6 Chapter 6 Solution Set

Problems

6.1 Derive that following alternative form of the Lippmann Schwinger equation Eq.(6.7a)

$$U(\mathbf{r},\nu)=U^{(in)}(\mathbf{r},\nu)+\int d^3r'\,G_+(\mathbf{r},\mathbf{r}')V(\mathbf{r}')U^{(in)}(\mathbf{r}',\nu)$$

where G_+ is the full Green function of background with embedded scatterer.

We start with two defining equations for the (total) field and full Green function

$$[\nabla^2_{r'}+k_0^2-V(\mathbf{r}')]U(\mathbf{r}',\nu)=0,\quad [\nabla^2_{r'}+k_0^2-V(\mathbf{r}')]G_+(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}').$$

Applying standard Green function techniques we then obtain

$$\int_{\partial\tau} dS'[U^{(in)}(\mathbf{r}',\nu)\frac{\partial}{\partial n'}G_+(\mathbf{r},\mathbf{r}')-G_+(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}U^{(in)}(\mathbf{r},\nu)]=U(\mathbf{r},\nu),\quad \mathbf{r}\in\tau.$$

We now apply standard Green function techniques to the equation satisfied by $U^{(in)}(\mathbf{r}',\nu)$ and $G_+(\mathbf{r},\mathbf{r}')$ to obtain

$$\begin{aligned}&\int_{\partial\tau} dS'[U^{(in)}(\mathbf{r}',\nu)\frac{\partial}{\partial n'}G_+(\mathbf{r},\mathbf{r}')-G_+(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}U^{(in)}(\mathbf{r},\nu)]\\&=U^{(in)}(\mathbf{r},\nu)+\int_\tau d^3r'\,G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')U^{(in)}(\mathbf{r}',\nu),\quad \mathbf{r}\in\tau.\end{aligned}$$

Substituting our earlier result in the above equation and letting $\tau\to\infty$ then yields the alternative form of the LS equation.

6.2 Prove that the full outgoing wave Green function $G_+(\mathbf{r},\mathbf{r}_0)$ is a symmetric function of its arguments.

This is proven in an identical manner as was done in the homogeneous background case in Section 2.8.4. The only difference is that here we use the fact that the Green function is outgoing at infinity to make the surface integral vanish.

6.3 Use Theorem 6.2 to compute the scattering amplitude of a scattering potential of the general form

$$V(\mathbf{r})=\sum_{m=1}^{M}V_m(\mathbf{r}-\mathbf{X}_m)$$

in terms of the scattering amplitudes of the component potentials $V_m(\mathbf{r})$.

According to the theorem each component scattering potential produces a scattering amplitude given by

$$f_m(\mathbf{s},\mathbf{s}_0;\mathbf{X}_m) = e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_m} f_m(\mathbf{s},\mathbf{s}_0;0)$$

where $f_m(\mathbf{s},\mathbf{s}_0;0)$ is the scattering amplitude for the potential centered at the origin. Since scattering is a linear transformation from the incident to scattered fields the overall scattering amplitude of the set of potentials is then given by

$$f(\mathbf{s},\mathbf{s}_0) = \sum_{m=1}^{M} f_m(\mathbf{s},\mathbf{s}_0;\mathbf{X}_m) = \sum_{m=1}^{M} e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_m} f_m(\mathbf{s},\mathbf{s}_0;0)$$

An important special case of this is when the scattering potentials are all identical. In this case the overall scattering potential is given by

$$f(\mathbf{s},\mathbf{s}_0) = f_0(\mathbf{s},\mathbf{s}_0)\sum_{m=1}^{M} e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_m}$$

where f_0 is the scattering amplitude of the single scattering potential centered at the origin. The above special case occurs in X-ray crystallography and is the basis for structure determination using X-ray scattering experiments.

