Fundamentals of Particle Physics

Supplementary materials:
Solutions of Problems

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Contents

Preface

This document gives detailed solutions to the problems and exercises in the first edition of Fundamental of Particle Physics.

Equation numbers starting with an S are specific to this document. Other equation numbers, such as 1.1, refer to the equations in the book. Notations are those used in the book.

1 Particle physics landscape

1.1. As $\int_V \mathrm{d}^3\boldsymbol{x}\, \boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{x}) = \int_V \mathrm{d}^3\boldsymbol{x}\, \rho(\boldsymbol{x})/\epsilon_0 = Q/\epsilon_0$, the Gauss's theorem, yields $\oint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} = Q/\epsilon_0$. With $\rho(\boldsymbol{x}) = q_0\delta^3(\boldsymbol{x})$, $Q = q_0$. Since the volume of integration is arbitrary, so is the surface, and using the surface of a sphere with radius $|\boldsymbol{r}|$, $\mathrm{d}\boldsymbol{S} = |\boldsymbol{r}|\,\mathrm{d}\theta\,|\boldsymbol{r}|\sin\theta\,\mathrm{d}\phi\,\hat{\boldsymbol{n}}$ yields

$$\oint_S \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} = |\boldsymbol{r}|^2\int_0^\pi \sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,|\boldsymbol{E}|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}} = |\boldsymbol{r}|^2 4\pi|\boldsymbol{E}|.$$

Therefore,

$$|\boldsymbol{E}| = \frac{q_0}{4\pi\epsilon_0|\boldsymbol{r}|^2},\ \text{i.e.}\quad \boldsymbol{E} = \frac{q_0}{4\pi\epsilon_0|\boldsymbol{r}|^2}\frac{\boldsymbol{r}}{|\boldsymbol{r}|},$$

the electric field being radial for symmetry reasons. As the force due to the electric field for a charge q is $\boldsymbol{f} = q\boldsymbol{E}$, we conclude

$$\boldsymbol{f} = \frac{qq_0}{4\pi\epsilon_0|\boldsymbol{r}|^2}\frac{\boldsymbol{r}}{|\boldsymbol{r}|}.$$

1.2. The Fourier transform of

$$(\nabla^2 - \lambda^2)f(\boldsymbol{r}) = -\delta(\boldsymbol{r}) \tag{S1.1}$$

is $(-|\boldsymbol{k}|^2 - \lambda^2)\tilde{f}(\boldsymbol{k}) = -1$, leading to $\tilde{f}(\boldsymbol{k}) = 1/(|\boldsymbol{k}|^2 + \lambda^2)$. Therefore,

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\iiint \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{|\boldsymbol{k}|^2+\lambda^2}\,\mathrm{d}\boldsymbol{k} = \frac{1}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\phi\iint\frac{e^{i|\boldsymbol{k}||\boldsymbol{r}|\cos\theta}}{|\boldsymbol{k}|^2+\lambda^2}\sin\theta\,\mathrm{d}\theta\,|\boldsymbol{k}|^2\,\mathrm{d}|\boldsymbol{k}|,$$

i.e.

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}|\boldsymbol{k}|\frac{|\boldsymbol{k}|^2}{|\boldsymbol{k}|^2+\lambda^2}\int_{-1}^1\mathrm{d}(\cos\theta)\,e^{i|\boldsymbol{k}||\boldsymbol{r}|\cos\theta} = \frac{1}{2\pi^2}\int_0^\infty\mathrm{d}|\boldsymbol{k}|\frac{|\boldsymbol{k}|^2}{|\boldsymbol{k}|^2+\lambda^2}\frac{\sin(|\boldsymbol{k}||\boldsymbol{r}|)}{|\boldsymbol{k}||\boldsymbol{r}|}.$$

Setting $u = |\boldsymbol{k}||\boldsymbol{r}|$, $f(\boldsymbol{r})$ reads

$$f(\boldsymbol{r}) = \frac{1}{2\pi^2}\frac{1}{|\boldsymbol{r}|}\int_0^\infty\mathrm{d}u\,\frac{u}{u^2+\lambda'^2}\sin u = \frac{1}{4\pi^2}\frac{1}{|\boldsymbol{r}|}\int_{-\infty}^\infty\mathrm{d}u\,\frac{u}{u^2+\lambda'^2}\sin u = \frac{1}{4\pi^2}\frac{1}{|\boldsymbol{r}|}\Im(I)$$

where $\lambda' = \lambda|\mathrm{r}|$ and

$$I = \int_{-\infty}^\infty \mathrm{d}u\frac{u}{u^2+\lambda'^2}e^{iu}.$$

The calculation of I is very similar to the examples presented in Appendix F of the book. We first calculate the integral in the complex plane

$$I_z = \oint_C \mathrm{d}z\frac{z}{z^2+\lambda'^2}e^{iz} = \lim_{R\to\infty}\left\{\int_{-R}^R\mathrm{d}u\frac{u}{u^2+\lambda'^2}e^{iu} + \int_0^\pi\mathrm{d}\theta\,iRe^{i\theta}\frac{Re^{i\theta}}{R^2e^{2i\theta}+\lambda'^2}e^{iR\cos\theta-R\sin\theta}\right\}$$

with a contour defined by a half-circle of radius R in the positive side of the imaginary axis since $\lambda' > 0$ (same contour as the first figure of the Appendix). As $\sin\theta > 0$, the second integral is zero when $R \to \infty$. The poles are $z_1 = i\lambda'$ and $z_2 = -i\lambda'$, and the residue theorem leads to

$$\int_{-\infty}^{\infty} \mathrm{d}u \frac{u}{u^2 + \lambda'^2} e^{iu} + 0 = 2i\pi \frac{z_1}{2z_1} e^{iz_1} = i\pi e^{-\lambda'}.$$

Therefore,

$$f(\boldsymbol{r}) = \frac{1}{4\pi^2} \frac{1}{|\boldsymbol{r}|} \pi e^{-\lambda'} = \frac{e^{-\lambda|\boldsymbol{r}|}}{4\pi|\boldsymbol{r}|}. \tag{S1.2}$$

The solution of Poisson's equation $\nabla^2 V = -\frac{e}{\epsilon_0}\delta(\boldsymbol{r})$ is obtained from the general solution (S1.2) of Eq. (S1.1) with $\lambda = 0$ and $f(\boldsymbol{r}) = \frac{\epsilon_0}{e} V(\boldsymbol{r})$, i.e.

$$V(\boldsymbol{r}) = \frac{e}{4\pi\epsilon_0|\boldsymbol{r}|}.$$

Similarly, the comparison of the general solution (S1.2) to the Yukawa potential $\varphi(\boldsymbol{r}) = \frac{g}{4\pi}\frac{e^{-r/r_0}}{r}$ leads to the identification $\lambda = 1/r_0$, which using Eq. (S1.1) shows that the Yukawa potential is the solution of

$$\left(\nabla^2 - \frac{1}{r_0^2}\right)\varphi(\boldsymbol{r}) = -g\delta(\boldsymbol{r}).$$

1.3. The result is straightforward:

$$\iint I(\theta)\cos\theta \,\mathrm{d}\Omega = \int_0^1 I_v \,(\cos\theta)^3 \,\mathrm{d}(\cos\theta) \int_0^{2\pi} \mathrm{d}\phi = I_v \left[\frac{x^4}{4}\right]_0^1 2\pi = I_v \frac{\pi}{2}.$$

1.4. If the particle lifetime is τ, when it travels with a velocity $v = \beta c$, its apparent time in the lab frame is $\gamma\tau$, and hence, it travels over a distance $l = \beta c\gamma\tau$. As the particle energy is $E = \gamma mc^2$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, it follows that $\beta^2 = \frac{E^2-(mc^2)^2}{E^2}$. Therefore,

$$\left(\frac{l}{c\tau}\right)^2 = \frac{E^2 - (mc^2)^2}{E^2} \frac{E^2}{(mc^2)^2},$$

which leads to the result

$$E = mc^2 \sqrt{1 + \left(\frac{l}{c\tau}\right)^2}.$$

1.5. As the Lorentz force is $\boldsymbol{f} = q\boldsymbol{v} \times \boldsymbol{B}$, the power $P = \boldsymbol{f} \cdot \boldsymbol{v} = 0$, and therefore, the kinetic energy T is constant since $P = \mathrm{d}T/\mathrm{d}t = 0$. As $T = (\gamma - 1)mc^2$, it implies that γ is constant. It follows from Newton's law that

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \gamma m \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\boldsymbol{v} \times \boldsymbol{B}.$$

Setting $\omega = q|\boldsymbol{B}|/(m\gamma)$, the equation above implies for $\boldsymbol{B}$ along the z-axis that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \omega\frac{\mathrm{d}y}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\omega\frac{\mathrm{d}x}{\mathrm{d}t}, \quad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = 0.$$

Thus, $z(t) = z(0) + v_z(0)t$. Setting $u = x + iy$, the evolution of u is governed by

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + i\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -i\omega\left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right) = -i\omega\frac{\mathrm{d}u}{\mathrm{d}t}.$$

Therefore, after an integration

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)_{t=0} e^{-i\omega t} = [v_x(0) + iv_y(0)]e^{-i\omega t},$$

and we conclude with another integration that

$$u(t) = \frac{i}{\omega}[v_x(0) + iv_y(0)]\left(e^{-i\omega t} - 1\right) + x(0) + iy(0).$$

The coordinates $x(t)$ and $y(t)$ are obtained by taking the real part and imaginary part of $u(t)$, respectively, yielding

$$\begin{aligned} x(t) &= x(0) + v_y(0)/\omega + [v_x(0)\sin(\omega t) - v_y(0)\cos(\omega t)]/\omega, \\ y(t) &= y(0) - v_x(0)/\omega + [v_y(0)\sin(\omega t) + v_x(0)\cos(\omega t)]/\omega. \end{aligned}$$

1.6. For small angles, $\phi \sim L/R$. Similarly, $R \sim s + R\cos(\phi/2)$. Therefore,

$$s = R\left(1 - \cos\frac{\phi}{2}\right) \sim R\frac{\phi^2}{8} = \frac{R}{8}\left(\frac{L}{R}\right)^2 = \frac{L^2}{8R}.$$

Since for a charge $|q| = 1$, $|\boldsymbol{p}|\cos\lambda = 0.3|\boldsymbol{B}|R$ [Eq. (1.30) in the book] and the pitch angle λ satisfies $\cos\lambda = |\boldsymbol{p}_\perp|/|\boldsymbol{p}|$ [see Eq. (1.28)], we deduce

$$|\boldsymbol{p}_\perp| = \frac{0.3|\boldsymbol{B}|L^2}{8s},$$

leading to the relative uncertainty $\sigma(|\boldsymbol{p}_\perp|)/|\boldsymbol{p}_\perp| = \sigma(s)/s = \sigma(s)\,8|\boldsymbol{p}_\perp|/(0.3BL^2)$.

1.7. The negative log-likelihood of measurements distributed according to a Gaussian law

$$G(r_i\,; \mu, \sigma_{r_i}) = \frac{1}{\sqrt{2\pi}\sigma_{r_i}} e^{-\frac{1}{2}\left(\frac{r_i-\mu}{\sigma_{r_i}}\right)^2}$$

is

$$\mathcal{L} = \frac{1}{2}\sum_i\left(\frac{r_i - \mu}{\sigma_{r_i}}\right)^2 + \text{constant}.$$

The best value of the average $\hat{\mu}$ is obtained for the minimum of $\mathcal{L}$, i.e.

$$\frac{\partial\mathcal{L}}{\partial\mu} = \sum_i \frac{r_i}{\sigma_{r_i}^2} - \hat{\mu}\sum_i\frac{1}{\sigma_{r_i}^2} = 0,$$

yielding

$$\hat{\mu} = \frac{\sum_i \frac{r_i}{\sigma_{r_i}^2}}{\sum_i \frac{1}{\sigma_{r_i}^2}}.$$

1.8. 1. The result is straightforward given that $N_{\mathrm{decay}}(t) = N(t) - N(t+\Delta t) = N_0\left(1 - e^{-\gamma\Delta t}\right)e^{-\gamma t}$.

2. If the number of counts n_k, i.e. the number of decays between t_k and $t_k + \Delta t$, has a standard deviation σ_k, the standard deviation of the random variable $\ln(n_k)$ satisfies

$$\sigma^2_{\ln n_k} = \left|\frac{\partial \ln(n_k)}{\partial n_k}\right|^2 \sigma_k^2 = \frac{1}{n_k^2}\sigma_k^2 = \frac{1}{n_k},$$

since $\sigma_k^2 = n_k$.

3. The χ^2 variable is defined by

$$\chi^2 = \sum_{k=1}^{5}\left(\frac{\ln n_k + \gamma t_k - \alpha}{\sigma_{\ln n_k}}\right)^2 = \sum_{k=1}^{5} n_k(\ln n_k + \gamma t_k - \alpha)^2,$$

where $t_k = (k-1)\Delta t$. The variables γ and α can be considered independent (or equivalently, γ and N_0). Therefore, we minimise χ^2 with respect to these variables, i.e.

$$\frac{\partial\chi^2}{\partial\alpha} = -2\sum_k n_k(\ln n_k + \gamma t_k - \alpha) = 0,$$
$$\frac{\partial\chi^2}{\partial\gamma} = 2\sum_k n_k(\ln n_k + \gamma t_k - \alpha)t_k = 0.$$

The substitution of α from the first equation into the second yields

$$\gamma = \frac{\left(\sum_k n_k \ln n_k\right)\left(\sum_k n_k t_k\right) - \left(\sum_k n_k\right)\left(\sum_k n_k \ln n_k t_k\right)}{\left(\sum_k n_k\right)\left(\sum_k n_k t_k^2\right) - \left(\sum_k n_k t_k\right)^2}$$
$$= \frac{1}{\Delta T}\frac{\left(\sum_k n_k \ln n_k\right)\left(\sum_k n_k(k-1)\right) - \left(\sum_k n_k\right)\left(\sum_k n_k \ln n_k(k-1)\right)}{\left(\sum_k n_k\right)\left(\sum_k n_k(k-1)^2\right) - \left(\sum_k n_k(k-1)\right)^2}.$$

Using the values of the table, one finds $\gamma = 0.039$ or equivalently $\tau = 1/\gamma = 25.45$, which is close to the actual value.

2 Preliminary concepts: special relativity and quantum mechanics

2.1. The result is immediate since $x' \cdot x' = x'_\sigma x'^\sigma = g_{\rho\sigma} x'^\rho x'^\sigma$. Given that $x'^\sigma = \Lambda^\sigma{}_\nu x^\nu$ and $x'^\rho = \Lambda^\rho{}_\mu x^\mu$, it follows that $x' \cdot x' = g_{\rho\sigma} \Lambda^\rho{}_\mu x^\mu \Lambda^\sigma{}_\nu x^\nu$, while $x \cdot x$ is obviously $g_{\mu\nu} x^\mu x^\nu$.

2.2. Eq. (2.14), i.e. $g_{\rho\sigma} \Lambda^\sigma{}_\nu \Lambda^\rho{}_\mu = g_{\mu\nu}$, is equivalent to $\Lambda_{\rho\nu} \Lambda^\rho{}_\mu = g_{\mu\nu}$. The multiplication by $g^{\nu\sigma}$ yields

$$
\begin{aligned}
g^{\nu\sigma} \Lambda_{\rho\nu} \Lambda^\rho{}_\mu &= g^{\nu\sigma} g_{\mu\nu} \\
\Lambda_\rho{}^\sigma \Lambda^\rho{}_\mu &= \delta^\sigma_\mu .
\end{aligned}
$$

By definition, the inverse of Λ satisfies

$$\left(\Lambda^{-1}\right)^\sigma{}_\rho \Lambda^\rho{}_\mu = \delta^\sigma_\mu .$$

Therefore, we conclude

$$\left(\Lambda^{-1}\right)^\sigma{}_\rho = \Lambda_\rho{}^\sigma . \tag{S2.1}$$

2.3. Let us consider a Lorentz transformation along the x-axis,

$$
\begin{aligned}
ct &= \gamma ct' + \beta\gamma x', \\
x &= \beta\gamma ct' + \gamma x', \\
y &= y', \\
z &= z'.
\end{aligned}
$$

As $\mathrm{d}\Omega = c\,\mathrm{d}t\,\mathrm{d}V = c\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, its expression as a function of the transformed variables is

$$\mathrm{d}\Omega = \begin{vmatrix} \frac{\mathrm{d}t}{\mathrm{d}t'} & \frac{\mathrm{d}t}{\mathrm{d}x'} \\ \frac{\mathrm{d}x}{\mathrm{d}t'} & \frac{\mathrm{d}x}{\mathrm{d}x'} \end{vmatrix} c\,\mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \begin{vmatrix} \gamma & \beta\gamma/c \\ \beta\gamma c & \gamma \end{vmatrix} c\,\mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$$

The Jacobian (i.e. the determinant above) is thus $\gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1$. Therefore,

$$\mathrm{d}\Omega = c\,\mathrm{d}t\,\mathrm{d}V = c\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = c\,\mathrm{d}t'\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z',$$

showing that $\mathrm{d}\Omega$ is a Lorentz scalar.

2.4. As 1 b = 10^{-24} cm^2, 1 μb = 10^{-34} m^2. A cross-section σ expressed in natural units (in GeV^{-2}) would be $\sigma/(10^9 e)^2$ j^{-2}. The quantity $\hbar c$ has the units j $\cdot$ m. Therefore, σ in μb is obtained by

$$\sigma\,(\mu\mathrm{b}) = \left(\frac{\hbar c}{10^9 e}\right)^2 \times 10^{-34} \sigma\,(\mathrm{GeV}^{-2}) \simeq 389\,\sigma\,(\mathrm{GeV}^{-2}).$$

2.5. The law of transformation of velocities is given in Eq. (2.24). Let us apply it to particle

(1), i.e. $\boldsymbol{w} = \boldsymbol{v_1}$ in the rest frame of the particle (2), i.e. $\boldsymbol{\beta} = \boldsymbol{v_2}/c$. This yields the velocity of particle (1) in the rest frame of the particle (2),

$$\boldsymbol{v_1}^{(2)} = \frac{\boldsymbol{v_1} + (\gamma - 1)(\boldsymbol{v_2} \cdot \boldsymbol{v_1})\frac{\boldsymbol{v_2}}{|\boldsymbol{v_2}|^2} - \boldsymbol{v_2}\gamma}{\gamma\left(1 - \frac{\boldsymbol{v_2}\cdot\boldsymbol{v_1}}{c^2}\right)}.$$

It follows that

$$\boldsymbol{v_1}^{(2)} \cdot \boldsymbol{v_2} = \frac{\boldsymbol{v_1} \cdot \boldsymbol{v_2} - |\boldsymbol{v_2}|^2}{1 - \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{c^2}} = \frac{(\boldsymbol{v_1} - \boldsymbol{v_2}).\boldsymbol{v_2}}{1 - \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{c^2}}, \quad \boldsymbol{v_1}^{(2)} \times \boldsymbol{v_2} = \frac{\boldsymbol{v_1} \times \boldsymbol{v_2}}{\gamma\left(1 - \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{c^2}\right)}.$$

As

$$|\boldsymbol{v_1}^{(2)}|^2 = \left(\frac{\boldsymbol{v_1}^{(2)} \cdot \boldsymbol{v_2}}{|\boldsymbol{v_2}|}\right)^2 + \left(\frac{\boldsymbol{v_1}^{(2)} \times \boldsymbol{v_2}}{|\boldsymbol{v_2}|}\right)^2 = \frac{1}{\left(1 - \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{c^2}\right)^2}\left[\left(\frac{(\boldsymbol{v_1} - \boldsymbol{v_2}) \cdot \boldsymbol{v_2}}{|\boldsymbol{v_2}|}\right)^2 + \frac{(\boldsymbol{v_1} \times \boldsymbol{v_2})^2}{\gamma^2|\boldsymbol{v_2}|^2}\right]$$

and $\gamma^2 = 1/(1 - |\boldsymbol{v_2}|^2/c^2)$, we conclude

$$|\boldsymbol{v_1}^{(2)}|^2 = \frac{1}{\left(1 - \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{c^2}\right)^2}\left[((\boldsymbol{v_1} - \boldsymbol{v_2}) \cdot \boldsymbol{v_2})^2 + \frac{(\boldsymbol{v_1} \times \boldsymbol{v_2})^2}{c^2}(c^2 - |\boldsymbol{v_2}|^2)\right]\frac{1}{|\boldsymbol{v_2}|^2}. \tag{S2.2}$$

Using the identity given in the text with $\boldsymbol{a} = \boldsymbol{v_1} - \boldsymbol{v_2}$ and $\boldsymbol{b} = \boldsymbol{v_2}$, we obtain

$$((\boldsymbol{v_1} - \boldsymbol{v_2}) \times \boldsymbol{v_2})^2 = |\boldsymbol{v_1} - \boldsymbol{v_2}|^2|\boldsymbol{v_2}|^2 - ((\boldsymbol{v_1} - \boldsymbol{v_2}) \cdot \boldsymbol{v_2})^2\,,$$

showing that

$$((\boldsymbol{v_1} - \boldsymbol{v_2}) \cdot \boldsymbol{v_2})^2 = |\boldsymbol{v_1} - \boldsymbol{v_2}|^2|\boldsymbol{v_2}|^2 - (\boldsymbol{v_1} \times \boldsymbol{v_2})^2.$$

The insertion of this result into Eq. (S2.2) leads to

$$|\boldsymbol{v_1}^{(2)}|^2 = \frac{(\boldsymbol{v_1} - \boldsymbol{v_2})^2 - \left(\frac{\boldsymbol{v_1}\times\boldsymbol{v_2}}{c}\right)^2}{\left(1 - \frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{c^2}\right)^2}.$$

Hence, the result announced in the text.

2.6. Addition of two spin 1.

1. Looking at Clebsch-Gordan tables, we find

$$|S = 2, S_z = 0\rangle = \frac{1}{\sqrt{6}}|S_{1_z} = +1; S_{2_z} - 1\rangle + \sqrt{\frac{2}{3}}|0; 0\rangle + \frac{1}{\sqrt{6}}| - 1; +1\rangle.$$

2. From the tables we have

$$|S_{1_z} = -1; S_{1_z} = +1\rangle = \frac{1}{\sqrt{6}}|S = 2, S_z = 0\rangle - \frac{1}{\sqrt{2}}|1, 0\rangle + \frac{1}{\sqrt{3}}|0, 0\rangle.$$

   So we form $|2, 0\rangle$ with probability 1/6, $|1, 0\rangle$ with probability 1/2, and $|0, 0\rangle$ with probability 1/3.

