is antisymmetric in i and j, and $\mathcal{T}_{i1}\mathcal{T}_{j2}$, which is symmetric in i and j. Then this term and the other two are equal to zero (see (1.4.60)). A similar argument shows that the coefficient of C^3 is also equal to zero. Therefore, D is given by (8.3.16).

•Problem 4. Take the derivative of $\mathbf{t}.\mathbf{t} = t_i t_i$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{t}.\mathbf{t}) = \frac{\mathrm{d}}{\mathrm{d}s}(t_i t_i) = 2t_i \frac{\mathrm{d}t_i}{\mathrm{d}s} = 2\mathbf{t}.\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}$$

•Problem 5. Refer to Fig. 8.2. Draw a line perpendicular to $\mathbf{\Delta t}$ bisecting $\Delta\theta$. Then

$$\sin\frac{\Delta\theta}{2} = \frac{|\mathbf{\Delta t}|/2}{|\mathbf{t}|} = \frac{1}{2}|\mathbf{\Delta t}| \approx \frac{\Delta\theta}{2}$$

The approximation is valid for small Δs. Then, $\Delta\theta$ goes to $|\mathbf{\Delta t}|$ as Δs goes to zero.

•Problem 6. From (8.5.7)

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \mathbf{0}$$

which means that $\mathbf{t}$ is a constant vector, say $\mathbf{e}$. Then, from (8.5.4)

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \mathbf{e}$$

and

$$\mathbf{r} = \mathbf{a}s + \mathbf{e}$$

where $\mathbf{a}$ is another constant vector (see (8.4.24)).

•Problem 7. From (8.5.12)

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \mathbf{0}$$

which means that $\mathbf{b}$ is a constant vector. We also know that

$$\mathbf{t}.\mathbf{b} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}.\mathbf{b} = 0$$

because the two vectors on the left are orthogonal. Using these two equations we obtain the following result

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{r}.\mathbf{b}) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}.\mathbf{b} + \mathbf{r}.\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \mathbf{r}.\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = 0$$

To compute the derivatives write in component form as in Problem 8.4. The last equation shows that $\mathbf{r}.\mathbf{b}$ is equal to a constant, which is the equation of a plane.

•Problem 8.

(a) Let

$$\mathbf{r} = (a\cos u, a\sin u, b\ u)$$

From (8.5.3)

$$\dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = (-a\sin u, a\cos u, b)$$

From (8.5.5)

$$\mathbf{t} = \frac{1}{|\dot{\mathbf{r}}|}(-a\sin u, a\cos u, b); \qquad |\dot{\mathbf{r}}| = \sqrt{a^2+b^2}$$

To get $\mathrm{d}\mathbf{t}/\mathrm{d}s$ use (8.5.2)

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{|\dot{\mathbf{r}}|^2}\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}u} = -\frac{a}{a^2+b^2}(\cos u, \sin u, 0)$$

Then, because $a > 0$

$$\kappa = \left|\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right| = \frac{a}{a^2+b^2}$$

$$\mathbf{n} = \frac{1}{\kappa}\left|\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}\right| = -(\cos u, \sin u, 0)$$

$$\mathbf{b} = \mathbf{t}\times\mathbf{n} = \frac{1}{|\dot{\mathbf{r}}|}(b\sin u, -b\cos u, a)$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{|\dot{\mathbf{r}}|^2}b(\cos u, \sin u, 0) = -\frac{b}{|\dot{\mathbf{r}}|^2}\mathbf{n}$$

Comparison of this result with (8.5.12) shows that

$$\tau = \frac{b}{a^2+b^2}$$

(b) Proceeding as in (a)

$$\mathbf{r} = (a\cos u, -a\sin u, b\ u)$$

$$\mathbf{t} = \frac{1}{|\dot{\mathbf{r}}|}(-a\sin u, -a\cos u, b)$$

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{1}{a^2+b^2}(-a\cos u, a\sin u, 0)$$

$$\kappa = \frac{a}{a^2+b^2}$$

$$\mathbf{n} = (-\cos u, \sin u, 0)$$

$$\mathbf{b} = -\frac{1}{|\dot{\mathbf{r}}|}(b\sin u, b\cos u, a)$$

$$\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \frac{b}{|\dot{\mathbf{r}}|^2}\mathbf{n}$$

$$\tau = -\frac{b}{a^2+b^2}$$

For $b > 0$, $\tau < 0$ and the helix is left-handed.

•Problem 9. Intermediate results. Using (8.5.10)

$$(\mathbf{b}\times\mathbf{t})_i = (\mathbf{t}\times\mathbf{n}\times\mathbf{t})_i = \epsilon_{ijk}\epsilon_{jmn}t_m n_n t_k = t_m t_m n_i - t_i n_n t_n = (\mathbf{t}.\mathbf{t})n_l - (\mathbf{n}.\mathbf{t})t_l = n_i$$

$$(\mathbf{b}\times\mathbf{n})_i = (\mathbf{t}\times\mathbf{n}\times\mathbf{n})_i = \epsilon_{ijk}\epsilon_{jmn}t_m n_n n_k = t_m n_m n_i - t_i n_n n_n = -t_i$$

Then

$$\mathbf{b}\times\mathbf{t} = \mathbf{n}; \qquad\qquad \mathbf{b}\times\mathbf{n} = -\mathbf{t}$$

Also, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ ($\mathbf{a}$ arbitrary). Then

$$\mathbf{d}\times\mathbf{t} = \kappa\mathbf{b}\times\mathbf{t} = \kappa\mathbf{n} = \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s}$$

$$\mathbf{d}\times\mathbf{n} = \tau\mathbf{t}\times\mathbf{n} + \kappa\mathbf{b}\times\mathbf{n} = \tau\mathbf{b} - \kappa\mathbf{t} = \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}$$

$$\mathbf{d}\times\mathbf{b} = \tau\mathbf{t}\times\mathbf{b} = -\tau\mathbf{n} = \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s}$$

Equations (8.5.7), (8.5.10), (8.5.14) and (8.5.12) were used.

•Problem 10. Consider a narrow ray tube with square cross section (similar to that in Fig. 8.7) with vertex at the source. The intersections of the tube with two spheres of radii r_o and r_1 centered at the origin have areas proportional to r_0^2 and r_1^2 (see 10.11.5). Then, using (8.7.10)

$$A_o^2 r_o^2 = A_1^2 r_1^2$$

or

$$\frac{A_1}{A_o} = \frac{r_o}{r_1}$$

•Problem 11. In the first term of (8.7.18) change the dummy index j to l, interchange i and k and use the symmetry of c_{ijkl} (see (4.5.11))

$$c_{ijkp}T_{,j}U_kU_i = c_{klip}T_{,l}U_iU_k = c_{ipkl}T_{,l}U_kU_i$$

This is equal to the second term.

•Problem 12. Start with (8.7.3) and use (8.7.25). This gives

$$\frac{2}{A}\frac{1}{c^2}\frac{\mathrm{d}A}{\mathrm{d}t} + \nabla^2 T = 0$$

Multiply this equation by $c^2A/2$

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{1}{2}c^2 A\nabla^2 T = 0$$

If in (8.7.26) we set $\alpha^2\rho = 1$, we get the equation above. This is the same relation that exists between the right-hand sides of (8.7.6) and (8.7.20) (with $c = \alpha$, $|\mathbf{U}| = \mathbf{A}$).

•Problem 13. Multiply both sides of the first equality in (8.7.43) scalarly with $\mathbf{t}$. This gives (8.7.42). The second equality follows from the definition of $\mathbf{n}$ (see (8.5.7)).

•Problem 14. Multiply (8.7.53a,b) by $\cos\theta$ and $\sin\theta$, respectively, add the corresponding results, and use (8.7.47).

$$-\kappa\cos^2\theta - \kappa\sin^2\theta = -\kappa = \frac{1}{c}\cos\theta\nabla c\,.\,\mathbf{e}^I + \frac{1}{c}\sin\theta\nabla c\,.\,\mathbf{e}^{II} = \frac{1}{c}\nabla c\,.\,\mathbf{n}$$

This is (8.5.28).

•Problem 15. Equations (8.7.47) and (8.7.48) can be written as

$$\begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{e}^I \\ \mathbf{e}^{II} \end{pmatrix}; \qquad \mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Then

$$\begin{pmatrix} \mathbf{e}^I \\ \mathbf{e}^{II} \end{pmatrix} = \mathbf{A}^{-1} = \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}; \qquad \mathbf{A}^{-1} = \mathbf{A}^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

•Problem 16. The raypath is symmetric with respect to the point $(x/2, 0)$. Therefore, if t indicates the total traveltime

$$\frac{t}{2} = \frac{1}{\alpha}\sqrt{H^2 + x^2/4}$$

which means that

$$t^2 = t_o^2 + \frac{x}{\alpha^2}; \qquad t_o = \frac{2H}{\alpha}$$

t_o is the vertical traveltime.

CHAPTER 9

•Problem 1. Similar to Problem 7.10. The required change of variable is $ax = u$, so that $\mathrm{d}x = \mathrm{d}u/a$. After this step change u to x.

•Problem 2. Consider

$$I = \int_{-\infty}^{\infty} f(x - x_o)\varphi(x)dx = \int_{-\infty}^{\infty} f(x)\varphi(x + x_o)dx$$

where the second integral is obtained by the change of variables $x - x_o = u$, so that $\mathrm{d}x = \mathrm{d}u$ and the integration limits are $\pm\infty$. If the δ were a regular function, (9.2.12) would follow immediately (see A-3), but in the context of distribution theory it is a definition (see A-10).

