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Problems for Chapter 8 of *Advanced Mathematics for Applications*

Sequences and Series

by Andrea Prosperetti

Notes: The semi-factorials of odd and even numbers are given by

$$(2n+1)!! \,=\, 1\cdot 3\cdot 5\ldots(2n-1)(2n+1)\,, \qquad (2n)!! \,=\, 2\cdot 4\cdot 6\ldots(2n-2)(2n) \,=\, 2^n n!\,.$$

The connection between the factorial and the Γ function (p, 442) is $n! = \Gamma(n+1)$. Stirling's formula for $\Gamma(z)$ is given on p. 445.

1 Convergence

1. For what range of the parameter p are the series

$$\sum_1^\infty \frac{1}{n^p}\,, \qquad \sum_1^\infty \frac{1}{n^p \log n}\,,$$

   convergent?

2. For what range of the parameter p are the series

$$\sum_1^\infty \frac{1}{n(\log n)^p}\,, \qquad \sum_1^\infty \frac{1}{n\log n\,(\log\log n)^p}\,,$$

   convergent?

3. For what range of the parameter p is the series

$$1+\sum_1^\infty \left(\frac{(2n-1)!!}{(2n)!!}\right)^p$$

   convergent?

4. Show how the radius of convergence of the hypergeometric series given in (2.5.45) p. 48 is determined by the ratio test.

5. The series

$$x+1+x^3+x^2+x^5+x^4+\ldots$$

   is a rearrangement of the geometric series $\sum_{n=0}^\infty x^n$ and therefore it certainly converges for $0<x<1$. An application of the ratio test illustrates the point that a_{n+1}/a_n does not have to have a defined limit to draw conclusions on the convergence of the series.

6. Prove that, if $\sum n a_n$ is convergent, so is $\sum a_n$.

7. By using (a) the ratio test and, (b) the root test with the Stirling formula, verify the convergence of the Bessel series

$$\sum_{n=0}^{\infty} \frac{(-z^2)^n}{2^{2n}(n!)^2}$$

for all values of z.

8. Show, by using the M-test (p. 227), that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

is uniformly convergent for all values of x.

9. Show that the two series

$$S_1(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2} \qquad \text{and} \qquad S_2(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+n^4x^2}$$

are uniformly convergent for all values of x. However, while S_1' can be obtain by term-by-term differentiation, $S_2'(0)$ cannot and, in fact, it does not exist.

10. Determine the radius of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n\,, \qquad \sum_{n=0}^{\infty} z^{n!}\,, \qquad \sum_{n=0}^{\infty} z^{2^n}\,, \qquad \sum_{n=0}^{\infty} [3+(-1)^n]^n\, z^n\,, \qquad \sum_{n=0}^{\infty} (n+a^n) z^n\,, \qquad \sum_{n=0}^{\infty} z^n \cos in\,.$$

Usee of Stirling's formula to approximate $n!$ will be useful in some cases.

11. Show that the two sequences having terms

$$F_n = \frac{n\sqrt{x}}{1+n^2x^3}\,, \qquad F_n = \frac{n^2x}{1+n^3x^3}\,,$$

are examples of sequences for which term-by-term integration fails.

12. Given that the radius of convergence of the (complex) series $\sum_{n=0}^{\infty} c_n z^n$ is R, with $0 < R < \infty$, determine the radius of convergence of the following series:

$$\sum_{n=1}^{\infty} n^k c_n z^n\,, \qquad \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n\,, \qquad \sum_{n=1}^{\infty} n^n c_n z^n\,, \qquad \sum_{n=0}^{\infty} (1+z_0^n)\, c_n z^n\,.$$

13. If the radii of convergence of the (complex) series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are R_a and R_b, respectively, with $0 < R_a, R_b < \infty$, what can be said about the radius of convergence of the following series

$$\sum_{n=1}^{\infty} (a_n \pm b_n) z^n\,, \qquad \sum_{n=0}^{\infty} a_n b_n z^n\,, \qquad \sum_{n=1}^{\infty} (a_n/b_n) z^n\,?$$

