Chapter 2

Question 1

(i) (a) $\frac{dy}{dx} = ky$ and (b) $y = c\,\mathbf{e}^{(k\,x)}$. Differentiating (b)

```
> diff(c*exp(k*x),x);
```

$$c\,k\,\mathbf{e}^{(k\,x)}$$

Substituting y into dy/dx, then $\frac{dy}{dx} = k\,c\,\mathbf{e}^{k}\,x$ therefore $k\,c\,\mathbf{e}^{k}\,x = k\,c\,\mathbf{e}^{k}\,x$. Hence true.

(ii) (a) $\frac{dy}{dx} = -\frac{x}{y}$, (b) $y = x^2 + y^2$=c. From y then

```
> diff(y(x),x)=diff(x^2,x)+diff(y(x)^2,x);
```

$$\frac{\partial}{\partial x}\,\mathrm{y}(x) = 2\,x + 2\,\mathrm{y}(x)\left(\frac{\partial}{\partial x}\,\mathrm{y}(x)\right)$$

which is equal to zero, since c is a constant. Solving we have

```
> solve(2*x+2*y(x)*diff(y(x),x)=0,diff(y(x),x));
```

$$-\frac{x}{\mathrm{y}(x)}$$

Therefore $-\frac{x}{y} = -\frac{x}{y}$. Hence, true.

(iii) (a) $\frac{dy}{dx} = -\frac{2\,y}{x}$, (b) $y = \frac{a}{x^2}$. From (b) differentiate with respect to x, then

```
> diff(a/x^2,x);
```

$$-2\,\frac{a}{x^3}$$

But $y = \frac{2\,a}{x^2}$ so that $\frac{dy}{dx} = -\frac{2\,y}{x}$. Therefore $-\frac{2\,y}{x} = -\frac{2\,y}{x}$. Hence true.

Question 2

Solving and checking limiting values

The first series of equations solve for $p(t)$. It is then checked for the limit t->0 and limit t ->infinity. In the second case, however, we first need to make assumptions about the sign of parameters k and a. The limit can be taken if we assume these parameters are both positive.

```
> eq1:=diff(p(t),t)=k*p(t)*(a-ln(p(t)));
```

$$eq1 := \frac{\partial}{\partial t}\,\mathrm{p}(t) = k\,\mathrm{p}(t)\,(a - \ln(\mathrm{p}(t)))$$

```
> dsolve(eq1,p(t));
```

$$\mathrm{p}(t) = \mathbf{e}^{(-\mathbf{e}^{(-t\,k - \_C1\,k)} + a)}$$

```
> dsolve({eq1,p(0)=p0},p(t));
```

$$p(t) = e^{\left(-e^{\left(-t\,k+\ln(a-\ln(p0))\right)}+a\right)}$$

```
> p:=t->exp(-exp(-t*k+ln(a-ln(p0)))+a);
```

$$p := t \rightarrow e^{\left(-e^{\left(-t\,k+\ln(a-\ln(p0))\right)}+a\right)}$$

```
> p(0);
```

$$p0$$

```
> limit(p(t),t=infinity);
```

$$\lim_{t \rightarrow \infty} e^{\left(-e^{\left(-t\,k+\ln(a-\ln(p0))\right)}+a\right)}$$

```
> assume(k>0,a>0);
> limit(p(t),t=infinity);
```

$$e^{a\sim}$$

The stationary values are obtained.

```
> solve(k*p*(a-ln(p))=0,p);
```

$$e^{a\sim}$$

```
> slope:=diff(k*p*(a-ln(p)),p);
```

$$slope := k\sim\,(a\sim - \ln(p)) - k\sim$$

```
> solve(slope=0,p);
```

$$e^{(a\sim - 1)}$$

```
> subs(p=%,slope);
>
```

$$k\sim\,(a\sim - \ln(e^{(a\sim - 1)})) - k\sim$$

```
> simplify(%);
```

$$0$$

Properties of the growth equation

The next set of equations considers a numerical example in order to plot the Gompertz equation to see what it looks like.

```
> Gompequ:=k*p*(a-ln(p));
```