3. Since S_z is conserved, we can form
   - a state $|S = 1, S_z = +1\rangle$ from $|S_{1_z} = 0; S_{2_z} = +1\rangle$ or $| + 1; 0\rangle$;
   - a state $|1, 0\rangle$ from $|0; 0\rangle$, $| - 1; +1\rangle$ or $| + 1; -1\rangle$;
   - a state $|1, -1\rangle$ from $|0; -1\rangle$ or $| - 1; 0\rangle$.

3 Collisions and decays

3.1. Without loss of generality, let us first show that $\mathrm{d}^3\boldsymbol{p}/E = \mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z/E$ is Lorentz invariant under a boost in the z-direction. The components in the boosted frame are

$$\begin{aligned} E' &= \gamma E - \beta\gamma p_z, \\ p'_x &= p_x, \\ p'_y &= p_y, \\ p'_z &= -\beta\gamma E + \gamma p_z, \end{aligned}$$

where $E = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$. Therefore,

$$\frac{\mathrm{d}p'_z}{E'} = \frac{\mathrm{d}p'_z}{\mathrm{d}p_z}\frac{\mathrm{d}p_z}{E'} = \left(-\beta\gamma\frac{2p_z}{2E} + \gamma\right)\frac{\mathrm{d}p_z}{E'} = \frac{E'}{E}\frac{\mathrm{d}p_z}{E'} = \frac{\mathrm{d}p_z}{E}.$$

Hence, $\mathrm{d}^3\boldsymbol{p}/E = \mathrm{d}^3\boldsymbol{p}'/E'$. The Lorentz invariant phase space in Eq. (3.3), i.e.

$$\mathrm{d}\Phi = \delta^{(4)}(\Delta p)\prod_{k=1}^{n_f}\frac{\mathrm{d}^3\boldsymbol{p}'_k}{(2\pi)^3 2E'_k},$$

also includes a delta function ensuring 4-momentum conservation, denoted here $\delta^{(4)}(\Delta p)$. From the Fourier transform in Eq. (E.9),

$$\delta^{(4)}(\Delta p) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\, e^{i\Delta p\cdot x}.$$

The term in the exponential is a 4-scalar, while d^4x is the 4-volume, already shown to be invariant in Problem 2.3. Therefore, $\mathrm{d}\Phi$ is indeed Lorentz invariant.

3.2. The check is straightforward: in the centre-of-mass frame, when masses are neglected, the incoming particles have 4-momentum $p_1 = (|\boldsymbol{p}^*|, \boldsymbol{p}^*)$ and $p_2 = (|\boldsymbol{p}^*|, -\boldsymbol{p}^*)$. The 4-momentum of the outgoing particles $p_3 = (|\boldsymbol{p}'^*|, \boldsymbol{p}'^*)$ and $p_4 = (|\boldsymbol{p}'^*|, -\boldsymbol{p}'^*)$ are constrained by the energy conservation, which imposes

$$|\boldsymbol{p}^*| + |\boldsymbol{p}^*| = |\boldsymbol{p}'^*| + |\boldsymbol{p}'^*|,$$

i.e. $|\boldsymbol{p}'^*| = |\boldsymbol{p}^*|$.Therefore, $s = (p_1 + p_2)^2 = 4|\boldsymbol{p}^*|^2$, while

$$t = (p_1 - p_3)^2 = -2p_1\cdot p_3 = -2(|\boldsymbol{p}^*|^2 - \boldsymbol{p}^*\cdot\boldsymbol{p}'^*) = -2(|\boldsymbol{p}^*|^2 - |\boldsymbol{p}^*|^2\cos\theta),$$

Similarly,

$$u = (p_1 - p_4)^2 = -2p_1\cdot p_4 = -2(|\boldsymbol{p}^*|^2 + \boldsymbol{p}^*\cdot\boldsymbol{p}'^*) = -2(|\boldsymbol{p}^*|^2 + |\boldsymbol{p}^*|^2\cos\theta).$$

3.3. This exercise is a simple application of formulas such as in Eq (3.50), i.e.

$$\begin{aligned}\mathrm{d}\Phi(P \to p_1 p_2 p_3 p_4) &= \tfrac{dm_{12}^2}{2\pi}\ \mathrm{d}\Phi(p_{12} \to p_1 p_2)\ \mathrm{d}\Phi(P \to p_{12} p_3 p_4) \\ &= \tfrac{dm_{12}^2}{2\pi}\ \mathrm{d}\Phi(p_{12} \to p_1 p_2) \tfrac{dm_{34}^2}{2\pi}\ \mathrm{d}\Phi(p_{34} \to p_3 p_4)\ \mathrm{d}\Phi(P \to p_{12} p_{34}).\end{aligned}$$

3.4. Let us evaluate the sign of $\delta = \left(\sum_{k=1}^N p_k\right)^2 - \left(\sum_{k=1}^N m_k\right)^2$. Denoting the 4-momentum $p_k = (E_k, \boldsymbol{p_k})$ with $|\boldsymbol{p_k}| = \sqrt{E_k^2 - m_k^2}$, we have

$$\left(\sum_{k=1}^N p_k\right)^2 = \sum_{k=1}^N m_k^2 + 2\sum_{k'>k} p_k \cdot p_{k'}, \qquad \left(\sum_{k=1}^N m_k\right)^2 = \sum_{k=1}^N m_k^2 + 2\sum_{k'>k} m_k m_{k'}.$$

Therefore, $\delta = 2\sum_{k'>k} \delta_{kk'}$ with

$$\delta_{kk'} = E_k E_{k'} - \boldsymbol{p_k} \cdot \boldsymbol{p_{k'}} - m_k m_{k'} = E_k E_{k'} - m_k m_{k'} - \sqrt{E_k^2 - m_k^2}\sqrt{E_{k'}^2 - m_{k'}^2}\cos\theta_{kk'},$$

where $\theta_{kk'}$ is the angle between $\boldsymbol{p_k}$ and $\boldsymbol{p_{k'}}$. As $-\cos\theta_{kk'} \geq -1$,

$$\delta_{kk'} \geq E_k E_{k'} - m_k m_{k'} - \sqrt{E_k^2 - m_k^2}\sqrt{E_{k'}^2 - m_{k'}^2}.$$

The right-hand side is the difference between two positive terms, respectively, $A = E_k E_{k'} - m_k m_{k'}$ and $B = \sqrt{E_k^2 - m_k^2}\sqrt{E_{k'}^2 - m_{k'}^2}$. If $A^2 \geq B^2$, then $A \geq B$ since the square root function is an increasing function and A, B are positive quantities. In such a case, $\delta_{kk'}$ would be larger than $A - B \geq 0$. Therefore, we determine the sign of $A^2 - B^2$:

$$\begin{aligned}A^2 - B^2 &= E_k^2 E_{k'}^2 + m_k^2 m_{k'}^2 - 2E_k E_{k'} m_k m_{k'} - (E_k^2 - m_k^2)(E_{k'}^2 - m_{k'}^2) \\ &= -2E_k E_{k'} m_k m_{k'} + E_k^2 m_{k'}^2 + E_{k'}^2 m_k^2 \\ &= (E_k m_{k'} - E_{k'} m_k)^2 \\ &> 0\end{aligned}$$

In conclusion, $\delta_{kk'} \geq 0$, which implies $\delta \geq 0$.

3.5. π^+ decay in $\mu^+\nu_\mu$.

1. The decay is isotropic in the rest frame (it is true only for $1 \to 2$ decays). Therefore, we have $\mathrm{d}N/\,\mathrm{d}\Omega^* = N_0/4\pi$ constant. Integrating over ϕ gives $\mathrm{d}N/\,\mathrm{d}\cos\theta^* = N_0/2$.
2. Using Eqs. (3.32) and (3.33) with $s = m_\pi^2$ and neglecting the neutrino mass yields

$$E_\mu^* = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}, \quad E_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = |\boldsymbol{p^*}|.$$

   With the numerical values, we obtain $E_\mu^* \simeq 110\ \mathrm{MeV}/c^2$ and $|\boldsymbol{p^*}| \simeq 30$ MeV.
3. Let $\mathcal{R}$ be the lab frame and $\mathcal{R}^*$ the rest frame of π^+. We start from

$$E_\mu = \gamma_{\mathcal{R}/\mathcal{R}^*}\left(E_\mu^* - \boldsymbol{\beta}_{\mathcal{R}/\mathcal{R}^*} \cdot \boldsymbol{p_\mu^*}\right).$$

   The lab frame is boosted *backwards* with respect to the rest frame of π^+, so we have $\boldsymbol{\beta}_{\mathcal{R}/\mathcal{R}^*} = -\beta\, \boldsymbol{e_z}$ if we define $\boldsymbol{e_z}$ as the unit vector along the π^+ momentum in $\mathcal{R}$. As a result

$$E_\mu = \gamma\left(E_\mu^* + \beta\, p_\mu^* \cos\theta^*\right). \tag{S3.1}$$

The minimal energy is obtained for backward muons ($\cos\theta^* = -1$) and is about 168 MeV; the maximal energy is for forward muons ($\cos\theta^* = +1$) and is about 272 MeV.

4. All quantities are constant in Eq. (S3.1). so we simply get

$$\frac{\mathrm{d}N}{\mathrm{d}E_\mu} = \frac{\mathrm{d}N}{\mathrm{d}\cos\theta^*}\frac{\mathrm{d}\cos\theta^*}{\mathrm{d}E_\mu} = \frac{N_0}{2}\frac{1}{\gamma\beta p_\mu^*} \simeq 5.7\ \mathrm{MeV}^{-1}.$$

To draw it in a 1-MeV binning, we would expect $\sim$ 5.7 events per bin, for all bins between 168 and 272 MeV. However, the number of events should be *integers*! Therefore, it will look like Poisson fluctuations around a mean value of 5.7 events per bin.

4 Conservation rules and symmetries

4.1. Forbidden reactions.

(1) $p + \bar{p} \to \gamma$ is excluded by 4-momentum conservation since in the centre-of-mass frame of the initial state, $m_p = \sqrt{m_p^2 + |\boldsymbol{p_p}|^2} + |-\boldsymbol{p_p}|$ cannot be satisfied because necessarily the photon momentum $\boldsymbol{p_\gamma} = -\boldsymbol{p_p}$ cannot vanish.
(2) $n \to p + \gamma$ is forbidden by the electric charge conservation.
(3) $\Lambda^0 \to \pi^+ + e^- + \bar{\nu}_e$ violates the baryon number conservation.
(4) $K^- \to \pi^0 + e^-$ violates the lepton number conservation.
(5) $p \to n + e^+ + \nu_e$ is kinematically forbidden since $m_p < m_n$
(6) $\gamma \to e^+ + e^-$ is excluded by 4-momentum conservation.
(7) $\pi^0 \to \gamma\gamma\gamma$ does not conserve the charge-conjugation parity (which is not violated by the electromagnetic interaction).
(8) $p + \nu_\mu \to n + \mu^+$ violates the lepton number conservation ($L_{\nu_\mu} = +1$ whereas $L_{\mu^+} = -1$).
(9) $p + \bar{p} \to \Lambda^0 + \Lambda^0$ violates the baryon number conservation.

4.2. Reaction $\pi^- + p \to \Delta^0 \to \pi^0 + n$.

1. The total angular momentum projection is conserved during the whole process so $J_z^{\rm i} = S_z^{\Delta} = J_z^{\rm f}$. For the initial particles, $J_z^{\rm i} = S_z^{p} + S_z^{\pi^-} + L_z$. Pions are spinless so $S_z^{\pi^-} = 0$. More tricky is the value of L_z. Even if we could show, as in the next question, that $L = 1$, we can prove in full generality that $L_z = 0$. Indeed,
   - "classically": given that $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$, we know that $\boldsymbol{L}$ is orthogonal to $\boldsymbol{p}$. By definition $\boldsymbol{p}$ is aligned with the z-axis so we can only have a 0 projection on that axis;
   - with spherical harmonics: the only possible values are for the total orbital momentum are $L = 1$ or $L = 2$. Looking at the expression of spherical harmonics in Table 2.1, p. 68, we see that the only $Y_L^m(\theta, \phi)$ that are non-zero at $\theta = 0$ are Y_1^0 and Y_2^0. Therefore, we necessarily have $L_z \equiv m = 0$.

   Finally we conclude $J_z^{\rm i} = S_z^{p} = \pm 1/2$.
2. Looking at the Δ^0, the total angular momentum has to be $\boldsymbol{3/2}$. For the final particles, we have to combine $\boldsymbol{S_\pi} = \boldsymbol{0}$, $\boldsymbol{S_n} = \boldsymbol{1/2}$ and the orbital angular momentum, $\boldsymbol{l}$, to form a 3/2 spin representation. The constraint $\boldsymbol{0} \otimes \boldsymbol{1/2} \otimes \boldsymbol{l} = \boldsymbol{1/2} \otimes \boldsymbol{l} = \boldsymbol{3/2}$ requires $\boldsymbol{l} = \boldsymbol{1}$ or $\boldsymbol{2}$. Looking now at parity conservation, we have $\eta_\Delta = \eta_\pi \eta_n (-1)^l$. Therefore, L has to be an odd number, i.e. $l = 1$.
3. We are in the $|3/2, +1/2\rangle$ state which decomposes into

$$|3/2, +1/2\rangle = \frac{1}{\sqrt{3}}|L_z = +1; S_z^n = -1/2\rangle + \sqrt{\frac{2}{3}}|L_z = 0; S_z^n = +1/2\rangle$$

   (the pion is spinless, so only the orbital momentum and the neutron spin matter). We thus have $L_z = +1$ with probability 1/3 and $L_z = 0$ with probability 2/3. Now, we

use the spherical harmonics to get the angular distribution of the neutron: recall that $|Y_L^m(\theta, \phi)|^2$ gives the probability density to have the particle at (θ, ϕ).
– case $L_z = +1$: we use Y_1^1 and the probability density is $|Y_1^1|^2 = 3\sin^2\theta/(8\pi)$;
– case $L_z = 0$: we use Y_1^0 and the probability density is $|Y_1^0|^2 = 3\cos^2\theta/(4\pi)$.

The angular distribution is finally given by

$$|Y_1^1|^2 \times 1/3 + |Y_1^0|^2 \times 2/3 = \frac{1 + 3\cos^2\theta}{8\pi}.$$

4. If $S_z(\Delta^0) = -1/2$, we have the same situation as the previous question if we reverse the z-axis. It corresponds to the change $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. We see that the probability density is not changed. We thus expect the same result.

5 From wave functions to quantum fields

5.1. The check is straightforward using the explicit representation in Eq. (5.30).

5.2. It is immediate: the insertion of the wave function $\psi_i(x) = v_i(p)e^{+ip\cdot x}$ into the Dirac equation leads to

$$(i\gamma^\mu \partial_\mu - m)v_i(p)e^{+ip\cdot x} = 0 \Leftrightarrow (i\gamma^\mu(ip_\mu) - m)v_i(p)e^{+ip\cdot x} = 0$$
$$(\gamma^\mu p_\mu + m)v_i(p) = 0.$$

5.3. The transformation in Eq. (5.40) that must be applied to spinors under the Lorentz transformation with the transformation parameters $\omega_{\sigma\rho}$ involves the matrices $S^{\sigma\rho}$ given in Eq. (5.39), where, as usual, the Einstein notation is used. As explained in Appendix D, the boost parameters corresponding to a rapidity $\boldsymbol{\zeta} = \zeta\boldsymbol{p}/|\boldsymbol{p}|$ corresponds to $\omega_{0i} = \zeta_i$ [see Eq. (D.26)]. Since ω_{0i} is antisymmetric, i.e. $\omega_{0i} = -\omega_{i0}$, as well as S^{0i}, the expression of the transformation (5.40) takes the form

$$S(\Lambda) = \exp\left(-\frac{i}{2}\omega_{0i}S^{0i} - \frac{i}{2}\omega_{i0}S^{i0}\right) = \exp\left(-i\omega_{0i}S^{0i}\right) = \exp\left(\frac{1}{2}\zeta\frac{p_i}{|\boldsymbol{p}|}\gamma^0\gamma^i\right),$$

where $S^{0i} = \frac{i}{2}\gamma^0\gamma^i$ has been used. Inserting the expression of the γ matrices in the Dirac representation, it follows that

$$\frac{p_i}{|\boldsymbol{p}|}\gamma^0\gamma^i = \frac{p_i}{|\boldsymbol{p}|}\begin{pmatrix} \mathbb{0} & \sigma^i \\ \sigma^i & \mathbb{0} \end{pmatrix} = \begin{pmatrix} \mathbb{0} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbb{0} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} S(\Lambda) &= \exp\left[\frac{\zeta}{2}\begin{pmatrix} \mathbb{0} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbb{0} \end{pmatrix}\right] \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{\zeta}{2|\boldsymbol{p}|}\right)^{2k}\begin{pmatrix} \mathbb{0} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & \mathbb{0} \end{pmatrix}^{2k} + \frac{1}{(2k+1)!}\left(\frac{\zeta}{2|\boldsymbol{p}|}\right)^{2k+1}\begin{pmatrix} \mathbb{0} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & \mathbb{0} \end{pmatrix}^{2k+1}. \end{aligned}$$

Given the properties of the Pauli matrices, it is easy to check that

$$\begin{pmatrix} \mathbb{0} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & \mathbb{0} \end{pmatrix}^{2k} = |\boldsymbol{p}|^{2k}\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix}, \qquad \begin{pmatrix} \mathbb{0} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & \mathbb{0} \end{pmatrix}^{2k+1} = |\boldsymbol{p}|^{2k+1}\begin{pmatrix} \mathbb{0} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbb{0} \end{pmatrix}$$

[see for instance Eq. (5.57)]. We conclude that

$$\begin{aligned} S(\Lambda) &= \cosh\left(\frac{\zeta}{2}\right)\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix} + \sinh\left(\frac{\zeta}{2}\right)\begin{pmatrix} \mathbb{0} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbb{0} \end{pmatrix} \\ &= \cosh\left(\frac{\zeta}{2}\right)\begin{pmatrix} \mathbb{1} & \tanh\left(\frac{\zeta}{2}\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \\ \tanh\left(\frac{\zeta}{2}\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} & \mathbb{1} \end{pmatrix}. \end{aligned}$$

According to Eq. (2.44),

$$\cosh\left(\frac{\zeta}{2}\right) = \sqrt{\frac{\cosh\zeta + 1}{2}} = \sqrt{\frac{\gamma + 1}{2}} = \sqrt{\frac{E+m}{2m}},$$
$$\tanh\left(\frac{\zeta}{2}\right) = \frac{\sinh\left(\frac{\zeta}{2}\right)\cosh\left(\frac{\zeta}{2}\right)}{\cosh^2\left(\frac{\zeta}{2}\right)} = \frac{\sinh\zeta}{\cosh\zeta + 1} = \frac{\beta\gamma}{\gamma+1} = \frac{|\boldsymbol{p}|}{E+m},$$

where $\gamma = E/m$ and $\beta\gamma = |\boldsymbol{p}|/m$ have been used. Therefore,

$$S(\Lambda) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \mathbb{1} & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} & \mathbb{1} \end{pmatrix}$$
$$= \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_x - ip_y}{E+m} \\ 0 & 1 & \frac{p_x+ip_y}{E+m} & -\frac{p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_x-ip_y}{E+m} & 1 & 0 \\ \frac{p_x+ip_y}{E+m} & -\frac{p_z}{E+m} & 0 & 1 \end{pmatrix}.$$

The spinors for a particle at rest are given by Eq. (5.52) up to the normalisation defined on page 139, i.e. $\pm\sqrt{E+m} = \pm\sqrt{2m}$ for a particle at rest. They read

$$\psi_1 = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \psi_2 = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \psi_3 = -\sqrt{2m}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \psi_4 = \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

The application of matrix $S(\Lambda)$ yields the spinors defined in Eq. (5.63).