6.4 Use the angular spectrum expansion of the scattered field given in Eq.(6.33) and the angular spectrum expansion of the outgoing wave multipole fields given in Eq.(3.49) of Chapter 3 to derive a multipole expansion of the scattered field including expressions for the multipole moments in terms of the scattering amplitude.

We can expand the scattering amplitude in spherical harmonics in the form of Eq.(6.40b)

$$f(\mathbf{s},\mathbf{s}_0) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m(\mathbf{s}_0)Y_l^m(\mathbf{s}), \quad f_l^m(\mathbf{s}_0) =< Y_l^m, f(\mathbf{s},\mathbf{s}_0) >_{\Omega_s}$$

which, when used in Eq.(6.33) yields

$$\begin{aligned} U_+^{(s)}(\mathbf{r},\mathbf{s}_0) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m(\mathbf{s}_0)\{\frac{ik_0}{2\pi}\int_{-\pi}^{\pi} d\beta \int_{C_\pm} d\alpha \sin\alpha\, Y_l^m(\mathbf{s})e^{ik_0\mathbf{s}\cdot\mathbf{r}}\} \\ &= i^{l+1}k_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m(\mathbf{s}_0)h_l^+(k_0r)Y_l^m(\hat{\mathbf{r}})e^{ik_0\mathbf{s}\cdot\mathbf{r}}, \end{aligned}$$

where we have used Eq.(3.49). On comparing the above multipole expansion with Eq.(6.40a) we then obtain

$$q_l^m = -i^l f_l^m(\mathbf{s}_0),$$

which is the required relationship between the multipole moments and the expansion coefficients of the scattering amplitude.

6.5 Use the scattering amplitude of a homogeneous sphere in the angular spectrum expansion given in Section 6.5 to compute the multipole expansion of the scattered field. You will need to make use of the angular spectrum expansions of the multipole fields given in Section 3.4.2. Verify that the expansion you obtained agrees with the one obtained in Section 6.3.

We showed in Section 6.5 that the scattered field admits the angular spectrum expansion

$$U_+^{(s)}(\mathbf{r}, \mathbf{s}_0) = \frac{ik_0}{2\pi} \int_{-\pi}^{\pi} d\beta \int_{C_\pm} d\alpha \sin\alpha \, f(\mathbf{s}, \mathbf{s}_0) e^{ik_0 \mathbf{s}\cdot\mathbf{r}}$$

which, on using the scattering amplitude of a homogeneous sphere found in Example 6.1 yields

$$\begin{aligned} U_+^{(s)}(\mathbf{r}, \mathbf{s}_0) &= \frac{ik_0}{2\pi} \int_{-\pi}^{\pi} d\beta \int_{C_\pm} d\alpha \sin\alpha \, \overbrace{-\frac{4\pi i}{k_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_l Y_l^m(\mathbf{s}) Y_l^{m*}(\mathbf{s}_0)}^{f(\mathbf{s},\mathbf{s}_0)} e^{ik_0 \mathbf{s}\cdot\mathbf{r}} \\ &= 2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_l Y_l^{m*}(\mathbf{s}_0) \int_{-\pi}^{\pi} d\beta \int_{C_\pm} d\alpha \sin\alpha \, Y_l^m(\mathbf{s}) e^{ik_0 \mathbf{s}\cdot\mathbf{r}} \end{aligned}$$

The angular spectrum integral in the above equation was obtained in Section 3.4.2 where it was shown to equal $2\pi i^l h_l^+(k_0 r) Y_l^m(\hat{\mathbf{r}})$. The above expression for the scattered field thus reduces to

$$U_+^{(s)}(\mathbf{r}, \mathbf{s}_0) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l R_l Y_l^{m*}(\mathbf{s}_0) h_l^+(k_0 r) Y_l^m(\hat{\mathbf{r}}).$$