•Problem 3. Rearrange the argument of the Dirac's delta as

$$r - c(t - t_o) = c[t_o - (t - r/c)]$$

and use(9.2.7b).

•Problem 4. Start with (9.4.14)

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi + \nabla\times\boldsymbol{\psi}$$

Then

$$\nabla.\mathbf{u}(\mathbf{x}, t) = \nabla.\nabla\phi + \nabla.\nabla\times\boldsymbol{\psi}$$

The second term on the right is zero (see (9.3.5)) and

$$\nabla.\nabla\phi = (\nabla\phi)_{i,i} = \phi_{,ii} = \nabla^2\phi$$

From (9.4.14) we also get

$$\nabla\times\mathbf{u}(\mathbf{x}, t) = \nabla\times\nabla\phi + \nabla\times\nabla\times\boldsymbol{\psi}$$

The first term on the right is equal to zero (see (1.4.61)). From the definition of Laplacian of a vector (see (1.4.53))

$$\nabla\times\nabla\times\boldsymbol{\psi} = -\nabla^2\boldsymbol{\psi} + \nabla(\nabla.\boldsymbol{\psi}) = -\nabla^2\boldsymbol{\psi}$$

because of (9.4.15)

•Problem 5. Start with

$$\phi = \phi(x_1, x_3, t); \qquad \boldsymbol{\psi} = (0, \psi(x_1, x_3, t), 0)$$

Then

$$\nabla\phi = (\phi_{,1}, 0, \phi_{,3})$$

and

$$(\nabla\times\boldsymbol{\psi})_1 = \epsilon_{132}\psi_{,3} = -\psi_{,3}$$

$$(\nabla\times\boldsymbol{\psi})_2 = 0$$

$$(\nabla\times\boldsymbol{\psi})_3 = \epsilon_{312}\psi_{,1} = \psi_{,1}$$

$$\nabla\times\boldsymbol{\psi} = (-\psi_{,3}, 0, \psi_{,1})$$

Apply (9.4.1)

$$\mathbf{u} = \nabla\phi + \nabla\times\psi$$

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3}$$

$$u_2 = 0$$

$$u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1}$$

•Problem 6. Find the terms in (9.4.2) As in Problem 9.5

$$\nabla\times\mathbf{u} = (-u_{2,3}, 0, u_{2,1}) \equiv \mathbf{v}$$

$$(\nabla\times\mathbf{v})_1 = \epsilon_{123}u_{2,12} = 0$$

$$(\nabla\times\mathbf{v})_2 = \epsilon_{213}u_{2,11} - \epsilon_{231}u_{2,33} = -\left(u_{2,11} + u_{2,33}\right)$$

$$(\nabla\times\mathbf{v})_3 = -\epsilon_{321}u_{2,32} = 0$$

Also

$$\nabla.\mathbf{u} = (0,0,0); \qquad \ddot{\mathbf{u}} = (0, \ddot{u}_2, 0)$$

Introducing these results in (9.4.2) gives

$$\frac{\partial^2 u_2}{\partial t^2} = \beta^2\nabla^2 u_2; \qquad \frac{\partial^2 u_2}{\partial x_2^2} \equiv 0$$

•Problem 7. The dependence of $\mathbf{W}$ on r is through $1/r$. Let

$$\mathbf{v} = \left(\frac{-1}{r}, 0, 0\right)$$

Then

$$(\nabla\times\mathbf{v})_1 = 0$$

$$(\nabla\times\mathbf{v})_2 = \epsilon_{231}\left(\frac{-1}{r}\right)_{,3} = -\frac{\partial}{\partial x_3}\frac{1}{r}$$

$$(\nabla\times\mathbf{v})_3 = \epsilon_{321}\left(\frac{-1}{r}\right)_{,2} = \frac{\partial}{\partial x_2}\frac{1}{r}$$

•Problem 8. Write (9.5.6) in component form. This gives three equations similar to (9.5.5). From the equation for the first component we get $(\mathbf{\Psi})_1 = 0$ (use (9.2.14) with $\Phi = 0$). For the other two components the solutions are similar to (9.5.10). Written in vector form the solution to (9.5.6) is given by (9.5.11).

•Problem 9. Use $\partial(x_j - \xi_j)/\partial x_i = \delta_{ij}$ and (9.5.13).

$$\frac{\partial^2}{\partial x_i x_j}\frac{1}{r} = \frac{\partial}{\partial x_i}\frac{\partial}{x_j}\frac{1}{r} = \frac{\partial}{\partial x_i}\left(\frac{-1}{r^2}\frac{dr}{dx_j}\right) = -\frac{\partial}{\partial x_i}\left(\frac{1}{r^2}\gamma_j\right)$$

$$= \frac{2}{r^3}\gamma_i\gamma_j - \frac{1}{r^2}\frac{\partial}{x_i}\left(\frac{x_j-\xi_j}{r}\right) = \frac{2}{r^3}\gamma_i\gamma_j - \frac{\delta_{ij}}{r^3} - \frac{1}{r^2}(x_j-\xi_j)\left(\frac{-1}{r^2}\right)\gamma_i = \frac{3\gamma_i\gamma_j-\delta_{ij}}{r^3}$$

•Problem 10.

(a)

$$(3\gamma_i\gamma_j - \delta_{ij})\gamma_i = 3\gamma_i\gamma_i\gamma_j - \gamma_j = 3|\mathbf{\Gamma}|^2\gamma_j - \gamma_j = 2\gamma_j$$

γ_j is a constant, so it does represent $\mathbf{\Gamma}$.

(b) Unless the dot product of two vectors it is equal to zero, it is not possible to make general statements about the angle between the vectors (see Problem 9.11).

•Problem 11. Let $v_i = 3\gamma_i\gamma_j - \delta_{ij}$ (j fixed) and θ the angle between v_i and γ_i. Then $\cos\theta$ is given by

$$\cos\theta = \frac{v_i\gamma_i}{|\mathbf{v}||\mathbf{\Gamma}|} = \frac{2\gamma_j}{|\mathbf{v}|}$$

where the numerator comes from Problem 9.10a. In addition

$$|\mathbf{v}|^2 = 9\gamma_i\gamma_i\gamma_j^2 + \delta_{ii} - 6\gamma_j^2 = 3\left(\gamma_j^2+1\right)$$

Recall that $\delta_{ii} = 3$. Therefore, $\cos\theta \neq 0$, and v_i is not perpendicular to γ_i (see Problem 9.10b).

Now consider $\mathbf{v}\times\mathbf{\Gamma}$. If $\mathbf{v}$ is parallel to $\mathbf{\Gamma}$ their vector product should be zero.

$$(\mathbf{v}\times\mathbf{\Gamma})_k = \epsilon_{kil}(3\gamma_i\gamma_j - \delta_{ij})\gamma_l = 3(\mathbf{\Gamma}\times\mathbf{\Gamma})_k\gamma_j - \epsilon_{kjl}\gamma_l = -\epsilon_{kjl}\gamma_l$$

Let $j = 1$. Then

$$(\mathbf{v}\times\mathbf{\Gamma}) = (-\epsilon_{11l}\gamma_l, -\epsilon_{213}\gamma_3, -\epsilon_{312}\gamma_2) = (0, \gamma_3, -\gamma_2)$$

This vector product is different from zero for arbitrary $\mathbf{\Gamma}$. Similar results are obtained for $j = 2, 3$.

•Problem 12. Start with the integral in (9.5.16) an integrate by parts as indicated below

$$\int_{r/\alpha}^{r/\beta} \tau T(t-\tau)\mathrm{d}\tau = \int_{r/\alpha}^{r/\beta} \underbrace{\tau}_{u}\underbrace{J''(t-\tau)\mathrm{d}\tau}_{dv} = -\tau J'(t-\tau)\Big|_{r/\alpha}^{r/\beta} + \int_{r/\alpha}^{r/\beta} J'(t-\tau)\mathrm{d}\tau$$

$$= \frac{r}{\alpha}J'(t-r/\alpha) - \frac{r}{\beta}J'(t-r/\beta) - J(t-\tau)\Big|_{r/\alpha}^{r/\beta} = \frac{r}{\alpha}J'(t-r/\alpha) + J(t-r/\alpha) - \frac{r}{\beta}J'(t-r/\beta) - J(t-r/\beta)$$

Introducing this expression in (9.5.16) gives the terms in $1/r^3$ and $1/r^2$ in (9.5.19). The terms in $1/r$ are obtained by replacing T with J''.

•Problem 13. Given

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega$$

we know from (5.4.25) that $A(\omega)$ is the Fourier transform of $f(t)$. Apply this fact to the expressions obtained by taking the first and second derivatives of both sides of the integral with respect to time

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathrm{i}\omega A(\omega)]e^{\mathrm{i}\omega t}\mathrm{d}\omega$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[-\omega^2 A(\omega)]e^{\mathrm{i}\omega t}\mathrm{d}\omega$$

Therefore

$$\mathcal{F}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = \mathrm{i}\omega A(\omega)$$

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right\} = -\omega^2 A(\omega)$$

Now apply (11.6.16) to $f(t)$ and $\mathrm{d}f/\mathrm{d}t$

$$\int_{-\infty}^{\infty}|f(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}|A(\omega)|^2\mathrm{d}\omega$$

$$\int_{-\infty}^{\infty}\left|\frac{\mathrm{d}f}{\mathrm{d}t}\right|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|A(\omega)|^2\mathrm{d}\omega$$

The time integrals give the energies of f and $\mathrm{d}f/\mathrm{d}t$ (see §11.6.2), which depend on the frequency content of $f(t)$. For example, if $|A(\omega)|$ is negligible for $\omega > 1$, then $f(t)$ will have more energy than $\mathrm{d}f/\mathrm{d}t$.

