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Problems for Chapter 17 of *Advanced Mathematics for Applications*

Analytic Functions

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1 Basic properties

1. Express in polar form the numbers

$$\pm 1\,, \qquad \pm i\,, \qquad 1 \pm i\,, \qquad -1 \pm i\,.$$

2. Find the real and imaginary parts of the following complex numbers

$$\cos(2+i)\,, \qquad \sin 2i\,, \qquad \cosh(2-i)\,, \qquad \sinh e^{i}\,.$$

3. Show that $w(z) = |z|^2$ is not analytic. For what values of z is it differentiable?

4. Are the following functions the real part of an analytic function?

$$u = 2x(1-y)\,, \qquad u = \frac{y}{x^2+y^2}\,,$$

$$u = \sinh x \sin y\,, \qquad u = 2x - x^3 + 3xy^2\,.$$

   If so, determine their harmonic conjugate

5. By differentiating the polar form of the Cauchy-Riemann relations and eliminating the mixed partials derive the form of the Laplace operator in plane polar coordinates.

6. Show that the families of curves $\mathrm{Re}\, z^2 = c_1$ and $\mathrm{Im}\, z^2 = c_2$ are orthogonal to each other.

7. Show that

$$|\sin(x+iy)|^2 = \cosh^2 y - \cos^2 x = \sinh^2 y + \sin^2 x$$

   and find the corresponding formula for $|\cos(x+iy)|^2$. Use the first formula to find the limit as $y \to \infty$ of $|sin(x+iy)|e^{-|y|}$.

8. Find the real part of $\cos(z+3i)$.

9. Evaluate the integral

$$\int_{-i}^{i} |z|\, \mathrm{d}z$$

   along (a) the straight line $\mathrm{Re}\, z = 0$, $-1 \le \mathrm{Im}\, z \le 1$, (b) along the left half of the unit circle, and (c) along the right half of the unit circle. Are the results different? Why?

10. Evaluate the integral
$$\int_0^{2+i} z^2 \,\mathrm{d}z$$
along (a) the segment joining the integration limits, and (b) along a contour consisting of the segments $0 \le \operatorname{Re} z \le 2$, $\operatorname{Im} z = 0$ and $\operatorname{Re} z = 2$, $0 \le \operatorname{Im} z \le 1$.

11. Evaluate the integral
$$\int |z|\overline{z}\,\mathrm{d}z$$
along the closed path consisting of $-1 \le \operatorname{Re} z \le 1$, $\operatorname{Im} z = 0$ and the semicircle $|z| = 1$, $\operatorname{Im} z > 0$.

12. Let z_0 be a simple root of the analytic function $\psi(z)$. Calculate the residue of $\phi(z)/\psi^2(z)$ at z_0 if $\phi(z_0) \neq 0$.

2 Multivalued functions

1. Find the values of the following expressions:
$$(-2)^{\sqrt{2}}\,, \qquad 2^i\,, \qquad 1^{-i}\,, \qquad \left(\frac{1-i}{\sqrt{2}}\right) 1+i\,, \qquad (3-4i)^{1+i}\,.$$

2. Find all values of the following expressions:
$$\cos^{-1} 2\,, \qquad \sin^{-1} i\,, \qquad \tan^{-1}(1+2i)\,, \qquad \cosh^{-1} 2i\,, \qquad \tanh^{-1}(1-i)\,.$$

3. Are the functions $\sqrt{z^2}$ and $(\sqrt{z})^2$ equal? Are they multivalued?

4. What is the value of $\sin^{-1} z$ when the variable z describes the segment of the straight line joining the origin to the point $1+i$, if the initial value of $\sin^{-1} z$ equals 0?