5.4. Covariance of the Dirac equation.

1. $[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C]$. and similarly with anti-commutators.
2. If $\mu = \nu$, $S^{\mu\nu} = 0$. Therefore $[S^{\mu\nu}, S^{\sigma\rho}] = 0$. On the other hand, the right-hand side of the Lie algebra then reads
$$i(g^{\mu\sigma}S^{\mu\rho} - g^{\mu\sigma}S^{\mu\rho} + g^{\mu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\mu\sigma}) = 0.$$
The Lie algebra is thus trivially satisfied. The same conclusion is obviously obtained if $\sigma = \rho$.
3. When $\sigma \neq \rho$, $S^{\sigma\rho} = \frac{i}{2}\gamma^\sigma\gamma^\rho$. Applying the identity of the first question leads to
$$[S^{\mu\nu}, S^{\sigma\rho}] = \frac{i}{2}[S^{\mu\nu}, \gamma^\sigma\gamma^\rho] = \frac{i}{2}\left([S^{\mu\nu}, \gamma^\sigma]\gamma^\rho + \gamma^\sigma[S^{\mu\nu}, \gamma^\rho]\right).$$
4. Fort $\mu \neq \nu$, $S^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu$
$$\begin{aligned}[S^{\mu\nu}, S^{\sigma\rho}] &= -\tfrac{1}{4}([\gamma^\mu\gamma^\nu, \gamma^\sigma]\gamma^\rho + \gamma^\sigma[\gamma^\mu\gamma^\nu, \gamma^\rho]) \\ &= \tfrac{1}{4}([\gamma^\sigma, \gamma^\mu\gamma^\nu]\gamma^\rho + \gamma^\sigma[\gamma^\rho, \gamma^\mu\gamma^\nu]) \\ &= \tfrac{1}{4}(\{\gamma^\sigma, \gamma^\mu\}\gamma^\nu\gamma^\rho - \gamma^\mu\{\gamma^\sigma, \gamma^\nu\}\gamma^\rho \\ &\qquad \gamma^\sigma\{\gamma^\rho, \gamma^\mu\}\gamma^\nu - \gamma^\sigma\gamma^\mu\{\gamma^\rho, \gamma^\nu\}) \\ &= \tfrac{1}{2}(g^{\sigma\mu}\gamma^\nu\gamma^\rho - g^{\sigma\nu}\gamma^\mu\gamma^\rho + g^{\rho\mu}\gamma^\sigma\gamma^\nu - g^{\rho\nu}\gamma^\sigma\gamma^\mu).\end{aligned}$$

As

$$\gamma^a\gamma^b = \gamma^a\gamma^b - \gamma^b\gamma^a + \gamma^b\gamma^a + \gamma^a\gamma^b - \gamma^a\gamma^b = [\gamma^a, \gamma^b] + \{\gamma^a, \gamma^b\} - \gamma^a\gamma^b,$$

using the Clifford algebra (5.29), we deduce the identity (with $a, b = 0, 1, 2, 3$)

$$\gamma^a\gamma^b = \frac{1}{2}[\gamma^a, \gamma^b] + \frac{1}{2}\{\gamma^a, \gamma^b\} = \frac{1}{2}[\gamma^a, \gamma^b] + g^{ab}.$$

It follows that

$$\begin{aligned}[S^{\mu\nu}, S^{\sigma\rho}] = \tfrac{1}{2}\big(&g^{\sigma\mu}\tfrac{1}{2}[\gamma^\nu, \gamma^\rho] + g^{\sigma\mu}g^{\nu\rho} - g^{\sigma\nu}\tfrac{1}{2}[\gamma^\mu, \gamma^\rho] - g^{\sigma\nu}g^{\mu\rho} \\ &+ g^{\rho\mu}\tfrac{1}{2}[\gamma^\sigma, \gamma^\nu] + g^{\rho\mu}g^{\sigma\nu} - g^{\rho\nu}\tfrac{1}{2}[\gamma^\sigma, \gamma^\mu] - g^{\rho\nu}g^{\sigma\mu}\big). \\ = -i(&g^{\sigma\mu}S^{\nu\rho} - g^{\sigma\nu}S^{\mu\rho} + g^{\rho\mu}S^{\sigma\nu} - g^{\rho\nu}S^{\sigma\mu}),\end{aligned}$$

where the symmetry of the g tensor has been used. Since $S^{\mu\nu}$ is antisymmetric, the previous equation yields the Lie algebra $\mathfrak{so}(1,3)$ of the Lorentz group, i.e.

$$[S^{\mu\nu}, S^{\sigma\rho}] = i(g^{\nu\sigma}S^{\mu\rho} - g^{\mu\sigma}S^{\nu\rho} + g^{\mu\rho}S^{\nu\sigma} - g^{\nu\rho}S^{\mu\sigma}).$$

5.5. Bilinear forms. Under a Lorentz transformation, we saw in Supplement 5.3 that

$$x \to x' = \Lambda^\mu{}_\nu\, x, \qquad \psi(x) \to \psi'(x') = S(\Lambda)\psi(x), \qquad \overline{\psi}(x) \to \overline{\psi'}(x') = \overline{\psi}(x)S(\Lambda)^{-1},$$

where the matrix $S(\Lambda)$ is

$$S(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\sigma\rho}S^{\sigma\rho}\right), \qquad S^{\sigma\rho} = \frac{i}{4}[\gamma^\sigma, \gamma^\rho].$$

1. $\overline{\psi}(x)\gamma^5\psi(x)$: As γ^5 anti-commutes with any γ^μ matrix, it commutes with $S^{\sigma\rho}$ and thus with $S(\Lambda)$, giving

$$\overline{\psi'}(x')\gamma^5\psi(x') = \overline{\psi}(x)S(\Lambda)^{-1}\gamma^5 S(\Lambda)\psi(x) = \overline{\psi}(x)S(\Lambda)^{-1}S(\Lambda)\gamma^5\psi(x) = \overline{\psi}(x)\gamma^5\psi(x).$$

This is the behaviour of a Lorentz scalar.

2. $\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x)$:

$$\overline{\psi'}(x')\gamma^5\gamma^\mu\psi(x') = \overline{\psi}(x)S(\Lambda)^{-1}\gamma^5\gamma^\mu S(\Lambda)\psi(x) = \overline{\psi}(x)\gamma^5 S(\Lambda)^{-1}\gamma^\mu S(\Lambda)\psi(x),$$

which, using the constraint (5.36), p. 131, becomes

$$\overline{\psi'}(x')\gamma^5\gamma^\mu\psi(x') = \overline{\psi}(x)\gamma^5\Lambda^\mu{}_\nu\gamma^\nu\psi(x) = \Lambda^\mu{}_\nu\,\overline{\psi}(x)\gamma^5\gamma^\nu\psi(x).$$

This shows that $\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x)$ is a Lorentz vector.

3. $\overline{\psi}(x)\gamma^\mu\gamma^\nu\psi(x)$:

$$\begin{aligned}\overline{\psi'}(x')\gamma^\mu\gamma^\nu\psi'(x') &= \overline{\psi}(x)S(\Lambda)^{-1}\gamma^\mu\gamma^\nu S(\Lambda)\psi(x) \\ &= \overline{\psi}(x)S(\Lambda)^{-1}\gamma^\mu S(\Lambda)S(\Lambda)^{-1}\gamma^\nu S(\Lambda)\psi(x) \\ &= \overline{\psi}(x)\Lambda^\mu{}_\sigma\gamma^\sigma\Lambda^\nu{}_\rho\gamma^\rho\psi(x) \\ &= \Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho\overline{\psi}(x)\gamma^\sigma\gamma^\rho\psi(x),\end{aligned}$$

where Eq. (5.36) has been used. The bilinear form $\overline{\psi}(x)\gamma^\mu\gamma^\nu\psi(x)$ is thus a rank-2 tensor.

Under the parity transformation, we have

$$x = (t, \boldsymbol{x}) \to x' = (t, -\boldsymbol{x}), \qquad \psi(x) \to \gamma^0\psi(x), \qquad \overline{\psi}(x) \to \overline{\psi'}(x') = \overline{\gamma^0\psi}(x) = \overline{\psi}(x)\gamma^0.$$

4. $\overline{\psi}(x)\gamma^5\psi(x)$ transforms into

$$\overline{\psi'}(x')\gamma^5\psi'(x') = \overline{\psi}(x)\gamma^0\gamma^5\gamma^0\psi(x) = -\overline{\psi}(x)\gamma^5\gamma^0\gamma^0\psi(x) = -\overline{\psi}(x)\gamma^5\psi(x).$$

This is thus a pseudo-scalar.

5. $\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x)$ transforms into

$$\overline{\psi'}(x')\gamma^5\gamma^\mu\psi'(x') = \overline{\psi}(x)\gamma^0\gamma^5\gamma^\mu\gamma^0\psi(x) = -\overline{\psi}(x)\gamma^5\gamma^0\gamma^\mu\gamma^0\psi(x).$$

For $\mu = 0$, $\gamma^0\gamma^\mu\gamma^0 = \gamma^\mu$, while for $\mu \neq 0$, $\gamma^0\gamma^\mu\gamma^0 = -\gamma^\mu\gamma^0\gamma^0 = -\gamma^\mu$. Therefore

$$\overline{\psi'}(x')\gamma^5\gamma^\mu\psi'(x') = \begin{cases} -\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x), \; \mu = 0 \\ +\overline{\psi}(x)\gamma^5\gamma^\mu\psi(x), \; \mu \neq 0 \end{cases}$$

This is the behaviour of a pseudo-vector.

5.6. Determination of the helicity states.

1. By definition of the helicity eigenstates, we have $\hat{h}\psi_\lambda = \lambda\psi_\lambda$, with $\psi_\lambda = u_\lambda e^{-ip\cdot x}$, and where

$$\hat{h} = \hat{\boldsymbol{S}}.\frac{\hat{\boldsymbol{P}}}{|\boldsymbol{p}|} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma}.\frac{\hat{\boldsymbol{P}}}{|\boldsymbol{p}|} & \mathbb{0} \\ \mathbb{0} & \boldsymbol{\sigma}.\frac{\hat{\boldsymbol{P}}}{|\boldsymbol{p}|} \end{pmatrix}.$$

In the helicity operator, $\hat{\boldsymbol{P}} = -i\boldsymbol{\nabla}$ acts on $e^{-ip.x}$ as $\hat{\boldsymbol{P}}e^{-ip.x} = \boldsymbol{p}e^{-ip.x}$. Looking at the upper bi-spinor of u_λ, we then obtain

$$\frac{1}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\phi_\lambda = \lambda\phi_\lambda.$$

2. In polar coordinates we have $\boldsymbol{p} = p\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, so that

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} = \begin{pmatrix} \cos\theta & \sin\theta\cdot e^{-i\phi} \\ \sin\theta\cdot e^{i\phi} & -\cos\theta \end{pmatrix}.$$

For $\lambda = +1/2$, we have to solve

$$\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} - \mathbb{1}\right)\phi_{\frac{1}{2}} = 2\begin{pmatrix} -\sin^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot e^{-i\varphi} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot e^{i\phi} & -\cos^2\frac{\theta}{2} \end{pmatrix}\phi_{\frac{1}{2}} = 0,$$

from which we can check that

$$\phi_{\frac{1}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

is a solution.

3. We just need to find the expression for the lower part of u_λ. From the eigenvalue equation, we know that $\boldsymbol{\sigma}.\boldsymbol{p}\;\phi_\lambda = 2\lambda|\boldsymbol{p}|\phi_\lambda$ with $\lambda = 1/2$, so simply

$$\frac{\boldsymbol{\sigma}.\boldsymbol{p}}{E+m}\,\phi_{\frac{1}{2}} = \frac{|\boldsymbol{p}|}{E+m}\,\phi_{\frac{1}{2}}.$$

4. Let us make as few computations as possible. First, note that the eigenstates are defined up to a phase and that $\hat{h} \to -\hat{h}$ under parity.

- $\psi_{-\frac{1}{2}}$: we can apply the parity transformation on $\phi_{\frac{1}{2}}$ to get $\phi_{-\frac{1}{2}}$. Parity transformation reads $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. Therefore,

$$\phi_{-\frac{1}{2}} = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix}.$$

 For the lower component, we proceed as in question 3 with $\lambda = -1/2$. We end up with the result in Eq. (5.74) of the book on p. 144, up to a sign.

- $\psi_{+\bar{\frac{1}{2}}}$ and $\psi_{-\bar{\frac{1}{2}}}$: the general formula of a v-spinor is

$$v_\lambda = \sqrt{E+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}.\boldsymbol{p}}{E+m}\chi_\lambda \\ \chi_\lambda \end{pmatrix}.$$

 For the helicity eigenstates of antiparticles, we start again from $\hat{h}\psi_\lambda = \lambda\psi_\lambda$, but now, we apply $\hat{\boldsymbol{P}}$ to $e^{+ip.x}$ as $\hat{\boldsymbol{P}}e^{+ip.x} = -\boldsymbol{p}e^{+ip.x}$. As a result, the eigenvalue equation reads

$$\boldsymbol{\sigma}.\boldsymbol{p}\,\chi_\lambda = -2\lambda|\boldsymbol{p}|\chi_\lambda.$$

 We can simply take $\chi_{\frac{1}{2}} = \phi_{-\frac{1}{2}}$ and $\chi_{-\frac{1}{2}} = \phi_{\frac{1}{2}}$.

5.7. For $\boldsymbol{p} = p\boldsymbol{e}_z$, as $\boldsymbol{\epsilon}(\boldsymbol{p}, i)\cdot\boldsymbol{p} = 0$, the set of 4-polarisation vectors (5.129) is obviously reduced to Eq, (5.122). Then, denoting the 4-polarisation vectors in the basis (5.129) with primed symbols, we have $\epsilon'^\mu(\lambda) = \Lambda^\mu{}_\sigma\epsilon^\sigma(\lambda)$. In this basis, the left-hand side of Eq. (5.128) reads

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon'^\mu(\lambda)\epsilon'^\nu(\lambda)^* = \Lambda^\mu{}_\sigma\left(\Lambda^\nu{}_\rho\right)^*\sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon^\sigma(\lambda)\epsilon^\rho(\lambda)^* = \Lambda^\mu{}_\sigma\left(\Lambda^\nu{}_\rho\right)^* g^{\sigma\rho} = \Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho g^{\sigma\rho}$$

since the elements of the SO(1, 3) group (i.e. the Lorentz group) are real matrices. Now, to prove that $\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho g^{\sigma\rho} = g^{\mu\nu}$, we notice that $\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho g^{\sigma\rho} = \Lambda^\mu{}_\sigma g^{\sigma\rho}(\Lambda^\intercal)_\rho{}^\nu$, which in matrices notation reads $\Lambda g\Lambda^\intercal$. Since Λ^{-1} is also a Lorentz transformation, it satisfies the definition (2.14), i.e. $(\Lambda^{-1})^\intercal g\Lambda^{-1} = g$. Let's multiply $A = \Lambda g\Lambda^\intercal$ by $g = (\Lambda^{-1})^\intercal g\Lambda^{-1}$ from the right. It yields

$$Ag = \Lambda g\Lambda^\intercal(\Lambda^{-1})^\intercal g\Lambda^{-1} = \Lambda gg\Lambda^{-1} = \Lambda\Lambda^{-1} = \mathbb{1}.$$

Therefore, $A = g^{-1} = g$, which proves that $\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho g^{\sigma\rho} = g^{\mu\nu}$. We conclude

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon'^\mu(\lambda)\epsilon'^\nu(\lambda)^* = g^{\mu\nu},$$

which shows that the 4-polarisation vectors in the basis (5.129) also satisfy Eq. (5.128).

6 A brief overview of Quantum Electrodynamics

6.1. The Lagrangian (6.15) is reproduced below:

$$\mathcal{L}_D = \sum_{\alpha,\beta} \bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta}\partial_\mu\psi_\beta - m\delta_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta.$$

Applying the Euler-Lagrange equation to ψ_β yields:

$$\left.\begin{array}{l} \frac{\partial\mathcal{L}_D}{\partial\psi_\beta} = \sum_\alpha -m\delta_{\alpha\beta}\bar{\psi}_\alpha \\ \frac{\partial\mathcal{L}_D}{\partial(\partial_\mu\psi_\beta)} = \sum_\alpha \bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta} \end{array}\right\} - m\bar{\psi}_\beta - \sum_\alpha \partial_\mu\bar{\psi}_\alpha i\gamma^\mu_{\alpha\beta} = 0,$$

which is simply the β component of the Dirac adjoint equation (5.43).

6.2. The electromagnetic Lagrangian can be written

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{0\nu}F^{0\nu} - \frac{1}{4}F_{j\nu}F^{j\nu} = -\frac{1}{4}F_{0\nu}F^{0\nu} - \frac{1}{4}F_{j0}F^{j0} - \frac{1}{4}F_{ji}F^{ji},$$

where i and $j = 1, 2$, or 3. As $F_{00} = 0$, $F_{0\nu}$ is reduced to F_{0i}. Since F is antisymmetric $F^{j0} = -F^{0j}$, it follows that

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{0i}F^{0i} - \frac{1}{4}F_{0j}F^{0j} - \frac{1}{4}F_{ji}F^{ji} = -\frac{1}{2}F_{0i}F^{0i} - \frac{1}{4}F_{ji}F^{ji}.$$

Note that $F_{0i} = -F^{0i}$ because we lower one time-component and one spatial-component, whereas $F_{ij} = F^{ij}$ because two spatial-components are lowered. Therefore,

$$\mathcal{L}_\gamma = +\frac{1}{2}\sum_i \left(F^{0i}\right)^2 - \frac{1}{4}\sum_{i,j}\left(F^{ji}\right)^2.$$

Using the components of the electric and magnetic fields given in Eq. (2.40), this expression is equivalent to

$$\mathcal{L}_\gamma = \frac{1}{2}(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2),$$

with $c = 1$.

6.3. Canonical quantisation of the electromagnetic field.

1. After the insertion of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, the Lagrangian (6.20) with $\zeta = 1$ reads

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)^2 \\ &= -\tfrac{1}{4}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \tfrac{1}{4}(\partial_\nu A_\mu)(\partial^\nu A^\mu) + \tfrac{1}{4}(\partial_\mu A_\nu)(\partial^\nu A^\mu) + \tfrac{1}{4}(\partial_\nu A_\mu)(\partial^\mu A^\nu) - \tfrac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu).\end{aligned}$$

As μ and ν are dummy indices, they can be swapped in the second and fourth terms, yielding

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu).$$

But

$$\partial_\mu[A_\nu(\partial^\nu A^\mu)-A^\mu(\partial_\nu A^\nu)] = (\partial_\mu A_\nu)(\partial^\nu A^\mu)+A_\nu\partial_\mu(\partial^\nu A^\mu)-(\partial_\mu A^\mu)(\partial_\nu A^\nu)-A^\mu\partial_\mu(\partial_\nu A^\nu),$$

where the second and fourth terms vanish since

$$A_\nu\partial_\mu(\partial^\nu A^\mu) = A_\nu\partial^\nu\partial_\mu A^\mu = A^\nu\partial_\nu\partial_\mu A^\mu = A^\mu\partial_\mu\partial_\nu A^\nu.$$

Therefore,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu[A_\nu(\partial^\nu A^\mu) - A^\mu(\partial_\nu A^\nu)].$$

2. The second term is a 4-divergence which does not contribute to the action. Therefore, it can be ignored, i.e.

$$\mathcal{L} \to \mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu = -\frac{1}{2}\partial_0 A^\nu\partial^0 A_\nu - \frac{1}{2}\partial_i A^\nu\partial^i A_\nu.$$

The canonically conjugate field Π_ν belonging to A^ν thus reads

$$\Pi_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\nu)} = -\frac{1}{2}\partial^0 A_\nu - \frac{1}{2}(\partial_0 A^\nu)\frac{\partial(\partial^0 A_\nu)}{\partial(\partial_0 A^\nu)} = -\frac{1}{2}\partial^0 A_\nu - \frac{1}{2}(\partial^0 A_\nu)\frac{\partial(\partial_0 A^\nu)}{\partial(\partial_0 A^\nu)} = -\partial^0 A_\nu.$$

Consequently,

$$\left[A_\mu(x), \mathring{A}_\nu(y)\right] = -\left[A_\mu(x), \Pi_\nu(y)\right] = -ig_{\mu\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}),$$
$$\left[\mathring{A}_\mu(x), \mathring{A}_\nu(y)\right] = \left[\Pi_\mu(x), \Pi_\nu(y)\right] = 0.$$

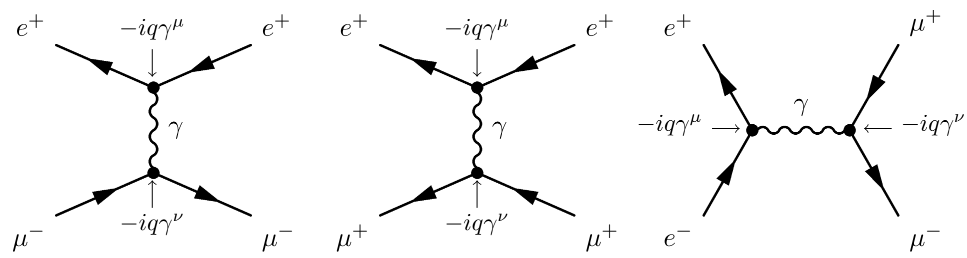
6.4. Trace of the product of four γ matrices.

$$\begin{aligned}
\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\eta\gamma^\nu) &= \mathrm{Tr}(\gamma^\mu\gamma^\eta\gamma^\nu\gamma^\rho) \\
&= -\mathrm{Tr}(\gamma^\mu\gamma^\eta\gamma^\rho\gamma^\nu) + 2g^{\rho\nu}\mathrm{Tr}(\gamma^\mu\gamma^\eta) \\
&= -\mathrm{Tr}(\gamma^\mu\gamma^\eta\gamma^\rho\gamma^\nu) + 8g^{\rho\nu}g^{\mu\eta} \\
&= \mathrm{Tr}(\gamma^\mu\gamma^\rho\gamma^\eta\gamma^\nu) - 8g^{\eta\rho}g^{\mu\nu} + 8g^{\rho\nu}g^{\mu\eta} \\
&= -\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\eta\gamma^\nu) + 8g^{\rho\mu}g^{\eta\nu} - 8g^{\eta\rho}g^{\mu\nu} + 8g^{\rho\nu}g^{\mu\eta},
\end{aligned}$$

and therefore,

$$\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\eta\gamma^\nu) = 4\,(g^{\rho\mu}g^{\eta\nu} - g^{\eta\rho}g^{\mu\nu} + g^{\rho\nu})\,g^{\mu\eta}.$$

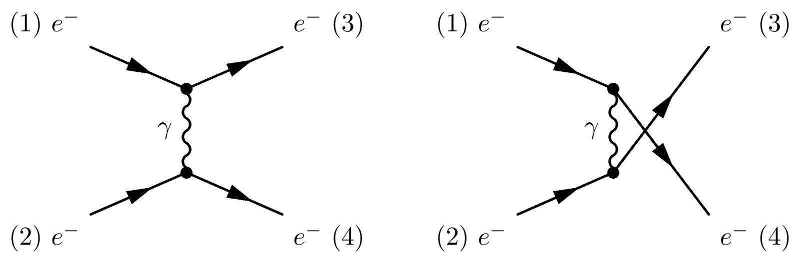
6.5. Feynman diagrams and amplitudes of reactions with $e^\pm$ and $\mu^\pm$. We note in all this exercise $q = -e$ the charge of the *electron* (or the muon), not the positron (nor μ^+). The Feynman diagrams of the first three reactions (at lowest order) are:



1. $e^+(k) + e^-(p) \to \mu^+(k') + \mu^-(p')$.
   It's a t-channel and
$$i\mathcal{M} = \bar{u}_\mu(p') \cdot (-iq\gamma^\mu) \cdot u_\mu(p) \times \frac{-ig_{\mu\nu}}{t + i\varepsilon} \times \bar{v}_e(k) \cdot (-iq\gamma^\nu) \cdot v_e(k'),$$
   where $t = (k - k')^2 = (p' - p)^2$. The amplitude should be a $\mathbb{C}$-number. Therefore, one should always check that all γ matrices are correctly contracted between an adjoint spinor and a spinor.
2. $e^+(k) + \mu^+(p) \to e^+(k') + \mu^+(p')$.
   Same as 1, except for the muon current:
$$i\mathcal{M} = \bar{v}_\mu(p) \cdot (-iq\gamma^\mu) \cdot v_\mu(p') \times \frac{-ig_{\mu\nu}}{t + i\varepsilon} \times \bar{v}_e(k) \cdot (-iq\gamma^\nu) \cdot v_e(k').$$
3. $e^+(k) + e^-(p) \to \mu^+(k') + \mu^-(p')$.
   It's a s-channel and
$$i\mathcal{M} = \bar{v}_e(k) \cdot (-iq\gamma^\mu) \cdot u_e(p) \times \frac{-ig_{\mu\nu}}{s + i\varepsilon} \times \bar{u}_\mu(p') \cdot (-iq\gamma^\nu) \cdot v_\mu(k'),$$
   where $s = (k + p)^2 = (k' + p')^2$.
4. $e^+ + \mu^- \to e^- + \mu^+$.
   Not allowed at any order: the process violates the individual lepton numbers, which are conserved in QED processes. Note that the conservation is ensured by the interaction Lagrangian in QED, which has no term of the kind $\bar{\psi}_f \gamma^\mu \psi_{f'} A_\mu$, where f and f' are two different fermions.