•Problem 14. Start with

$$\mathcal{F}\{h'(t)\} = \int_{-\infty}^{\infty}h'(t)e^{-\mathrm{i}\omega t}\mathrm{d}t$$

Integrate by parts and assume that $h(\pm\infty) = 0$. Then

$$\mathcal{F}\{h'(t)\} = h(t)e^{-\mathrm{i}\omega t}\Big|_{-\infty}^{\infty} + \mathrm{i}\omega\int_{-\infty}^{\infty}h(t)e^{-\mathrm{i}\omega t}\mathrm{d}t = \mathrm{i}\omega\mathcal{F}\{h(t)\}$$

or

$$\mathcal{F}\{h'(t)\} = \mathrm{i}\omega\mathcal{F}\{h(t)\}$$

Let us apply this result to our problem.

$$T(\omega) = \mathcal{F}\{T(t)\} = \mathcal{F}\{J''(t)\} = \mathcal{F}\left\{\frac{\mathrm{d}J'}{\mathrm{d}t}\right\} = \mathrm{i}\omega\mathcal{F}\{J'(t)\} = -\omega^2\mathcal{F}\{J(t)\}$$

Then

$$\mathcal{F}\{J'(t)\} = -\frac{\mathrm{i}}{\omega}T(\omega)$$

and

$$\mathcal{F}\{J(t)\} = -\frac{1}{\omega^2}T(\omega)$$

We also need

$$\mathcal{F}\{h(t-t_o)\} = e^{-\mathrm{i}\omega t_o}\mathcal{F}\{h(t)\}$$

(see (6.5.68) where $h = J, J', J''$ and $t_o = r/\alpha, r/\beta$. Introducing these results in (9.5.19) gives (9.5.20).

•Problem 15.

$$J'(t) = \int_0^t J''(\tau)\mathrm{d}\tau = \int_0^t a^2\tau\mathrm{e}^{-a\tau}\mathrm{d}\tau = \mathrm{e}^{-a\tau}(-a\tau - 1)\Big|_0^t = \left(1 - \mathrm{e}^{-at} - at\mathrm{e}^{-at}\right)H(t)$$

$$J(t) = \int_0^t J'(\tau)\mathrm{d}\tau = t + \frac{1}{a}\mathrm{e}^{-at} - \frac{1}{a} + I$$

where

$$I = -\int_0^t a\tau\mathrm{e}^{-a\tau}\mathrm{d}\tau = -\frac{1}{a}J'(t) = -\frac{1}{a} + \frac{1}{a}\mathrm{e}^{-at} + t\mathrm{e}^{-at}$$

Combining these results gives (9.5.21). To justify the $H(t)$, consider a function $g(t) = f(t)H(t)$ with $f(0) = 0$. The derivative of $g(t)$ is $g'(t) = f'(t)H(t) + f(t)\delta(t)$ (see (A.30)), but $f(t)\delta(t) = f(0)\delta(t) = 0\delta(t) = 0$ (see (A.20)-(A.22)), so that this term does not contribute to the derivative. This applies to $J(t)$ and $J'(t)$ because they are both zero at the origin.

•Problem 16.

$$\mathcal{F}\{J''(t)\} = \int_0^\infty a^2 t\mathrm{e}^{-at}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \int_0^\infty a^2 t\mathrm{e}^{-at}\cos\omega t\ \mathrm{d}t - \mathrm{i}\int_0^\infty a^2 t\mathrm{e}^{-at}\sin\omega t\ \mathrm{d}t$$

$$a^2\frac{a^2-\omega^2}{a^2+\omega^2} - \mathrm{i}a^2\frac{2a\omega^2}{a^2+\omega^2}$$

$$|\mathcal{F}\{J''(t)\}| = \frac{a^2}{a^2+\omega^2} = \frac{1}{1+(\omega/a)^2}$$

Note the combination at in $J''(t)$ an the combination ω/a in its Fourier transform. This means that the narrower the function in one domain, the broader it is in the other domain (see Problem 7.10).

•Problem 17.

(a) Plot of $|\cos\theta|$. Draw a line segment from the origin making an angle θ with the x_3 axis and having length $r = |\cos\theta|$. This gives the point Q in Fig. 9.5a. The projection of Q on the x_1 and x_3 axes will be indicated with x_Q and y_Q. The Cartesian coordinates of Q are

$$x_Q = r\sin\theta = |\cos\theta|\sin\theta$$

$$y_Q = r\cos\theta = |\cos\theta|\cos\theta$$

For $-\pi/2 \le \theta \le \pi/2$ (or $x_3 > 0$), $|\cos\theta| = \cos\theta$, and

$$x_Q = \cos\theta\sin\theta$$

$$y_Q = \cos^2\theta$$

Then

$$x_Q^2 + \left(y_Q - \frac{1}{2}\right)^2 = \cos^2\theta\sin^2\theta + \cos^4\theta - \cos^2\theta + \frac{1}{4} = \frac{1}{4}$$

and

$$x_Q^2 + \left(y_Q - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

This is the equation of a circle with center at $(0, 1/2)$ and radius $1/2$.

For $\pi/2 \le \theta \le 3\pi/2$ (or $x_3 < 0$), $|\cos\theta| = -\cos\theta$, and

$$x_Q = -\cos\theta\sin\theta$$

$$y_Q = -\cos^2\theta$$

Then

$$x_Q^2 + \left(y_Q + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

This is the equation of a circle with center at $(0, -1/2)$ and radius 1/2.

(b) Plot of $|\sin\theta|$. In this case

$$x_Q = r\sin\theta = |\sin\theta|\sin\theta$$

$$y_Q = r\cos\theta = |\sin\theta|\cos\theta$$

For $0 \le \theta \le \pi$ (or $x_1 > 0$), $|\sin\theta| = \sin\theta$, and

$$x_Q = r\sin\theta = \sin^2\theta$$

$$y_Q = r\cos\theta = \sin\theta\cos\theta$$

Then

$$\left(x_Q - \frac{1}{2}\right)^2 + y_Q^2 = \left(\frac{1}{2}\right)^2$$

This is the equation of a circle with center at $(1/2, 0)$ and radius 1/2.

For $\pi \le \theta \le 2\pi$ (or $x_1 < 0$), $|\sin\theta| = -\sin\theta$, and

$$x_Q = r\sin\theta = -\sin^2\theta$$

$$y_Q = r\cos\theta = -\sin\theta\cos\theta$$

Then

$$\left(x_Q + \frac{1}{2}\right)^2 + y_Q^2 = \left(\frac{1}{2}\right)^2$$

This is the equation of a circle with center at $(-1/2, 0)$ and radius 1/2.

•Problem 18. In (9.6.1), let $t_{ij} = \gamma_i\gamma_j$. The time dependence comes from scalar quantities and can be ignored. After a rotation of coordinates

$$t'_{ij} = a_{im}a_{jn}t_{mn} = a_{im}\gamma_m a_{jn}\gamma_n = \gamma'_i\gamma'_j$$

Therefore, Green's function is a tensor-valued function.

•Problem 19. To avoid dealing with the convolution apply the Fourier transform in the time domain to both sides of (9.7.2):

$$u_i(\mathbf{x}, \omega) \equiv u_i = F_j(\omega)G_{ij}(\mathbf{x}, \omega; \boldsymbol{\xi}, 0) \equiv G_{ij}F_j$$

Here u_i and F_j are vectors. Start with

$$u'_i = G'_{ij}F'_j$$

write u'_i and F'_j in terms of u_m and F_n

$$a_{im}u_m = G'_{ij}a_{jn}F_n$$

use the first equation equation eliminate u_m and rearrange

$$(a_{im}G_{mn} - G'_{ij}a_{jn})F_n = 0$$

This implies

$$a_{im}G_{mn} = G'_{ij}a_{jn}$$

contract with a_{pn}

$$a_{im}a_{pn}G_{mn} = G'_{ij}a_{jn}a_{pn} = G'_{ij}\delta_{jp}$$

Therefore

$$G'_{ip} = a_{im}a_{pn}G_{mn}$$

This equation is similar to (1.4.10), which means that G_{ij} is a tensor.

•Problem 20. Start with (9.9.1) in the frequency domain

$$u_k = M_{ij}G_{ki,j} \equiv M_{ij}s_{kij}$$

Here u_k is a vector and $s_{kij} = G_{ki,j}$ is a tensor (see the two previous problems and §1.4.3). Start with

$$u'_k = M'_{ij}s'_{kij}$$

then

$$a_{kl}u_l = a_{kl}a_{im}a_{jn}s_{lmn}M'_{ij}$$

and

$$a_{kl}s_{lmn}M_{mn} = a_{kl}a_{im}a_{jn}s_{lmn}M'_{ij}$$

so that

$$\left(a_{im}a_{jn}M'_{ij} - M_{mn}\right)a_{kl}s_{lmn} = 0$$

Then

$$M_{mn} = a_{im}a_{jn}M'_{ij}$$

which means that M_{mn} is a tensor (see (1.4.12)).