The easiest way to insure that the above result agrees with the result obtained in Section 6.3 is to evaluate the scattering amplitude. If make use of the asymptotic expansion of the spherical Hankel function given in Example 6.1 which then yields the desired result:

$$\begin{aligned} U_+^{(s)}(r\mathbf{r}, \mathbf{s}_0) &\sim 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l R_l Y_l^{m*}(\mathbf{s}_0) \overbrace{(-i)^{l+1} \frac{e^{ik_0 r}}{k_0 r}}^{h_l^+(k_0 r)} Y_l^m(\hat{\mathbf{r}}) \\ &= -\frac{4\pi i}{k_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l R_l Y_l^{m*}(\mathbf{s}_0) Y_l^m(\hat{\mathbf{r}}) \frac{e^{ik_0 r}}{r} \end{aligned}$$

which is the expression for the scattering amplitude found in Section 6.3.

6.6 Use the scattering amplitude of a homogeneous cylinder in the 2D angular spectrum expansion to compute the multipole expansion of the scattered field. You will need to make use of the angular spectrum expansions of the 2D multipole fields found in Problem 4.13 of Chapter 4. Verify that the expansion you obtained agrees with the one obtained in Section 6.3.

The 2D angular spectrum expansion of a general outgoing wave field is found from Eq.(4.39b) of Section 4.6.3 to be of the form

$$U_+(\mathbf{r},\omega) = \sqrt{\frac{k_0}{2\pi}} e^{i\pi/4} \int_{C_\pm} d\alpha\, A(\mathbf{s},\omega) e^{ik_0 \mathbf{s}\cdot\mathbf{r}}$$

where $A(\mathbf{s},\omega)$ is the radiation pattern of the field U_+. The scattered field from a 2D scattering potential having scattering amplitude $f(\mathbf{s},\mathbf{s}_0)$ is then obtained by simply replacing A in the above equation by the scattering amplitude. For a homogeneous cylinder the scattering amplitude is found from Example 6.2 to be

$$f(\mathbf{s},\mathbf{s}_0) = \sqrt{\frac{2}{\pi k_0}} e^{-i\frac{\pi}{4}} \sum_{l=-\infty}^{\infty} R_l e^{il(\phi-\phi_0)}$$

thus yielding the angular spectrum expansion

$$U_+(\mathbf{r},\omega) = \frac{1}{\pi} \sum_{l=-\infty}^{\infty} R_l \int_{C_\pm} d\alpha\, e^{il(\alpha-\alpha_0)} e^{ik_0 \mathbf{s}\cdot\mathbf{r}}.$$

The angular spectrum expansion of the 2D outgoing wave multipole fields was found in Eq.(4.6) of the solution to Problem 4.13 of Chapter 4 to be

$$H_l^+(k_0 r) e^{il\phi} = \frac{(-i)^l}{\pi} \int_{C_\pm} d\alpha\, e^{il\alpha} e^{ik_0 \mathbf{s}\cdot\mathbf{r}}.$$

On making use of this result in the above angular spectrum expansion of the scattered field we obtain

$$U_+(\mathbf{r},\omega) = \sum_{l=-\infty}^{\infty} i^l R_l H_l^+(k_0 r) e^{il(\phi-\phi_0)}$$

where we have set $\alpha_0 = \phi_0$. Finally, we use the fact that (see Example 6.2)

$$b_l(\mathbf{s}_0) = R_l a_{0l} = i^l R_l e^{-il\phi_0},$$

so that the expansion reduces to that found in Section 6.3.

6.7 Express the multipole moments of the scattered field in terms of its boundary value over a sphere that completely surrounds the scattering volume.