6.6. Reactions with e^+ and e^-.

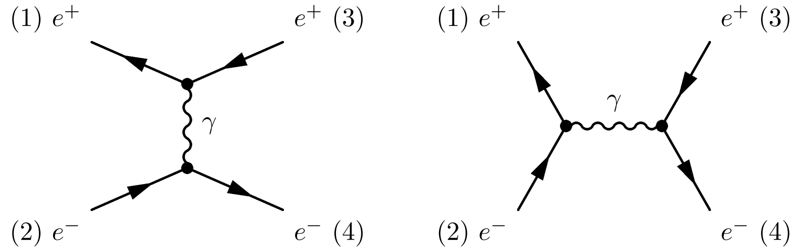
1. $e^- + e^- \to e^- + e^-$ (Møller scattering).
   There are two diagrams:



To check if they are equivalent, look at the *hidden* part (that is, the vertices and propagators – everything but external lines) of the diagrams. If they are different (e.g. the momenta flowing in the propagators are different), then the two diagrams are not equivalent, and we need to add/subtract them. Here we have a t- and a u-channel that have $p_1 - p_3$ and $p_1 - p_4$, respectively, flowing in the photon propagator (assuming a downward flow). We can retrieve one from the other by exchanging the two outgoing electrons so, we put a minus sign between the two amplitudes since fermions are exchanged. Note that a diagram where the incoming electrons would be interchanged would be equivalent to the second diagram since the 4-momentum in the photon propagator would be $p_2 - p_3 = p_4 - p_1$ by 4-momentum conservation (the sign of the 4-momentum in the propagator is physically meaningless since the photon can be seen as going upward or downward).

2. $e^+ + e^- \to e^+ + e^-$ (Bhabba scattering).

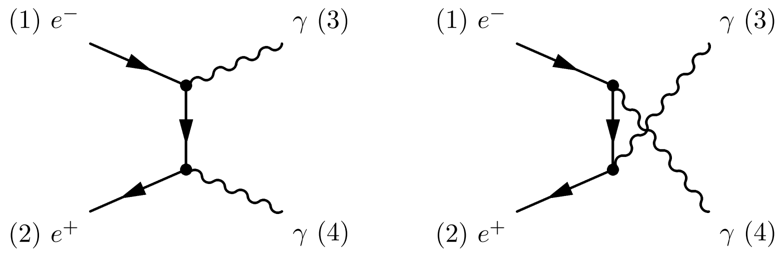
   There are two diagrams:



   An s- and a t-channel. The label (1) and (3) are used for positrons, while (2) and (4) are for electrons. We transform the first diagram into the second by exchanging the outgoing electron (4) with the incoming positron (1). As the exchange concerns fermions, the two amplitudes must be subtracted. Note that the interchange of (3) and (4) should not be considered since (3) would no longer label a positron but an electron. When new diagrams are envisaged, the labelling convention used in the first diagram should always be followed.

3. $e^- + e^+ \to \gamma + \gamma$ (pair annihilation).

   There are two diagrams:



   A t- and a u-channel. We transform one into the other with the exchange of the two outgoing photons. Since they are bosons, the amplitudes must be added.

6.7. To show that the absolute amplitude squared (6.93) of the scattering $e^- + \mu^- \to e^- + \mu^-$,

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}\left[(s - m_e^2 - m_\mu^2)^2 + (u - m_e^2 - m_\mu^2)^2 + 2t(m_e^2 + m_\mu^2)\right],$$

is equivalent to

$$\overline{|\mathcal{M}|^2} = \frac{4e^4}{t^2}\left[(s - m_e^2 - m_\mu^2)(m_e^2 + m_\mu^2 - u) + t(m_e^2 + m_\mu^2) + \frac{t^2}{2}\right],$$

it is easier to start from the second formula,

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}\left[2(s - m_e^2 - m_\mu^2)(m_e^2 + m_\mu^2 - u) + 2t(m_e^2 + m_\mu^2) + t^2\right].$$

Given that $s+t+u=\sum_{i=1}^{4} m_i^2 = 2m_e^2+2m_\mu^2$, the Mandelstam variable t satisfies

$$\begin{aligned} t^2 &= (2m_e^2+2m_\mu^2-s-u)^2 \\ &= (m_e^2+m_\mu^2-s+m_e^2+m_\mu^2-u)^2 \\ &= (m_e^2+m_\mu^2-s)^2+(m_e^2+m_\mu^2-u)^2+2(m_e^2+m_\mu^2-s)(m_e^2+m_\mu^2-u) \\ &= (s-m_e^2-m_\mu^2)^2+(u-m_e^2-m_\mu^2)^2-2(s-m_e^2-m_\mu^2)(m_e^2+m_\mu^2-u). \end{aligned}$$

The insertion of t^2 into the second formula leads to the first.

6.8. Eq. (6.98) is a simple consequence of 4-momentum conservation. In the reaction, $e^- + \mu^- \to e^- + \mu^-$, for an initial muon at rest, 4-momentum conservation imposes

$$\begin{cases} E_e + m_\mu = E_e' + E_\mu', \\ \boldsymbol{p_e} = \boldsymbol{p_e'} + \boldsymbol{p_\mu}. \end{cases}$$

When the electron mass is neglected, the second equation implies

$$\begin{aligned} |\boldsymbol{p_e}-\boldsymbol{p_e'}|^2 &= |\boldsymbol{p_\mu}|^2 \\ E_e^2+E_e'^2-2E_eE_e'\cos\theta &= E_\mu^2-m_\mu^2. \end{aligned}$$

Inserting E_μ from energy conservation yields

$$\begin{aligned} E_e^2+E_e'^2-2E_eE_e'\cos\theta &= (E_e+m_\mu-E_e')^2-m_\mu^2 \\ &= E_e^2+E_e'^2-2E_eE_e'-2m_\mu E_e-2m_\mu E_e'. \end{aligned}$$

It follows that

$$E_e' = \frac{E_e m_\mu}{m_\mu + E_e(1-\cos\theta)}.$$

6.9. As $\not{p}^2 = p^2 = \not{p}'^2 = p'^2 = m^2$ and $k^2=0$, $A = \mathrm{Tr}\left(\not{p}'\gamma^\mu(2p^\nu+\not{k}\gamma^\nu)\not{p}(2p_\nu+\gamma_\nu\not{k})\gamma_\mu\right)$ reads

$$\begin{aligned} A &= 4p^2\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma_\mu\right)+2\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{k}\not{p}^2\gamma_\mu\right)+2\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{p}^2\not{k}\gamma_\mu\right)+\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{k}\gamma^\nu\not{p}\gamma_\nu\not{k}\gamma_\mu\right) \\ &= 4m^2\left[\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma_\mu\right)+\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{k}\gamma_\mu\right)\right]+\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{k}\gamma^\nu\not{p}\gamma_\nu\not{k}\gamma_\mu\right) \end{aligned}$$

The identity (G.6) tells us that $\gamma^\mu\not{p}\gamma_\mu = p_\nu\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu p_\nu = -2\not{p}$, $\gamma^\mu\not{k}\gamma_\mu = -2\not{k}$, and $\gamma^\nu\not{p}\gamma_\nu = -2\not{p}$. Consequently,

$$\begin{aligned} A &= 4m^2\left[-2\mathrm{Tr}\left(\not{p}'\gamma^\nu\right)(p_\nu+k_\nu)\right]-2p_\rho\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{k}\gamma^\rho\not{k}\gamma_\mu\right) \\ &= -8m^2\left[4p'^\nu(p_\nu+k_\nu)\right]+4p_\rho\mathrm{Tr}\left(\not{p}'\not{k}\gamma^\rho\not{k}\right), \end{aligned}$$

where the identity (6.85) and (G.8) have been used. Therefore, with the help of Eq. (6.89), it follows that

$$\begin{aligned} A &= -32m^2(p'\cdot p+p'\cdot k)+4\mathrm{Tr}\left(\not{p}'\not{k}\not{p}\not{k}\right) \\ &= -32m^2(p'\cdot p+p'\cdot k)+16(p'\cdot k\, p\cdot k-p'\cdot p\, k^2+p'\cdot k\, k\cdot p) \\ &= 32[-m^2p'\cdot p+p'\cdot k(p\cdot k-m^2)]. \end{aligned}$$

Similarly, $B = \mathrm{Tr}(\gamma^\mu(2p^\nu+\not{k}\gamma^\nu)(2p_\nu+\gamma_\nu\not{k})\gamma_\mu)$ becomes

$$\begin{aligned} B &= 4p^2\mathrm{Tr}(\gamma^\mu\gamma_\mu)+2\mathrm{Tr}(\gamma^\mu\not{p}\not{k}\gamma_\mu)+2\mathrm{Tr}(\gamma^\mu\not{k}\not{p}\gamma_\mu)+\mathrm{Tr}(\gamma^\mu\not{k}\gamma^\nu\gamma_\nu\not{k}\gamma_\mu) \\ &= 16m^2\mathrm{Tr}(\mathbb{1})+8p\cdot k\mathrm{Tr}(\mathbb{1})+8k\cdot p\mathrm{Tr}(\mathbb{1})+4\mathrm{Tr}(\gamma^\mu\not{k}^2\gamma_\mu) \\ &= 64m^2+64p\cdot k. \end{aligned}$$

6.10. For $Q^2 = -q^2 \gg m^2$,

$$\begin{aligned}
f(Q^2) &= 6\int_0^1 \mathrm{d}z\, z(1-z)\ln\left[1+\tfrac{Q^2}{m^2}z(1-z)\right] \\
&= 6\int_0^1 \mathrm{d}z\, z(1-z)\ln\left[\tfrac{Q^2}{m^2}z(1-z)\left(1+\tfrac{1}{\frac{Q^2}{m^2}z(1-z)}\right)\right] \\
&\simeq 6\int_0^1 \mathrm{d}z\, z(1-z)\ln\left[\tfrac{Q^2}{m^2}z(1-z)\right] \\
&= 6\int_0^1 \mathrm{d}z\, z(1-z)\ln\left(\tfrac{Q^2}{m^2}\right) + 6\int_0^1 \mathrm{d}z\, z(1-z)\ln\left[z(1-z)\right] \\
&= 6\times\tfrac{1}{6}\ln\left(\tfrac{Q^2}{m^2}\right) + 6\int_0^1 \mathrm{d}z\, z(1-z)\ln z + 6\int_0^1 \mathrm{d}z\, z(1-z)\ln(z-1) \\
&= \ln\left(\tfrac{Q^2}{m^2}\right) + 6\times\tfrac{-5}{36} + 6\times\tfrac{-5}{36} \\
&= \ln\left(\tfrac{Q^2}{m^2}\right) - \tfrac{5}{3}.
\end{aligned}$$

Therefore,

$$f(Q^2) - f(Q_0^2) = \ln\left(\frac{Q^2}{m^2}\right) - \ln\left(\frac{Q_0^2}{m^2}\right) = \ln\left(\frac{Q^2}{Q_0^2}\right),$$

and Eq. (6.115) becomes Eq. (6.116).

6.11. The ultra-relativistic regime of the reaction $e^-(p) + e^+(k) \to \mu^-(p') + \mu^+(k')$.

1. In the ultra-relativistic regime, helicity and chirality are the same since masses are neglected. In Section 6.5, p. 207, for a reaction described by an s-channel diagram (which is the case, see Fig. 6.6), we listed the allowed configurations in Fig. 6.10. Only four amplitudes can contribute to the process, i.e.

$$\mathcal{M}_{LR\to LR}, \quad \mathcal{M}_{LR\to RL}, \quad \mathcal{M}_{RL\to LR}, \quad \mathcal{M}_{RL\to RL},$$

   where $\mathcal{M}_{ab\to cd}$ denotes the amplitude for e^- with the helicity (or chirality) a, e^+ with b, μ^- with c and μ^+ with d.

2. The s-channel amplitude reads

$$\begin{aligned}
i\mathcal{M} &= \bar{v}_{\mathrm{e}}(k)\cdot(-iq\gamma^\mu)\cdot u_{\mathrm{e}}(p)\times\tfrac{-ig_{\mu\nu}}{s+i\epsilon}\times\bar{u}_{\mathrm{mu}}(p')\cdot(-iq\gamma^\nu)\cdot v_{\mathrm{mu}}(k') \\
&= \tfrac{ig_{\mu\nu}}{s}\,(j^\mu)_{\mathrm{e}}\,(j^\nu)_{\mathrm{mu}},
\end{aligned} \tag{S6.1}$$

   with $s = (p+k)^2$, $(j^\mu)_{\mathrm{e}} = q\bar{v}_{\mathrm{e}}(k)\gamma^\mu u_{\mathrm{e}}(p)$, and $(j^\nu)_{\mathrm{mu}} = q\bar{u}_{\mathrm{mu}}(p')\gamma^\nu v_{\mathrm{mu}}(k')$. Let us use the centre-of-mass frame and orient the z-axis in the direction of propagation of the electron. As the initial particles are the same, so is the energy $E_{e^-} = E_{e^+} = \sqrt{s}/2$. Similarly the final particles are the same, so $E_{\mu^-} = E_{\mu^+} = \sqrt{s}/2$. As a result, all leptons have the same energy $E = \sqrt{s}/2$. Since we neglect all masses, all leptons have the momentum $|\boldsymbol{p}_\ell| = E$.

   Let us start with the amplitude $\mathcal{M}_{LR\to LR}$. Using the spinor formula (5.96), rotation invariance allows to set $\phi = 0$. For the electron $(\theta_{e^-}, \phi_{e^-}) = (0, 0)$ while for the positron $(\theta_{e^+}, \phi_{e^+}) = (\pi, 0)$, giving, respectively, for $u_{\mathrm{e}}(p)$ and $v_{\mathrm{e}}(k)$,

$$u_{L\mathrm{e}} \approx u_{-\frac{1}{2}} \approx \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_{R\mathrm{e}} \approx v_{+\frac{1}{2}} \approx \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}. \tag{S6.2}$$

For the muon, for simplicity, we orient the x-axis in the muon direction of propagation. Hence, we have $(\theta_{\mu^-}, \phi_{\mu^-}) = (\theta, 0)$. The anti-muon is emitted back-to-back so we take $(\theta_{\mu^+}, \phi_{\mu^+}) = (\pi - \theta_{\mu^-}, \pi + \phi_{\mu^-}) = (\pi - \theta, \pi)$. Then, the spinors read, respectively, for $u_{\text{mu}}(p')$ and $v_{\text{mu}}(k')$,

$$u_{L\text{mu}} \approx u_{-\frac{1}{2}} \approx \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}, \qquad v_{R\text{mu}} \approx v_{+\frac{1}{2}} \approx \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}, \tag{S6.3}$$

using $s \equiv \sin\frac{\theta}{2}$ and $c \equiv \cos\frac{\theta}{2}$.

To compute the electron current $(j^\mu)_{\text{e}}^{LR} = q(v_{Re}^\dagger \gamma^0 \gamma^\mu u_{Le})$ for $\mu = 0 \dots 3$, note that $\gamma^0\gamma^\mu$ are

$$(\gamma^0)^2 = \mathbb{1}, \quad \gamma^0\gamma^i = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix},$$

namely

$$\gamma^0\gamma^1 = \begin{pmatrix} &&&1 \\ &&1& \\ &1&& \\ 1&&& \end{pmatrix}, \quad \gamma^0\gamma^2 = i\begin{pmatrix} &&&-1 \\ &&1& \\ &-1&& \\ 1&&& \end{pmatrix}, \quad \gamma^0\gamma^3 = \begin{pmatrix} &&1& \\ &&&-1 \\ 1&&& \\ &-1&& \end{pmatrix}.$$

This yields $(j^0)_{\text{e}}^{LR} = (j^3)_{\text{e}}^{LR} = 0$, $(j^1)_{\text{e}}^{LR} = -2qE$, $(j^2)_{\text{e}}^{LR} = 2iqE$, i.e.

$$(j^\mu)_{\text{e}}^{LR} = q(v_{R\text{e}}^\dagger \gamma^0 \gamma^\mu u_{L\text{e}}) = qE\begin{pmatrix} 0 \\ -2 \\ 2i \\ 0 \end{pmatrix} = q\sqrt{s}\begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix}.$$

Similarly, we find

$$(j^\mu)_{\text{mu}}^{LR} = q(u_{L\text{mu}}^\dagger \gamma^0 \gamma^\mu v_{R\text{mu}}) = qE\begin{pmatrix} 0 \\ -2\cos\theta \\ -2i \\ 2\sin\theta \end{pmatrix} = q\sqrt{s}\begin{pmatrix} 0 \\ -\cos\theta \\ -i \\ \sin\theta \end{pmatrix}.$$

Therefore, given that $q = -e$, the amplitude is

$$\begin{aligned} \mathcal{M}_{LR\to LR} &= \frac{g_{\mu\nu}}{s}\,(j^\mu)_{\text{e}}^{LR}\,(j^\nu)_{\text{mu}}^{LR} \\ &= \frac{1}{s}\Big((j^0)_{\text{e}}^{LR}\,(j^0)_{\text{mu}}^{LR} - (j^1)_{\text{e}}^{LR}\,(j^1)_{\text{mu}}^{LR} - (j^2)_{\text{e}}^{LR}\,(j^2)_{\text{mu}}^{LR} - (j^3)_{\text{e}}^{LR}\,(j^3)_{\text{mu}}^{LR}\Big) \\ &= -e^2(\cos\theta + 1). \end{aligned} \tag{S6.4}$$

For the three other amplitudes, no need to redo the whole calculation. Since parity is conserved by QED processes, the amplitude of the parity transformed process is equal to the original amplitude up to a phase. The parity transformation inverts helicity. Therefore, we should have

$$\mathcal{M}_{RL\to RL} = \eta_{RL}\mathcal{M}_{LR\to LR}, \qquad \mathcal{M}_{RL\to LR} = \eta_{LR}\mathcal{M}_{LR\to RL},$$

with $|\eta_{LR}| = |\eta_{RL}| = 1$. Hence, only $\mathcal{M}_{LR\to RL}$ needs to be evaluated, imposing to re-evaluate the muon current for the RL helicities. Following the same approach as before, one finds

$$u_{R\text{mu}} \approx u_{+\frac{1}{2}} \approx \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}, \qquad v_{L\text{mu}} \approx v_{-\frac{1}{2}} \approx \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}. \tag{S6.5}$$

This yields

$$(j^\mu)^{RL}_{\text{mu}} = q(u^\dagger_{R\text{mu}}\gamma^0\gamma^\mu v_{L\text{mu}}) = q\sqrt{s}\begin{pmatrix} 0 \\ -\cos\theta \\ i \\ \sin\theta \end{pmatrix},$$

and thus

$$\mathcal{M}_{LR\to RL} = \frac{g_{\mu\nu}}{s}\,(j^\mu)^{LR}_{\text{e}}\,(j^\nu)^{RL}_{\text{mu}} = e^2(1-\cos\theta). \tag{S6.6}$$

3. The four processes differ by observable quantities. Therefore, there are no interferences. We then have

$$\begin{aligned}\overline{|\mathcal{M}_{\text{tot}}|^2} &= \frac{1}{4}\left(|\mathcal{M}_{LR\to LR}|^2 + |\mathcal{M}_{RL\to RL}|^2 + |\mathcal{M}_{LR\to RL}|^2 + |\mathcal{M}_{RL\to LR}|^2\right) \\ &= \frac{e^4}{2}\left[(1+\cos\theta)^2 + (1-\cos\theta)^2\right] \\ &= e^4\left(1+\cos^2\theta\right).\end{aligned}$$

4. Using the Mandelstam variables,

$$\begin{aligned} s &\equiv (p+k)^2 = 4E^2 = 4|\boldsymbol{p}|^2, \\ t &\equiv (p-p')^2 = -(\boldsymbol{p}-\boldsymbol{p}')^2 = -2|\boldsymbol{p}|^2(1-\cos\theta), \\ u &\equiv (p-k')^2 = -(\boldsymbol{p}-\boldsymbol{k}')^2 = -2|\boldsymbol{p}|^2(1+\cos\theta),\end{aligned}$$

it follows that $(1+\cos\theta)^2 = 4u^2/s^2$ and $(1-\cos\theta)^2 = 4t^2/s^2$, and thus

$$\overline{|\mathcal{M}_{\text{tot}}|^2} = 2e^4\frac{t^2+u^2}{s^2}.$$

5. Starting from Eq. (S6.1), we have

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\left[\bar{v}_{\text{e}}(k)\gamma^\mu u_{\text{e}}(p)\bar{u}_{\text{mu}}(p')\gamma_\mu v_{\text{mu}}(k')\right]^*\left[\bar{v}_{\text{e}}(k)\gamma^\nu u_{\text{e}}(p)\bar{u}_{\text{mu}}(p')\gamma_\nu v_{\text{mu}}(k')\right].$$

Therefore, the unpolarised amplitude squared is

$$\overline{|\mathcal{M}_{\text{tot}}|^2} = \frac{e^4}{s^2}L^{\mu\nu}(\text{e})\text{L}_{\mu\nu}(\text{mu}),$$

with

$$\begin{aligned} L^{\mu\nu}(\text{e}) &= \tfrac{1}{2}\textstyle\sum_{r,r'}\left[\bar{v}_{\text{e}}(k)\gamma^\mu u_{\text{e}}(p)\right]^*\left[\bar{v}_{\text{e}}(k)\gamma^\nu u_{\text{e}}(p)\right] = \tfrac{1}{2}\text{Tr}\left[\not{p}\gamma^\mu\not{k}\gamma^\nu\right], \\ L_{\mu\nu}(\text{mu}) &= \tfrac{1}{2}\textstyle\sum_{s,s'}\left[\bar{u}_{\text{mu}}(p')\gamma_\mu v_{\text{mu}}(k')\right]^*\left[\bar{u}_{\text{mu}}(p')\gamma_\nu v_{\text{mu}}(k')\right] = \tfrac{1}{2}\text{Tr}\left[\not{k}'\gamma_\mu\not{p}'\gamma_\nu\right],\end{aligned}$$

where the trace results from Eq. (6.83), when masses are neglected. Now, the trace of the product of four γ matrices is given by Eq. (6.88). This yields

$$\overline{|\mathcal{M}_{\rm tot}|^2} = \frac{e^4}{s^2} 8\left[(p \cdot k')(k' \cdot p') + (p \cdot p')(k \cdot k')\right].$$

Since $t = -2p \cdot p' = -2k \cdot k'$ and $u = -2k \cdot p' = -2k' \cdot p$, it follows that

$$\overline{|\mathcal{M}_{\rm tot}|^2} = 2e^4 \frac{t^2 + u^2}{s^2}.$$

6.12. The non-relativistic Hamiltonian in the presence of a magnetic field reads

$$\hat{H} = \frac{(\hat{p} - q\boldsymbol{A})^2}{2m} = \frac{\hat{p}^2}{2m} - \frac{q}{m}\boldsymbol{A} \cdot \hat{\boldsymbol{p}} + \frac{q^2 A^2}{2m} \simeq \frac{\hat{p}^2}{2m} - \frac{q}{m}\boldsymbol{A} \cdot \hat{\boldsymbol{p}}$$

where the term in q^2 is neglected. We identify the non-relativistic kinetic energy $\hat{p}^2/(2m)$ and the additional energy given by

$$E_B = -\frac{q}{m}\boldsymbol{A} \cdot \hat{\boldsymbol{p}}.$$

But, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{A}$ can be chosen satisfying the gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$. Therefore $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$ and E_B become

$$E_B = -\frac{q}{2m}(\boldsymbol{B} \times \boldsymbol{r}) \cdot \hat{\boldsymbol{p}} = -\frac{q}{2m}(\boldsymbol{r} \times \hat{\boldsymbol{p}}) \cdot \boldsymbol{B} = -\frac{q}{2m}\hat{\boldsymbol{L}} \cdot \boldsymbol{B} = -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B},$$

where the circular shift property of the scalar triple product has been applied in the second step. We finally find the expected expression for the magnetic moment $\hat{\boldsymbol{\mu}}$.