•Problem 21. Start with (9.13.3)-(9.13.8). Using

$$\bar{\mathbf{M}}\mathbf{\Gamma} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} \gamma_3 \\ 0 \\ -\gamma_1 \end{pmatrix}$$

$$\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma} = \gamma_1\gamma_3 - \gamma_3\gamma_1 = 0$$

$$\mathrm{tr}(\bar{\mathbf{M}}) = 0$$

$$\bar{\mathbf{M}}^T\mathbf{\Gamma} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} -\gamma_3 \\ 0 \\ \gamma_1 \end{pmatrix}$$

$$\bar{\mathbf{M}}^T\mathbf{\Gamma} + \bar{\mathbf{M}}\mathbf{\Gamma} = \mathbf{0}$$

we obtain

$$\mathbf{A}^N = \mathbf{A}^{I\alpha} = \mathbf{A}^{FP} = \mathbf{0}$$

$$\mathbf{A}^{I\beta} = -\bar{\mathbf{M}}^T\mathbf{\Gamma} - 2\bar{\mathbf{M}}\mathbf{\Gamma} = (-\gamma_3, 0, \gamma_1)^T$$
$$\mathbf{A}^{FS} = -\bar{\mathbf{M}}\mathbf{\Gamma} = (-\gamma_3, 0, \gamma_1)^T$$

The last two vectors are perpendicular to $\mathbf{\Gamma}$ and thus represent S wave motion.

•Problem 22. Start with (9.13.3) - (9.13.8). Because $\bar{\mathbf{M}}$ is the identity matrix

$$\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma} = \mathbf{\Gamma}^T\mathbf{\Gamma} = 1$$

$$\bar{\mathbf{M}}^T\mathbf{\Gamma} = \bar{\mathbf{M}}\mathbf{\Gamma} = \mathbf{\Gamma}$$

Also

$$\mathrm{tr}(\bar{\mathbf{M}}) = 3$$

Then

$$\mathbf{A}^{I\alpha} = \mathbf{A}^{FP} = \mathbf{\Gamma}$$

and all the other terms are equal to zero.

•Problem 23. Use (9.2.6)

$$r = \left[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2\right]^{1/2}$$

Then

$$\frac{\partial r}{\partial \xi_j} = \frac{1}{2r}\frac{\partial}{\partial \xi_j}(x_j - \xi_j)^2 = -\frac{x_j - \xi_j}{r} = -\gamma_j$$

(see (9.5.13)) and

$$\frac{\partial \gamma_i}{\partial \xi_j} = \frac{\partial}{\partial \xi_j}\frac{x_j - \xi_j}{r} = -\frac{1}{r}\frac{\partial \xi_i}{\partial \xi_j} - \frac{x_j - \xi_j}{r^2}\frac{\partial r}{\partial \xi_j} = \frac{1}{r}\delta_{ij} + \frac{1}{r}\frac{x_j - \xi_j}{r}\gamma_j = \frac{1}{r}(\gamma_i\gamma_j - \delta_{ij})$$

•Problem 24. Refer to Fig. 9.10. The components of $\mathbf{\Gamma}$ are the components of the point B in spherical coordinates. Because $\mathbf{\Theta}$ is perpendicular to $\mathbf{\Gamma}$ and both vectors are in the same plane, the angles between $\mathbf{\Theta}$ and the x_1 and x_3 axes are ϕ and $\theta + \pi/2$, respectively (assume that $\mathbf{\Theta}$ has been translated to the origin). Therefore, the coordinates of $\mathbf{\Theta}$ are obtained from those of $\mathbf{\Gamma}$ with θ replaced by $\theta + \pi/2$ (recall that $\sin(\theta + \pi/2) = \cos\theta$ and $\cos(\theta + \pi/2) = -\sin\theta$). For $\mathbf{\Phi}$ the corresponding angles are $\phi + \pi/2$ and $\pi/2$. To verify that the three vectors form a right-handed coordinate system perform the appropriate vector products.

•Problem 25. Start with (9.9.18a)

$$\mathcal{R}^P(\theta, \phi) = \gamma_i \dot{M}_{ij}\gamma_j$$

The conditions for extremal values are

$$\frac{\partial \mathcal{R}^P}{\partial \theta} = 0; \qquad \frac{\partial \mathcal{R}^P}{\partial \phi} = 0$$

For a symmetric moment tensor we have

$$\frac{\partial \mathcal{R}^P}{\partial \theta} = \frac{\partial \gamma_i}{\partial \theta}\dot{M}_{ij}\gamma_j + \gamma_i\dot{M}_{ij}\frac{\partial \gamma_j}{\partial \theta} = \frac{\partial \gamma_i}{\partial \theta}\dot{M}_{ij}\gamma_j + \gamma_j\dot{M}_{ji}\frac{\partial \gamma_i}{\partial \theta} = 2\frac{\partial \gamma_i}{\partial \theta}\dot{M}_{ij}\gamma_j = 2\frac{\partial \mathbf{\Gamma}^T}{\partial \theta}\dot{\mathbf{M}}\mathbf{\Gamma} = 0 \quad (1)$$

and

$$\frac{\partial \mathcal{R}^P}{\partial \phi} = 2\frac{\partial \gamma_i}{\partial \phi}\dot{M}_{ij}\gamma_j = 2\frac{\partial \mathbf{\Gamma}^T}{\partial \phi}\dot{\mathbf{M}}\mathbf{\Gamma} = 0 \tag{2}$$

Now use

$$\frac{\partial \mathbf{\Gamma}^T}{\partial \theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) = \mathbf{\Theta}^T \tag{3}$$

and

$$\frac{\partial \mathbf{\Gamma}^T}{\partial \phi} = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0) = \sin\theta\mathbf{\Phi}^T \tag{4}$$

(Harkrider, 1976; R. Herrmann, personal comm., 1994) Therefore, from (1), (3) and (9.9.18b) we obtain

$$\mathbf{\Theta}^T\dot{\mathbf{M}}\mathbf{\Gamma} = \mathcal{R}^{SV} = 0$$

and from (2), (4) and (9.9.18c)

$$\mathbf{\Phi}^T\dot{\mathbf{M}}\mathbf{\Gamma} = \sin\theta\mathcal{R}^{SH} = 0$$

•Problem 26.

$$\bar{\mathbf{M}}'\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\mathbf{R}^T\bar{\mathbf{M}}'\mathbf{R} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

•Problem 27. The rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} \mathbf{t}^T \\ \mathbf{b}^T \\ \mathbf{P}^T \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} = \mathbf{R}^T$$

Then

$$\mathbf{R}^T\mathbf{M}^{dc} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{R}^T\mathbf{M}^{dc}\mathbf{R} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

•Problem 28. From (9.12.4a)

$$\mathcal{R}^P = \sin 2\theta\cos\phi$$

Then

$$\frac{\partial \mathcal{R}^P}{\partial \theta} = 2\cos 2\theta\cos\phi = 0$$

and

$$\frac{\partial \mathcal{R}^P}{\partial \phi} = -\sin 2\theta\sin\phi = 0$$

The pairs (θ, ϕ) that satisfy these two conditions simultaneously are $(\pi/4, 0)$, $(-\pi/4, 0)$ $(\pi/2, \pi/2)$ (this is not in the hint) and $(0, \pi/2)$. The first two pairs define the directions of $\mathbf{t}$ and $\mathbf{p}$ and correspond to extremal values of $\mathcal{R}^P$ (equal to 1 and -1, respectively). From (9.12.4b,c), $\mathcal{R}^{SV}$ and $\mathcal{R}^{SH}$ are zero for these two directions. The third pair corresponds to the direction of $\mathbf{b}$ and in this case the three radiation patterns are equal to zero. For the last pair $\mathcal{R}^P$ and $\mathcal{R}^{SV}$ are equal to zero, but not $\mathcal{R}^{SH}$, which is equal to -1. On the other hand $\sin\theta\mathcal{R}^{SH} = 0$, in agreement with the last equation in Problem 9.25.

•Problem 29. In (9.13.1), using (9.13.2), there are combinations of the following types

$$\gamma_i \bar{M}_{ij}\gamma_j\gamma_k = (\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma})(\mathbf{\Gamma})_k$$

$$\delta_{ij}\bar{M}_{ij}\gamma_k = \bar{M}_{ii}\gamma_k = \mathrm{tr}(\bar{\mathbf{M}})(\mathbf{\Gamma})_k$$

$$\delta_{kj}\bar{M}_{ij}\gamma_i = \bar{M}_{ik}\gamma_i = (\bar{\mathbf{M}}^T\mathbf{\Gamma})_k$$

$$\delta_{ki}\bar{M}_{ij}\gamma_j = \bar{M}_{kj}\gamma_j = (\bar{\mathbf{M}}\mathbf{\Gamma})_k$$

Introducing these results in (9.13.1) immediately gives (9.13.3).

•Problem 30. Introduce (9.13.9) in the integral in (9.13.3) and integrate by parts

$$\int_{r/\alpha}^{r/\beta} \underbrace{\tau}_{u}\underbrace{J'(t-\tau)\mathrm{d}\tau}_{dv} = -\tau J(t-\tau)\Big|_{r/\alpha}^{r/\beta} + \int_{r/\alpha}^{r/\beta} J(t-\tau)\mathrm{d}\tau$$

$$= \frac{r}{\alpha}E(t-r/\alpha) - \frac{r}{\beta}E(t-r/\beta) + \int_{r/\alpha}^{r/\beta} J(t-\tau)\mathrm{d}\tau$$

where (9.13.10) was used. Now let $t - \tau = u$, so that $\mathrm{d}\tau = -\mathrm{d}u$. With this change of variable the integral becomes

$$\int_{r/\alpha}^{r/\beta} J(t-\tau)\mathrm{d}\tau = -\int_{t-r/\alpha}^{t-r/\beta} J(u)\mathrm{d}u = \int_0^{t-r/\alpha} J(u)\mathrm{d}u - \int_0^{t-r/\beta} J(u)\mathrm{d}u = G(t-r/\alpha) - G(t-r/\beta)$$

where (9.13.10) was used. Introducing these expression in (9.13.3) gives (9.13.11).