This was done in Section 4.8.2 for radiated fields and the same solution applies here for a scattered field. Thus, following the procedure employed in that section we express the scattered field over the surface of a sphere having radius a in the multipole expansion

$$U_+^{(s)}(\mathbf{r},\nu)|_{r=a} = -ik_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_l^m(\nu) h_l^+(k_0 a) Y_l^m(\hat{\mathbf{r}})$$

from which we find that

$$q_l^m(\nu) = \frac{i}{k_0} \frac{u_l^m(\nu)}{h_l^+(k_0 a)},$$

where

$$u_l^m(\nu) = \int d\Omega\, U_+^{(s)}(\mathbf{r},\nu)|_{r=a} Y_l^{m*}(\hat{\mathbf{r}}).$$

6.8 Compute the 2D Born approximation of the scattering amplitude of a homogeneous scatterer with wavenumber k_1 and having a radius a_0 and centered at $\mathbf{X}_0$. Verify that this scattering amplitude is in agreement with Theorems 6.4 and 6.5 but does not satisfy the optical theorem 6.6.

Within the Born approximation the 2D scattering amplitude of a potential centered at $\mathbf{X}_0$ is given by

$$f_{X_0}(\mathbf{s},\mathbf{s}_0) = -\sqrt{\frac{1}{8\pi k_0}} e^{i\frac{\pi}{4}} \int d^2r\, V(\mathbf{r}-\mathbf{X}_0) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}.$$

On making the change of variable $\mathbf{r}' = \mathbf{r} - \mathbf{X}_0$ we then obtain

$$f_{X_0}(\mathbf{s},\mathbf{s}_0) = -\sqrt{\frac{1}{8\pi k_0}} e^{i\frac{\pi}{4}} e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_0} \int d^2r\, V(\mathbf{r}) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}},$$

where we have then replaced the dummy variable $\mathbf{r}'$ in the integral by $\mathbf{r}$. We conclude from this that this scattering amplitude is in agreement with Theorems 6.4 and 6.5.

For a homogeneous scatterer with wavenumber k_1 and having a radius a_0 the above integral becomes

$$\int d^2r\, V(\mathbf{r}) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}} = V_0 \int_0^{a_0} r dr \int_0^{2\pi} d\phi\, e^{-ik_0|\mathbf{s}-\mathbf{s}_0| r\cos\phi},$$

where $V_0 = k_1^2 - k_0^2$ and we have aligned the x axis with $\mathbf{s}-\mathbf{s}_0$ so that $(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r} = r\cos\phi$. The integral over ϕ is $2\pi J_0(k_0|\mathbf{s}-\mathbf{s}_0]r)$ so that

$$\int_0^{a_0} r dr \int_0^{2\pi} d\phi\, e^{-ik_0|\mathbf{s}-\mathbf{s}_0| r\cos\phi} = 2\pi \int_0^{a_0} r dr J_0(k_0|\mathbf{s}-\mathbf{s}_0|r).$$

If we now make use of the recursion relationship

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$$

and make a simple change of variable in the r.h.s. of the above equation we obtain

$$2\pi \int_0^{a_0} r dr J_0(k_0|\mathbf{s}-\mathbf{s}_0|r) = 2\pi a_0 \frac{J_1(k_0|\mathbf{s}-\mathbf{s}_0|a_0)}{k_0|\mathbf{s}-\mathbf{s}_0|}$$

which then yields our final result

$$f_{X_0}(\mathbf{s},\mathbf{s}_0) = -\sqrt{\frac{\pi}{2k_0}} e^{i\frac{\pi}{4}} a_0 e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_0} \frac{J_1(k_0|\mathbf{s}-\mathbf{s}_0|a_0)}{k_0|\mathbf{s}-\mathbf{s}_0|}. \tag{6.1}$$

6.9 Repeat problem 6.8 for the 3D case of a sphere of radius a_0 centered at $\mathbf{X}_0$.

This problem is done in a entirely parallel manner as employed in the preceding problem. In particular, within the Born approximation the 3D scattering amplitude of a potential centered at $\mathbf{X}_0$ is given by

$$f_{X_0}(\mathbf{s}, \mathbf{s}_0) = -\frac{1}{4\pi} \int d^3r\, V(\mathbf{r} - \mathbf{X}_0) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}},$$

which on making the change of variable $\mathbf{r}' = \mathbf{r} - \mathbf{X}_0$ we then obtain

$$f_{X_0}(\mathbf{s}, \mathbf{s}_0) = -\frac{1}{4\pi} e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_0} \int d^3r\, V(\mathbf{r}) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$

where we have then replaced the dummy variable $\mathbf{r}'$ in the integral by $\mathbf{r}$. We conclude from this that this scattering amplitude is in agreement with Theorems 6.4 and 6.5.