$$Gompequ := k\sim\,p\,(a\sim - \ln(p))$$

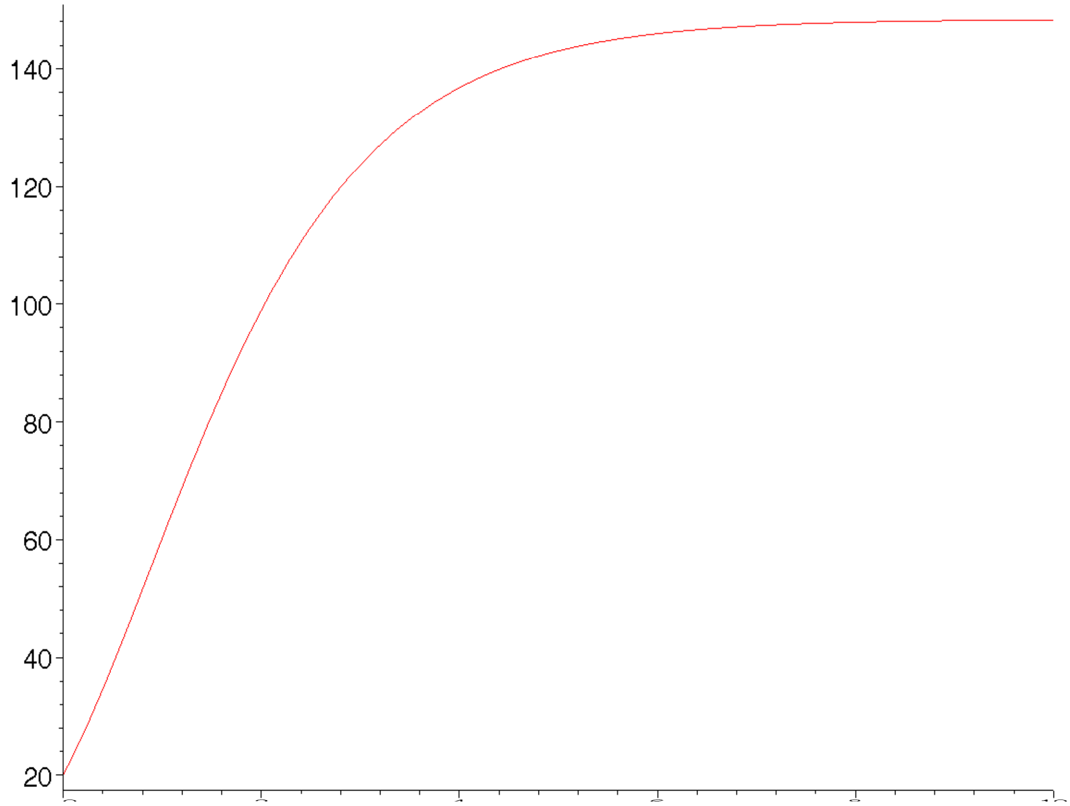
```
> Gompequ1:=subs({k=0.8,a=5},Gompequ);
```

$$Gompequ1 := .8\,p\,(5 - \ln(p))$$

```
> Gompequ2:=subs({k=0.8,a=5,p0=20},p(t));
```

$$Gompequ2 := e^{\left(-e^{\left(-.8\,t+\ln(5-\ln(20))\right)}+5\right)}$$

```
> plot(Gompequ2,t=0..10);
```




```
> limit(Gompequ2,t=infinity);
```

148.4131591

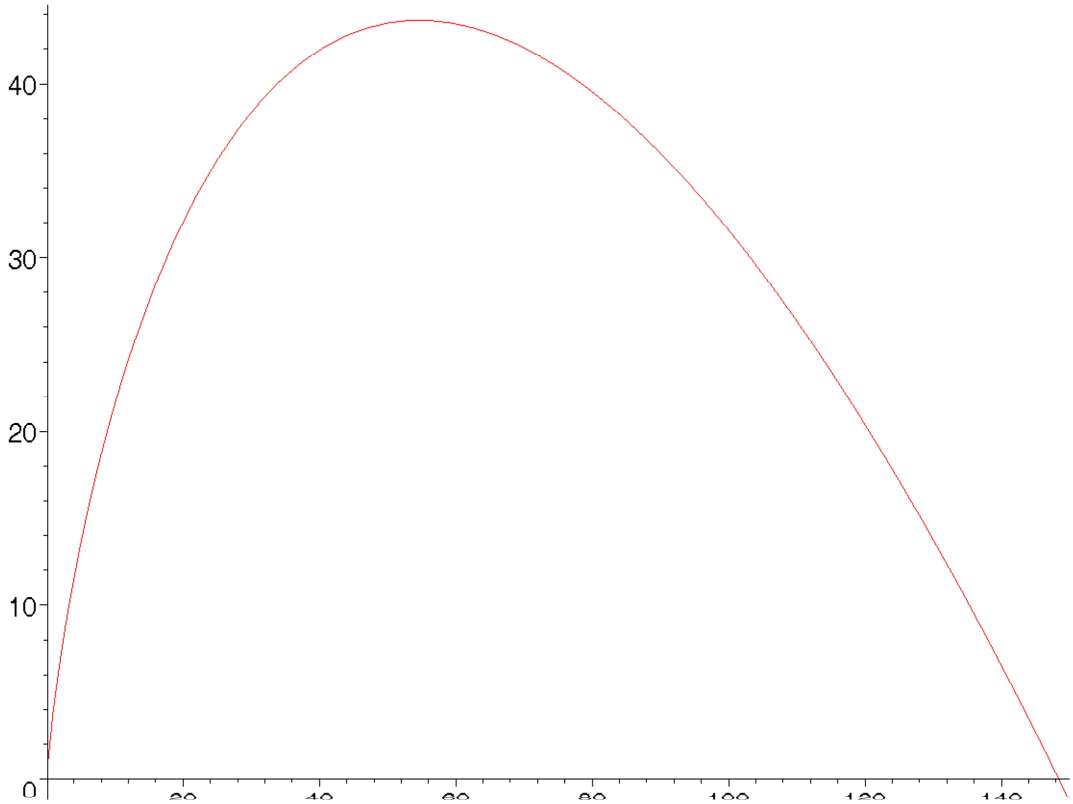
Since the Gompertz equation is at a maximum when $p = \mathbf{e}^4$, then

```
> evalf(exp(4));
```

54.59815003

with a plot,

```
> plot(Gompequ1,p=0.1..150);
```



It is to be noted that the Gompertz equation has two stationary values, one at zero and the other at $\mathbf{e}^{a}$. Second, it has very similar properties to that of the logistic equation. Third, however, the sustainable yield is not half the carrying capacity, since the sustainable yield is given by $\mathbf{e}^{(a-1)}$, which is much less than half the carrying capacity.