•Problem 31. The starting point are equations (9.13.4)-(9.13.8). Use $\bar{\mathbf{M}}^T = \bar{\mathbf{M}}$, $\mathbf{\Gamma}^T\mathbf{\Gamma} = 1$, and the orthogonality of $\mathbf{\Gamma}$, $\mathbf{\Theta}$ and $\mathbf{\Phi}$. Recall that $\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma}$ is a scalar, so that $(\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma})\mathbf{\Gamma}$ is orthogonal to $\mathbf{\Theta}$ and $\mathbf{\Phi}$. Therefore, this term as well as $\mathrm{tr}(\bar{\mathbf{M}})\mathbf{\Gamma}$ cancel out for the projections in the $\mathbf{\Theta}$ and $\mathbf{\Phi}$ directions.

To apply (9.13.15)-(9.13.17) to $\bar{\mathbf{M}}^{dc}$, note that its trace is equal to zero. In addition, for any two vectors $\mathbf{v}$ and $\mathbf{w}$ we have

$$\mathbf{v}^T\bar{\mathbf{M}}^{dc}\mathbf{w} = v_1w_3 + v_3w_1$$

and

$$\mathbf{\Gamma}^T\bar{\mathbf{M}}^{dc}\mathbf{\Gamma} = 2\gamma_1\gamma_3 = 2\sin\theta\cos\phi\cos\theta = \sin 2\theta\cos\phi$$

$$\mathbf{\Theta}^T\bar{\mathbf{M}}^{dc}\mathbf{\Gamma} = \theta_1\gamma_3 + \theta_3\gamma_1 = \cos^2\theta\cos\phi - \sin^2\theta\cos\phi = \cos 2\theta\cos\phi$$

$$\mathbf{\Phi}^T\bar{\mathbf{M}}^{dc}\mathbf{\Gamma} = \phi_1\gamma_3 + \phi_3\gamma_1 = -\cos\theta\sin\phi$$

$$\mathcal{R}^{N\Gamma} = \pm 9 \sin 2\theta \cos\phi$$
$$\mathcal{R}^{N\Theta} = \pm(-6 \cos 2\theta \cos\phi)$$
$$\mathcal{R}^{N\Phi} = \pm 6 \cos\theta \sin\phi$$

$$\mathcal{R}^{I\alpha\Gamma} = 4 \sin 2\theta \cos\phi$$
$$\mathcal{R}^{I\alpha\Theta} = -2 \cos 2\theta \cos\phi$$
$$\mathcal{R}^{I\alpha\Phi} = 2 \cos\theta \sin\phi$$

$$\mathcal{R}^{I\beta\Gamma} = -3 \sin 2\theta \cos\phi$$
$$\mathcal{R}^{I\beta\Theta} = 3 \cos 2\theta \cos\phi$$
$$\mathcal{R}^{I\beta\Phi} = -3 \cos\theta \sin\phi$$

Similar equations are given by Aki and Richards (1980). The equations for the far field are given in (9.12.4).

CHAPTER 10

•Problem 1. Introduce (10.2.10) and (10.2.11a) in (10.2.9), replace $\bar{v}_i$ by (10.2.11b). From the left side we get

$$\int_{-\infty}^{\infty} \mathrm{d}t \int_V [u_i \delta_{in} \delta(\mathbf{x}, -t; \boldsymbol{\xi}, -\tau) - G_{in}(\mathbf{x}, -t; \boldsymbol{\xi}, -\tau) f_i] \mathrm{d}V = u_n(\boldsymbol{\xi}, \tau) - \int_{-\infty}^{\infty} dt \int_V G_{in}(\mathbf{x}, -t; \boldsymbol{\xi}, -\tau) f_i \mathrm{d}V \tag{1}$$

The right-hand side of (10.2.12) is obtained immediately, with the first integral coming from (1). The effect of the temporal δ is to replace t with τ.

•Problem 2. Using (9.2.3) we can write

$$\delta(\boldsymbol{\xi} - \boldsymbol{\sigma}) = \delta(\xi_1 - \sigma_1)\delta(\xi_2 - \sigma_2)\delta(\xi_3 - \sigma_3)$$

Introduce this expression in the first integral in (10.4.3)

$$I = \int_V \delta(\xi_1 - \sigma_1)\delta(\xi_2 - \sigma_2)\delta(\xi_3 - \sigma_3) \frac{\partial}{\partial \xi_q} G_{np} \ \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

Then, using (A.27) or the first equality in (A-31) we can write

$$\int \delta(\xi_q - \sigma_q) \frac{\partial}{\partial \xi_q} G_{np} \ \mathrm{d}\xi_q = - \int \frac{\partial}{\partial \xi_q} \delta(\xi_q - \sigma_q) G_{np} \ \mathrm{d}\xi_q = - \int \delta_{,q}(\xi_q - \sigma_q) G_{np} \ \mathrm{d}\xi_q$$

Therefore,

$$I = - \int_V \delta_{,q}(\boldsymbol{\xi} - \boldsymbol{\sigma}) G_{np} \ \mathrm{d}V_\xi$$

•Problem 3. Total body force: is given by the integral over V of $e_p(\boldsymbol{\xi}, t)$. Start with (10.4.7) with $[T_p] = 0$

$$\int_V e_p(\boldsymbol{\xi}, \tau) \mathrm{d}V = - \int_\Sigma [u_i(\boldsymbol{\sigma}, \tau)] c_{ijpq}(\boldsymbol{\sigma}) \left\{ \int_V \delta_{,q}(\boldsymbol{\xi} - \boldsymbol{\sigma}) dV_\xi \right\} \nu_j \mathrm{d}\Sigma_\sigma$$

The volume integral involves the δ only, which using Gauss theorem can be written as

$$\int_V \delta_{,q}(\boldsymbol{\xi} - \boldsymbol{\sigma}) \mathrm{d}V_\xi = \int_S \delta(\boldsymbol{\xi} - \boldsymbol{\sigma}) n_q \mathrm{d}S_\xi = 0$$

where S and n_q are as in (10.2.9). The integral on the right is zero because the δ is nonzero over Σ only and because V and S do not have a common point (Burridge and Knopoff, 1964).

Total moment: here we are interested in the torque caused by a force, which is given by the vector product of the vector $\mathbf{r} = (x_1, x_2, x_3)$ and the force (e.g., Arya, 1990). For a force $\mathbf{f}$ we have

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$$

The q-th component of the torque, τ_q, is given by $\tau_q = \epsilon_{qrp} x_r f_p$. In our case the force is the equivalent body force. To get the total moment integrate over V. Use ξ_r instead of x_r

$$\tau_q = \int_V \epsilon_{qrp} \xi_r f_p \mathrm{d}V_\xi = - \int_\Sigma \epsilon_{qrp} [u_i(\boldsymbol{\sigma}, \tau)] c_{ijpq}(\boldsymbol{\sigma}) \left\{ \int_V \xi_r \delta_{,q}(\boldsymbol{\xi} - \boldsymbol{\sigma}) dV_\xi \right\} \nu_j \mathrm{d}\Sigma_\sigma$$

Consider the volume integral

$$\int_V \xi_r \delta_{,q}(\boldsymbol{\xi} - \boldsymbol{\sigma}) \mathrm{d}V_\xi = -\int_V \xi_{r,q} \delta(\boldsymbol{\xi} - \boldsymbol{\sigma}) \mathrm{d}V_\xi = -\sigma_{r,q} = -\delta_{rq}$$

where δ_{rq} is Kronecker's delta. The first equality follows from an argument similar to that used in Problem 10.2. Then

$$\tau_q = \int_\Sigma \epsilon_{qrp}[u_i]c_{ijpq}\delta_{rq}\nu_j \mathrm{d}\Sigma_\sigma = \int_\Sigma \epsilon_{qrp}[u_i]c_{ijpr}\nu_j \mathrm{d}\Sigma_\sigma = 0$$

The last result follows from the fact that the integral involves the product of a symmetric and antisymmetric tensor (see (4.5.3b)) (after Burridge and Knopoff, 1964).

•Problem 4. Use (A.27) or the first equality in (A.31)

$$\int \xi_3 \frac{\partial}{\partial \xi_3} \delta(\xi_3) d\xi_3 = -\int \xi_{3,3} \delta(\xi_3) d\xi_3 = -\int \delta(\xi_3) d\xi_3 = -1$$

because $\xi_{3,3} = 1$.

•Problem 5. Green's function $G_{np}(\mathbf{x}, t; \boldsymbol{\sigma}, \tau)$ satisfies (10.2.2) with f_i given by the product of deltas in (10.2.10) (after obvious modifications)

$$(c_{njrq}G_{rp,q})_{,j} + \delta_{np}\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau) = \rho \frac{\partial^2 G_{np}}{\partial t^2}$$

Now let us take as origin time the time at which the source acts. The time $t' = t - \tau$ is known as the elapsed time and Green's function becomes $G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\sigma}, 0) = G_{np}(\mathbf{x}, t'; \boldsymbol{\sigma}, 0)$. Then

$$\frac{\partial^2 G_{np}}{\partial t^2} = \frac{\partial^2 G_{np}}{\partial t'^2}$$

and

$$(c_{njrq}G_{rp,q})_{,j} + \delta_{np}\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t') = \rho \frac{\partial^2 G_{np}}{\partial t'^2}$$

(after Haberman, 1983). Therefore, as long as the boundary conditions of the problem are time independent, the origin time can be chosen arbitrarily (Aki and Richards, 1980).

