Introduction to Experimental Mathematics
Catalog of projects

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We list a number of projects for groups work that may be useful testing grounds for the methods taught in the book. In each case, we do not have a complete understanding of the problem under study, but we also cannot guarantee that the problems are open.

We would be grateful to obtain reports on any progress or suggestions for similar projects.

1 Coloring generalized LEGO-structures

The smallest number of color needed to color buildings with $a \times b$ LEGO bricks (the LEGO-chromatic number (Example 8.1.2)) has been studied in the book. As an interesting variation, think of the brick as a $a \times b \times c$-box (often with $c = 1$) which may be placed in any of 6 different directions, as long as the corners are in $\mathbb{Z}^3$, and ask the same question: what is the smallest number $\widetilde{\chi}_{a,b,c}$ so that any such building with $a \times b \times c$-bricks can be colored with $\widetilde{\chi}_{a,b,c}$ colors so that no two adjacent bricks have the same color? An argument provided in 2013 to the authors by Eigil Rischel shows that 150 colors will suffice for any fixed choice of $a, b, c \in \mathbb{N}$, but surely that can be dramatically improved. In fact, we know of no examples where more than 6 colors are necessary.

1. How can $\widetilde{\chi}_{a,b,c}$ be bounded above and below for small values of a, b, c?
2. Are there good estimates on $\widetilde{m}_{a,b,c}$ defined as the largest number so that there exists a building where every brick has at least $\widetilde{m}_{a,b,c}$ neighbors?

2 Y-functions

Consider an increasing function $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$, having the property that $f(0) = 1$. If X is an arbitrary finite set, set $f^*(X) = f(|X|)$.

Choose any four finite subsets $X_1, ..., X_4$ of $\mathbb{N}$. For some subset $A \subseteq \{1, 2, 3, 4\}$ set

$$g(A) = f^*(\cap_{i \in A} X_i).$$

We abbreviate $12 = \{1, 2\}$, etc., so that for example, $g(12) = f^*(X_1 \cap X_2)$

We say that f is a Y-function if

$$g(13)^3 g(14)^3 g(34)^3 g(23) g(24) \leq g(1) g(12) g(3)^2 g(4)^2 g(134)^4 g(234)$$

no matter how you choose $X_1, \ldots, X_4$.

Group theory leads to the following examples of Y-functions:

(I) $f(n) = n!$

(II) $f(n) = q^n$

(III) $f(n) = q^{\binom{n}{2}}(q-1)(q^2-1)\cdots(q^n-1)$.

where q in (ii) is an arbitrary integer, and q in (iii) is a prime power.

1. Find examples of increasing functions which are not Y-functions.
2. Examine how for various examples of Y-functions one may choose $X_1, \ldots, X_4$ – all different – with

$$\frac{g(13)^3 g(14)^3 g(34)^3 g(23) g(24)}{g(1) g(12) g(3)^2 g(4)^2 g(134)^4 g(234)}$$

as close to 1 as possible.

3. The given examples of Y-functions grow very quickly; is there a lower limit for how quickly a Y-function can grow?

- Project idea: Jørn Børling Olsson
- Material:
  - T.H. Chan and R. Yeung: On a Relation Between Information Inequalities and Group Theory. IEEE Trans. Inform. Theory **48** (2002).

3 Packing bricks in $\mathbb{R}^n$

Note that for any $n \in \mathbb{N}$ and any vector $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$ we have

$$n^n(x_1 \cdots x_n) \leq (x_1 + \cdots + x_n)^n$$

which can be interpreted as expressing that the volume of n^n little bricks (parallelepipeds) of dimension

$$x_1 \times x_2 \times \cdots \times x_n$$

does not exceed the volume of one large cube of dimension

$$(x_1 + \cdots + x_n) \times \cdots \times (x_1 + \cdots + x_n).$$

A question, raised by Hoffman, is: for which n can these n^n bricks be packed inside the cube irrespective of the given dimensions? For $n = 2$ a general solution is easily found, but already with $n = 3$ it is non-trivial to find such a concrete packing.

1. Can packings for larger n be located experimentally?

2. How many packings exist, as a function of n, up to rigid transformations?

3. Is there a computable threshold on the "eccentricity" of the bricks so that the number of packings is finite under this threshold, but infinite above?

- Material:
  - D.G. Hoffman: Packing problems and inequalities. In: The mathematical Gardner, Wadsworth International, Belmont, Calif, 1981.

4 HJ-permutations

We consider permutations in S_n as maps on $\mathbb{Z}/n$. In recent work of Helfgott and Juschenko, the number $\eta(\pi)$ for such a permutation is defined as

$$\eta(\pi) = |\{x \in \mathbb{Z}/n : \pi(x+1) = 2\pi(x)\}|.$$

For instance, the permutation of S_{11}

$$\pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \mathbf{7} & \mathbf{3} & \mathbf{6} & 1 & \mathbf{4} & 8 & 2 & 0 & \mathbf{5} & \mathbf{10} & \mathbf{9} \end{pmatrix}$$

has $\eta(\pi) = 7$ because the entries in bold satisfy the condition given. In cycle form we have

$$\pi = (0,7)(1,3)(2,6)(5,8)(9,10)$$

so we note that the order of π is 2.

1. Fix $N \in \mathbb{N}$. What is the maximal value $\eta(\pi)$ for π a permutation of order N?

2. Fix $N \in \mathbb{N}$. Does there exist a sequence of permutations $\pi_k \in S_{n_k}$ so that $\eta(\pi_k)/n_k \to 1$ while the order remains bounded by N?

- Material:
  - Harald Helfgott & Kate Juschenko: Soficity, short cycles and the Higman group (arXiv:1512.02135)

5 Probability distributions on graphs

This project is inspired by the following problem: given n points in the plane, determine a point and a minimal circle with center in this point such that every point is in the interior or on the boundary of the circle. There exists an algorithm to solve this problem, but the corresponding problem in information theory is open: given a subset $S \subseteq M_1^+(F)$ of probability distributions on a finite set F, find a probability distribution p on F and a radius R_{min} such that $D(q\|p) \leq R_{min}$ for all $q \in S$, and $R_{\min}$ is the smallest number with this property. Here, $D(q\|p) = \sum_{x \in F} q(x) \log(q(x)/p(x))$ is the *Kullback-Leibler divergence* between q and p, which is a natural measure of distance between probability distributions (one could think of it as a non-symmetric metric).

The general problem is probably too difficult to tackle experimentally, so this project only concerns the following special type of probability distribution. Let F be a *co-tree*, i.e., a finite set equipped with an order such that every non-maximal element has a unique follower. Let S be the set of probability distributions $q \in M_1^+(F)$ such that $q(x) \leq q(y)$ if and only if $x \geq y$. A simple special case is a *linearly ordered* set.

1. How does one determine the central probability distribution p and the minimal radius $R_{\min}$ when F is a linearly ordered set?

2. How does one find them if F is a general co-tree?

3. How does one find them if F contains a single non-maximal element without a unique follower?

- Project idea: Henrik Densing Petersen