Question 3

(i) $\frac{dy}{dx} = x(1-y^2)$ -1<y<1 then $\frac{dy}{1-y^2} = x\ dx$. Therefore integrating both sides

```
> int(1/(1-y^2), y);
```

$$\operatorname{arctanh}(y)$$

```
> convert(%,ln);
```

$$\frac{1}{2}\ln(y+1) - \frac{1}{2}\ln(1-y)$$

```
> int(x, x);
```

$$\frac{1}{2}x^2$$

Hence, $\operatorname{arctanh}(y) = \frac{x^2}{2} + c$, where c is the constant of integration. Solving for y we obtain

```
> solve(1/2*ln(y+1)-1/2*ln(1-y)=(x^2/2)+c,y);
```

$$\frac{\mathbf{e}^{(x^2+2c)} - 1}{1 + \mathbf{e}^{(x^2+2c)}}$$

(ii) $\frac{dy}{dx} = y^2 - 2\,y + 1$. But $y^2 - 2\,y + 1 = (1 - y)^2$. Hence $\frac{dy}{1 - y^2} = dx$. Integrating both sides

```
> int(1/(1-y)^2, y);
```

$$\frac{1}{1-y}$$

```
> int(1, x);
```

$$x$$

```
> solve(1/(1-y)+c=x,y);
```

$$\frac{-1-c+x}{-c+x}$$

(iii) $\frac{dy}{dx} = \frac{y^2}{x^2}$ hence $\frac{dy}{y^2} = \frac{dx}{x^2}$. Integrating both sides

```
> int(1/y^2, y);
```

$$-\frac{1}{y}$$

```
> int(1/x^2, x);
```

$$-\frac{1}{x}$$

```
> solve(-1/y=(-1/x)+c,y);
```

$$-\frac{x}{-1+c\,x}$$

Question 4

(i) $\frac{dy}{dx} = x^2 - 2\,x + 1$ and y=1 when x=0. Hence $dy = (x^2 - 2\,x + 1)\,dx$. Integrating both sides

```
> int(1, y);
```

$$y$$

```
> int((x^2-2*x+1), x);
```

$$\frac{1}{3}x^3 - x^2 + x$$

Adding a constant of integration and solving for y we obtain

```
> solve(y=1/3*x^3-x^2+x+c,y);
```

$$\frac{1}{3}x^3 - x^2 + x + c$$

Using initial values we can solve for c

```
> subs({y=1,x=0},solve(y=1/3*x^3-x^2+x+c,c));
```

$$1$$

Therefore $y = \frac{x^3}{3} - x^2 + x + 1$.

(ii) $\frac{dy}{dx} = \frac{3\,x^2+4\,x+2}{2\,(y-1)}$, y=-1 when x=0. Re-arranging gives $2\,(y-1)\,dy = (3\,x^2+4\,x+2)\,dx$ and integrating both sides

```
> int(2*(y-1), y);
```

$$-2\,y+y^2$$

```
> int((3*x^2+4*x+2), x);
```

$$x^3+2\,x^2+2\,x$$

Adding the constant of integration we obtain the equation $y^2-2\,y = x^3+2\,x^2+2\,x+c$, and solving for c we obtain

```
> subs({y=-1,x=0},solve(-2*y+y^2=x^3+2*x^2+2*x+c,c));
```

$$3$$

Hence, solving for y gives

```
> solve(-2*y+y^2=x^3+2*x^2+2*x+3,y);
```

$$1+\sqrt{4+2\,x+x^3+2\,x^2},\ 1-\sqrt{4+2\,x+x^3+2\,x^2}$$

Question 5

Bernoulli equations are those which can be expressed as follows

$$\frac{dy}{dx}+\mathrm{f}(x)\,y = \mathrm{h}(x)\,y^{\alpha}$$

Maplve can solve such equations directly with the **dsolve** command.

(i) $\frac{dy}{dx}-y=-y^2$

```
> dsolve(diff(y(x),x)-y(x)=-y(x)^2,y(x));
```

$$\mathrm{y}(x) = \frac{1}{1+\mathbf{e}^{(-x)}\,\_C1}$$

(ii) $\frac{dy}{dx}-y=x\,y^2$

```
> dsolve(diff(y(x),x)-y(x)=x*y(x)^2,y(x));
```

$$\mathrm{y}(x) = -\frac{1}{x-1-\mathbf{e}^{(-x)}\,\_C1}$$

(iii) $\frac{dy}{dx} = 2\,y-\mathbf{e}^x\,y^2$

```
> dsolve(diff(y(x),x)=2*y(x)-exp(x)*y(x)^2,y(x));
```