5. For the following multivalued functions determine whether the condition stated is sufficient to uniquely determine a specific branch:
$$w(z) = (z-1)\log z\,, \quad w(1) = 0\,; \qquad w(z) = z^{iz}\,, \quad w(1) = 1\,;$$
$$w(z) = z^{z/2}\,, \quad w(1) = 1\,; \qquad w(z) = z^{z}\,, \quad w'(1) = 1\,.$$

6. Let $w(z)$ be the particular branch of the function $\log(i+z)$ satisfying the condition $w(1-i) = 0$. Find the value of $w(-1-i)$ when the domain of definition of w is the complex plane cut (a) along the negative imaginary axis from $-\infty$ to $-i$, and (b) along the imaginary axis from $-\infty$ to $+i$.

7. Let $w(z)$ be a regular branch of the function $\log(1-z^2)$ defined in the complex plane cut along the semi-infinite lines $\operatorname{Im} y = 0$ and $-\infty < \operatorname{Re} z < -1$, $1 < \operatorname{Re} z < \infty$ and satisfying the condition $w(0) = 0$. Calculate the values of $w(z)$ at the points $\pm i$, $(1 \pm i)/\sqrt{2}$.

8. In the complex plane cut along the negative real semi-axis define the functions
$$w_1(z) = (\log z)^2\,, \quad w_1(1) = 0\,; \qquad w_2(z) = z^{\alpha}\,, \quad w_2(1) = 1\,;$$
$$w_3(z) = \frac{z-1}{\log z}\,, \quad w_3(1) = 1\,; \qquad w_4(z) = z^{1/2}\log z\,, \quad w_4(2) > 0\,.$$
Let N be an arbitrary positive integer and calculate
$$w_j(-N+i0) - w_j(-N-i0)\,, \qquad j = 1, 2, 3, 4\,.$$

9. In the complex lane cut along the real segment $-1 < \mathrm{Re}\, z < 1$ define

$$w(z) \,=\, (z+1)^{\alpha}(z-1)^{\alpha}\,, \qquad w(2) \,>\, 0\,,$$

where α is real. For $0 < x < 1$ and $\mathrm{Im}\, z = 0$ calculate

$$g(x) \,=\, w(x+i0) - w(x-i0)\,.$$

10. In the complex lane cut along the negative real semi-axis define

$$w(z) \,=\, z^{z}$$

and calculate

$$g(x) \,=\, w(x+i0) - w(x-i0)\,.$$

11. The point z describes a complete circuit in the positive direction around the circle $|z| = 2$ starting and anding at $z = 2$. Assuming that the initial value of $\arg w(z)$ is 0 and that it varies continuously, find the final value of $\arg w(z)$ when

$$(a) \quad w(z) \,=\, \sqrt{z-1}\,, \qquad (b) \quad w(z) \,=\, (z-1)^{1/3}\,, (c) \quad w(z) \,=\, \sqrt{z^2-1}\,.$$

12. The point z describes a complete circuit in the positive direction around the circle $|z| = 2$ starting and anding at $z = 2$. Assuming that the initial value of $\arg w(z)$ is 0 and that it varies continuously, find the final value of $\arg w(z)$ when

$$(a) \quad w(z) \,=\, \sqrt{z^2+2z-3}\,, \qquad (b) \quad w(z) \,=\, \sqrt{\frac{z-1}{z+1}}\,.$$

13. In the complex plane cut along the negative real semi-axis define the functions

$$w_1(z) \,=\, (\log\, z)^2\,, \quad w_1(1) \,=\, 0\,; \qquad w_2(z) \,=\, z^{\alpha}\,, \quad w_2(1) \,=\, 1\,;$$

$$w_3(z) \,=\, \frac{z-1}{\log\, z}\,, \quad w_3(1) \,=\, 1\,; \qquad w_4(z) \,=\, z^{1/2}\log\, z\,, \quad w_4(2) > 0\,.$$

