Digital Logic Design: a rigorous approach ©

Chapter 3: Sequences and Series

Guy Even   Moti Medina

School of Electrical Engineering Tel-Aviv Univ.

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Book Homepage:
http://www.eng.tau.ac.il/~guy/Even-Medina

Sequences

Definition

An infinite sequence is a function f whose domain is $\mathbb{N}$ or $\mathbb{N}^+$.

Instead of denoting a sequences by a function $f : \mathbb{N} \rightarrow \mathbb{R}$, one usually writes $\{f(n)\}_{n=0}^{\infty}$ or $\{f_n\}_{n=0}^{\infty}$. Sometimes sequences are only defined for $n \geq 1$.

Example

- $\{g(n)\}_{n=0}^{\infty}$ - the Fibonacci sequence
- $\{d_n\}_{n=0}^{\infty}$ where d_n is the nth digit of $\pi \approx 3.1415926$.

finite sequences are n-tuples

Definition

A prefix of $\mathbb{N}$ is a set $\{i \in \mathbb{N} \mid i \leq n\}$, for some $n \in \mathbb{N}$. One could similarly consider prefixes of $\mathbb{N}^+$.

Definition

A finite sequence is a function f whose domain is a prefix of $\mathbb{N}$ or $\mathbb{N}^+$.

Note that if the domain of a sequence f is $\{i \in \mathbb{N} \mid i < n\}$ or $\{i \in \mathbb{N}^+ \mid i \leq n\}$, then f is simply an n-tuple.

important sequences

- arithmetic sequences
- geometric sequences
- harmonic sequences

arithmetic sequences

The simplest sequence is the sequence $(0, 1, \ldots)$ defined by $f(n) = n$.

An arithmetic sequence is specified by two parameters: a_0 - the first element in the sequence and d- the difference between successive elements.

Definition

The arithmetic sequence $\{a_n\}_{n=0}^{\infty}$ specified by the parameters a_0 and d is defined by

$$a_n \triangleq a_0 + n \cdot d.$$

Equivalent definition by recursion: The first element is simply a_0. The reduction rule: $a_{n+1} = a_n + d$.

Claim

$\{a_n\}_{n=0}^{\infty}$ is an arithmetic sequence iff $\exists d \forall n : a_{n+1} - a_n = d$.

arithmetic sequence - examples

1. The sequence of even numbers $\{e_n\}_{n=0}^{\infty}$ is defined by
$$e_n \triangleq 2n .$$
The sequence $\{e_n\}_{n=0}^{\infty}$ is an arithmetic sequence since $e_{n+1} - e_n = 2$, thus the difference between consecutive elements is constant, as required.
2. The sequence of odd numbers $\{\omega_n\}_{n=0}^{\infty}$ is defined by
$$\omega_n \triangleq 2n + 1 .$$
The sequence $\{\omega_n\}_{n=0}^{\infty}$ is also an arithmetic sequence since $\omega(n+1) - \omega(n) = 2$.
3. If $\{a_n\}_{n=0}^{\infty}$ is an arithmetic sequence with a difference d, then $\{b_n\}_{n=0}^{\infty}$ defined by $b_n = a_{2n}$ is also an arithmetic sequence. Indeed, $b_{n+1} - b_n = a_{2n+2} - a_{2n} = 2d$.

geometric sequences

The simplest example of a geometric sequence is the sequence of powers of 2: $(1, 2, 4, 8, \ldots)$. In general, a geometric sequence is specified by two parameters: b_0 - the first element and q - the ratio or quotient between successive elements.

Definition

The geometric sequence $\{b_n\}_{n=0}^{\infty}$ specified by the parameters b_0 and q is defined by

$$b_n \triangleq b_0 \cdot q^n.$$

One can also define the geometric sequence $\{b_n\}_{n=0}^{\infty}$ by recursion. The first element is simply b_0. The recursion step is $b_{n+1} = q \cdot b_n$.

Claim

$\{b_n\}_{n=0}^{\infty}$ is a geometric sequence iff $\exists q \forall n : a_{n+1}/a_n = q$.

geometric sequence - examples

1. The sequence of powers of 3 $\{3^n\}_{n=0}^{\infty}$ is a geometric sequence.
2. If $\{b_n\}_{n=0}^{\infty}$ is a geometric sequence with a quotient q, then $\{c_n\}_{n=0}^{\infty}$ defined by $c_n = b_{2n}$ is also a geometric sequence. Indeed, $b_{n+1}/b_n = a_{2n+2}/a_{2n} = q^2$.
3. If $\{b_n\}_{n=0}^{\infty}$ is a geometric sequence with a quotient q and $b_n > 0$, then the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_n \triangleq \log b_n$ is an arithmetic sequence.
4. If $q = 1$ then the sequence $b_n = a_0 \cdot q^n$ is constant.

harmonic sequence

Definition

The harmonic sequence $\{c_n\}_{n=1}^{\infty}$ is defined by $c_n \triangleq \frac{1}{n}$, for $n \geq 1$.

Note that the first index in the harmonic sequence is $n = 1$. The harmonic sequence is simply the sequence $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$.

Series

The sum of a sequence is called a series. We are interested in the sum of the first n elements of sequences.
Given a sequence $\{a_i\}_i$ we are interested in sums

$$\sum_{i \leq n} a_i .$$

We generalize the formula

$$\sum_{i=1}^{n} i = \frac{1}{2} \cdot n(n+1).$$

Theorem

If $a_n \triangleq a_0 + n \cdot d$ *and* $S_n \triangleq \sum_{i=0}^{n} a_i$,
Then

$$S_n = a_0 \cdot (n+1) + d \cdot \frac{n \cdot (n+1)}{2} .$$

geometric series

Theorem

Assume that $q \neq 1$. Let

$$b_n \triangleq b_0 \cdot q^n \text{ and } \qquad S_n \triangleq \sum_{i=0}^{n} b_i.$$

Then,

$$S_n = b_0 \cdot \frac{q^{n+1} - 1}{q - 1}. \tag{1}$$

Example

- $\sum_{i=0}^{n-1} 2^i = 2^n - 1.$
- $\sum_{i=1}^{n} 2^{-i} = \frac{1}{2} \cdot \frac{1-2^{-n}}{1-1/2} = 1 - 2^{-n}$

harmonic series

Theorem

Let

$$H_n \triangleq \sum_{i=1}^{n} \frac{1}{i}.$$

Then, for every $k \in \mathbb{N}$

$$1 + \frac{k}{2} \leq H_{2^k} \leq k + 1. \tag{2}$$

The theorem is useful because it tells us that H_n grows logarithmically in n. In particular, H_n tends to infinity as n grows. Compare with $\int_1^n \frac{1}{x} dx$.