$$\mathrm{y}(x) = 3\,\frac{1}{\mathbf{e}^x+3\,\mathbf{e}^{(-2\,x)}\,\_C1}$$

Question 6

First we clear p, a and b.

```
> p:='p'; a:='a'; b:='b';
```

$$p := p$$

$$a := a$$

$$b := b$$

The logistic equation can be expressed:

$$\frac{dp}{dt} - a\,p = -b\,p^2$$

which is a Bernoulli equation with $f(t) = -a$, $h(t) = -b$ and α=2.

```
> dsolve(diff(p(t),t)-a*p(t)=-b*p(t)^2,p(t));
```

$$p(t) = \frac{a}{b + e^{(-a\,t)}\,\_C1\,a}$$

In order to solve for $p(t)$ for initial $p(0) = p0$ it is necessary to clear the value of p from the

```
> p:='p';p0:=p0;
```

$$p := p$$

$$p0 := p0$$

```
> dsolve({diff(p(t),t)-a*p(t)=-b*p(t)^2,p(0)=p0},p(t));
```

$$p(t) = \frac{a}{b + \frac{e^{(-a\,t)}\,(a - p0\,b)}{p0}}$$

```
> simplify(%);
```

$$p(t) = \frac{p0\,a}{p0\,b + e^{(-a\,t)}\,a - e^{(-a\,t)}\,p0\,b}$$

Multiplying numerator and denominator by $e^{(a\,t)}$, this can be further expressed in the form,

$$p(t) = \frac{p0\,a\,e^{(a\,t)}}{a - b\,p0 + b\,p0\,e^{(a\,t)}}$$

Question 7

The question gives the results: $R(0) = 6.68$, $R(t) = 6.08$ and $\lambda = 1.245\;10^{(-4)}$. Then

$$t = \frac{1\,\ln\left(\frac{R(0)}{R(t)}\right)}{\lambda} =$$

```
> ln(6.68/6.08)/(1.245*10^(-4));
```

$$755.9300562$$

Therefore table dates from approximately

```
> 1977-756;
```

$$1221$$

Hence the table dates from about 1220 AD (13th Century). However, King Arthur ruled in the 5th Century, and so the table could not be the authentic round table of King Arthur.

Question 8

The question gives the results: $R(0) = 6.68$, $R(1950) = 4.09$ and $\lambda = 1.2445\ 10^{(-4)}$. Then

$$t - t0 = \frac{1 \ln\left(\frac{R(0)}{R(t)}\right)}{\lambda} =$$

```
> ln(6.68/4.09)/(1.245*10^(-4));
```

$$3940.345524$$

Hence, $t0$ =

```
> 1950-3940;
```

$$-1990$$

So Hammurabi reigned about 1990 BC. (Note: Historians put the reign of Hammurabi at about 1792-1750 BC.)

Question 9

(i)

Assume $f_1(t) = \mathbf{e}^{(r\,t)}$ and $f_2(t) = t\,\mathbf{e}^{(r\,t)}$ are linearly *dependent*, then

$$b_1\,\mathbf{e}^{(r\,t)} + b_2\,t\,\mathbf{e}^{(r\,t)} = 0$$

$$(b_1 + b_2\,t)\,\mathbf{e}^{(r\,t)} = 0$$

which implies $b_1 + b_2\,t = 0$ for all t only if $b_1 = b_2 = 0$. Hence, $\mathbf{e}^{(r\,t)}$ and $t\,\mathbf{e}^{(r\,t)}$ must be linearly independent.

(ii)

Assume $f_1(t) = \mathbf{e}^{(r\,t)}$, $f_2(t) = t\,\mathbf{e}^{(r\,t)}$ and $f_3(t) = t^2\,\mathbf{e}^{(r\,t)}$ are linearly *dependent*, then

$$b_1\,\mathbf{e}^{(r\,t)} + b_2\,t\,\mathbf{e}^{(r\,t)} + b_3\,t^2\,\mathbf{e}^{(r\,t)} = 0$$

$$(b_1 + b_2\,t + b_3\,t^2)\,\mathbf{e}^{(r\,t)} = 0$$

implies $b_1 + t\,b_2 + t^2\,b_3 = 0$ for all t only if $b_1 = b_2 = b_3 = 0$. Hence, $\mathbf{e}^{(r\,t)}$, $t\,\mathbf{e}^{(r\,t)}$ and $t^2\,\mathbf{e}^{(r\,t)}$ must be linearly independent.

Question 10

```
> r:=alpha+I*beta;
```

$$r := \alpha + I\,\beta$$

```
> s:=alpha-I*beta;
```

$$s := \alpha - I\,\beta$$

```
> exp(r*t);
```

$$\mathbf{e}^{((\alpha + I\,\beta)\,t)}$$

Using the Convert command on the above output gives the result in terms of cos and sin. The same can be done with $\mathbf{e}^{(s\,t)}$.

```
> convert(exp((alpha+I*beta)*t),trig);
```

$$(\cosh(\alpha\,t) + \sinh(\alpha\,t))\,(\cos(\beta\,t) + I\sin(\beta\,t))$$

```
> convert(cosh(alpha*t)+sinh(alpha*t),exp);
```

$$\mathbf{e}^{(\alpha\,t)}$$

So that $e^{(r\,t)}$ can be expressed $e^{(\alpha\,t)}\,(\cos(\beta\,t)+I\sin(\beta\,t))$. Similarly,

```
> exp(s*t);
```

$$e^{((\alpha-I\beta)\,t)}$$

```
> convert(exp((alpha-I*beta)*t),trig);
```

$$(\cosh(\alpha\,t)+\sinh(\alpha\,t))\,(\cos(\beta\,t)-I\sin(\beta\,t))$$

which can be expressed $(e^{(\alpha\,t)})(\cos(\beta\,t)-I\sin(\beta\,t))$.