6.13. Starting from the definition of

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right),$$

we have for $k, j = 1, 2, 3$,

$$\begin{aligned}
\sigma^{kj} &= \tfrac{i}{2}\left[\begin{pmatrix} \mathbb{0} & \sigma^k \\ -\sigma^k & \mathbb{0} \end{pmatrix}\begin{pmatrix} \mathbb{0} & \sigma^j \\ -\sigma^j & \mathbb{0} \end{pmatrix} - \begin{pmatrix} \mathbb{0} & \sigma^j \\ -\sigma^j & \mathbb{0} \end{pmatrix}\begin{pmatrix} \mathbb{0} & \sigma^k \\ -\sigma^k & \mathbb{0} \end{pmatrix}\right] \\
&= \tfrac{i}{2}\begin{pmatrix} -(\sigma^k\sigma^j - \sigma^j\sigma^k) & \mathbb{0} \\ \mathbb{0} & -(\sigma^k\sigma^j - \sigma^j\sigma^k) \end{pmatrix} \\
&= \tfrac{i}{2}\begin{pmatrix} -2i\sum_l \epsilon_{kjl}\sigma^l & \mathbb{0} \\ \mathbb{0} & -2i\sum_l \epsilon_{kjl}\sigma^l \end{pmatrix} \\
&= \textstyle\sum_l \epsilon_{kjl}\begin{pmatrix} \sigma^l & \mathbb{0} \\ \mathbb{0} & \sigma^l \end{pmatrix}
\end{aligned},$$

where ϵ_{kjl} is the usual antisymmetric tensor appearing in the commutator of the Pauli matrices ($\epsilon_{kjl} = 1$ for cyclic permutation of 123, $= -1$ for anti-cyclic permutation, 0 otherwise). Similarly,

$$\begin{aligned}
\sigma^{0j} &= \tfrac{i}{2}\left[\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{0} & \sigma^j \\ -\sigma^j & \mathbb{0} \end{pmatrix} - \begin{pmatrix} \mathbb{0} & \sigma^j \\ -\sigma^j & \mathbb{0} \end{pmatrix}\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}\right] \\
&= i\begin{pmatrix} \mathbb{0} & \sigma^j \\ \sigma^j & \mathbb{0} \end{pmatrix}.
\end{aligned}$$

6.14. In Eq. (6.124), given that in the low energy limit the spinor $u \simeq \binom{u_a}{0}$, the term $\overline{u_f}\sigma^{kj}u_i\partial_j A_k$ is explicitly

$$\overline{u_f}\sigma^{kj}u_i\partial_j A_k = \sum_l \epsilon_{kjl}(u^\dagger_{a_f}, 0)\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \sigma^l & \mathbb{0} \\ \mathbb{0} & \sigma^l \end{pmatrix}\begin{pmatrix} u_{a_i} \\ 0 \end{pmatrix}\partial_j A_k = \sum_l \epsilon_{kjl}\, u^\dagger_{a_f}\sigma^l u_{a_i}\partial_j A_k. \tag{S6.7}$$

Let us focus, for example, on the component $k = 1$ of A_k. It is contracted by

$$\bar{u}_f\sigma^{1j}u_i\partial_j = \bar{u}_f\sigma^{11}\partial_1 u_i + \bar{u}_f\sigma^{12}\partial_2 u_i + \bar{u}_f\sigma^{13}\partial_3 u_i,$$

which according to Eq. (S6.7) is

$$\begin{aligned} \bar{u}_f\sigma^{1j}u_i\partial_j &= \textstyle\sum_l \epsilon_{11l}\, u^\dagger_{a_f}\sigma^l\partial_1 u_{a_i} + \sum_l \epsilon_{12l}\, u^\dagger_{a_f}\sigma^l\partial_2 u_{a_i} + \sum_l \epsilon_{13l}\, u^\dagger_{a_f}\sigma^l\partial_3 u_{a_i} \\ &= u^\dagger_{a_f}\left(\textstyle\sum_l \epsilon_{11l}\sigma^l\partial_1 + \sum_l \epsilon_{12l}\sigma^l\partial_2 + \sum_l \epsilon_{13l}\sigma^l\partial_3\right)u_{a_i}. \end{aligned}$$

The properties of the antisymmetric tensor ϵ are such that $\epsilon_{11l} = 0$, and ϵ_{12l} and ϵ_{13l} are non-zero only for $l = 3$ and 2, respectively. Therefore,

$$\begin{aligned} \bar{u}_f\sigma^{1j}u_i\partial_j &= u^\dagger_{a_f}\left(\sigma^3\partial_2 - \sigma^2\partial_3\right)u_{a_i} \\ &= -u^\dagger_{a_f}\,[\boldsymbol{\sigma}\times\boldsymbol{\partial}]^1\,u_{a_i}. \end{aligned}$$

We would have reach a similar result for the other components. Therefore,

$$\overline{u_f}\sigma^{kj}u_i\partial_j A_k = u^\dagger_{a_f}\left(\sum_l \epsilon_{kjl}\,\sigma^l\partial_j A_k\right)u_{a_i} = -u^\dagger_{a_f}\left([\boldsymbol{\sigma}\times\boldsymbol{\partial}]^k A_k\right)u_{a_i}.$$

7 From hadrons to partons

7.1. Spinless scattering: the Rutherford cross-section of $e^-(k) + \mathrm{p}(p) \to e^-(k') + \mathrm{p}(p')$.

1. Using the Feynman rules for scalars, the amplitude is given by a t-channel diagram and reads

$$i\mathcal{M} = \left[-i(-e)(k^\mu + k'^\mu)\right] \frac{-ig_{\mu\nu}}{q^2} \left[-ie(p^\nu + p'^\nu)\right],$$

with $q = k - k'$. Consequently,

$$\mathcal{M} = -\frac{e^2}{q^2}(k^\mu + k'^\mu)(p_\mu + p'_\mu) = -\frac{e^2}{q^2}(k \cdot p + k \cdot p' + k' \cdot p + k' \cdot p').$$

For a proton at rest and neglecting its recoil, energy conservation implies $E + M = E' + M$, so $E = E'$, and then, the 4-momenta are $k = (E, \boldsymbol{k})$, $k' = (E, \boldsymbol{k'})$, $p = p' = (M, \boldsymbol{0})$. Therefore, the scalar products in the amplitude are all equal to EM. When the electron mass is neglected, as

$$q^2 = (k - k')^2 \simeq -2k \cdot k' = -2(E^2 - E^2 \cos\theta) = -4E^2 \sin^2 \frac{\theta}{2},$$

with θ the scattering angle of the electrons, the amplitude finally reads

$$\mathcal{M} = \frac{e^2}{4E^2 \sin^2 \frac{\theta}{2}} 4EM = \frac{4\pi\alpha M}{E \sin^2 \frac{\theta}{2}}.$$

2. Formula (3.44) is appropriate to express differential cross-sections in the lab frame since

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{64\pi s}\frac{\overline{|\mathcal{M}|^2}}{|\boldsymbol{p}^*|^2}\frac{\mathrm{d}t}{\mathrm{d}\Omega}.$$

Since particles are considered spinless, $\overline{|\mathcal{M}|^2} = |\mathcal{M}|^2$. The quantity $\boldsymbol{p}^*$ is the momentum evaluated in the centre-of-mass frame and reads, with the help of Eq. (3.34),

$$|\boldsymbol{p}^*|^2 = \frac{(s - M^2)(s - M^2)}{4s}.$$

As $s = (p + k)^2 = M^2 + 2EM$, $|\boldsymbol{p}^*|^2 = E^2M^2/s$. Moreover, $t = q^2 = -2E^2(1 - \cos\theta)$ implies $\mathrm{d}t = 2E^2\,\mathrm{d}(\cos\theta)$. Given that $\mathrm{d}\Omega = 2\pi\,\mathrm{d}(\cos\theta)$, $\mathrm{d}t = E^2\,\mathrm{d}\Omega/\pi$, it follows that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi s}\frac{s}{E^2M^2}\left(\frac{4\pi\alpha M}{E\sin^2\frac{\theta}{2}}\right)^2 \frac{E^2}{\pi} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}}.$$

This is the well-known Rutherford formula, usually derived using a spinless particle (historically alpha) in a static Coulomb field.

7.2. The Mott cross-section with a spinless proton and spin 1/2 electron, $e^-(k) + \mathrm{p}(p) \to e^-(k') + \mathrm{p}(p')$.

1. The proton current is the same as in Problem 7.1, but this time, the electron current is the Dirac current. The amplitude reads

$$i\mathcal{M} = [\bar{u}_{r'}(k')(ie\gamma^\mu)u_r(k)]\,\frac{-ig_{\mu\nu}}{q^2}\,[-ie(p^\nu + p'^\nu)]\,,$$

with, as before, $q^2 = -4EE'\sin^2(\theta/2)$ since the electron mass is neglected. The unpolarised amplitude squared is now given by

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{q^4}L^{\mu\nu}(p_\mu + p'_\mu)(p_\nu + p'_\nu),$$

where

$$\begin{aligned} L^{\mu\nu} &= \tfrac{1}{2}\textstyle\sum_{r,r'}[\bar{u}_{r'}(k')\gamma^\mu u_r(k)]^*\,[\bar{u}_{r'}(k')\gamma^\mu u_r(k)] \\ &= \tfrac{1}{2}\mathrm{Tr}\,(\not{k}\gamma^\mu\not{k}'\gamma^\nu) \\ &= 2\,(k^\mu k'^\nu - k\cdot k' g^{\mu\nu} + k^\nu k'^\mu)\,, \end{aligned}$$

where the formulas (6.83) and (6.88) have been successively used. Inserting $L^{\mu\nu}$ into $\overline{|\mathcal{M}|^2}$ formula leads to

$$\begin{aligned} \overline{|\mathcal{M}|^2} = \tfrac{e^4}{q^4}[&(k\cdot p)(k'\cdot p) + (k\cdot p)(k'\cdot p') + (k\cdot p')(k'\cdot p) + (k\cdot p')(k'\cdot p') \\ &- M^2(k\cdot k') - (k\cdot k')(p\cdot p')]. \end{aligned}$$

Now, the proton recoil is no longer neglected, but the colliding proton is at rest. Therefore, $k = (E, \boldsymbol{k})$, $k' = (E', \boldsymbol{k}')$, $p = (M, \boldsymbol{0})$ and $p' = k + p - k' = (E + M - E', \boldsymbol{k} - \boldsymbol{k}')$. It follows that

$$k\cdot k' = 2EE'\sin^2\tfrac{\theta}{2},\ p\cdot p' = M(E + M - E'),\ k\cdot p = EM,\ k'\cdot p = E'M,$$

and

$$k\cdot p' = -2EE'\sin^2\tfrac{\theta}{2} + EM,\ k'\cdot p' = 2EE'\sin^2\tfrac{\theta}{2} + E'M.$$

Inserting these scalar products in $\overline{|\mathcal{M}|^2}$ formula yields

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \tfrac{8e^4}{q^4}EE'\left(2M^2 + EM\sin^2\tfrac{\theta}{2} - E'M\sin^2\tfrac{\theta}{2} - 2EE'\sin^4\tfrac{\theta}{2} - 2M^2\sin^2\tfrac{\theta}{2}\right) \\ &= \tfrac{8e^4}{q^4}EE'\left[2M^2\cos^2\tfrac{\theta}{2} + M\sin^2\tfrac{\theta}{2}\left(E - E' - 2\tfrac{EE'}{M}\sin^2\tfrac{\theta}{2}\right)\right]. \end{aligned}$$

However, according to Eq. (7.9),

$$E - E' = 2\frac{EE'}{M}\sin^2\frac{\theta}{2}. \tag{S7.1}$$

Therefore

$$\overline{|\mathcal{M}|^2} = \frac{16e^4}{q^4}EE'M^2\cos^2\frac{\theta}{2},$$

with $q^4 = 16E^2E'^2\sin^4(\theta/2)$.

2. To get the differential cross-section, we proceed as in Problem 7.1, i.e.

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi s}\frac{\overline{|\mathcal{M}|^2}}{|\boldsymbol{p}^*|^2}\frac{dt}{d\Omega}.$$

The momentum $|\boldsymbol{p}^*|^2$ is still given by E^2M^2/s. However, even if t has the same expression, the calculation of $dt/d\Omega$ requires some precautions. This time, E' also depends on θ [see Eq. (S7.1)], so $dt/d\Omega$ is no longer E^2/π. It is easier to calculate $dt/d\Omega$ using

$$t = (k - k')^2 = (p' - p)^2 = 2M^2 - 2p' \cdot p = 2M^2 - 2(E + M - E')M,$$

so that

$$\frac{dt}{d\Omega} = 2M\frac{dE'}{2\pi\, d(\cos\theta)}.$$

Actually, the calculation is exactly the same as that done on page 199 in the book, where the role played by the muon is now played by the proton. This leads to the expression (6.99), which is now

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{\overline{|\mathcal{M}|^2}}{M^2}\left(\frac{E'}{E}\right)^2.$$

The insertion of the expression of $\overline{|\mathcal{M}|^2}$ yields the Mott differential cross-section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}\cos^2\frac{\theta}{2}.$$

7.3. The general elementary cross-section formula applied to the reaction $e^-(k) + \mathrm{p}(p) \to e^-(k') + \mathrm{p}(p')$ with the assumptions $k = (E, \boldsymbol{k})$, $k' = (E', \boldsymbol{k}')$, $|\boldsymbol{k}| = E$ and $|\boldsymbol{k}'| = E'$, $p = (M, \boldsymbol{0})$ is

$$d\sigma = \frac{1}{4|p \cdot k|}(2\pi)^4\delta^{(4)}(p' + k' - p - k)\overline{|\mathcal{M}|^2}\frac{d^3\boldsymbol{p}'}{(2\pi)^3 2p'^0}\frac{d^3\boldsymbol{k}'}{(2\pi)^3 2E'}.$$

Writing $d^3\boldsymbol{k}' = E'^2\, dE'\, d\Omega$ and $p \cdot k = EM$ yields

$$\frac{d\sigma}{dE'\, d\Omega} = \frac{1}{8M(2\pi)^2}\delta^{(4)}(p' + k' - p - k)\overline{|\mathcal{M}|^2}\frac{E'}{E}\frac{d^3\boldsymbol{p}'}{2p'^0}.$$

To integrate over d^4p', we first use Eq. (E.6), i.e.

$$\int_{-\infty}^{+\infty} d^4p'\ \delta(p'^2 - M^2)\theta(p'^0) = \int_{-\infty}^{+\infty}\frac{d^3\boldsymbol{p}'}{2p'^0},$$

yielding

$$\begin{aligned}
\frac{d\sigma}{dE'\, d\Omega} &= \int d^4p'\ \delta(p'^2 - M^2)\theta(p'^0)\frac{1}{8M(2\pi)^2}\delta^{(4)}(p' + k' - p - k)\overline{|\mathcal{M}|^2}\frac{E'}{E}\\
&= \frac{1}{8M(2\pi)^2}\delta((p + q)^2 - M^2)\overline{|\mathcal{M}|^2}\frac{E'}{E}\\
&= \frac{1}{8M(2\pi)^2}\delta(q^2 + 2p \cdot q)\overline{|\mathcal{M}|^2}\frac{E'}{E}\\
&= \frac{1}{8M(2\pi)^2}\delta(q^2 + 2M(E - E'))\overline{|\mathcal{M}|^2}\frac{E'}{E}.
\end{aligned}$$

Finally, using a property of the δ-function, we conclude

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'\,\mathrm{d}\Omega} = \frac{1}{16M^2(2\pi)^2}\delta\left(\frac{q^2}{2M} + E - E'\right)\overline{|\mathcal{M}|^2}\frac{E'}{E}. \tag{S7.2}$$

7.4. With $\psi_{p'}(x) = u(p')e^{-ip'\cdot x}$ and $\psi_p(x) = u(p)e^{-ip\cdot x}$, the expression of the proton current $j^\mu_{\text{f.s.}} = \overline{\psi_{p'}}(x)\Gamma^\mu\psi_p(x)$ is

$$j^\mu_{\text{f.s.}}(x) = e^{iq\cdot x}\bar{u}(p')\left[f_1(q^2)\gamma^\mu + f_2(q^2)(p'+p)^\mu + f_3(q^2)q^\mu\right]u(p),$$

with $q = p' - p$. According to the Gordon decomposition (7.12),

$$\bar{u}(p')\,(p'+p)^\mu\,u(p) = 2M\,\bar{u}(p')\gamma^\mu u(p) - i\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p),$$

where M is the proton mass, the current becomes

$$j^\mu_{\text{f.s.}}(x) = e^{iq\cdot x}\bar{u}(p')\left[\left(f_1(q^2) + 2Mf_2(q^2)\right)\gamma^\mu - if_2(q^2)\sigma^{\mu\nu}q_\nu + f_3(q^2)q^\mu\right]u(p),$$

which, after defining $\mathcal{F}_1$ and $\mathcal{F}_2$, such as $\mathcal{F}_1(q^2) = f_1(q^2) + 2Mf_2(q^2)$ and $f_2(q^2) = -\frac{\kappa}{2M}\mathcal{F}_2(q^2)$, reads

$$j^\mu_{\text{f.s.}}(x) = e^{iq\cdot x}\bar{u}(p')\left[\mathcal{F}_1(q^2)\gamma^\mu + i\frac{\kappa}{2M}\mathcal{F}_2(q^2)\sigma^{\mu\nu}q_\nu + f_3(q^2)q^\mu\right]u(p).$$

Its conservation $\partial_\mu j^\mu_{\text{f.s.}} = 0$ implies

$$q_\mu\,\bar{u}(p')\left[\mathcal{F}_1(q^2)\gamma^\mu + i\frac{\kappa}{2M}\mathcal{F}_2(q^2)\sigma^{\mu\nu}q_\nu + f_3(q^2)q^\mu\right]u(p) = 0.$$

Due to the Dirac equation in momentum space, i.e.

$$(\not{p} - m)u(p) = 0, \qquad \bar{u}(p')(\not{p}' - m) = 0,$$

the term $\bar{u}(p')\mathcal{F}_1(q^2)\not{q}u(p) = 0$. Moreover, $\sigma^{\mu\nu}$ is an antisymmetric tensor which implies $q_\mu\sigma^{\mu\nu}q_\nu = -q_\mu\sigma^{\nu\mu}q_\nu = -q_\nu\sigma^{\mu\nu}q_\mu$. Therefore, $q_\mu\sigma^{\mu\nu}q_\nu = 0$. As a result, the current conservation reduces to the constraint

$$f_3(q^2)q^2\bar{u}(p')u(p) = 0,$$

for all p, p' and $q = p - p'$, imposing $f_3(q^2) = 0$. The current is thus

$$j^\mu_{\text{f.s.}}(x) = e^{iq\cdot x}\bar{u}(p')\left[\mathcal{F}_1(q^2)\gamma^\mu + i\frac{\kappa}{2M}\mathcal{F}_2(q^2)\sigma^{\mu\nu}q_\nu\right]u(p).$$

7.5. Estimation of the proton radius from the form factor.

1. For a charge distribution $\rho(\boldsymbol{r})$ with a spherical symmetry [$\rho(\boldsymbol{r}) = \rho(r)$], the form factor reads

$$G(\boldsymbol{q}) = \int \rho(\boldsymbol{r})e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r} = \int \mathrm{d}r\,\rho(r)r^2\int_0^\pi \mathrm{d}\theta\,\sin\theta e^{i|\boldsymbol{q}|r\cos\theta}\int_0^{2\pi}\mathrm{d}\varphi$$

The integral over θ is $\left[-\frac{e^{i|\boldsymbol{q}|r\cos\theta}}{i|\boldsymbol{q}|r}\right]_0^\pi = 2\sin(qr)/(|\boldsymbol{q}|r)$ while that over φ is simply 2π. Therefore,

$$G(|\boldsymbol{q}|) = \frac{4\pi}{|\boldsymbol{q}|}\int r\rho(r)\sin(|\boldsymbol{q}|r)\,\mathrm{d}r.$$

2. The density normalisation requires

$$\int \rho(r)\, r^2\, \mathrm{d}r \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi = A \int e^{-\alpha r}\, r^2\, \mathrm{d}r \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi = 4\pi A \int e^{-\alpha r}\, r^2\, \mathrm{d}r = 1.$$

With two successive integrations by part, the integral is $2\alpha^3$. Therefore, $A = \alpha^3/(8\pi)$.