For a homogeneous scatterer with wavenumber k_1 and having a radius a_0 the above integral becomes

$$\int d^3r\, V(\mathbf{r}) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}} = 2\pi V_0 \int_0^{a_0} r^2 dr \int_0^{\pi} d\theta\, e^{-ik_0|\mathbf{s}-\mathbf{s}_0| r\cos\theta}$$

where $V_0 = k_1^2 - k_0^2$ and we have aligned the z axis with $\mathbf{s} - \mathbf{s}_0$ so that $(\mathbf{s} - \mathbf{s}_0) \cdot \mathbf{r} = r\cos\phi$. The integral over θ is $2j_0(k_0|\mathbf{s} - \mathbf{s}_0|r)$ so that

$$\int_0^{a_0} r^2 dr \int_0^{\pi} d\theta\, e^{-ik_0|\mathbf{s}-\mathbf{s}_0| r\cos\phi} = 2\int_0^{a_0} r^2 dr j_0(k_0|\mathbf{s} - \mathbf{s}_0|r).$$

If we now make use of the recursion relationship

$$\frac{d}{dx}[x^2 j_1(x)] = x^2 j_0(x)$$

and make a simple change of variable in the r.h.s. of the above equation we obtain

$$2\int_0^{a_0} r^2 dr j_0(k_0|\mathbf{s} - \mathbf{s}_0|r) = 2a_0^2 \frac{j_1(k_0|\mathbf{s} - \mathbf{s}_0|a_0)}{k_0|\mathbf{s} - \mathbf{s}_0|}$$

which then yields our final result

$$f_{X_0}(\mathbf{s}, \mathbf{s}_0) = -e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{X}_0} a_0^2 \frac{j_1(k_0|\mathbf{s} - \mathbf{s}_0|a_0)}{k_0|\mathbf{s} - \mathbf{s}_0|}. \tag{6.2}$$

6.10 Compute the generalized scattering amplitude of a homogeneous sphere for the case of an incident free multipole field $j_l(kr)Y_l^m(\hat{\mathbf{r}})$ by using the technique given in Section 6.5.

We found the plane wave expansion of the free multipole fields in Example 3.4 to be

$$j_l(kr)Y_l^m(\hat{\mathbf{r}}) = \frac{(-i)^l}{4\pi} \int d\Omega_s\, Y_l^m(\mathbf{s}) e^{ik\mathbf{s}\cdot\mathbf{r}},$$

Thus yielding the plane wave amplitude

$$A(\mathbf{s}_0, \nu) = \frac{(-i)^l}{4\pi} Y_l^m(\mathbf{s}_0).$$

The plane wave scattering amplitude of a homogeneous sphere was found in Example 6.1 to be

$$f(\mathbf{s},\mathbf{s}_0) = -\frac{4\pi i}{k_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_l Y_l^m(\mathbf{s})Y_l^{m*}(\mathbf{s}_0).$$

Thus, using Eq.(6.35b) we obtain

$$f(\mathbf{s},\nu) = \int_{4\pi} d\Omega_{s_0}\, A(\mathbf{s}_0,\nu) f(\mathbf{s},\mathbf{s}_0)$$

$$= \int_{4\pi} d\Omega_{s_0}\, \overbrace{\frac{(-i)^l}{4\pi}Y_l^m(\mathbf{s}_0)}^{A(\mathbf{s}_0,\nu)}\overbrace{-\frac{4\pi i}{k_0}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} R_{l'} Y_{l'}^{m'}(\mathbf{s})Y_{l'}^{m'^*}(\mathbf{s}_0)}^{f(\mathbf{s},\mathbf{s}_0)}$$

$$= \frac{(-i)^{(l+1)}}{k_0} R_l Y_l^m(\mathbf{s}).$$

6.11 Compute the field scattered from an infinite Dirichlet plane (a plane over which the field vanishes) located at $z = 0$ due to an incident wave field radiated by a source $Q(\mathbf{r})$ located in the l.h.s. $z < 0$. Express your answer in terms of the outgoing wave Green function $G_+(\mathbf{r}-\mathbf{r}')$.