•Problem 6. To get the moment tensor density use (10.6.2) and (10.5.2). The only nonzero contributions come from $[u_3]$ and $\nu_3 = 1$

$$c_{33pq} = \lambda\delta_{33}\delta_{pq} + \mu(\delta_{3p}\delta_{3q} + \delta_{3q}\delta_{3p}) = \lambda\delta_{pq} + 2\mu\delta_{3p}\delta_{3q}$$

Because $\delta_{pq} \neq 0$ for (p, q) equal to $(1, 1)$, $(2, 2)$ and $(3, 3)$ and $\delta_{3p}\delta_{3q}$ is nonzero for $(p, q) = (3, 3)$, we have

$$m_{11} = m_{22} = \lambda[u_3]; \qquad m_{33} = (\lambda + 2\mu)[u_3]$$

or

$$m_{ij} = [u_3](\lambda\delta_{ij} + 2\mu\delta_{i3}\delta_{j3})$$

To get the moment tensor use (10.6.5) and replace the integral by the area A times the average value of $[u_3]$. The basic features of this solution were derived by Burridge and Knopoff (1964).

•Problem 7. Introduce (10.4.7) in (10.6.13) (assuming $[T_p] = 0$)

$$\acute{M}_{pq} = -\int_\Sigma [u_i]c_{ijpl}\left\{\int_{Vo}(\xi_q - \bar{\xi}_q)\delta_{,l}(\boldsymbol{\xi} - \boldsymbol{\sigma})dV_\xi\right\}\nu_j \mathrm{d}\Sigma_\sigma = \int_\Sigma [u_i]c_{ijpq}\nu_j \mathrm{d}\Sigma_\sigma = \acute{M}_{qp} = M_{pq}$$

The second equality comes from the fact that the volume integral contributes a factor $-\delta_{ql}$ (see Problem 10.3, total moment). The term $\bar{\xi}_q$ is a constant and its derivative with respect to ξ_l gives zero. The symmetry of $\acute{M}_{pq}$ follows from the symmetry of c_{ijpq}. Finally, the last integral is the definition of M_{pq}, as can be seen from (10.6.5) and (10.6.2).

•Problem 8.

(a) Start with $\mathbf{b} = \mathbf{t} \times \mathbf{p}$ (see (9.11.6)) and use (10.7.15) and (10.7.16)

$$\mathbf{b} = \frac{1}{2}(\mathbf{s} + \boldsymbol{\nu}) \times (\mathbf{s} - \boldsymbol{\nu}) = \boldsymbol{\nu} \times \mathbf{s}$$

This result follows from $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for arbitrary $\mathbf{a}$ and $\mathbf{b}$.

(b) Using $\mathbf{s}.\boldsymbol{\nu} = 0$ and $\mathbf{s}.\mathbf{s} = \boldsymbol{\nu}.\boldsymbol{\nu} = 1$ we obtain

$$M_{ij}p_j = \frac{1}{\sqrt{2}}(s_i\nu_j + s_j\nu_i)(s_j - \nu_j) = -\frac{1}{\sqrt{2}}(s_i - \nu_i) = -p_i$$

so that $\mathbf{p}$ is an eigenvector of M_{ij} with eigenvalue equal to -1.

$$M_{ij}t_j = \frac{1}{\sqrt{2}}(s_i\nu_j + s_j\nu_i)(s_j + \nu_j) = \frac{1}{\sqrt{2}}(s_i + \nu_i) = t_i$$

so that $\mathbf{t}$ is an eigenvector of M_{ij} with eigenvalue equal to 1.
Now using $\mathbf{b} \perp \boldsymbol{\nu}$ and $\mathbf{b} \perp \mathbf{s}$ (see part (a)) we obtain

$$M_{ij}b_j = (s_i\nu_j + s_j\nu_i)b_j = s_i\nu_j b_j + \nu_i s_j b_j = 0$$

so that $\mathbf{b}$ is an eigenvector of M_{ij} with eigenvalue equal to 0.

•Problem 9. For a zero trace symmetric moment tensor from (9.13.4), (9.13.5) and (9.13.7) we obtain the following expressions for the projections along $\boldsymbol{\Gamma}$, $\boldsymbol{\Theta}$ and Φ

$$\boldsymbol{\Gamma}^T\mathbf{A}^N = 9\boldsymbol{\Gamma}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (1)$$
$$\boldsymbol{\Theta}^T\mathbf{A}^N = -6\boldsymbol{\Theta}^T\bar{\mathbf{M}}\Gamma \quad (2)$$
$$\boldsymbol{\Phi}^T\mathbf{A}^N = -6\boldsymbol{\Phi}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (3)$$

$$\boldsymbol{\Gamma}^T\mathbf{A}^{I\alpha} = 4\boldsymbol{\Gamma}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (4)$$
$$\boldsymbol{\Theta}^T\mathbf{A}^{I\alpha} = -2\boldsymbol{\Theta}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (5)$$
$$\boldsymbol{\Phi}^T\mathbf{A}^{I\alpha} = -2\boldsymbol{\Phi}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (6)$$

$$\boldsymbol{\Gamma}^T\mathbf{A}^{I\beta} = 3\boldsymbol{\Gamma}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (7)$$
$$\boldsymbol{\Theta}^T\mathbf{A}^{I\beta} = -3\boldsymbol{\Theta}^T\bar{\mathbf{M}}\boldsymbol{\Gamma} \quad (8)$$

$$\mathbf{\Phi}^T \mathbf{A}^{I\beta} = -3\mathbf{\Phi}^T \bar{\mathbf{M}}\mathbf{\Gamma} \tag{9}$$

To get u^s_Γ use the coefficients of (1), (4) and (7) in the expression in brackets in (10.10.3). This gives

$$u^s_\Gamma = \frac{1}{4\pi\rho}\frac{D_o}{r^2}\left[\frac{9}{2}\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)+\frac{4}{\alpha^2}-\frac{3}{\beta^2}\right]\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma} = \frac{1}{8\pi\rho}\frac{D_o}{r^2}\left(\frac{3}{\beta^2}-\frac{1}{\alpha^2}\right)\mathbf{\Gamma}^T\bar{\mathbf{M}}\mathbf{\Gamma}$$

In a similar way, using (2), (5), (8) and (3), (6), (9) we obtain

$$u^s_\Theta = \frac{1}{4\pi\rho}\frac{D_o}{r^2}\left[-3\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)-\frac{2}{\alpha^2}+\frac{3}{\beta^2}\right]\mathbf{\Theta}^T\bar{\mathbf{M}}\mathbf{\Gamma} = \frac{1}{4\pi\rho}\frac{D_o}{r^2}\frac{1}{\alpha^2}\mathbf{\Theta}^T\bar{\mathbf{M}}\mathbf{\Gamma}$$

and

$$u^s_\Phi = \frac{1}{4\pi\rho}\frac{D_o}{r^2}\frac{1}{\alpha^2}\mathbf{\Phi}^T\bar{\mathbf{M}}\mathbf{\Gamma}$$

•Problem 10.

(a) Refer to Fig. 9.10. $\mathbf{R}$ is the product of two rotation matrices

$$\mathbf{R} = \mathbf{R}_2\mathbf{R}_1$$

$\mathbf{R}_1$ is a counterclockwise rotation of angle ϕ about the x_3 axis. This puts the x'_1 axis and $\mathbf{\Theta}$ on the same plane and the x'_2 axis and $\mathbf{\Phi}$ along the same direction

$$\mathbf{R}_1 = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{R}_2$ is a clockwise rotation of angle θ about the x'_2 axis. This puts the x''_3 axis and $\mathbf{\Gamma}$ along the same direction

$$\mathbf{R}_2 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Multiplication of the two matrices gives (10.10.8). The right-hand side of that equation follows from (9.9.16).