3. Let us calculate $G(|\boldsymbol{q}|)$:

$$G(\boldsymbol{q}) = \frac{\alpha^3}{2|\boldsymbol{q}|} \int_0^\infty r e^{-\alpha r} \sin(|\boldsymbol{q}|r)\, \mathrm{d}r = \frac{\alpha^3}{2|\boldsymbol{q}|} \Im \left(\int_0^\infty r e^{(-\alpha + i|\boldsymbol{q}|)r}\, \mathrm{d}r \right).$$

An integration by part eliminates r, yielding

$$G(\boldsymbol{q}) = -\frac{\alpha^3}{2|\boldsymbol{q}|} \Im \left(\int_0^\infty \frac{e^{(-\alpha+i|\boldsymbol{q}|)r}}{-\alpha + i|\boldsymbol{q}|}\, \mathrm{d}r \right) = \frac{\alpha^3}{2|\boldsymbol{q}|} \Im \left(\frac{1}{(-\alpha + i|\boldsymbol{q}|)^2} \right) = \frac{\alpha^4}{(\alpha^2 + |\boldsymbol{q}|)^2} = \frac{1}{\left(1 + \frac{|\boldsymbol{q}|)^2}{\alpha^2}\right)^2}.$$

This matches the dipole formula in Eq. (7.24) for $\alpha = 0.71$ GeV (given $Q^2 \simeq |\boldsymbol{q}^2|$). The mean value of the proton radius is then obtained from

$$\langle r^2 \rangle = \int r^2 \rho(r)\, \mathrm{d}^3\boldsymbol{r} = \frac{\alpha^3}{2} \int_0^\infty r^4 e^{-\alpha r}\, \mathrm{d}r$$

giving, after multiple integrations by part, $24/\alpha^5$. Therefore,

$$r_p = \sqrt{\langle r^2 \rangle} = \frac{\sqrt{12}}{\alpha} = \sqrt{\frac{12}{0.71}}\ \mathrm{GeV}^{-1} = \sqrt{\frac{12}{0.71}} \times 0.197\ \mathrm{fm},$$

giving $r_p = 0.8$ fm.

7.6. We start from

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^\mu p^\nu + \frac{W_3}{M^2} q^\mu q^\nu + \frac{W_4}{M^2}(p^\mu q^\nu + p^\nu q^\mu),$$

and impose $q_\mu W^{\mu\nu} = 0$, i.e.

$$\begin{aligned} q_\mu W^{\mu\nu} &= -W_1 q^\nu + \tfrac{W_2}{M^2} q \cdot p p^\nu + \tfrac{W_3}{M^2} q^2 q^\nu + \tfrac{W_4}{M^2}(q \cdot p q^\nu + q^2 p^\nu) \\ &= q^\nu \left[-W_1 + \tfrac{W_3}{M^2} q^2 + \tfrac{W_4}{M^2} q \cdot p\right] + p^\nu \left[\tfrac{W_2}{M^2} q \cdot p + \tfrac{W_4}{M^2} q^2\right] \\ &= 0. \end{aligned}$$

As it is zero for all q and p, this implies each bracket above is zero. Therefore,

$$W_4 = -\frac{q \cdot p}{q^2} W_2$$

and

$$W_3 = \frac{M^2}{q^2} W_1 - \frac{q \cdot p}{q^2} W_4 = \frac{M^2}{q^2} W_1 + \left(\frac{q \cdot p}{q^2}\right)^2 W_2.$$

It follows that

$$\begin{aligned} W^{\mu\nu} &= -W_1 g^{\mu\nu} + \tfrac{W_2}{M^2} p^\mu p^\nu + \left[\tfrac{M^2}{q^2} W_1 + \left(\tfrac{q \cdot p}{q^2}\right)^2 W_2\right] \tfrac{q^\mu q^\nu}{M^2} - \left[\tfrac{q \cdot p}{q^2} W_2\right] \tfrac{(p^\mu q^\nu + p^\nu q^\mu)}{M^2} \\ &= W_1 \left(-g^{\mu\nu} + \tfrac{q^\mu q^\nu}{q^2}\right) + \tfrac{W_2}{M^2} \left[p^\mu p^\nu + \left(\tfrac{q \cdot p}{q^2}\right)^2 q^\mu q^\nu - \tfrac{q \cdot p}{q^2}(p^\mu q^\nu + p^\nu q^\mu)\right] \\ &= W_1 \left(-g^{\mu\nu} + \tfrac{q^\mu q^\nu}{q^2}\right) + \tfrac{W_2}{M^2} \left(p^\mu - \tfrac{p \cdot q}{q^2} q^\mu\right)\left(p^\nu - \tfrac{p \cdot q}{q^2} q^\nu\right). \end{aligned}$$

7.7. Deep inelastic scattering cross-section of neutrinos on nucleons. According to the text,

$$F_2^{\nu p}(x) = 2x[d(x) + \bar{u}(x)],$$

where d and $\bar{u}$ are the PDFs of the d quark and $\bar{u}$ quark (from the sea) in the proton. This is coming from the charge-current interaction of neutrino where the interaction with ν_μ on quarks produces

$$\nu_\mu + d \to \mu^- + u, \qquad \nu_\mu + \bar{u} \to \mu^- + \bar{d}.$$

(The contribution of quarks heavier than u and d are neglected here). Note that the charge-current interaction of $\nu_\mu + u$ or $\nu_\mu + \bar{u}$ is impossible. As neutron also contains d quarks (and $\bar{u}$ from the sea), we expect

$$F_2^{\nu p}(x) = 2x[d^n(x) + \bar{u}^n(x)],$$

where d^n and $\bar{u}^n$ are the PDFs of quarks in the neutron. The isospin symmetry tells us that $d^n = u^p \equiv u$ and $\bar{u}^n = \bar{d}^p \equiv d$, and thus

$$F_2^{\nu n}(x) = 2x[u(x) + \bar{d}(x)].$$

Therefore, the nucleon structure function is

$$F_2^{\nu N}(x) = \frac{1}{2}\left[F_2^{\nu p}(x) + F_2^{\nu n}(x)\right] = x[u(x) + d(x) + \bar{u}(x) + \bar{d}(x)].$$

Similarly, we saw in Eqs. (7.36) and (7.37) that

$$F_2^{ep}(x) = x\left[\left(\tfrac{2}{3}\right)^2 (u(x) + \bar{u}(x)) + \left(\tfrac{1}{3}\right)^2 \left(d(x) + \bar{d}(x)\right)\right],$$
$$F_2^{en}(x) = x\left[\left(\tfrac{2}{3}\right)^2 \left(d(x) + \bar{d}(x)\right) + \left(\tfrac{1}{3}\right)^2 (u(x) + \bar{u}(x))\right],$$

such as

$$F_2^{eN}(x) = \frac{1}{2}\left[F_2^{ep}(x) + F_2^{en}(x)\right] = \frac{x}{2}\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right][u(x) + d(x) + \bar{u}(x) + \bar{u}(x)]\,.$$

Therefore,

$$\frac{F_2^{eN}}{F_2^{\nu N}} = \frac{1}{2}\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = \frac{5}{18},$$

where $2/3$ and $1/3$ are coming from the electric charge of quarks. In other words,

$$\frac{F_2^{eN}}{F_2^{\nu N}} = \frac{1}{2}\left[q_u^2 + q_d^2\right].$$

If q_u or q_d were integers, we would have at least $q_u^2 \geq 1$ or $q_d^2 \geq 1$ since the proton is electrically charged, implying $F_2^{eN}/F_2^{\nu N} \geq 1/2$. As measurements are consistent with $5/18 < 1/2$, necessarily, one of the quarks has a fractional charge.

8 Quantum Chromodynamics

8.1. Applications of the isospin symmetry.

1. In the reaction $p + p \to \pi^+ + d$, in terms of isospin representations, the initial state $\mathbf{2} \otimes \mathbf{2}$ forms a total $\mathbf{3}$ (isospin 1) or $\mathbf{1}$ (isospin 0) representation. Looking at the third component of isospin T_3, we have $T_3(pp) = 1/2 + 1/2 = 1$. Therefore, total isospin 0 is excluded, leaving isospin 1 as the only possibility. In the final state, since deuteron has isospin 0, the pion needs to have isospin 1, as the isospin of the final state must be 1. Pions are thus members of an isospin triplet.
2. The particles involved in the reactions $p + p \to \pi^+ + d$ and $n + n \to \pi^- + d$ have isospin representations

$$\begin{pmatrix} p \\ n \end{pmatrix} \sim \mathbf{2}, \quad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \sim \mathbf{3}, \quad \begin{pmatrix} d \end{pmatrix} \sim \mathbf{1}.$$

   The reaction $n + n \to \pi^- + d$ is the isospin conjugated of $p + p \to \pi^+ + d$ reaction (in each representation, we take the particle with opposite $T_3 : p \to n$, $\pi^+ \to \pi^-$, $d \to d$). Since isospin is considered here to be a symmetry of strong interactions, the cross sections are the same.
3. In terms of isospin states, we have

$$p = \left|1/2, +1/2\right\rangle, \quad n = \left|1/2, -1/2\right\rangle, \quad d = \left|0, 0\right\rangle, \quad \pi^\epsilon = \left|1, \epsilon\right\rangle \text{ for } \epsilon \in \{\pm 1, 0\}.$$

   Therefore, the reactions involve the isospin transitions

$$\begin{aligned} p + p \to \pi^+ + d : &\quad \left|1/2, +1/2\right\rangle \otimes \left|1/2, +1/2\right\rangle \longrightarrow \left|1, +1\right\rangle \otimes \left|0, 0\right\rangle, \\ n + n \to \pi^- + d : &\quad \left|1/2, -1/2\right\rangle \otimes \left|1/2, -1/2\right\rangle \longrightarrow \left|1, -1\right\rangle \otimes \left|0, 0\right\rangle, \\ n + p \to \pi^0 + d : &\quad \left|1/2, -1/2\right\rangle \otimes \left|1/2, +1/2\right\rangle \longrightarrow \left|1, 0\right\rangle \otimes \left|0, 0\right\rangle. \end{aligned}$$

   Using the Clebsch-Gordan table in Fig. 2.1, p. 72, we have

$$\left|1/2, -1/2\right\rangle \otimes \left|1/2, +1/2\right\rangle = \frac{1}{\sqrt{2}}\Big(\left|1, 0\right\rangle + \left|0, 0\right\rangle\Big),$$

   where the right-hand side is written in terms of total isospin. With total isospin, the reactions are described by

$$\begin{aligned} p + p \to \pi^+ + d : &\quad \left|1, +1\right\rangle \longrightarrow \left|1, +1\right\rangle, \\ n + n \to \pi^- + d : &\quad \left|1, -1\right\rangle \longrightarrow \left|1, -1\right\rangle, \\ n + p \to \pi^0 + d : &\quad \frac{1}{\sqrt{2}}\Big(\left|1, 0\right\rangle + \left|0, 0\right\rangle\Big) \longrightarrow \left|1, 0\right\rangle. \end{aligned}$$

From the $n+p$, state we thus have 50% probability of producing the $|1,0\rangle$ state ($\pi^0 d$) and 50% probability of producing the $|0,0\rangle$ state (ηd). As a result, we expect

$$\sigma(np \to \pi^0 d) = \frac{1}{2}\sigma(pp \to \pi^+ d).$$

Note: the formalism from this question can also be used to answer the first two questions.

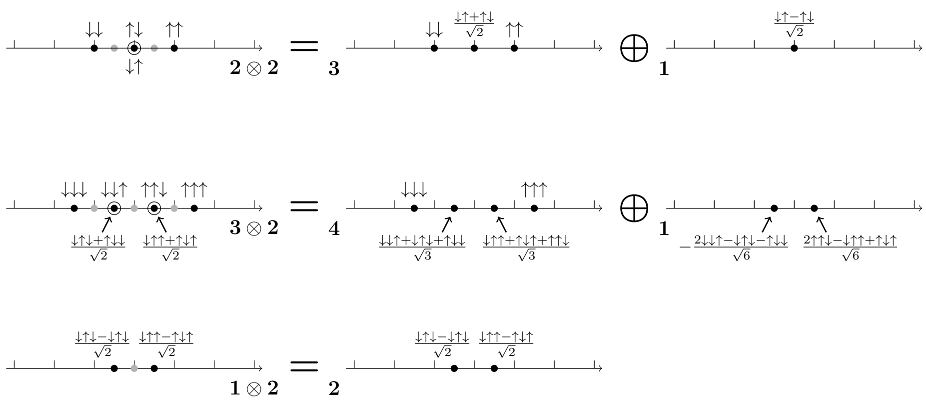
8.2. Weight diagrams with $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. When we start from the fundamental representation $\mathbf{3}$ (the triangle) and superimpose on each corner another triangle (i.e. we do $\mathbf{3} \otimes \mathbf{3}$), we obtain the diagram in Fig. 8.6 located on the left-hand side of the equal sign of the first row. Starting from the highest weight uu and applying the ladder operators I_- and V_- (Fig. 8.5 tells us the action of the ladder operators) generates the states corresponding to the second diagram of the first row (after normalisation). The third diagram is obtained by requiring its states to be orthogonal to that of the second diagram. For instance, the state $(ud+du)/\sqrt{2}$ in the second diagram must be orthogonal to a linear combination of du and ud, which have degeneracy 2 in the first diagram. We obtain $(ud-du)/\sqrt{2}$, i.e. the state in the third diagram. The diagrams in the second and third rows are obtained using the same recipe. We end up with the well-known decomposition $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = \mathbf{10}+\mathbf{8}+\mathbf{8}+\mathbf{1}$.

Let us determine the missing states in Fig. 8.6. Let us focus on the octet in the second row in Fig. 8.6. Starting from the state $(2uus - usu - suu)/\sqrt{6}$, we can apply V_- to get the state in the lower-right corner. We find $-(2ssu - uss - sus)/\sqrt{6}$. We then apply I_- to get the state in the lower-left corner, obtaining $-(2ssd - dss - sds)/\sqrt{6}$. Similarly, for the octet in the last row, starting from $(usu - suu)/\sqrt{2}$, we obtain:

$$\frac{usu - suu}{\sqrt{2}} \xrightarrow{V_-} \frac{uss - sus}{\sqrt{2}} \xrightarrow{I_-} \frac{dss - sds}{\sqrt{2}}.$$

For the decuplet, we start from uuu and apply twice V_-, obtaining successively $(suu + usu + uus)/\sqrt{3}$ and $(ssu + sus + uss)/\sqrt{3}$. Finally, starting from ddd and applying twice U_- yields $(sdd + dsd + dds)/\sqrt{3}$ and $(ssd + sds + dss)/\sqrt{3}$.

8.3. In the context of the spin, let us called the two states $\uparrow$ and $\downarrow$. The action of the ladder $I_\pm$ is obviously $I_-|\uparrow\rangle = |\downarrow\rangle$ and $I_+|\downarrow\rangle = |\uparrow\rangle$. In the weight diagrams, the representations are represented along the I_z axis, the fundamental representation $\mathbf{2}$ having its member in $I_z = \pm 1/2$. The construction proceeds as in the previous problem. The result is:



The first row is straightforward to obtain. The second row starts from the **3** representation (in grey in the first diagram of the second row), and we superimpose the fundamental representation on top of each grey point to obtain the black points of the first diagram. Then, we apply the ladder operator I_- to the highest weight $\uparrow\uparrow\uparrow$ (successively three times), which generates the second diagram of the second row. The third diagram is obtained by requiring the orthogonality of the states with those of the second diagram. For instance, let's take the state at $I_z = 1/2$. According to the first diagram of the second row, it must be made of a linear combination of $\uparrow\uparrow\downarrow$ and $\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow$ (those states located at $I_z = 1/2$ in the first diagram), i.e. $a \uparrow\uparrow\downarrow + b(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow)$. Moreover, it must be orthogonal to the state located at $I_z = 1/2$ in the second diagram of the second row, i.e.

$$\left\langle a \uparrow\uparrow\downarrow + b(\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow) \middle| \frac{\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow}{\sqrt{3}} \right\rangle = \frac{1}{\sqrt{3}}(a + 2b) = 0,$$

implying $b = -a/2$. Since the state must be normalised, we also have the constraint $a^2 + 2b^2 = 1$, giving $a = 2/\sqrt{6}$ and $b = -1/\sqrt{6}$. The other state in the third diagram at $I_z = -1/2$ is obtained by applying the ladder operator I_-,

$$I_- \frac{2\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow}{\sqrt{6}} = -\frac{2\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow - \uparrow\downarrow\downarrow}{\sqrt{6}}.$$

8.4. The Δ baryons belong to the spin 3/2 decuplet (see Fig. 8.1). Comparing Fig. 8.1 with Fig. 8.6, the flavour content of the Δ^+ baryon is $(uud + udu + duu)/\sqrt{3}$, which for a spin-up state [see Eq. (8.21)] gives

$$\begin{aligned}|\Delta^+, \uparrow\rangle &= \tfrac{uud+udu+duu}{\sqrt{3}} \otimes \tfrac{\downarrow\uparrow\uparrow+\uparrow\downarrow\uparrow+\uparrow\uparrow\downarrow}{\sqrt{3}} \\ &= \tfrac{1}{3}\big(u\downarrow u\uparrow d\uparrow + u\uparrow u\downarrow d\uparrow + u\uparrow u\uparrow d\downarrow + \\ &\quad u\downarrow d\uparrow u\uparrow + u\uparrow d\downarrow u\uparrow + u\uparrow d\uparrow u\downarrow + \\ &\quad d\downarrow u\uparrow u\uparrow + d\uparrow u\downarrow u\uparrow + d\uparrow u\uparrow u\downarrow\big).\end{aligned}$$

Similarly, Λ^0 is a member of the baryon spin 1/2 octet located at the centre. There are two baryons in this location (see Fig. 8.1): Σ^0, which belongs to an isospin triplet with its partner $\Sigma^\pm$, and Λ^0, which is necessarily a singlet of isospin. Looking at the flavour contents given in Fig. 8.6, it is easy to check that $(usd + sud - dsu - sdu)/2$ in $\mathbf{8}_{MS}$ (the octet in the second row) and $[2(uds - dus) + usd - sud - (dsu - sdu)]/\sqrt{12}$ in $\mathbf{8}_{MA}$ (the octet in the third row) is an isosinglet since the action of $I_\pm$ on these states yields 0. The symmetric combination under the exchange of the first two quarks in $\mathbf{8}_{MS}$ must be combined with the symmetric combination under the exchange of the first two spins, while the anti-symmetric combination in $\mathbf{8}_{MA}$ must be combined with the anti-symmetric combination of spins. Therefore,

$$\begin{aligned}|\Lambda^0, \uparrow\rangle &= \tfrac{1}{\sqrt{2}} \big(|\mathbf{8}_{MS}\rangle \otimes |\mathbf{2}_{MS}\rangle + |\mathbf{8}_{MA}\rangle \otimes |\mathbf{2}_{MA}\rangle\big) \\ &= \tfrac{1}{\sqrt{2}} \left(\tfrac{usd+sud-dsu-sdu}{2} \otimes \tfrac{2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow}{\sqrt{6}} + \tfrac{2(uds-dus)+usd-sud-(dsu-sdu)}{\sqrt{12}} \otimes \tfrac{\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow}{\sqrt{2}}\right) \\ &= \tfrac{1}{4\sqrt{3}} \big[usd(2\uparrow\uparrow\downarrow -2\downarrow\uparrow\uparrow) + sud(2\uparrow\uparrow\downarrow -2\uparrow\downarrow\uparrow) \\ &\quad -dsu(2\uparrow\uparrow\downarrow -2\downarrow\uparrow\uparrow) - sdu(2\uparrow\uparrow\downarrow -2\uparrow\downarrow\uparrow) \\ &\quad +2uds(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) - 2dus(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\big].\end{aligned}$$

8.5. By definition, the magnetic moment of Λ^0 is

$$\mu_\Lambda \equiv \langle \Lambda^0 \uparrow | \boldsymbol{\mu} | \Lambda^0 \uparrow \rangle .$$

Using the expansion of $|\Lambda^0, \uparrow\rangle$ obtained in the previous problem, only the combinations of flavour and spin with identical initial and final states can contribute to the calculation. For instance, the first term in the expansion is

$$\begin{aligned}
&\left(\frac{1}{4\sqrt{3}}\right)^2 \langle usd\, 2 \uparrow\uparrow\downarrow | \boldsymbol{\mu} | usd\, 2 \uparrow\uparrow\downarrow\rangle \\
&= \tfrac{1}{48} 4 \langle usd \uparrow\uparrow\downarrow | \boldsymbol{\mu_q} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{\mu_q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \boldsymbol{\mu_q},\ | usd \uparrow\uparrow\downarrow\rangle \\
&= \tfrac{1}{12}(\mu_u + \mu_s - \mu_d).
\end{aligned}$$

Therefore, we globally find

$$\begin{aligned}
\mu_\Lambda = \tfrac{1}{12}\ \Big[&(\mu_u + \mu_s - \mu_d) + (-\mu_u + \mu_s + \mu_d) + (\mu_s + \mu_u - \mu_d) + (\mu_s - \mu_u + \mu_d) \\
&+ (\mu_d + \mu_s - \mu_u) + (-\mu_d + \mu_s + \mu_u) + (\mu_s + \mu_d - \mu_u) + (\mu_s - \mu_d + \mu_u) \\
&+ (\mu_u - \mu_d + \mu_s) + (-\mu_u + \mu_d + \mu_s) + (\mu_d - \mu_u + \mu_s) + (-\mu_d + \mu_u + \mu_s)\Big] \\
= \mu_s.&
\end{aligned}$$

Given that the measured value is $\mu_\Lambda = -0.61\mu_N$, we deduce from Eq. (8.23)

$$m_s = -\frac{1}{3}\frac{M}{\mu_s}\mu_N = 0.55M = 513\ \mathrm{MeV}/c^2,$$

where $M = 938\ \mathrm{MeV}/c^2$ is the proton mass.