Let N be an arbitrary positive integer and calculate

$$w_j(-N+i0) - w_j(-N-i0)\,, \qquad j \,=\, 1,\, 2,\, 3,\, 4\,.$$

14. In the complex lane cut along the real segment $-1 < \mathrm{Re}\, z < 1$ define

$$w(z) \,=\, (z+1)^{\alpha}(z-1)^{\alpha}\,, \qquad w(2) \,>\, 0\,,$$

where α is real. For $0 < x < 1$ and $\mathrm{Im}\, z = 0$ calculate

$$g(x) \,=\, w(x+i0) - w(x-i0)\,.$$

15. In the complex lane cut along the negative real semi-axis define

$$w(z) \,=\, z^{z}$$

and calculate

$$g(x) \,=\, w(x+i0) - w(x-i0)\,.$$

16. The point z describes a complete circuit in the positive direction around the circle $|z| = 2$ starting and anding at $z = 2$. Assuming that the initial value of $\arg w(z)$ is 0 and that it varies continuously, find the final value of $\arg w(z)$ when

$$(a) \quad w(z) \;=\; \sqrt{z-1}\,, \qquad (b) \quad w(z) \;=\; (z-1)^{1/3}\,, (c) \quad w(z) \;=\; \sqrt{z^2-1}\,.$$

17. The point z describes a complete circuit in the positive direction around the circle $|z| = 2$ starting and anding at $z = 2$. Assuming that the initial value of $\arg w(z)$ is 0 and that it varies continuously, find the final value of $\arg w(z)$ when

$$(a) \quad w(z) \;=\; \sqrt{z^2+2z-3}\,, \qquad (b) \quad w(z) \;=\; \sqrt{\frac{z-1}{z+1}}\,.$$

18. The point z describes a complete circuit in the positive direction around the circle $|z| = 2$ starting and anding at $z = 2$. Assuming that the initial value of $\operatorname{Im} w(z) = 0$ and that $w(z)$ varies continuously, find the final value of $w(z)$ when

$$(a) \quad w(z) \;=\; 2\log z\,, \qquad (b) \quad w(z) \;=\; \log z - \log(z+1)\,, (c) \quad w(z) \;=\; \log z + \log(z+1)\,.$$

3 Riemann surfaces

1. Describe the Riemann surface associated with the function

$$w(z) \;=\; z^{1/3}$$

2. Describe the Riemann surface associated with the function

$$w(z) \;=\; \left(\frac{z}{1+z}\right)^{1/2}\,.$$

3. Describe the Riemann surface associated with the function

$$w(z) \;=\; \log\, z^n\,.$$

4. Describe the Riemann surface associated with the function

$$w(z) \;=\; \log\, e^z$$

5. Describe the Riemann surface associated with the function

$$w(z) \;=\; \left[\sqrt{z} + \sqrt{z-1}\right]^{1/3}\,.$$

6. Describe the Riemann surface associated with the function

$$w(z) \;=\; \log\,\frac{z^2+2z+2}{z^2-2z+2}\,.$$

7. What values can the integral

$$\int_1^{z_0} \frac{\mathrm{d}z}{z^{1/N}}$$

where N is a positive integer, have for an arbitrary path extending from a definite one of the (N distinct) points $+1$ to a definite one of the (N distinct) points $z_0 \neq 0$? The integral is defined on a Riemann surface having N sheets.

8. Describe the Riemann surfaces associated to the functions

$$z = \left(\frac{w-a}{w-b}\right)^{1/n} , \qquad z = \frac{1}{2}\left(w + \frac{1}{w}\right) , \qquad z = \frac{w}{(1-w)^2} ,$$

or their inverses; n is a positive integer. In particular, find domains of univalence of each function.

9. Describe the Riemann surfaces associated to the functions

$$z = \frac{1}{2}\left(w^n + \frac{1}{w^n}\right) , \qquad z = \frac{w}{(1+w^n)^2} ,$$

or their inverses; n is a positive integer. In particular, find domains of univalence of each function.

4 Contour integration

4.1 General

1. Evaluate the integral

$$\frac{1}{2\pi i} \oint_{|z|=1} z^n e^{2/z} \, \mathrm{d}z$$

where the unit circle is described in the positive direction and n is a positive or negative integer.