(b) Apply $\mathbf{R}$ to $\mathbf{\Theta}$

$$\mathbf{R}\mathbf{\Theta} = \begin{pmatrix} \mathbf{\Theta}^T \\ \mathbf{\Phi}^T \\ \mathbf{\Gamma}^T \end{pmatrix}\mathbf{\Theta} = \begin{pmatrix} \mathbf{\Theta}^T\mathbf{\Theta} \\ \mathbf{\Phi}^T\mathbf{\Theta} \\ \mathbf{\Gamma}^T\mathbf{\Theta} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Similarly,

$$\mathbf{R}\mathbf{\Phi} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad \mathbf{R}\mathbf{\Gamma} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(c) Inverse matrix. Because $\mathbf{R}$ is a rotation matrix, its inverse is given by

$$\mathbf{R}^{-1} = \mathbf{R}^T = (\mathbf{\Theta}|\mathbf{\Phi}|\mathbf{\Gamma}) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

where the vertical bars indicate matrix partitioning. Recall that $\mathbf{\Theta}$, $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ are column vectors.

$$\mathbf{R}\mathbf{R}^{-1} = \begin{pmatrix} \mathbf{\Theta}^T \\ \mathbf{\Phi}^T \\ \mathbf{\Gamma}^T \end{pmatrix} (\mathbf{\Theta}|\mathbf{\Phi}|\mathbf{\Gamma}) = \mathbf{I}$$

•Problem 11.

$$J(\tau) = H(\tau)\left[\tau + \frac{2}{a}\left(\mathrm{e}^{-a\tau} - 1\right) + \tau\mathrm{e}^{-a\tau}\right] = \frac{1}{a}\left[H(\tau)\left(a\tau + e^{-a\tau} - 1\right) - J'(\tau)\right]$$

(see (9.5.21) and (9.5.22)). Then, using $J(0) = 0$ and (9.5.21) we obtain

$$G(t) = D_o \int_0^t J(\tau)\mathrm{d}\tau = \frac{D_o}{a}\left(\frac{a}{2}\tau^2 - \frac{1}{a}\mathrm{e}^{-a\tau} - \tau\right)\Big|_0^t - \frac{D_o}{a}J(t) =$$

$$\frac{D_o}{2a^2}\left(-2at\mathrm{e}^{-at} - 6\mathrm{e}^{-at} + a^2t^2 - 4at + 6\right) = \frac{D_o}{2a^2}\left[6\left(1 - \mathrm{e}^{-at}\right) + at\left(at - 4 - 2\mathrm{e}^{-at}\right)\right]$$

$G(0) = 0$ and as discussed in Problem 9.15, we can multiply the expression above by $H(t)$.

•Problem 12. Refer to Fig. 10.14. Let $\Delta = \theta$ and $r_o = r$. We are interested in the area $\mathrm{d}\sigma$ of the surface element with corners B, C, D, F. Then, from (10.11.8) and (10.11.6) we obtain

$$\mathrm{d}\sigma = \overline{BC}\ \overline{BD} = r\sin\theta\ \delta\phi\ r\delta\theta = r^2\sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi$$

In the last step an obvious change was made.

CHAPTER 11

•Problem 1. Write (11.4.10) as follows

$$v_R^2 - v_I^2 = \frac{M_R}{\rho} = m_R \tag{1}$$

$$2v_R v_I = \frac{M_I}{\rho} = m_I \tag{2}$$

Solve (2) for v_R and introduce in (1)

$$\frac{m_I^2}{4v_I^2} - v_I^2 = m_R$$

Operate

$$v_I^4 + m_R v_I^2 - \frac{1}{4}m_i^2 = 0$$

Solve for v_I^2

$$v_I^2 = \frac{1}{2}\left(\pm\sqrt{m_R^2 + m_I^2} - m_R\right) = \frac{1}{2\rho}(|M| - M_R)$$

The + sign was chosen so that v_I is real. Then

$$v_I = \sqrt{\frac{|M| - M_R}{2\rho}}$$

Introducing v_I^2 in (1) and solving for v_R gives (11.4.11a).

•Problem 2. Start with the following general result valid for any α

$$\mathcal{F}\left\{e^{-\alpha|t|}\right\} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

(Papoulis, 1962) and use

$$\mathcal{F}\{f(t)\} = F(\omega); \qquad \mathcal{F}\{F(t)\} = 2\pi f(-\omega)$$

(see (A.51)). Then

$$\mathcal{F}\left\{\frac{2\alpha}{\alpha^2 + t^2}\right\} = 2\pi e^{-\alpha|\omega|}$$

so that

$$\mathcal{F}^{-1}\left\{e^{-\alpha|\omega|}\right\} = \frac{1}{\pi}\frac{\alpha}{\alpha^2 + t^2}$$

Then, using (6.5.68) and $\alpha = T/2Q$ we obtain the last factor in (11.5.5).

•Problem 3. From Standard Mathematical Tables (1981, W. Beyer editor, CRC Press, p. 376, No. 615)

$$\int_0^\infty \frac{x^a}{(m + x^b)^c}dx = m^{(a+1-bc)/b}\left[\frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c - \frac{a+1}{b}\right)}{\Gamma(c)}\right]; \qquad a > -1; b > 0; m > 0;\ c > \frac{a+1}{b}$$

where Γ is the gamma function. In our case, $m = 1$, $b = 2$, $c = 1$, and the last condition above requires $a < 1$. Using

$$\Gamma(1) = 1; \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

(e.g., Arfken, 1985) we get

$$I(s) = \frac{1}{2}\frac{\pi}{\sin(\pi(s+1)/2)} = \frac{1}{2}\frac{\pi}{\cos \pi s/2}$$

If s goes to 1, $I(s)$ goes to ∞. This result also follows from the fact that $\Gamma(0) = \infty$. The reason the lower limit of $I(s)$ is 0 while it is $-\infty$ in (11.6.21) is that $\alpha(\omega)$ must be an even function (see (11.5.17b)). Therefore, the integral in (11.6.21) is $2I(s)$.

•Problem 4.

$$\int_{-\infty}^{\infty} \frac{\alpha(\omega')}{\omega - \omega'} \mathrm{d}\omega' = \int_{-\infty}^{0} \frac{\alpha(\omega')}{\omega - \omega'} \mathrm{d}\omega' + \int_{0}^{\infty} \frac{\alpha(\omega')}{\omega - \omega'} \mathrm{d}\omega' \equiv I_1 + I_2$$

Consider I_1. Let $\omega' = -u$, so that $\mathrm{d}\omega' = -\mathrm{d}u$ and use $\alpha(u) = \alpha(-u)$ (see (11.5.17b)). This gives

$$I_1 = -\int_{\infty}^{0} \frac{\alpha(u)}{\omega + u} \mathrm{d}u \equiv \int_{0}^{\infty} \frac{\alpha(\omega')}{\omega + \omega'} \mathrm{d}\omega'$$

In the first integral the dummy variable u was changed to ω'. Then

$$I_1 + I_2 = \int_0^{\infty} \alpha(\omega') \left(\frac{1}{\omega + \omega'} + \frac{1}{\omega - \omega'} \right) \mathrm{d}\omega' = 2\omega \int_0^{\infty} \frac{\alpha(\omega')}{\omega^2 - \omega'^2} d\omega'$$

•Problem 5. Start with

$$I = \mathcal{P} \int_{-R}^{R} \frac{d\omega'}{\omega' - \omega} = \lim_{\delta \to 0} \left(\int_{-R}^{\omega - \delta} \frac{d\omega'}{\omega' - \omega} + \int_{\omega + \delta}^{R} \frac{d\omega'}{\omega' - \omega} \right); \qquad -R < \omega < R$$

The second equality corresponds to the definition of principal value. The third integral is elementary and gives

$$\int_{\omega+\delta}^{R} \frac{d\omega'}{\omega' - \omega} = \ln(\omega' - \omega)\Big|_{\omega+\delta}^{R} = \ln(R - \omega) - \ln \delta$$

In the second integral let $\omega' = -y$. Then $\mathrm{d}\omega' = -\mathrm{d}y$; $\omega' - \omega = -(y + \omega)$ and the integration limits become R and $\delta - \omega$, so that

$$\int_{-R}^{\omega - \delta} \frac{d\omega'}{\omega' - \omega} = \int_{R}^{\delta - \omega} \frac{\mathrm{d}y}{y + \omega} = \ln \delta - \ln(R + \omega)$$

Adding these two results gives

$$I = \ln(R - \omega) - \ln(R + \omega) = \ln \frac{R - \omega}{R + \omega}$$

Finally, as R goes to ∞, I goes to $\ln(1) = 0$ (Byron and Fuller, 1970). This proves (11.7.16).

Using (11.7.16) we can write in general

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{f(\omega')-f(\omega_o)}{\omega'-\omega}\mathrm{d}\omega' = \mathcal{P}\int_{-\infty}^{\infty}\frac{f(\omega')}{\omega'-\omega}\mathrm{d}\omega' - f(\omega_o)\mathcal{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{\omega'-\omega} = \mathcal{P}\int_{-\infty}^{\infty}\frac{f(\omega')}{\omega'-\omega}\mathrm{d}\omega'$$

as long as $f(\omega_o)$ is bounded. This result was used in (11.7.17).

•Problem 6. From (11.7.6b)

$$\mathcal{I}\{n(\omega)\} = \frac{C}{\omega}\alpha(\omega)$$

Then,using (11.5.17b)

$$\mathcal{I}\{n(-\omega)\} = -\frac{C}{\omega}\alpha(-\omega) = -\frac{C}{\omega}\alpha(\omega) = -\mathcal{I}\{n(\omega)\}$$

This proves (11.7.20). To prove (11.7.21) start with

$$I = \int_{-\infty}^{\infty}\frac{\mathcal{I}\{n(\omega')\}}{\omega'(\omega'-\omega)}d\omega' = \int_{-\infty}^{0}\frac{\mathcal{I}\{n(\omega')\}}{\omega'(\omega'-\omega)}\mathrm{d}\omega' + \int_{0}^{\infty}\frac{\mathcal{I}\{n(\omega')\}}{\omega'(\omega'-\omega)}\mathrm{d}\omega'$$

(see (11.7.19)). Rewrite the second integral as done in Problem 11.4. This gives

$$\int_{-\infty}^{0}\frac{\mathcal{I}\{n(\omega')\}}{\omega'(\omega'-\omega)}\mathrm{d}\omega' = \int_{0}^{\infty}\frac{\mathcal{I}\{n(-\omega')\}}{(-\omega')(-\omega'-\omega)}\mathrm{d}\omega' = -\int_{0}^{\infty}\frac{\mathcal{I}\{n(\omega')\}}{\omega'(\omega'+\omega)}d\omega'$$

Then

$$I = -\int_{0}^{\infty}\frac{\mathcal{I}\{n(\omega')\}}{\omega'}\left(\frac{1}{\omega'+\omega}-\frac{1}{\omega'-\omega}\right)d\omega' = 2\omega\int_{0}^{\infty}\frac{\mathcal{I}\{n(\omega')\}}{\omega'(\omega'^2-\omega^2)}\mathrm{d}\omega'$$

Introducing this result and (11.7.4) in (11.7.19) gives (11.7.21).