This problem was posed in terms of a radiation problem as Problem 2.14 in Chapter 2. In that problem we computed the total field radiated by a source located in the l.h.s. in the presence of a Dirichlet plane at $z = 0$ and obtained the solution

$$U(\mathbf{r},\omega) = \int_{\tau_0} d^3r'\, G_D(\mathbf{r},\mathbf{r}',\omega)Q(\mathbf{r}',\omega), \quad z < 0,$$

where G_D is the Dirichlet Green function that vanished at $z = 0$. The above field is the radiated field throughout the half-space $z < 0$ that vanishes on the boundary $z = 0$; i.e., is the solution to the radiation problem in the presence of a perfectly conducting infinite plane located at $z = 0$. This field can be interpreted as being the sum of the incident wave radiated by Q in infinite free space and the scattered wave generated by the Dirichlet plane surface. On setting

$$G_D(\mathbf{r},\mathbf{r}') = G_+(\mathbf{r}-\mathbf{r}') - G_+(\mathbf{r}-\tilde{\mathbf{r}}'), \quad \mathbf{r}' = (x',y',z'),\ \tilde{\mathbf{r}}' = (x',y',-z')$$

we obtain

$$U(\mathbf{r},\omega) = \overbrace{\int_{\tau_0} d^3r'\, G_+(\mathbf{r}-\mathbf{r}',\omega)Q(\mathbf{r}',\omega)}^{U^{(in)}(\mathbf{r},\omega)}\overbrace{-\int_{\tau_0} d^3r'\, G_+(\mathbf{r}-\tilde{\mathbf{r}}',\omega)Q(\mathbf{r}',\omega)}^{U^{(s)}(\mathbf{r},\omega)}, \quad z < 0,$$

which is the required result.

6.12 Use the result obtained in the previous problem to derive the so-called "law-of-reflection" which states that a plane wave incident from the left-half space with unit propagation vector $\mathbf{s}_0$ onto an infinite plane Dirichlet surface located on the (x,y) plane will generate a reflected plane wave that propagates into the left-half space with unit wave vector $\widetilde{\mathbf{s}_0}=(s_{0_x},s_{0_y},-s_{0_z})$.

An incident plane wave to some region of space τ_0 is generated by a delta function source located asymptotically far from τ_0. Thus we take the source $Q(\mathbf{r},\omega)$ in the previous problem to be

$$Q(\mathbf{r},\omega)=\delta(\mathbf{r}-\mathbf{R}_0),$$

with $\mathbf{R}_0=(x_0,y_0,-z_0)$ located in the l.h.s. and z_0 arbitrarily large. This then generates an incident field given by

$$U^{(in)}(\mathbf{r},\omega)=\int_{\tau_0}d^3r'\,G_+(\mathbf{r}-\mathbf{r}',\omega)Q(\mathbf{r}',\omega)=G_+(\mathbf{r}-\mathbf{R}_0,\omega)=-\frac{e^{ikR_0}}{4\pi}e^{-ik\hat{\mathbf{R}}_0\cdot\mathbf{r}}.$$

This corresponds to an incident plane wave with amplitude $-\exp(ikR_0)/4\pi$ and unit propagation vector $\mathbf{s}_0=-\hat{\mathbf{R}}_0$ and $s_{0z}=\hat{\mathbf{z}}\cdot\mathbf{s}_0>0$. The plane wave thus propagates in the positive z direction.