Now form a linear combination of $e^{(r\,t)}$ and $e^{(s\,t)}$, in each case using the convert command to simplify the result.

```
> (1/2)*exp(r*t)+(1/2)*exp(s*t);
```

$$\frac{1}{2}\,e^{((\alpha+I\beta)\,t)}+\frac{1}{2}\,e^{((\alpha-I\beta)\,t)}$$

```
> convert(1/2*exp((alpha+I*beta)*t)+1/2*exp((alpha-I*beta)*t),t
  rig);
```

$$\frac{1}{2}(\cosh(\alpha\,t)+\sinh(\alpha\,t))\,(\cos(\beta\,t)+I\sin(\beta\,t))$$
$$+\frac{1}{2}(\cosh(\alpha\,t)+\sinh(\alpha\,t))\,(\cos(\beta\,t)-I\sin(\beta\,t))$$

```
> convert(cosh(alpha*t)+sinh(alpha*t),exp);
```

$$e^{(\alpha\,t)}$$

Which means the linear combination can be written,

$$\frac{1\,e^{(\alpha\,t)}\,2\cos(\beta\,t)}{2}=e^{(\alpha\,t)}\cos(\beta\,t)$$

```
> y1:=exp(alpha*t)*cos(beta*t);
```

$$y1:=e^{(\alpha\,t)}\cos(\beta\,t)$$

```
> (1/(2*I))*exp(r*t)+(-1/(2*I))*exp(s*t);
```

$$-\frac{1}{2}I\,e^{((\alpha+I\beta)\,t)}+\frac{1}{2}I\,e^{((\alpha-I\beta)\,t)}$$

```
> convert(-1/2*I*exp((alpha+I*beta)*t)+1/2*I*exp((alpha-I*beta)
  *t),trig);
```

$$-\frac{1}{2}I\,(\cosh(\alpha\,t)+\sinh(\alpha\,t))\,(\cos(\beta\,t)+I\sin(\beta\,t))$$
$$+\frac{1}{2}I\,(\cosh(\alpha\,t)+\sinh(\alpha\,t))\,(\cos(\beta\,t)-I\sin(\beta\,t))$$

But we already know the expression for the first bracket in each term is $e^{(\alpha\,t)}$, and so we can write the linear combination in the form

$$\frac{1\,I\,e^{(\alpha\,t)}\,(-2\,I\sin(\beta\,t))}{2}=e^{(\alpha\,t)}\sin(\beta\,t)$$

```
> y2:=exp(alpha*t)*sin(beta*t);
```

$$y2 := e^{(\alpha t)} \sin(\beta t)$$

Therefore $y = c1\ y1 + c2\ y2$, i.e.,

```
> y:=c1*y1+c2*y2;
```

$$y := c1\ e^{(\alpha t)} \cos(\beta t) + c2\ e^{(\alpha t)} \sin(\beta t)$$

Question 11

(i)

$$\frac{dP}{dt} = r\ P$$

(ii)

After clearing the variables, we set $P(0) = P0$ and solve the differential equation.

```
> P:='P';r:='r';P0:='P0';
```

$$P := P$$

$$r := r$$

$$P0 := P0$$

```
> dsolve({diff(P(t),t)=r*P(t),P(0)=P0},P(t));
```

$$P(t) = P0\ e^{(r t)}$$

(iii)

```
> evalf(subs({P0=2000,r=0.075,t=5},%));
```

$$P(5) = 2909.982830$$

Question 12

Half life is given by $t = \frac{\ln(2)}{.05}$, i.e.,

```
> evalf(ln(2)/0.05);
```