2. For a and b positive constants evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{(a + b\cos^2\phi)^2} .$$

3. For a and b real constants evaluate the integral

$$\int_0^{2\pi} \frac{\cos 2x}{a^2\cos^2 x + b^2 \sin^2 x} \, \mathrm{d}x .$$

4. For $c \neq \pm 1$ a complex constant evaluate the integral

$$\int_0^{2\pi} \frac{\cos^2 3x}{1 - 2c\cos x + c^2} \, \mathrm{d}x .$$

5. For a and b positive constants and n an arbitrary integer evaluate the integral

$$\int_0^{2\pi} e^{\cos\phi} \cos(n\phi - \sin\phi) \, \mathrm{d}\phi .$$

6. Reduce the evaluation of the integral

$$I_n^m(a,b) = \int_0^{\pi} \frac{\cos^n \phi}{(b + a\cos\phi)^n} \, \mathrm{d}\phi ,$$

where m and n are integers and $0 < a < b$ are real constants, to an expression which does not involve any integration.

7. Evaluate the integral

$$\int_0^{\infty} \frac{x^2+1}{x^4+1} \, \mathrm{d}x .$$

8. Evaluate the integral
$$\int_0^\infty \frac{x^2}{\cosh x}\,\mathrm{d}x\ .$$
9. For $0 < p < 1$ and $a > 0$ evaluate the integral
$$\int_0^\infty \frac{x^p}{x+a}\,\mathrm{d}x\ .$$
10. For $n > 1$ an integer, evaluate the integrals
$$I_1 \ = \ \int_0^\infty \frac{\mathrm{d}x}{1+x^n}\ , \qquad I_2 \ = \ \int_0^\infty \frac{x^n}{1+x^{2n}}\ .$$
11. For $|t| < \pi$ evaluate the integral
$$\int_{-\infty}^\infty \frac{x^2}{x^4 - 2x^2\cos t + 1}\,\mathrm{d}x\ .$$
12. For what range of values of p does the integral
$$\int_0^\infty \frac{x^p}{x^2 + 2x\cos t + 1}\,\mathrm{d}x\ ,$$
where $|t| < \pi$, exist? Evaluate it for p in this range.
13. Evaluate the integral
$$\int_0^\infty \frac{\log x}{x^2 + a^2}\,\mathrm{d}x$$
where a is a real constant.
14. By making a suitable change of variable, evaluate the integral
$$\int_0^\infty \frac{\log x}{\sqrt{x}(x+1)^2}\,\mathrm{d}x$$
where a is a real constant.
15. By making a suitable change of variable, evaluate the integral
$$\int_0^\infty \frac{\mathrm{d}x}{\cosh ax + \cos t}\,\mathrm{d}x$$
where $0 < a$ and $0 < t < \pi$ are real constants.
16. Evaluate the integral
$$\int_0^\infty \frac{x}{x^5+1}\,\mathrm{d}x$$
where a is a real constant.
17. By applying the contour C shown in the first line of Table 17.8 p. 466 to the integral
$$\oint_C \frac{e^{2iz}-1}{z^2}\,\mathrm{d}z\ ,$$
calculate the integral
$$\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2}\,\mathrm{d}x\ ,$$
where a and b are real positive constants.

18. By proceeding as in the next-to-the-last line of Table 17.8 p. 466 show in detail how the integral

$$\int_0^1 \frac{x^{1-p}(1-x)^p}{(1+x)^2}\,\mathrm{d}x$$

can be calculated. What are the allowed values of the real constant p such that the integral exists? Evaluate it for p in this range.

19. By using the contour shown in the next-to-the-last line of Table 17.8 p. 466 calculate the integral

$$\int_0^1 \left(\frac{x}{1-x}\right)^p \frac{\mathrm{d}x}{x+a}$$

where $a > 0$ is a real constant. What are the allowed values of the real constant p such that the integral exists? Evaluate it for p in this range.