•Problem 8. From (11.6.25)

$$\frac{\omega}{c} = \frac{\omega + c_\infty\breve{\alpha}}{c_\infty}$$

Then

$$\frac{c}{\omega} = \frac{c_\infty}{\omega + c_\infty\breve{\alpha}} = \frac{c_\infty}{\omega}\frac{1}{1+(c_\infty/\omega)\breve{\alpha}}$$

and

$$c = c_\infty\left(1+\frac{c_\infty}{\omega}\breve{\alpha}\right)^{-1}$$

Comparison of this expression with (11.8.2) shows that

$$\breve{\alpha}(\omega) = \frac{2}{\pi}\omega\alpha_o\frac{\ln(1/\alpha_1\omega)}{1-\alpha_1^2\omega^2} \approx \frac{2}{\pi}\omega\alpha_o\ \ln\frac{1}{\alpha_1\omega}$$

Introduce the last expression and (11.8.1) in (11.8.5)

$$2Q\alpha = 2Q\frac{\alpha_o\omega}{1+\alpha_1\omega} \approx 2Q\alpha_o\omega = \frac{\omega}{c_\infty} + \frac{2}{\pi}\omega\alpha_o\ \ln\frac{1}{\alpha_1\omega}$$

Then

$$2Q\alpha_o = \frac{1}{c_\infty} + \frac{2}{\pi}\alpha_o\ \ln\frac{1}{\alpha_1\omega}$$

Appendix A

•Problem A.1. Let

$$I = \langle \mathrm{sgn}(x), \check{\varphi}(x)\rangle = \int_{-\infty}^{\infty} \mathrm{sgn}(x)\varphi(-x)dx$$

Let $u = -x$. Then $\mathrm{d}u = -\mathrm{d}x$ and

$$I = -\int_{\infty}^{-\infty} \mathrm{sgn}(-x)\varphi(x)dx = \int_{-\infty}^{\infty} \mathrm{sgn}(-x)\varphi(x)dx = -\int_{-\infty}^{\infty} \mathrm{sgn}(x)\varphi(x)dx = -\langle \mathrm{sgn}(x), \varphi(x)\rangle$$

The dummy variable u was changed to x and $\mathrm{sgn}(-x) = -\mathrm{sgn}(x)$ was used.

•Problem A.2. Let $T = \mathrm{sgn}x$ and $D(\omega) = \hat{T} = 2/\mathrm{i}\omega$ (see (A.67)) and use the fact that T is odd. Then

$$\langle \hat{D}, \psi\rangle = \langle \hat{\hat{T}}, \psi\rangle = 2\pi\langle T, \check{\psi}\rangle = -2\pi\langle T, \psi\rangle$$

((A.52) and (A.18) were used). Therefore

$$\mathcal{F}\{D(x)\} = -2\pi T(\omega)$$

and

$$\mathcal{F}\left\{\frac{2}{\mathrm{i}x}\right\} = -2\pi\mathrm{sgn}\omega$$

or

$$\mathcal{F}\left\{\frac{1}{x}\right\} = -\mathrm{i}\pi\mathrm{sgn}\omega$$

•Problem A.3. Start with (A.59). Use it to get the Fourier transform of $\exp(-\mathrm{i}ta)$. Use an approach similar to that used in Problem A.2. Let $T = \delta_a$, $\hat{T} = D(\omega) = \exp(-\mathrm{i}\omega a)$. Then

$$\langle \hat{D}, \psi\rangle = \langle \hat{\hat{\delta}}, \psi\rangle = 2\pi\langle \delta_a, \check{\psi}\rangle = 2\pi\langle \delta_a, \psi(-\omega)\rangle = 2\pi\psi(-a) = 2\pi\langle \delta_{-a}, \psi\rangle$$

((A.16) was used) and

$$\hat{D} = 2\pi\delta_{-a} = 2\pi\delta(\omega + a)$$

The expression on the right-hand side is a symbolic notation. Therefore

$$\mathcal{F}\left\{\mathrm{e}^{-\mathrm{i}at}\right\} = 2\pi\delta(\omega + a)$$

If a is replaced by $-a$ we obtain

$$\mathcal{F}\left\{\mathrm{e}^{\mathrm{i}at}\right\} = 2\pi\delta(\omega - a)$$

Write $\cos at$ in exponential form

$$\cos at = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}at} + \mathrm{e}^{-\mathrm{i}at}\right)$$

Then

$$\mathcal{F}\{\cos at\} = \pi[\delta(\omega - a) + \delta(\omega + a)]$$

Write $\sin at$ in exponential form

$$\sin at = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}at} - \mathrm{e}^{-\mathrm{i}at}\right)$$

Then

$$\mathcal{F}\{\sin at\} = \frac{\pi}{\mathrm{i}}[\delta(\omega - a) - \delta(\omega + a)] = \mathrm{i}\pi[\delta(\omega + a) - \delta(\omega - a)]$$

Graphical representation of the transforms: For the cosine, a pair of spikes in the up direction located at $\omega = \pm a$. For the sine, a pair of imaginary spikes, one in the up direction at $\omega = -a$ and one in the down directions at $\omega = a$.

Appendix C

•Problem C.1. Using (9.2.3) we can write

$$\delta(\mathbf{x}-\mathbf{x}_o) = \delta(x_1 - x_{o1})\delta(x_2 - x_{o2})\delta(x_3 - x_{o3}) = \delta_{x_{o1}}\delta_{x_{o2}}\delta_{x_{o3}}$$

The expression to the right of the first equality is a symbolic notation while the last expression is the distribution notation. For each of the deltas we can use (A.59), but to satisfy the sign convention introduced in (5.4.26), in (A.58) we have to change the sign in the exponential. In addition k_j must replace ω. Then

$$\mathcal{F}\left\{\delta_{x_{oj}}\right\} = \mathrm{e}^{\mathrm{i}x_{oj}k_j}; \qquad j = 1, 2, 3$$

Therefore, formally we can write

$$\mathcal{F}\{\delta(\mathbf{x}-\mathbf{x}_o)\} = \mathrm{e}^{\mathrm{i}k_1 x_{o1}}\mathrm{e}^{\mathrm{i}k_2 x_{o2}}\mathrm{e}^{\mathrm{i}k_3 x_{o3}} = \mathrm{e}^{\mathrm{i}(k_1 x_{o1} + k_2 x_{o2} + k_3 x_{o3})} = \mathrm{e}^{\mathbf{ik.x}_o}$$

•Problem C.2. Here we are interested in the Fourier transform in the space domain (see (C.3)). This means that we must use (5.4.26). As done in the previous problem, the exponent in the integrand can be written as

$$\mathbf{ik.r} = \mathrm{i}(k_1 x_1 + k_2 x_2 + k_3 x_3)$$

As in Problem 9.13, we can write

$$\mathcal{F}\left\{\frac{\partial^2 G}{\partial x_j^2}\right\} = -k_j^2 \hat{G}(\mathbf{k}); \qquad j = 1, 2, 3$$

where $\hat{G}(\mathbf{k})$ indicates the Fourier transform of G in the space domain. To get this result take the derivative of (5.4.27) with respect to x_j. Then, because

$$\nabla^2 G = \frac{\partial^2 G}{\partial x_1^2} + \frac{\partial^2 G}{\partial x_2^2} + \frac{\partial^2 G}{\partial x_3^2}$$

we obtain

$$\mathcal{F}\{\nabla^2 G\} = -\left(k_1^2 + k_2^2 + k_3^2\right)\hat{G} = -|\mathbf{k}|^2 \hat{G} \equiv -k^2 \hat{G}$$

Appendix D

•Problem D.1. Let $r = (x_i x_i)^{1/2}$. Then

$$\nabla^2 \frac{1}{r} = \nabla^2 \left[(x_i x_i)^{-1/2}\right] = \left[(x_i x_i)^{-1/2}\right]_{,jj}$$

$$\left[(x_k x_k)^{-1/2}\right]_{,j} = -\frac{1}{2}(x_i x_i)^{-3/2}(x_k x_k)_{,j} = -(x_i x_i)^{-3/2} x_j$$

(use $(x_k x_k)_{,j} = 2x_j$) and

$$\left[(x_i x_i)^{-1/2}\right]_{,jk} = 3(x_i x_i)^{-5/2} x_j x_k - (x_i x_i)^{-3/2}\delta_{jk} = 3r^{-5} x_j x_k - r^{-3}\delta_{jk} = \left(3r^{-2} x_j x_k - \delta_{jk}\right) r^{-3}$$

Then, for $k = j$, $\delta_{jj} = 3$ and

$$\nabla^2 \frac{1}{r} = 0; \qquad r \neq 0$$

which is (D.5).

For (D.6) note that

$$r_{,i} = (\nabla|\mathbf{r}|)_i = \frac{x_i}{|\mathbf{r}|} = \frac{x_i}{r}$$

(see Problem 1.5f). Then

$$r_{,i} r_{,i} = \frac{1}{r^2}(x_i x_i) = 1$$