$$13.86294361$$

Question 13

(i)

```
> eq1:=x^2+2*x-15;
```

$$eq1 := x^2 + 2\ x - 15$$

```
> factor(eq1);
```

$$(x + 5)\ (x - 3)$$

```
> solve(eq1=0,x);
```

$$3, -5$$

(ii)

```
> fx:=diff(eq1,x);
```

$$fx := 2\ x + 2$$

```
> turn:=solve(fx=0,x);
```

$$turn := -1$$

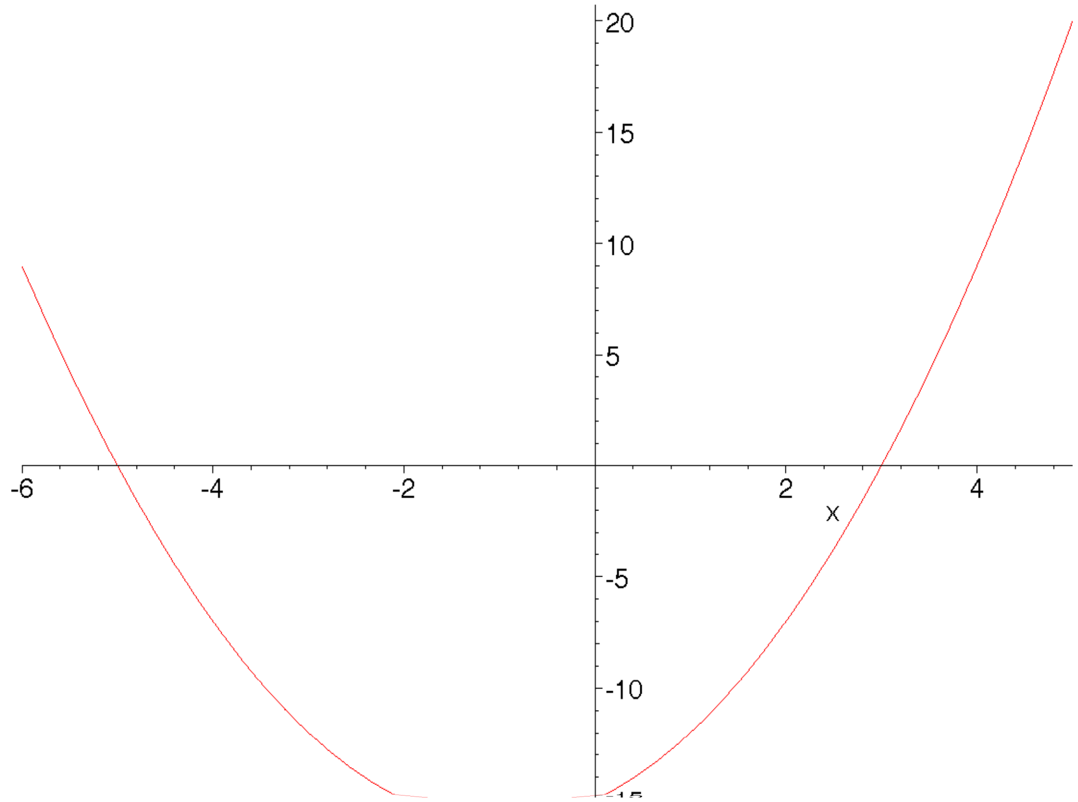
```
> sol=subs(x=-1,turn);
```

$$sol = -1$$

```
> fxx:=diff(fx,x);
```

$$fxx := 2$$

```
> plot(eq1,x=-6..5);
```




```
> slope5:=subs(x=-5,fx);
```

$$slope5 := -8$$

```
> slope3:=subs(x=3,fx);
```

$$slope3 := 8$$

Hence, $x = -5$ is an attractor since $f'(5) < 0$; and $x = 3$ is a repellor since $f'(3) > 0$.

Question 14

```
> k='k';k1='k1';k2='k2';k3='k3';
```

$$k\sim = k$$

$$k1 = k1$$

$$k2 = k2$$

$$k3 = k3$$

(i)

The equation is $k(t) = \left(\frac{a\,s}{n+\delta} + e^{(-(1-\alpha)(n+\delta)t)}\left(k0^{(1-\alpha)} - \frac{a\,s}{n+\delta}\right)\right)^{\left(\frac{1}{1-\alpha}\right)}$ and $a = 4$, $\alpha = .25$, $s = .1$, $\delta = .4$ and $n = .03$.

```
> k:=t->(a*s/(n+delta)+exp(-(1-alpha)*(n+delta)*t)*(k0^(1-alpha
)-a*s/(n+delta)))^(1/(1-alpha));
```

$$k := t \rightarrow \left(\frac{a\,s}{n+\delta} + e^{(-(1-\alpha)(n+\delta)t)}\left(k0^{(1-\alpha)} - \frac{a\,s}{n+\delta}\right)\right)^{\left(\frac{1}{1-\alpha}\right)}$$

```
> k1:=subs({a=4,alpha=0.25,s=0.1,delta=0.4,n=0.03,k0=0.5},k(t))
  ;
```

$$k1 := (.9302325581 - .3356290006\,e^{(-.3225\,t)})^{1.333333333}$$

```
> k2:=subs({a=4,alpha=0.25,s=0.1,delta=0.4,n=0.03,k0=1.2},k(t))
  ;
```

$$k2 := (.9302325581 + .2162987929\,e^{(-.3225\,t)})^{1.333333333}$$

But $\frac{dk}{dt} = s\,a\,k^{\alpha} - (n+\delta)\,k$ hence

```
> kdot:=0.1*4*k^0.25-(0.03+0.4)*k;
```