20. By using the contour shown in the next-to-the-last line of Table 17.8 p. 466 calculate the integral

$$\int_0^1 \left(\frac{x}{1-x}\right)^p \frac{\mathrm{d}x}{(x+a)^2}$$

where $a > 0$ is a real constant. What are the allowed values of the real constant p such that the integral exists? Evaluate it for p in this range.

21. By using the contour shown in the third line of Table 17.8 p. 466 prove that, if $f(z)$ is a rational function with no poles on the positive real axis or at $z = 0$ and such that $f(z) \propto z^{-1}$ at infinity, then

$$\int_0^\infty \frac{f(x)}{(\log x)^2 + \pi^2}\,\mathrm{d}x \;=\; \sum_{k=1}^{N} \mathrm{Res}\left[\frac{f(z)}{\log z - i\pi},\; z = a_k\right]$$

where $a_1 = -1$ and $a_2, \ldots, a_N$ are the poles of $f(z)$ different from -1 and $\log z = \log|z| + i \arg z$ with $0 < \arg z < 2$. By using this result evaluate the integrals

$$\int_0^\infty \frac{\mathrm{d}x}{(x+2)[(\log x)^2 + \pi^2]}\,\mathrm{d}x\;, \qquad \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)[(\log x)^2 + \pi^2]}\,\mathrm{d}x\;,$$

with a a real constant.

22. Let $f(z)$ be analytic everywhere in the finite complex plane and let $R > 1$ and a be real numbers with $|a| < 1/2$. Prove the validity of the formula

$$\int_{|z|=R} \left(\frac{z}{z-1}\right)^a f(z)\,\mathrm{d}z \;=\; 2i\,\sin\pi a \int_0^1 \left(\frac{x}{1-x}\right)^a f(x)\,\mathrm{d}x\;.$$

23. By considering the integral of $z^{\alpha-1}$ along the path shown in the last line of Table 17.8 p. 463 with $n = 8$ show that

$$\int_0^\infty e^{-x\cos\lambda} x^{\alpha-1}\cos(x\sin\lambda)\,\mathrm{d}x \;=\; \Gamma(\alpha)\cos(\alpha\lambda)\,,$$

$$\int_0^\infty e^{-x\sin\lambda} x^{\alpha-1}\sin(x\sin\lambda)\,\mathrm{d}x \;=\; \Gamma(\alpha)\sin(\alpha\lambda)\,,$$

where $0 < \lambda < \pi/2$ and $0 < \alpha$.

24. Deduce the second Laplace formula for the Legendre polynomial $P_n(x)$ (Eq. (13.3.7) p. 320) by integrating

$$\oint_C \frac{\mathrm{d}z}{z^{n+1}\sqrt{(1-2xz+z^2}}$$

where the contour C is a circle centered at the origin, the radius of which is eventually made infinite.

4.2 Principal values

1. Evaluate the principal value of the following integral which does not exist in the ordinary sense:

$$\int_0^\infty \frac{x}{x^4 - 1} \, \mathrm{d}x \ .$$

2. Evaluate the principal value of the following integral which does not exist in the ordinary sense:

$$\int_0^\infty \frac{x^2}{x^4 - 1} \, \mathrm{d}x \ .$$

3. For $0 < p < 1$ evaluate the principal value of the following integral which does not exist in the ordinary sense:

$$\int_0^\infty \frac{x^p}{1 - x} \, \mathrm{d}x \ .$$

4.3 Fourier transforms

1. For a and b positive constants evaluate the integral

$$\int_0^\infty \frac{x \, \sin ax}{x^2 + b^2} \, \mathrm{d}x \ .$$

2. For a and b positive constants evaluate the integral

$$\int_0^\infty \frac{x^2 - b^2}{x^2 + b^2} \frac{\sin ax}{x} \, \mathrm{d}x \ .$$

3. Evaluate the principal value of the following integral which does not exist in the ordinary sense:

$$\int_{-\infty}^\infty \frac{x \, \cos x}{x^2 - 5x + 6} \, \mathrm{d}x \ .$$

4. Evaluate the principal value of the following integral which does not exist in the ordinary sense:

$$\int_{-\infty}^\infty \frac{\sin ax}{x(x - b)} \ ,$$

   where $b > 0$.