8.6. The reaction $e^+e^- \to q\bar{q}$

1. In this reaction, the initial state is colourless. As colour conservation is always satisfied, the $q\bar{q}$ pair has zero colour hypercharge and zero isospin.
2. By construction, the singlet $(\bar{r}r + \bar{g}r + \bar{b}b)/\sqrt{3}$ obtained in the product $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ has $Y_c = I_{3_c} = 0$. The states in the octet satisfying this constraint are $(\bar{r}r - \bar{g}g)/\sqrt{2}$ and $(\bar{r}r + \bar{g}g - 2\bar{b}b)/\sqrt{6}$ [see Eq. (8.52) obtained for gluons].
3. We saw on page 266 that the cross-section of the reaction is given by
$$\sigma_{e^+e^-\to q\bar{q}} = \frac{4\pi\alpha^2 Q^2}{3s} C = \sigma_0 Q^2 C,$$
where $C = 3$ is a colour factor due to the three possible colour-anti-colour combinations carried by the pair. It can also be interpreted as the number of possible states satisfying the colour constraint: we have $C = N_{\text{octet}} + N_{\text{singlet}} = 2 + 1 = 3$.
4. If the $q\bar{q}$ pair is produced in a colour octet state, even if $Y_c = I_{3_c} = 0$, it carries, by definition, a non-zero colour since it is not a singlet. Therefore, strictly speaking, the reaction $e^+e^- \to q\bar{q}$ cannot occur since the initial state is obviously a singlet of colour. Hence, other decay products must be present in the final state to form at the end a singlet. The presence of a gluon is a good candidate. We know that gluons are members of an octet, and according to the product $\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{8}$, if $q\bar{q}$ is in $\mathbf{8}$ and the gluon in $\mathbf{8}$, there is a possibility of forming a singlet with
$$e^+e^-\ (\text{singlet}) \to q\bar{q}\ (\text{octet}) + g\ (\text{octet}).$$

If finally, $q\bar{q}$ (octet) forms a meson, as mesons are colour singlets, for the same reason as before, the pair must radiate a gluon, i.e.

$$q\bar{q} \text{ (octet)} \to \text{meson } (q\bar{q} \text{ singlet}) + g \text{ (octet)}.$$

Therefore, we would have globally

$$e^+e^- \text{ (singlet)} \to \text{meson (singlet)} + g \text{ (octet)} + g \text{ (octet)}.$$

The other possibility is the direct production (with no consideration for parity or charge conjugation conservation)

$$e^+e^- \text{ (singlet)} \to \text{meson (singlet)}.$$

8.7. Colour factor for two quarks in $\bar{\mathbf{3}}$. To calculate the colour factor, let us take a member of $\bar{\mathbf{3}}$, for instance, $(rg - gr)/\sqrt{2}$ (the two other members lead to the same factor by colour symmetry). Following the calculations in Section 8.3.3, we obtain

$$f^{(t)}_{\frac{rg-gr}{\sqrt{2}}\to\frac{rg-gr}{\sqrt{2}}} = \frac{1}{2}\left(f^{(t)}_{rg\to rg} - f^{(t)}_{rg\to gr} - f^{(t)}_{gr\to rg} + f^{(t)}_{gr\to gr}\right) = \frac{1}{2}\left(-\frac{1}{6} - \frac{1}{2} - \frac{1}{2} - \frac{1}{6}\right) = -\frac{2}{3},$$

For a transition between a colour sextet state to a colour anti-triplet state, we take, for instance, rr as a member of the sextet. The calculation gives

$$f^{(t)}_{rr\to\frac{rg-gr}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\left(f^{(t)}_{rr\to rg} - f^{(t)}_{rr\to gr}\right)$$

By colour conservation, we know that necessarily, $f^{(t)}_{rr\to rg} = f^{(t)}_{rr\to gr} = 0$. This can be checked by using Eq. (8.70) since $f^{(t)}_{rr\to rg} = C^{1\,2}_{1\,1}$ and $f^{(t)}_{rr\to gr} = C^{2\,1}_{1\,1}$. There are no Gell-Mann matrices with $(\lambda_a)_{11}(\lambda_a)_{21} \neq 0$ [see Eq. (8.34)]

8.8. Fierz identity. As any Hermitian matrix M can be decomposed onto the basis made of $\{\mathbb{1}, \lambda_1, \ldots, \lambda_8\}$, i.e. $M = c_0\mathbb{1} + c_a\lambda_a$, and given that the Gell-Mann matrices are traceless, it follows that $\mathrm{Tr}(M) = c_0\mathrm{Tr}(\mathbb{1}) = 3c_0$, implying $c_0 = \mathrm{Tr}(M)/3$. Similarly, $\mathrm{Tr}(\lambda_a M) = c_a\mathrm{Tr}(\lambda_a^2) = 2c_a$ according to Eq. (8.35). Therefore, the component $M_{\alpha'\alpha}$ reads

$$M_{\alpha'\alpha} = \begin{cases} \frac{\mathrm{Tr}(M)}{3}\delta_{\alpha'\alpha} + \frac{\mathrm{Tr}(\lambda_a M)}{2}(\lambda_a)_{\alpha'\alpha} = \frac{1}{3}M_{\beta\beta'}\delta_{\beta'\beta}\delta_{\alpha\alpha'} + \frac{1}{2}(\lambda_a)_{\beta'\beta}M_{\beta\beta'}(\lambda_a)_{\alpha'\alpha} \\ M_{\beta\beta'}\delta_{\alpha'\beta}\delta_{\alpha\beta'} \end{cases}$$

which shows that

$$M_{\beta\beta'}\delta_{\alpha'\beta}\delta_{\alpha\beta'} = \frac{1}{3}M_{\beta\beta'}\delta_{\beta'\beta}\delta_{\alpha\alpha'} + \frac{1}{2}(\lambda_a)_{\beta'\beta}M_{\beta\beta'}(\lambda_a)_{\alpha'\alpha},$$

or equivalently (making the summation over a explicit),

$$M_{\beta\beta'}\left(\delta_{\alpha'\beta}\delta_{\alpha\beta'} - \frac{1}{3}\delta_{\beta'\beta}\delta_{\alpha\alpha'}\right) = M_{\beta\beta'}\frac{1}{2}\sum_{a=1}^{8}(\lambda_a)_{\beta'\beta}(\lambda_a)_{\alpha'\alpha}.$$

Since this equality must hold for all Hermitian matrices, this implies the Fierz identity.

$$\frac{1}{4}\sum_{a=1}^{8}(\lambda_a)_{\alpha'\alpha}\,(\lambda_a)_{\beta'\beta} = \frac{1}{2}\left(\delta_{\alpha'\beta}\delta_{\alpha\beta'} - \frac{1}{3}\delta_{\alpha'\alpha}\delta_{\beta'\beta}\right),$$

8.9. Alternative proof of Eq. (8.79). The Gell-Mann matrices being Hermitian, it follows from Eq. (8.70) that

$$|C^{\alpha\,\alpha'}_{\beta\beta'}|^2 = \frac{1}{16}\sum_a (\lambda_a)_{\alpha'\alpha}\,(\lambda_a)_{\beta'\beta}\sum_b (\lambda_b)^*_{\alpha'\alpha}\,(\lambda_b)^*_{\beta'\beta} = \frac{1}{16}\sum_{a,b}(\lambda_a)_{\alpha'\alpha}\,(\lambda_a)_{\beta'\beta}(\lambda_b)_{\alpha\alpha'}\,(\lambda_b)_{\beta\beta'}.$$

Therefore,

$$\sum_{\alpha,\alpha',\beta,\beta'}|C^{\alpha\,\alpha'}_{\beta\beta'}|^2 = \frac{1}{16}\sum_{a,b}\sum_{\alpha',\beta'}(\lambda_a\lambda_b)_{\alpha'\alpha'}(\lambda_a\lambda_b)_{\beta'\beta'} = \frac{1}{16}\sum_{a,b}(\mathrm{Tr}(\lambda_a\lambda_b))^2\,.$$

Given the normalisation of Gell-Mann matrices in Eq. (8.35), p. 268, we conclude

$$\sum_{\alpha,\alpha',\beta,\beta'}|C^{\alpha\,\alpha'}_{\beta\beta'}|^2 = \frac{1}{16}\sum_{a,b}(2\delta_{ab})^2 = \frac{1}{4}\sum_a \delta_{aa} = 2.$$

8.10. As $f_{abd} = f_{bda}$, making explicit the summation over a (that over b remains implicit below)

$$\sum_a f_{abd}\frac{1}{2^3}\mathrm{Tr}\,(\lambda_a\lambda_b\lambda_c) = \frac{1}{2^2}\mathrm{Tr}\left(\sum_a f_{bda}\frac{\lambda_a}{2}\lambda_b\lambda_c\right) = -i\frac{1}{2^2}\mathrm{Tr}\left(\left[\frac{\lambda_b}{2},\frac{\lambda_d}{2}\right]\lambda_b\lambda_c\right),$$

where Eq. (8.36) has been used in the last equality. Therefore,

$$\sum_a f_{abd}\frac{1}{2^3}\mathrm{Tr}\,(\lambda_a\lambda_b\lambda_c) = \frac{i}{2^4}\left[-\mathrm{Tr}(\lambda_b\lambda_d\lambda_b\lambda_c) + \mathrm{Tr}(\lambda_d\lambda_b\lambda_b\lambda_c)\right]. \qquad \text{(S8.1)}$$

The trace $\mathrm{Tr}(\lambda_b\lambda_d\lambda_b\lambda_c)$ is by definition $\sum_\beta(\lambda_b\lambda_d\lambda_b)_{\alpha'\beta}\,(\lambda_c)_{\beta\alpha'}$. Now, with the help of the Fierz identity (8.78),

$$\begin{aligned}
\textstyle\sum_b \frac{1}{2^3}(\lambda_b\lambda_d\lambda_b)_{\alpha'\beta} &= \textstyle\sum_b\sum_{\alpha\beta'}\frac{1}{2^3}(\lambda_b)_{\alpha'\alpha}(\lambda_d)_{\alpha\beta'}(\lambda_b)_{\beta'\beta}\\
&= \textstyle\sum_{\alpha,\beta'}\frac{1}{2}(\lambda_d)_{\alpha\beta'}\frac{1}{4}\sum_b(\lambda_b)_{\alpha'\alpha}(\lambda_b)_{\beta'\beta}\\
&= \textstyle\sum_{\alpha,\beta'}\frac{1}{2}(\lambda_d)_{\alpha\beta'}\frac{1}{2}\left(\delta_{\alpha'\beta}\delta_{\alpha\beta'} - \frac{1}{3}\delta_{\alpha'\alpha}\delta_{\beta'\beta}\right)\\
&= \textstyle\sum_\alpha \frac{1}{4}(\lambda_d)_{\alpha\alpha}\delta_{\alpha'\beta} - \frac{1}{12}(\lambda_d)_{\alpha\beta}\delta_{\alpha'\alpha}.
\end{aligned}$$

As $\sum_\alpha(\lambda_d)_{\alpha\alpha} = \mathrm{Tr}(\lambda_d) = 0$, we conclude

$$\sum_b \frac{1}{2^3}(\lambda_b\lambda_d\lambda_b)_{\alpha'\beta} = -\frac{1}{12}(\lambda_d)_{\alpha'\beta}.$$

Now, multiplying by $\frac{1}{2}(\lambda_c)_{\beta\alpha'}$ on both sides and summing over $\beta\alpha'$ yields

$$\sum_b \frac{1}{2^4}\mathrm{Tr}(\lambda_b\lambda_d\lambda_b\lambda_c) = -\frac{1}{12}\sum_{\alpha',\beta}(\lambda_d)_{\alpha'\beta}\frac{1}{2}(\lambda_c)_{\beta\alpha'} = -\frac{1}{12}\frac{1}{2}\mathrm{Tr}(\lambda_d\lambda_c) = -\frac{1}{12}\delta_{dc}.$$

For the second trace in Eq. (S8.1), we can use Eq. (8.88), i.e.

$$\begin{aligned}
\textstyle\sum_b \frac{1}{2^4}\mathrm{Tr}(\lambda_d\lambda_b\lambda_b\lambda_c) &= \textstyle\frac{1}{2^2}\sum_{\alpha,\beta,\delta}(\lambda_d)_{\alpha\beta}\frac{1}{2^2}\sum_b(\lambda_b\lambda_b)_{\beta\delta}(\lambda_c)_{\delta\alpha}\\
&= \textstyle\frac{1}{2^2}\sum_{\alpha,\beta,\delta}(\lambda_d)_{\alpha\beta}C_F\delta_{\beta\delta}(\lambda_c)_{\delta\alpha}\\
&= \textstyle\frac{C_F}{2^2}\sum_{\alpha,\beta}(\lambda_d)_{\alpha\beta}(\lambda_c)_{\beta\alpha}\\
&= \textstyle\frac{C_F}{2^2}\mathrm{Tr}(\lambda_d\lambda_c)\\
&= \textstyle\frac{C_F}{2}\delta_{dc}\\
&= \textstyle\frac{2}{3}\delta_{dc},
\end{aligned}$$

since $C_F = 4/3$. It follows from the calculation of both traces that

$$\sum_{a,b} f_{abd} \frac{1}{2^3} \mathrm{Tr}(\lambda_a \lambda_b \lambda_c) = i \left(\frac{1}{12} \delta_{dc} + \frac{2}{3} \delta_{dc} \right) = \frac{3}{4} i \delta_{dc},$$

which is the result given in Eq. (8.97).

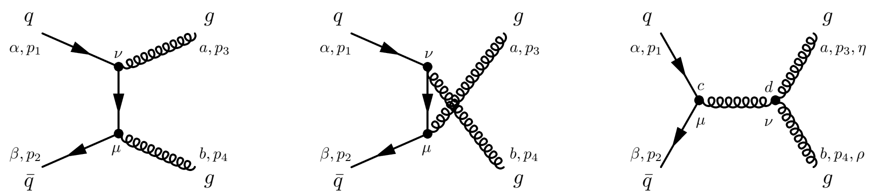
8.11. Colour factors for baryons. Let us follow the approach followed for mesons on page 310 in the book. As $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes \mathbf{3}$, we can re-use the result of the calculation done in Problem 8.7 for $\mathbf{6}$ and $\bar{\mathbf{3}}$. We found for the anti-triplet, $f_{\mathbf{6}} = -2/3 < 0$. For the sextet, let us take the state rr, $f_{\mathbf{6}} \equiv f^{(t)}_{rr \to rr} = 1/3 > 0$, using the result in Eq. (8.75). In QED, the potential between two particles of the same electric charge is repulsive $V_{qq}(r) \sim +\alpha/r$. Making the hypothesis of similar potential in QCD, we expect $V_{qq}(r) \sim +C\,\alpha_s/r$, where C is the colour factor. Therefore, if $C > 0$, the potential is repulsive and if $C < 0$, it is attractive. Two quarks placed in the sextet thus feel a repulsive force while it is attractive in the anti-triplet. In this naive approach, adding the third quark, we thus expect it will likely combine with the attractive anti-triplet to form the baryon. It is consistent with the fact that $\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}$ generates the singlet used by baryons while $\mathbf{6} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8}$ does not. Note that the colour state in the singlet reads

$$\begin{aligned}
\psi^{\text{baryon}}_{\text{colour}} &= \tfrac{1}{\sqrt{6}} (rgb - rbg + gbr - grb + brg - bgr) \\
&= \tfrac{1}{\sqrt{3}} \left(\tfrac{rg-gr}{\sqrt{2}} b + \tfrac{gb-bg}{\sqrt{2}} r + \tfrac{br-rb}{\sqrt{2}} g \right) \\
&= \tfrac{1}{\sqrt{3}} \left(b \tfrac{rg-gr}{\sqrt{2}} + r \tfrac{gb-bg}{\sqrt{2}} + g \tfrac{br-rb}{\sqrt{2}} \right) \\
&= \tfrac{1}{\sqrt{3}} \left(\tfrac{g\mathbf{b}r-r\mathbf{b}g}{\sqrt{2}} + \tfrac{b\mathbf{r}g-g\mathbf{r}b}{\sqrt{2}} + \tfrac{r\mathbf{g}b-b\mathbf{g}r}{\sqrt{2}} \right).
\end{aligned}$$

The three two-quark states with the $\sqrt{2}$ written with a small font size above are just those of the anti-triplet. We observe (look at the last three lines) that each pair of quarks is in the anti-triplet state, i.e. feeling the attractive state.

8.12. Quarks annihilation into gluons.

1. There are three diagrams:



2. As bosons are exchanged, the amplitude is the sum of the amplitudes associated with each diagram, $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$.
3. Using the notations $q(p_1, \text{colour } \alpha) + \bar{q}(p_2, \text{colour } \beta) \to g(p_3, \text{colour } a) + g(p_4, \text{colour } b)$ and applying the QCD Feynman rules, one finds

$$\begin{aligned}
i\mathcal{M}_1 &= \bar{v}(p_2) c^\dagger_\beta \left(-i g_s \gamma^\mu \tfrac{\lambda_b}{2} \right) \epsilon^*_\mu(p_4)\, i \tfrac{\not p_1 - \not p_3 + m}{(p_1-p_3)^2 - m^2} \left(-i g_s \gamma^\nu \tfrac{\lambda_a}{2} \right) \epsilon^*_\nu(p_3) u(p_1) c_\alpha \\
&= g_s^2 c^\dagger_\beta \tfrac{\lambda_b}{2} \tfrac{\lambda_a}{2} c_\alpha \left[i\mathcal{M}^{\text{QED}}_1 (q\bar{q} \to \gamma\gamma) \tfrac{1}{e^2 Q_q^2} \right],
\end{aligned}$$

where Q_q is the quark electric charge in units of e. Similarly,

$$\begin{aligned} i\mathcal{M}_2 &= \bar{v}(p_2)c_\beta^\dagger \left(-ig_s\gamma^\mu \tfrac{\lambda_a}{2}\right) \epsilon_\mu^*(p_3)\, i\tfrac{\not{p}_1-\not{p}_4+m}{(p_1-p_4)^2-m^2} \left(-ig_s\gamma^\nu \tfrac{\lambda_b}{2}\right) \epsilon_\nu^*(p_4)u(p_1)c_\alpha \\ &= g_s^2 c_\beta^\dagger \tfrac{\lambda_a}{2}\tfrac{\lambda_b}{2} c_\alpha \left[i\mathcal{M}_2^{\text{QED}}(q\bar{q} \to \gamma\gamma)\tfrac{1}{e^2 Q_q^2}\right]. \end{aligned}$$

The third amplitude is specific to QCD (due to its non-Abelian structure). It involves the three-gluon vertex. This vertex factor is given in the QCD Feynman rules assuming that the 4-momenta point towards the vertex. As $p_1 + p_2 = p_3 + p_4$, we have $(p_1 + p_2) + (-p_3) + (-p_4) = 0$. So the three momenta entering into the three-gluon vertex are $p_1 + p_2$, $-p_3$ and $-p_4$, leading to the factor (see the diagram above)

$$-g_s f^{dab} \left[g_{\nu\rho}(p_1 + p_2 + p_3)_\rho + g_{\eta\rho}(-p_3 + p_4)_\nu + g_{\rho\nu}(-p_4 - p_1 - p_2)_\eta \right].$$

The amplitude thus reads

$$\begin{aligned} i\mathcal{M}_3 = \;& \bar{v}(p_2)c_\beta^\dagger \left(-ig_s\gamma^\mu \tfrac{\lambda_c}{2}\right) u(p_1)c_\alpha \left(-i\tfrac{g_{\mu\nu}}{(p_1+p_2)^2}\delta^{cd}\right) \\ & \times\left(-g_s f^{dab} \left[g_{\nu\rho}(p_1 + p_2 + p_3)_\rho + g_{\eta\rho}(-p_3 + p_4)_\nu + g_{\rho\nu}(-p_4 - p_1 - p_2)_\eta \right]\right) \\ & \times\epsilon^{\eta *}(p_3)\epsilon^{\rho *}(p_4), \end{aligned}$$

with a summation over all repeated indices (including colour indices).

9 Weak interaction

9.1. $\mathrm{Sm}^* \to \mathrm{Sm} + \gamma$. The excited atom Sm^* can be considered at rest, and the recoil of Sm non-relativistic. If E_0 denotes the difference in energy level between Sm^* and Sm (both have the same mass), the conservation of energy-momentum leads to

$$\left.\begin{array}{ll} E_0 & = \frac{1}{2}mv^2 + E\gamma \\ 0 & = mv - E_\gamma/c \end{array}\right\} \frac{1}{2}mv^2 + mvc - E_0 = 0,$$

where v is the velocity of the recoiling atom. Therefore,

$$v = \frac{-mc + \sqrt{m^2c^2 + 2mE_0}}{m} \simeq -c + c\left(1 + \frac{E_0}{mc^2}\right) = \frac{E_0}{mc}.$$

It follows that $\Delta E = E_0 - E_\gamma$ is

$$\Delta E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{E_0^2}{mc^2} \simeq 3.3 \text{ eV},$$

using $mc^2 = 141.51$ GeV and $E_0 = 963$ keV.