$$kdot := .4\,k^{.25} - .43\,k$$

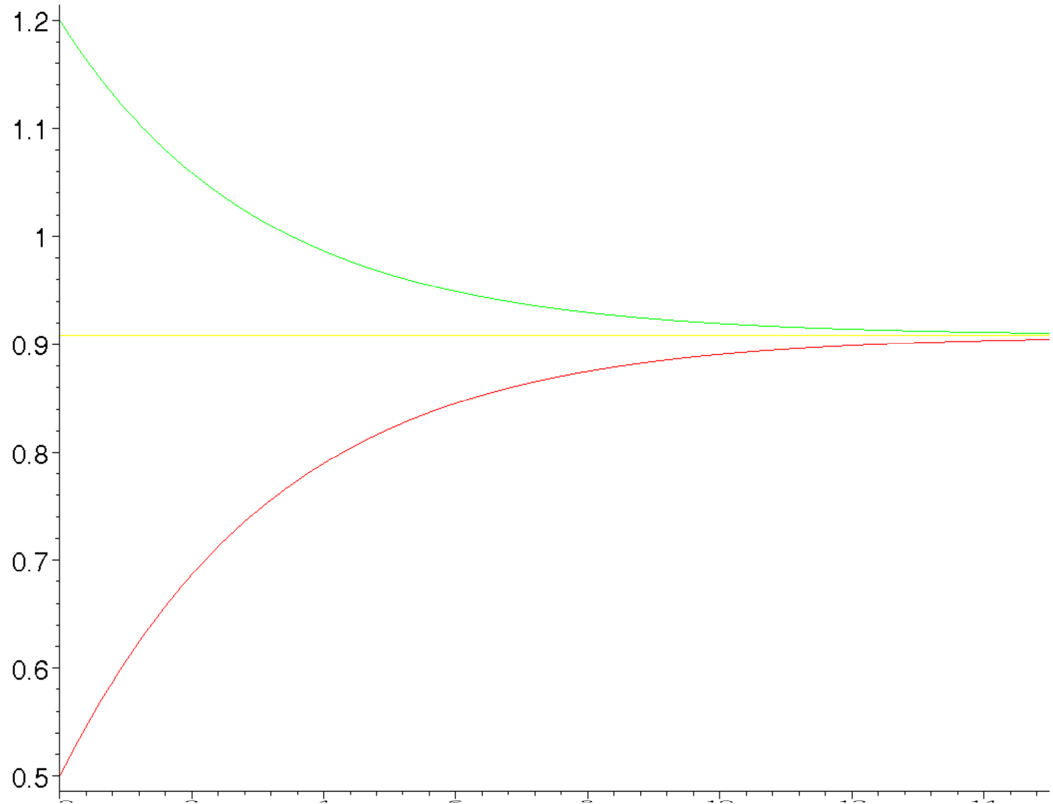
```
> sol:=solve(kdot=0,k);
```

$$sol := .9080756864,\ 0.$$

```
> k3:=subs({a=4,alpha=0.25,s=0.1,delta=0.4,n=0.03,k0=0.90807568
  64},k(t));
```

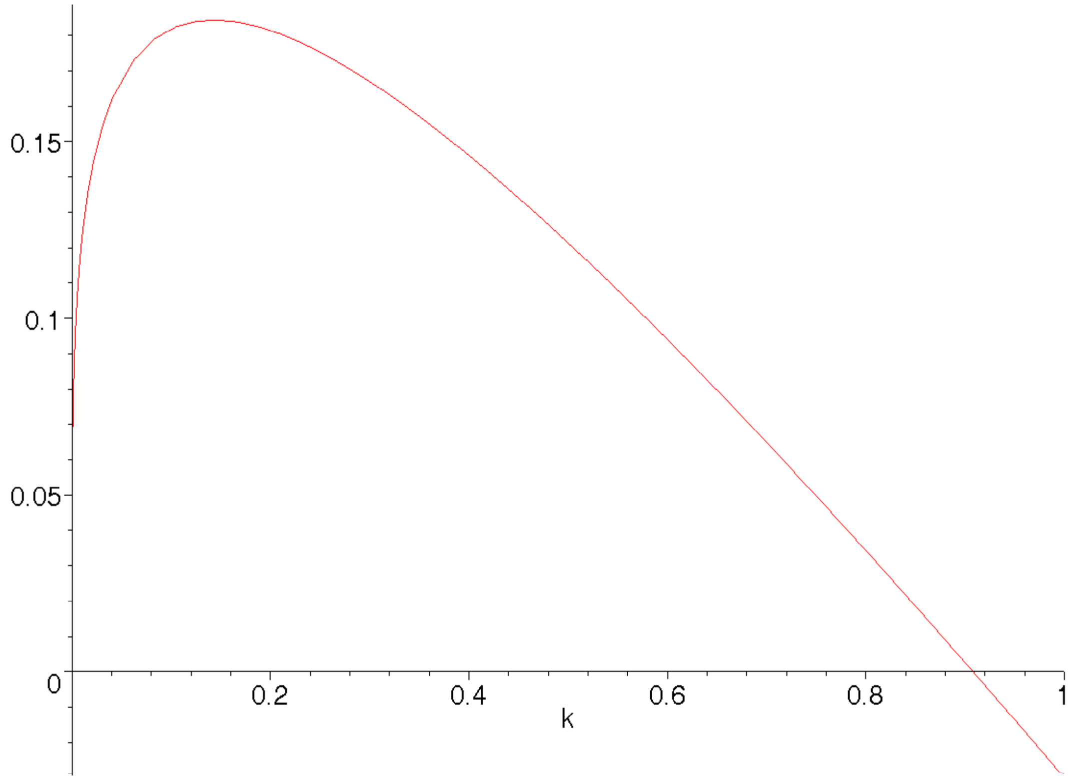
$$k3 := .9080756864$$

```
> plot([k1,k2,k3],t=0..15);
```



(ii)

```
> plot(kdot,k=0..1);
```



(iii)

```
> fk:=diff(kdot,k);
```

$$fk := .100 \frac{1}{k^{.75}} - .43$$

```
> fke=subs(k=0.908076,fk);
```

$$fke = -.3225000278$$

Let $f(k) = s\,a\,k^{\alpha} - (n + \delta)\,k$ then the linear approximation is given by:

```
> fprime=subs(k=k3,diff(kdot,k));
```

$$fprime = -.3225000000$$

Since |fprime| < 1 then $k = k3 = 0.908756864$ is a stable equilibrium.