5. Calculate the Fourier transform of the function $u(x) = |x|^{-\alpha}$ with $0 < \operatorname{Re} \alpha < 1$.

6. Calculate the Fourier transform of the function

$$u(x) \ = \ \frac{e^{-a|x|}}{\sqrt{|x|}} \ , \qquad 0 < a \, .$$

7. Calculate the Fourier transform of the functions

$$u(x) \ = \ \frac{\sinh ax}{\sinh \pi x} \ , \qquad , u(x) \ = \ \frac{\cosh ax}{\cosh \pi x} \ , \qquad -\pi < a < \pi \, .$$

4.4 Inverse Laplace transforms

1. Calculate the inverse Laplace transform of the functions

$$\frac{as+b}{(s+\alpha)(s+\beta)}\,, \qquad \frac{as+b}{(s+\alpha)^2}\,.$$

2. Calculate directly by contour integration the inverse Laplace transforms of the functions

$$\frac{1}{1+\sqrt{s}}\,, \qquad \frac{1}{s\sqrt{s+1}}\,, \qquad \frac{1}{1+\sqrt{s+1}}\,.$$

   Do not use the operational rules available for this type of transforms.

3. Calculate directly by contour integration the inverse Laplace transform of the function

$$\frac{1}{s}\,e^{-a\sqrt{s}}\,, \qquad a>0\,.$$

   Do not use the operational rules available for this type of transforms.

4. Calculate the inverse Laplace transform of the functions

$$\frac{1-e^{-as}}{s}\,, \qquad \frac{e^{-as}-e^{-bs}}{s}\,, \qquad \frac{e^{-as}-e^{-bs}}{s^2}$$

   where $0 \le a < b$.

5. Calculate the inverse Laplace transform of the functions

$$\log\frac{s+b}{s+a}\,, \qquad \log\frac{s^2+b^2}{s^2+a^2}\,.$$

5 Logarithmic residue

1. By proceeding as in Example 17.9.4 p. 469 determine the number of positive roots of the equation

$$x^3+cx^2+ax+b \,=\, 0$$

   in dependence of the real parameters a, b and c.

2. Determine the number of zeros of the function

$$f(z) \,=\, z^5+5z^3+2z$$

   (a) inside the circle $|z|=1$; (b) in the annulus $1\le|z|<2$; (c) in the annulus $2\le|z|<3$.

3. By letting $g(z)=-5z^5+1$ and $h(z)=z^8-2z$ determine, with the aid of Rouché's Theorem (p. 470) the number of zeros of the function

$$f(z) \,=\, z^8-5z^5-2z+1$$

   inside the circle $|z|=1$.

4. By letting $g(z)=-5z^5+1$ and $h(z)=z^8-2z$ determine, with the aid of Rouché's Theorem (p. 470) the number of zeros of the function

$$f(z) \,=\, z^8-5z^5-2z+1$$

   inside the circle $|z|=1$.

6 Conformal mapping

1. Consider the triangle delimited by the lines $\operatorname{Re} z = 1$, $\operatorname{Im} z = 1$ and $\operatorname{Re} z + \operatorname{Im} z = 1$. Find its image under the mapping $w = z^2$.

2. Find the images of the lines $\operatorname{Re} z = c_1 \neq 0$, $\operatorname{Im} z = c_2 \neq 0$ under the mapping $w = 1/z$.