2 Summation of series

1. Show by using summation by exact differences that

$$\sum_{n=1}^{\infty} \frac{1}{(a+n)(a+n+1)} = \frac{1}{a+1}\,.$$

2. Show by using summation by exact differences that

$$\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)(1-z^{n+1})} = \begin{cases} z(1-z)^{-2} & \text{if} \quad |z|<1 \\ (1-z)^{-2} & \text{if} \quad |z|>1 \end{cases} .$$

3. Calculate the radius of convergence of the following series and find their sum in closed form:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2-1}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{(2n+1)^2-1}$$

4. Show that

$$\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} = 1 - \left(\frac{1}{z}-1\right) \log \frac{1}{1-z} .$$

5. Calculate the radius of convergence of the following series and find their sum in closed form:

$$\sum_{n=1}^{\infty} \frac{n}{n+1} z^{n+1}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} .$$

6. Calculate in closed form the sums of the series

$$\sum_{n=0}^{\infty} \frac{1}{n^4-a^4}, \qquad -\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4-a^4}$$

7. Calculate the radius of convergence of the following series and find their sum in closed form:

$$\sum_{n=1}^{\infty} \frac{z^n}{n(4n^2-1)}, \qquad \sum_{n=1}^{\infty} \frac{(n-1)z^n}{(n+2)\,n!} .$$

8. Write the series

$$1 - \frac{1\cdot 3}{2\cdot 4} z^2 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} z^4 + \dots$$

in a compact form as an infinite summation and calculate its sum.

9. Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+a^2n^2} = \frac{\pi}{a} \coth \frac{\pi}{a} .$$

10. Show that

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{2^{n-1}} = \frac{4\sin\theta}{5-4\cos\theta} .$$

11. Show that

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cos n\theta = \left(2 \sin \frac{\theta}{2}\right)^{-1/2} \cos \frac{\pi-\theta}{4} .$$

12. Calculate in closed form the sums of the series

$$-\sum_{n=2}^{\infty} \frac{\cos(n\pi/2)}{n^2-1} \cos n\theta, \qquad -\sum_{n=2}^{\infty} \frac{n\,\cos(n\pi/2)}{n^2-1} \sin n\theta .$$

13. Calculate in closed form the sums of the series

$$-\sum_{n=2}^{\infty} \frac{\sin(n\pi/2)}{n^2-1} \sin n\theta\,, \qquad -\sum_{n=2}^{\infty} \frac{n\,\sin(n\pi/2)}{n^2-1} \cos n\theta\,.$$

14. Calculate in closed form the sums of the series

$$\sum_{n=2}^{\infty} \frac{\sin n\theta}{n^2-1}\,, \qquad -\sum_{n=2}^{\infty} \frac{\sin\theta}{n(n^2-1)}\,.$$

15. Calculate in closed form the sums of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1} \sin n\theta\,, \qquad \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}\,.$$

16. Calculate in closed form the sums of the series

$$\frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2+a^2}\,, \qquad \frac{1}{a} + 2a \sum_{n=1}^{\infty} (-1)^n \frac{\cos n\theta}{n^2+a^2}\,.$$

17. Calculate in closed form the finite sums

$$\sum_{n=1}^{n} n\,\sin n\theta\,, \qquad \sum_{n=1}^{n} n\,\cos n\theta\,.$$

18. Show by using the Laplace transform method that

$$\sum_{n=1}^{\infty} \log\left(1 + \frac{a^2}{\pi^2 n^2}\right) \;=\; \log\frac{\sinh a}{a}$$

19. Show by using the Laplace transform method that

$$\sum_{n=1}^{\infty} \left[\frac{a}{n} - \log\frac{n+a}{n}\right] \;=\; \gamma a + \log\Gamma(a+1)\,,$$

where γ is Euler's constant (p. 150).

20. Show by using the Laplace transform method that

$$\sum_{n=1}^{\infty} \tan^{-1}\frac{2}{n^2} \;=\; \frac{3}{4}\pi\,.\gamma a + \log\Gamma(a+1)\,,$$

21. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} \;=\; \frac{\pi}{a}\frac{1}{e^{\pi a}-e^{-\pi a}} - \frac{1}{2a^2}\,.$$

22. Show that the locus represented by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin nx\,\sin ny \;=\; 0$$

consists of two systems of lines at right angles dividing the (x,y)-plane into squares of area π^2.