6.13 Express the scattered (reflected) wave field found in Problem 6.11 in an angular spectrum expansion and interpret your result in terms of the law of reflection stated in the previous problem.

See Problem 4.10

6.14 Derive the Ricatti equation Eq.(6.72a) from the Helmholtz equation.

Using the representation of the field in terms of a complex phase given in Eq.(6.71) we find that

$$\nabla U=ik_0\nabla WU,\quad \nabla^2U=ik_0\nabla^2WU+ik_0\nabla W\cdot\nabla U=ik_0\nabla^2WU-k_0^2\nabla W\cdot\nabla WU$$

from which we obtain

$$\nabla^2U+k_0^2n^2U=[ik_0\nabla^2W-k_0^2\nabla W\cdot\nabla W+k_0^2n^2]U=0\rightarrow ik_0\nabla^2W-k_0^2\nabla W\cdot\nabla W+k_0^2n^2=0.$$

6.15 Derive the form of the Ricatti equation given in Eq.(6.75) from Eq.(6.72a).

This form of the equation is obtained by expressing the phase W in the form

$$W(\mathbf{r},\nu)=W_0(\mathbf{r},\nu)+\delta W(\mathbf{r},\nu)$$

where W_0 is the phase of the field in free space and δW the perturbation in the phase introduced by the presence of the scatterer. We now have to substitute the above decomposition of the phase into the Ricatti equation to obtain the form of this equation given in Eq.(6.75). On making this substitution we obtain

$$\overbrace{ik_0(\nabla^2W_0+\nabla^2\delta W)}^{ik_0\nabla^2W}+\overbrace{-k_0^2(\nabla W_0+\nabla\delta W)^2}^{-k_0^2\nabla W\cdot\nabla W}+k_0^2n^2=0$$

which simplifies to

$$\overbrace{ik_0\nabla^2W_0-k_0^2(\nabla W_0)^2}^{-k_0^2}+ik_0\nabla^2\delta W-k_0^2(\nabla\delta W)^2-2k_0^2\nabla W_0\cdot\nabla\delta W+k_0^2n^2=0$$

$$ik_0\nabla^2\delta W-k_0^2(\nabla\delta W)^2-2k_0^2\nabla W_0\cdot\nabla\delta W+k_0^2(n^2-1)=0$$

6.16 Derive Eq.(6.83).

Here we start with the Rytov Ansatz given in Eq.(6.82)

$$ik_0\delta W_R(\mathbf{r};\mathbf{s}_0) = e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}}F(\mathbf{r}).$$

where the phase perturbation is assumed to satisfy the linearized Ricatti equation Eq.(6.78b). We have that

$$\nabla\delta W_R = [-ik_0\mathbf{s}_0F + \nabla F]\frac{e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}}}{ik_0}$$
$$\nabla^2\delta W_R = [-2\mathbf{s}_0\cdot\nabla F + \frac{1}{ik_0}\nabla^2F + ik_0F]e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}}.$$

Eq.(6.83) results from using the above two expressions in the linearlized Ricatti equation and simplifying the resulting equation.

6.17 Derive Eq.(6.85).

For plane wave incidence we have that

$$U_R = e^{ik_0[\mathbf{s}_0\cdot\mathbf{r}+\delta W_R]} = e^{ik_0\mathbf{s}_0\cdot\mathbf{r}}e^{ik_0\delta W_R} = e^{ik_0\mathbf{s}_0\cdot\mathbf{r}}\left[1 + ik_0\delta W_R + \frac{1}{2}(ik_0\delta W_R)^2 + \cdots\right]$$
$$\sim e^{ik_0\mathbf{s}_0\cdot\mathbf{r}} + ik_0e^{ik_0\mathbf{s}_0\cdot\mathbf{r}}\delta W_R = e^{ik_0\mathbf{s}_0\cdot\mathbf{r}} + U_B^{(s)}(\mathbf{r};\mathbf{s}_0)$$

where we have made use of Eq.(6.84).