3. Which part of the plane is stretched and which part shrunk under the mappings
$$w = z^2, \qquad w = z^2 + 2z, \qquad w = \frac{1}{z}\,?$$

4. Find a linear transformation which maps the circle $|z-1| = 1$ onto the circle $w - 3i/2| = 2$.

5. Show that the transformation $w = (ze^{-\alpha} + e^{\alpha}/z)/2$, with α a real constant, maps the interior of the unit circle onto the exterior of the ellipse $(\operatorname{Re} w/\cosh\alpha)^2 + (\operatorname{Im} w/\sinh\alpha)^2 = 1$,

6. Show that the transformation $w = 2/\sqrt{z} - 1$ maps the domain exterior to the parabola $(\operatorname{Im} z)^2 = 4(1 - \operatorname{Re} z)$ conformally onto the domain $|w| < 1$. Why doesn't the same transformation map the interior of the parabola onto $|w| > 1$?

7. Show that, under the linear fractional mapping (p. 474)
$$w = \frac{az + b}{cz + f},$$
circles are mapped into circles, straight lines into straight lines and points symmetric with respect to a circle into points symmetric with respect to the image of that circle.

8. Consider two circles, one centered at the origin and having a radius R, the other one centered at $x = a$ with radius S. Suppose that $a + S < R$ so that the small circles is inside the one centered at the origin. Determine the fractional linear map (p. 474) which maps the annular region between the circles into the annular region between two concentric circles.

9. Find the image of the quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$ under the linear fractional mapping (p. 474)
$$w = \frac{z - i}{z + i}.$$

10. Find the image of the half-disk $|z| < 1$, $\operatorname{Im} z > 0$ under the linear fractional mapping (p. 474)
$$w = \frac{2z - i}{2 + iz}.$$

11. Find the image of the strip $0 < \operatorname{Re} z < 1$ under the linear fractional mapping (p. 474)
$$w = \frac{z-1}{z} \qquad \text{and} \qquad w = \frac{z-1}{z-2}.$$

12. Find the linear fractional mapping (p. 474) which carries the points $-1, \infty, i$ into the points
$$(a) \quad i, 1, 1+i, \qquad (b) \quad \infty, i, 1, \qquad (c) 0, \infty 1.$$

13. Find the linear fractional mapping (p. 474) which maps the upper half plane $\operatorname{Im} z > 0$ into (a) itself; (b) the lower half plane; (C) the right half plane.

Figure 1:

14. Use the Schwarz-Christoffel transformation (p. 476) to map the region shown in the left diagram of figure 1 into the upper half plane. [Use a transformation $w(z)$ which maps the points B, C and D onto the points $w = -1$, $-\lambda$ and $w = 0$, where $0 < \lambda < 1$ is to be determined.]

15. Use the Schwarz-Christoffel transformation (p. 476) to map the region shown in the right diagram of figure 1 into the upper half plane. [Transform the channel wall and the plane of symmetry into the upper half plane mapping the points B and C into the points -1 and 0.]

16. Use the Schwarz-Christoffel transformation (p. 476) to map the region shown in the left diagram of figure 2 into the upper half plane. [Map the points B, C and D into the points -1, 0 and λ where λ is to be determined.]

17. Use the Schwarz-Christoffel transformation (p. 476) to map the region shown in the right diagram of figure 2 into the upper half plane. In this way you can solve the problem of a line charge located as shown in the figure in the neighborhood of a semi-infinite conductor kept at 0 potential; in particular, you can find the electric field in the symmetry plane between the line charge and the conductor. [Map the region outside the inverted U-shaped region onto the upper half plane making the corners go into the points ± 1.]

Figure 2:

18. Find, by using a suitable conformal mapping, the solution of the two-dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2\pi\delta(x)\,\delta(y-b)$$

in the domain $-\infty < x < \infty$, $0 < y < d$ (with $b < d$, subject to $u(x, y = 0) = u(x, y = d) = 0$. This may represent the electrostatic potential between two large grounded parallel-plates in the presence of an (infinite) line charge.