23. Define

$$f(\theta) \;=\; \sum_{n=2}^{\infty} \frac{\cos n\theta}{n^2-1}\,, \qquad g(\theta) \;=\; \sum_{n=2}^{\infty} \frac{\sin n\theta}{n^2-1}\,.$$

Show that, if $0<\theta<2\pi$, the function $P(\theta)=f(\theta)+ig(\theta)$ satisfies the equation

$$\frac{\mathrm{d}P}{\mathrm{d}\theta}+iP \;=\; ie^{i\theta}\left[\frac{1}{2}i(\pi-\theta)-\log\left(2\sin\frac{\theta}{2}\right)\right]\,.$$

Deduce that

$$f(\theta) \;=\; \frac{1}{2}+\frac{1}{4}\cos\theta-\frac{1}{2}(\pi-\theta)\sin\theta\,, \qquad g(\theta) \;=\; \frac{1}{4}\sin\theta-\sin\theta\,\log\left(2\sin\frac{\theta}{2}\right)\,.$$

24. Show, by using the standard integral representation of the function J_0 (see (12.2.25) p. 308), that

$$\frac{1}{a^2}+2\sum_{n=1}^{\infty}(-1)^n\frac{J_0(n\pi)}{a^2-n^2\pi^2} \;=\; \frac{J_0(a)}{a\,\sin a}\,.$$

25. Show that

$$\sum_{n=1}^{\infty}(-1)^n nJ_{2n}(a) \;=\; -\frac{1}{4}aJ_1(a)\,, \qquad \sum_{n=0}^{\infty}\sin[(2n+1)\theta]\,J_{2n+1}(a) \;=\; \sin(a\sin\theta)\,.$$

26. Show that

$$\sum_{n=-\infty}^{\infty} J_0(an) \;=\; \begin{cases} 2a^{-1} & 0<a<2\pi \\ 2a^{-1}+4\sum_{k=1}^{m}(a^2-4\pi^2k^2)^{-1/2} & 2m\pi<a<2(m+1)\pi \end{cases}\,.$$

27. Show that, if $-1<a<1$,

$$\sum_{n=-\infty}^{\infty} J_0(n\pi)\,\cos n\pi a \;=\; \frac{2}{\pi\sqrt{1-a^2}}\,.$$

28. Show that

$$\frac{2}{\pi}\sum_{m=-\infty}^{\infty} K_0([(2m+1)a]) \;=\; \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{a^2+\pi^2n^2}}\,.$$

29. Show that

$$\sum_{n=1}^{\infty} J_n(na) \;=\; \frac{1}{2}\frac{a}{1-a}\,, \qquad \sum_{n=1}^{\infty} J_{2n}(2na) \;=\; \frac{1}{2}\frac{a^2}{1-a^2}\,.$$

3 Double series

1. Consider the double series

$$\sum_{m=1,n=1}^{\infty}{}' \frac{1}{m^2-n^2}$$

in which the prime signifies that the terms with $m=n$ are omitted. Show that

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}{}'\frac{1}{m^2-n^2}\right) \;=\; -\frac{3}{4}\sum_{m=1}^{\infty}\frac{1}{m^2}\,, \qquad \text{while} \qquad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}{}'\frac{1}{m^2-n^2}\right) \;=\; \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}\,.$$

2. Consider the double series

$$\sum_{m=0,n=0}^{\infty}{}' \frac{(m-n)(m+n-1)!}{2^{m+n}m!n!}$$

in which the prime signifies that the $(0,0)$ terms is omitted. Show that

$$\sum_{m=1}^{\infty}\left(\sum_{n=0}^{\infty}{}' \frac{(m-n)(m+n-1)!}{2^{m+n}m!n!}\right) = -1\,, \qquad \text{while} \qquad \sum_{n=1}^{\infty}\left(\sum_{m=0}^{\infty}{}' \frac{(m-n)(m+n-1)!}{2^{m+n}m!n!}\right) = 1\,.$$

3. Prove the following relation concerning the product of the two series indicated:

$$\left(\sum_{n=0}^{\infty}\frac{x^n}{(n!)^2}\right)\left(\sum_{n=0}^{\infty}\frac{(-x)^n}{(n!)^2}\right) = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{(n!)^2(2n)!}\,.$$