Question 15

```
> Y:='Y';s:='s';r:='r';
```

$$Y := Y$$

$$s := s$$

$$r := r$$

(i)

$\frac{dY}{dt}$ against Y is simply a linear line through the origin with slope $0 < \frac{s}{v}$. The phase line is horizontal with fixed point at the origin, and arrows indicating an ever increasing value of Y for some $0 < \mathrm{Y}(0)$.

(ii)

Given the differential equation $\frac{dY}{dt} - \frac{s\,Y(t)}{v} = 0$, we can solve as follows

```
> dsolve({diff(Y(t),t)-(s/v)*Y(t)=0,Y(0)=Y0},Y(t));
```

$$Y(t) = Y0\,e^{\left(\frac{s\,t}{v}\right)}$$

Question 16

```
> Y:='Y';x:='x';sol:='sol';y:='y';
```

$$Y := Y$$

$$x := x$$

$$sol := sol$$

$$y := y$$

Given $Y(t) = Y0\,e^{(r\,t)}$. Solving $\frac{dD}{dt} = k\,Y(t) = k\,Y0\,e^{(r\,t)}$. Since D is protected, we use x in its place.

```
> dsolve({diff(x(t),t)=k*Y0*exp(r*t),x(0)=x0},x(t));
```

$$x(t) = \frac{k\,Y0\,e^{(r\,t)}}{r} - \frac{k\,Y0 - x0\,r}{r}$$

```
> x:=k*Y0*exp(r*t)/r-(k*Y0-x0*r)/r;
```

$$x := \frac{k\,Y0\,e^{(r\,t)}}{r} - \frac{k\,Y0 - x0\,r}{r}$$

```
> y:=Y0*exp(r*t);
```

$$y := Y0\,e^{(r\,t)}$$

```
> x/y;
```

$$\frac{\frac{k\,Y0\,e^{(r\,t)}}{r} - \frac{k\,Y0 - x0\,r}{r}}{Y0\,e^{(r\,t)}}$$

```
> simplify(%);
```

$$\frac{\left(k\,Y0\,e^{(r\,t)} - k\,Y0 + x0\,r\right)e^{(-r\,t)}}{r\,Y0}$$

Which can be expressed,

$$\frac{k}{r} - \frac{e^{(-r\,t)}\,(-r\,x0 + k\,Y0)}{r\,Y0} = \frac{k}{r} + \left(\frac{x0}{Y0} - \frac{k}{r}\right)e^{(-r\,t)}$$

Or, in terms of D,

$$\frac{D(t)}{Y(t)} = \left(\frac{D0}{Y0} - \frac{k}{r}\right)e^{(-r\,t)} + \frac{k}{r}.$$

Question 17

Let i denote the nominal interest rate and r the real interest rate, if π denotes the rate of inflation, then $r = i - \pi$. Then

$$A\,e^{(25\,r)} = 2\,A$$

Hence

$$2 = e^{(25\,(i-.05))}$$

```
> solve(2=exp(25*(i-0.05)),i);
```

$$.07772588722$$

Therefore $i = 7.77\%$

Question 18

(a)

```
>
```

	A	B	C	D
1	g	$100\,\frac{\ln(2)}{g}$	g	$100\,\frac{\ln(2)}{g}$
2	2.7000	25.6721	2.4000	28.8811
3	5.0000	13.8629	2.0000	34.6573
4	2.5000	27.7258	-.2000	-346.5735
5	2.0000	34.6573	.2000	346.5735
6	1.4000	49.5105		

(b)

Negative growth rates indicate a decline, so the figure indicates the number of years for GDP to halve in value.

Question 19

(a)

Assuming exponential growth, then

$$P(1992) = 1162000000 = 667073000\,e^{(\lambda\,(1992-1960))}$$

```
> solve(1162000000=667073000*exp(lamda*(1992-1960)),lamda);
```

$$\frac{1}{32}\ln\left(\frac{1162000}{667073}\right)$$

```
> evalf(%);
```

$$.01734370164$$

(b)

China's population will double in

```
> evalf(ln(2)/0.0173);
```

$$40.06631102$$

(c)
At the beginning of the new millennium, China's population will be

```
> 1162000000*exp(0.0173*(2000-1992));
```

$$.1334481275\ 10^{10}$$

Question 20

```
> 5000*exp(0.05*40)+(2000/0.05)*(exp(0.05*40)-1);
```

292507.5245

i.e., £292,507.52.