9.2. Muon decay $\mu^-(p) \to e^-(k) + \bar{\nu}_e(k') + \nu_\mu(p')$.

1. Using the Feynman rules, the amplitude reads

$$i\mathcal{M} = \bar{u}_{\nu_\mu}(p')\left(i\frac{g_w}{\sqrt{2}}\gamma^\mu\frac{1-\gamma^5}{2}\right)u_\mu(p)\left(i\frac{-g_{\mu\nu} + q_\mu q_\nu/M_W^2}{q^2 - M_W^2}\right)\bar{u}_e(k)\left(-i\frac{g_w}{\sqrt{2}}\gamma^\nu\frac{1-\gamma^5}{2}\right)v_{\nu_e}(k'),$$

where $q = p - p' = k + k'$. Neglecting the mass of the outgoing particles, the Dirac equation implies $\not{k}u_e(k) = 0$ or equivalently $\bar{u}_e(k)\not{k} = 0$ and $\not{k}'v_{\nu_e}(k') = 0$. Therefore,

$$\begin{aligned} q_\nu\bar{u}_e(k)\left(-i\tfrac{g_w}{\sqrt{2}}\gamma^\nu\tfrac{1-\gamma^5}{2}\right)v_{\nu_e}(k') &= (k+k')_\nu\bar{u}_e(k)\left(-i\tfrac{g_w}{\sqrt{2}}\gamma^\nu\tfrac{1-\gamma^5}{2}\right)v_{\nu_e}(k') \\ &= -i\tfrac{g_w}{\sqrt{2}}\bar{u}_e(k)(\not{k} + \not{k}')\tfrac{1-\gamma^5}{2}v_{\nu_e}(k') \\ &= -i\tfrac{g_w}{\sqrt{2}}\left(\bar{u}_e(k)\not{k}\tfrac{1-\gamma^5}{2}v_{\nu_e}(k') + \bar{u}_e(k)\tfrac{1+\gamma^5}{2}\not{k}'v_{\nu_e}(k')\right) \\ &= 0. \end{aligned}$$

Hence, for $q^2 = (p-p')^2 \ll M_W^2$, the amplitude reduces to

$$\mathcal{M} = -\frac{g_w^2}{8M_W^2}\left[\bar{u}_{\nu_\mu}(p')\gamma^\mu(1-\gamma^5)u_\mu(p)\right]\left[\bar{u}_e(k)\gamma_\mu(1-\gamma^5)v_{\nu_e}(k')\right].$$

As $G_F/\sqrt{2} = g_w^2/(8M_W^2)$, the spin-averaged amplitude squared reads

$$\begin{aligned} \overline{|\mathcal{M}|^2} = \tfrac{G_F^2}{2}\tfrac{1}{2} &\textstyle\sum_{\text{spins}}\left[\bar{u}_{\nu_\mu}(p')\gamma^\mu(1-\gamma^5)u_\mu(p)\right]^*\left[\bar{u}_{\nu_\mu}(p')\gamma^\nu(1-\gamma^5)u_\mu(p)\right]\times \\ &\textstyle\sum_{\text{spins}}\left[\bar{u}_e(k)\gamma_\mu(1-\gamma^5)v_{\nu_e}(k')\right]^*\left[\bar{u}_e(k)\gamma_\mu(1-\gamma^5)v_{\nu_e}(k')\right]. \end{aligned}$$

The sum over the spins is easily performed using Eq. (6.84) with $\Gamma_1 = \gamma^\mu(1-\gamma^5)$, $\Gamma_2 = \gamma^\nu(1-\gamma^5)$, and neglecting the masses. It yields, given that $\gamma^0\Gamma_1^\dagger\gamma^0 = \gamma^0[\gamma^\mu(1-\gamma^5)]^\dagger\gamma^0 = \gamma^\mu(1-\gamma^5)$,

$$\overline{|\mathcal{M}|^2} = \frac{G_F^2}{4}\mathrm{Tr}\left[\not{p}\gamma^\mu(1-\gamma^5)\not{p}'\gamma^\nu(1-\gamma^5)\right]\mathrm{Tr}\left[\not{k}'\gamma_\mu(1-\gamma^5)\not{k}\gamma_\nu(1-\gamma^5)\right],$$

which according to Eqs. (9.8) and (9.9) reads

$$\overline{|\mathcal{M}|^2} = 64\,G_F^2\,(p\cdot k')(p'\cdot k).$$

2. The general formula for the decay width is given in Eq. (3.4), yielding

$$\mathrm{d}\Gamma = (2\pi)^4\frac{1}{2m_\mu}\delta^{(4)}(p'+k+k'-p)\overline{|\mathcal{M}|^2}\frac{\mathrm{d}^3\boldsymbol{p'}}{(2\pi)^3 2p'^0}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2k^0}\frac{\mathrm{d}^3\boldsymbol{k'}}{(2\pi)^3 2k^0}.$$

We can integrate over p' using Eq. (E.6), i.e. $\int\frac{\mathrm{d}^3\boldsymbol{p'}}{2p'^0} = \int \mathrm{d}^4p'\,\delta(p'^2)\theta(p'^0)$ (neglecting the mass of ν_μ). This yields in the muon rest frame,

$$\mathrm{d}\Gamma = \frac{8G_F^2}{(2\pi)^5 m_\mu}\delta[(p-k-k')^2]\theta(m_\mu-k^0-k'^0)(p\cdot k')[(p-k-k')\cdot k]\frac{\mathrm{d}^3\boldsymbol{k}}{k^0}\frac{\mathrm{d}^3\boldsymbol{k'}}{k'^0}.$$

Let us denote θ, the angle between $\boldsymbol{k}$ and $\boldsymbol{k'}$. We have

$$\begin{array}{lll} p^2 = m_\mu^2, & k^2 = 0, & k'^2 = 0,\\ p\cdot k = m_\mu k^0, & p\cdot k' = m_\mu k'^0, & k'\cdot k = k'^0k^0(1-\cos\theta),\\ \mathrm{d}^3\boldsymbol{k} = 4\pi(k^0)^2\,\mathrm{d}k^0, & & \mathrm{d}^3\boldsymbol{k'} = 2\pi(k'^0)^2\,\mathrm{d}(\cos\theta), \end{array}$$

where the integration over the angles of the solid angles has been performed in $\mathrm{d}^3\boldsymbol{k}$ and $\mathrm{d}^3\boldsymbol{k'}$ (since the amplitude depends only on the relative angle between $\boldsymbol{k}$ and $\boldsymbol{k'}$). Inserting the previous quantities in $\mathrm{d}\Gamma$ yields

$$\begin{aligned} \mathrm{d}\Gamma &= \tfrac{2G_F^2}{\pi^3}\;\delta\left[m_\mu^2 - 2m_\mu k^0 - 2m_\mu k'^0 + 2k^0k'^0(1-\cos\theta)\right]\theta(m_\mu-k^0-k'^0)\times\\ &\quad k'^0\left[m_\mu k^0 - k'^0k^0(1-\cos\theta)\right]k^0\,\mathrm{d}k^0 k'^0\,\mathrm{d}k'^0\,\mathrm{d}(\cos\theta)\\ &= \tfrac{G_F^2}{\pi^3}\;\delta\left[\tfrac{m_\mu^2}{2k^0k'^0} - \tfrac{m_\mu}{k'^0} - \tfrac{m_\mu}{k^0} + 1 - \cos\theta\right]\theta(m_\mu-k^0-k'^0)\times\\ &\quad k'^0\left[m_\mu k^0 - k'^0k^0(1-\cos\theta)\right]\mathrm{d}k^0\,\mathrm{d}k'^0\,\mathrm{d}(\cos\theta). \end{aligned}$$

After the integration over $\cos\theta$, the value

$$\cos\theta = 1 + \frac{m_\mu^2}{2k^0k'^0} - \frac{m_\mu}{k'^0} - \frac{m_\mu}{k^0} \tag{S9.1}$$

is fixed, giving

$$\begin{aligned} \tfrac{\mathrm{d}\Gamma}{\mathrm{d}k^0\,\mathrm{d}k'^0} &= \tfrac{G_F^2}{\pi^3}k^0k'^0\left[m_\mu - k'^0 + k'^0\cos\theta\right]\theta(m_\mu-k^0-k'^0)\\ &= \tfrac{G_F^2}{2\pi^3}k'^0 m_\mu\left[m_\mu - 2k'^0\right]\theta(m_\mu-k^0-k'^0). \end{aligned}$$

3. The kinematics is constrained by Eq. (S9.1), where necessarily $|\cos\theta|\le 1$. First, $\cos\theta \le 1$ implies

$$\frac{m_\mu^2}{2k^0k'^0} - \frac{m_\mu}{k'^0} - \frac{m_\mu}{k^0} \le 0 \Rightarrow k'^0 \ge \frac{m_\mu}{2} - k^0.$$

Secondly, $\cos\theta \geq -1$ implies

$$1 + \frac{m_\mu^2}{4k^0k'^0} - \frac{m_\mu}{2k'^0} - \frac{m_\mu}{2k^0} \geq 0 \Rightarrow \left(k^0 - \frac{m_\mu}{2}\right)\left(k'^0 - \frac{m_\mu}{2}\right) \geq 0.$$

Therefore, either $k^0 \geq \frac{m_\mu}{2}$ and $k'^0 \geq \frac{m_\mu}{2}$ or $k^0 \leq \frac{m_\mu}{2}$ and $k'^0 \leq \frac{m_\mu}{2}$. The first possibility would imply $k^0 + k'^0 \geq m_\mu$, which is ruled out by the presence of $\theta(m_\mu - k^0 - k'^0)$ that imposes $m_\mu \geq k^0 + k'^0$. In conclusion, necessarily,

$$k^0 \leq \frac{m_\mu}{2}, \qquad \frac{m_\mu}{2} - k^0 \leq k'^0 \leq \frac{m_\mu}{2}.$$

It follows that

$$\begin{aligned}\Gamma &= \tfrac{G_F^2}{2\pi^3} m_\mu \int_0^{\frac{m_\mu}{2}} \mathrm{d}k^0 \int_{\frac{m_\mu}{2}-k^0}^{\frac{m_\mu}{2}} k'^0 \left[m_\mu - 2k'^0\right] \theta(m_\mu - k^0 - k'^0)\ \mathrm{d}k'^0 \\ &= \tfrac{G_F^2}{\pi^3} m_\mu \int_0^{\frac{m_\mu}{2}} \mathrm{d}k^0 \int_{\frac{m_\mu}{2}-k^0}^{\frac{m_\mu}{2}} k'^0 \left[\tfrac{m_\mu}{2} - k'^0\right] \mathrm{d}k'^0.\end{aligned}$$

Let us change the integration variable for $x = \frac{m_\mu}{2} - k'^0$. It follows that

$$\begin{aligned}\Gamma &= \tfrac{G_F^2}{\pi^3} m_\mu \int_0^{\frac{m_\mu}{2}} \mathrm{d}k^0 \int_0^{k^0} \left(\tfrac{m_\mu}{2} - x\right) x\ \mathrm{d}x \\ &= \tfrac{G_F^2}{\pi^3} m_\mu \int_0^{\frac{m_\mu}{2}} \mathrm{d}k^0 \left(\tfrac{m_\mu (k'^0)^2}{4} - \tfrac{(k'^0)^3}{3}\right) \\ &= \tfrac{G_F^2}{192\pi^3} m_\mu^5.\end{aligned}$$

We conclude,

$$\tau_\mu = \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2 m_\mu^5}.$$

9.3. Eq. (9.37) is equivalent to

$$\delta_{\alpha\beta} = \sum_k |V_{\alpha k}|^2 |V_{\beta k}|^2 + \sum_{k\neq j} A_{kj},$$

where

$$A_{kj} = V_{\alpha k}^* V_{\beta k} V_{\alpha j} V_{\beta j}^*.$$

Note that A_{kj} is such that $A_{jk} = A_{kj}^*$. Therefore,

$$\sum_{k\neq j} A_{kj} = \sum_{k>j} A_{kj} + \sum_{k<j} A_{kj} = \sum_{k>j} A_{kj} + \sum_{k>j} A_{jk} = \sum_{k>j} 2\Re\left\{A_{kj}\right\},$$

and hence,

$$\sum_k |V_{\alpha k}|^2 |V_{\beta k}|^2 = \delta_{\alpha\beta} - \sum_{k>j} 2\Re\left\{A_{kj}\right\}.$$

Moreover, in Eq. (9.39),

$$\Re\left\{\sum_{k>j} A_{kj} \exp\left(-i\,2\pi\frac{L}{L_{kj}^{\mathrm{osc}}}\right)\right\} = \sum_{k>j} \Re\left\{A_{kj}\right\}\cos\left(-2\pi\frac{L}{L_{kj}^{\mathrm{osc}}}\right) - \Im\left\{A_{kj}\right\}\sin\left(-2\pi\frac{L}{L_{kj}^{\mathrm{osc}}}\right).$$

Therefore, Eq. (9.39) can be written

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta}(L) &= \delta_{\alpha\beta} - \textstyle\sum_{k>j} 2\Re\left\{A_{kj}\right\} + 2\sum_{k>j} \Re\left\{A_{kj}\right\}\cos\left(2\pi \frac{L}{L^{\rm osc}_{kj}}\right) + \Im\left\{A_{kj}\right\}\sin\left(2\pi\frac{L}{L^{\rm osc}_{kj}}\right) \\ &= \delta_{\alpha\beta} - 4\textstyle\sum_{k>j}\sin^2\left(2\pi\frac{L}{2L^{\rm osc}_{kj}}\right) + 2\sum_{k>j}\Im\left\{A_{kj}\right\}\sin\left(2\pi\frac{L}{L^{\rm osc}_{kj}}\right), \end{aligned}$$

which is Eq. (9.40).

9.4. We start from $M^\mu = \hat{C}^\dagger\gamma^0\gamma^\mu P_R\hat{C}$ in Eq. (9.51), with $\hat{C} = i\gamma^2$ from Eq. (5.84). Given that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, M^μ reads

$$M^\mu = -i\gamma^0\gamma^2\gamma^0\gamma^0\gamma^\mu P_R i\gamma^2 = \gamma^0\gamma^2\gamma^\mu P_R\gamma^2.$$

The calculation of $(M^\mu)^\dagger$ is easy given that $(\gamma^0)^2 = \mathbb{1}$, $P_R^\dagger = P_R$ and $\gamma^2\gamma^0 = -\gamma^0\gamma^2$. This leads to

$$(M^\mu)^\dagger = -\gamma^2 P_R\gamma^0\gamma^\mu\gamma^2.$$

Now, $(M^\mu)^\intercal = (M^\mu)^{\dagger *}$ and since $(\gamma^2)^* = -\gamma^2$ and all the other γ matrices are real in the Dirac representation (including γ^5), this yields

$$(M^\mu)^\intercal = -\gamma^2 P_R\gamma^0(\gamma^\mu)^*\gamma^2.$$

Therefore,

$$(M^{\mu=2})^\intercal = -\gamma^2 P_R\gamma^0(\gamma^2)^*\gamma^2 = \gamma^2 P_R\gamma^0\gamma^2\gamma^2 = -\gamma^2 P_R\gamma^0 = -\gamma^2\gamma^0 P_L = \gamma^0\gamma^2 P_L,$$

while

$$(M^{\mu\neq 2})^\intercal = -\gamma^2 P_R\gamma^0\gamma^\mu\gamma^2 = -\gamma^2 P_R\gamma^2\gamma^0\gamma^\mu = -(\gamma^2)^2 P_L\gamma^0\gamma^\mu = P_L\gamma^0\gamma^\mu = \gamma^0\gamma^\mu P_L.$$

Consequently, for all μ,

$$(M^\mu)^\intercal = \gamma^0\gamma^\mu P_L.$$

Therefore, the charge conjugate transformation of $\overline{\psi}_i\gamma^\mu P_R\psi_j$ is

$$\hat{C}\overline{\psi}_i\gamma^\mu P_R\psi_j\hat{C}^{-1} = -\psi_j^\dagger(M^\mu)^\intercal\psi_i = -\psi_j^\dagger\gamma^0\gamma^\mu P_L\psi_i = -\overline{\psi}_j\gamma^\mu P_L\psi_i,$$

which is Eq. (9.52).

9.5. With the definition of K_L and K_S in Eqs. (9.69a) and (9.69b) , we have

$$\begin{cases} |K_L\rangle = p\,|K^0\rangle - q\,|\overline{K}^0\rangle \\ |K_S\rangle = p\,|K^0\rangle + q\,|\overline{K}^0\rangle \end{cases} \Rightarrow \begin{cases} |K^0\rangle = \frac{|K_L\rangle + |K_S\rangle}{2p} \\ |\overline{K}^0\rangle = \frac{|K_S\rangle - |K_L\rangle}{2q} \end{cases}$$

Therefore, for $|K^0(0)\rangle = K^0$,

$$\begin{aligned} |K^0(t)\rangle &= \tfrac{1}{2p}\left[|K_L\rangle\, e^{-i\lambda_L t} + |K_S\rangle\, e^{-i\lambda_S t}\right] \\ &= \tfrac{1}{2p}\left[p\,|K^0\rangle\left(e^{-i\lambda_L t} + e^{-i\lambda_S t}\right) + q\,|\overline{K}^0\rangle\left(e^{-i\lambda_S t} - e^{-i\lambda_L t}\right)\right] \\ &= g_+(t)\,|K^0\rangle - \tfrac{q}{p} g_-(t)\overline{K}^0, \end{aligned} \qquad \text{(S9.2)}$$

where

$$g_\pm(t) = \frac{e^{-i\lambda_L t} \pm e^{-i\lambda_S t}}{2}.$$

Similarly, for $|\overline{K}^0(0)\rangle = |\overline{K}^0\rangle$,

$$\begin{aligned} |\overline{K}^0(t)\rangle &= \tfrac{1}{2q}\left[|K_s\rangle\, e^{-i\lambda_s t} - |K_L\rangle\, e^{-i\lambda_L t}\right] \\ &= g_+(t)\,|\overline{K}^0\rangle - \tfrac{p}{q} g_-(t)\,|K^0\rangle\,. \end{aligned} \tag{S9.3}$$

The probabilities of finding a given state at t are then

$$P_{K^0\to K^0}(t) = \left|\langle K^0|K^0(t)\rangle\right|^2 = |g_+(t)|^2, \tag{S9.4a}$$

$$P_{\overline{K}^0\to \overline{K}^0}(t) = \left|\langle \overline{K}^0|\overline{K}^0(t)\rangle\right|^2 = |g_+(t)|^2, \tag{S9.4b}$$

$$P_{K^0\to \overline{K}^0}(t) = \left|\langle \overline{K}^0|K^0(t)\rangle\right|^2 = \left|\frac{q}{p}\right|^2 |g_-(t)|^2, \tag{S9.4c}$$

$$P_{\overline{K}^0\to K^0}(t) = \left|\langle K^0|\overline{K}^0(t)\rangle\right|^2 = \left|\frac{p}{q}\right|^2 |g_-(t)|^2. \tag{S9.4d}$$

With $\lambda_L = m_L - i\Gamma_L/2$ and $\lambda_S = m_S - i\Gamma_S/2$,

$$\begin{aligned} |g_\pm(t)|^2 &= \frac{1}{4}\left(e^{-im_L t}e^{-\frac{\Gamma_L}{2}t} \pm e^{-im_S t}e^{-\frac{\Gamma_S}{2}t}\right)\left(e^{im_L t}e^{-\frac{\Gamma_L}{2}t} \pm e^{im_S t}e^{-\frac{\Gamma_S}{2}t}\right) \\ &= \frac{1}{4}\left[e^{-\Gamma_L t} + e^{-\Gamma_S t} \pm e^{-(\Gamma_L+\Gamma_S)t/2}\left(e^{i(m_L-m_S)t} + e^{-i(m_L-m_S)t}\right)\right] \\ &= \frac{1}{4}\left[e^{-\Gamma_L t} + e^{-\Gamma_S t} \pm e^{-(\Gamma_L+\Gamma_S)t/2} 2\cos\left((m_L-m_S)t\right)\right] \qquad \text{(S9.5a)} \\ &= \frac{1}{4}e^{-(\Gamma_L+\Gamma_S)t/2}\left[e^{-(\Gamma_L-\Gamma_S)t/2} + e^{(\Gamma_L-\Gamma_S)t/2} \pm 2\cos\left((m_L-m_S)t\right)\right] \\ &= \frac{1}{2}e^{-(\Gamma_L+\Gamma_S)t/2}\left[\cosh\left((\Gamma_L-\Gamma_S)t/2\right) \pm \cos\left((m_L-m_S)t\right)\right] \\ &= \frac{1}{2}e^{-\Gamma t}\left[\cosh\left(\Delta\Gamma t/2\right) \pm \cos\left(\Delta m t\right)\right], \qquad \text{(S9.6a)} \end{aligned}$$

where $\Gamma = (\Gamma_L + \Gamma_s)/2$, $\Delta\Gamma = \Gamma_L - \Gamma_S$, and $\Delta m = m_L - m_S > 0$. The insertion of Eq. (S9.5a) into Eqs. (S9.4a)-(S9.4d) leads to the expressions (9.72a)-(9.72c) in the book.

9.6. Transitions $A_{K^0\to f} = \langle f|T|K^0\rangle$ and $A_{\overline{K}^0\to f} = \langle f|T|\overline{K}^0\rangle$. Using the expressions (S9.2) and (S9.3) from the previous problem, we deduce

$$A_{K^0\to f}(t) = \langle f|T|K^0(t)\rangle = g_+(t)A_{K^0\to f} - \frac{q}{p}g_-(t)A_{\overline{K}^0\to f},$$

and hence,

$$\begin{aligned} \left|A_{K^0\to f}(t)\right|^2 &= \left|A_{K^0\to f}\right|^2\left[|g_+(t)|^2 + \left|\frac{q}{p}\frac{A_{\overline{K}^0\to f}}{A_{K^0\to f}}\right|^2 |g_-(t)|^2 - 2\Re\left(g_+^*(t)g_-(t)\frac{q}{p}\frac{A_{\overline{K}^0\to f}}{A_{K^0\to f}}\right)\right] \\ &= \left|A_{K^0\to f}\right|^2\left[|g_+(t)|^2 + \left|\lambda_f\right|^2 |g_-(t)|^2 - 2\Re\left(g_+^*(t)g_-(t)\lambda_f\right)\right]. \end{aligned}$$

with $\lambda_f = \frac{q}{p}\frac{A_{\overline{K}^0\to f}}{A_{K^0\to f}}$. Similarly,

$$\left|A_{\overline{K}^0\to f}(t)\right|^2 = \left|\frac{p}{q}\right|^2\left|A_{K^0\to f}\right|^2\left[|g_-(t)|^2 + \left|\lambda_f\right|^2 |g_+(t)|^2 - 2\Re\left(g_-^*(t)g_+(t)\lambda_f\right)\right].$$

The quantity $|g_\pm(t)|^2$ has already been calculated in the previous problem in Eq. (S9.6a).

A very similar calculation leads to

$$\begin{aligned} g_+^*(t)g_-(t) &= \tfrac{1}{4}\left(e^{im_L t}e^{-\frac{\Gamma_L}{2}t} + e^{im_S t}e^{-\frac{\Gamma_S}{2}t}\right)\left(e^{-im_L t}e^{-\frac{\Gamma_L}{2}t} - e^{-im_S t}e^{-\frac{\Gamma_S}{2}t}\right) \\ &= \tfrac{1}{4}\left[e^{-\Gamma_L t} - e^{-\Gamma_S t} - e^{i\Delta m t}e^{-\Gamma t} + e^{-i\Delta m t}e^{-\Gamma t}\right] \\ &= -\tfrac{1}{2}e^{-\Gamma t}\left[\sinh(\Delta\Gamma t/2) + i\sin(\Delta m t)\right], \end{aligned}$$

Therefore, inserting the expressions of $|g_\pm(t)|^2$ and $g_+^*(t)g_-(t)$, we deduce

$$\begin{aligned} \left|A_{K^0\to f}(t)\right|^2 &= \left|A_{K^0\to f}\right|^2 \tfrac{1}{2}e^{-\Gamma t}\Big[\cosh(\Delta\Gamma t/2) + \cos(\Delta m t) \\ &\quad + \left|\lambda_f\right|^2\left[\cosh(\Delta\Gamma t/2) - \cos(\Delta m t)\right] \\ &\quad + 2\Re\left(\lambda_f\left[\sinh(\Delta\Gamma t/2) + i\sin(\Delta m t)\right]\right)\Big] \\ &= \left|A_{K^0\to f}\right|^2 \tfrac{1}{2}e^{-\Gamma t}\Big[\left(1 + \left|\lambda_f\right|^2\right)\cosh(\Delta\Gamma t/2) + \left(1 - \left|\lambda_f\right|^2\right)\cos(\Delta m t) \\ &\quad + 2\Re\left(\lambda_f\right)\sinh(\Delta\Gamma t/2) - 2\Im\left(\lambda_f\right)\sin(\Delta m t)\Big]. \end{aligned}$$

Similarly,

$$\begin{aligned} \left|A_{\overline{K}^0\to f}(t)\right|^2 &= \left|\frac{p}{q}\right|^2\left|A_{K^0\to f}\right|^2 \tfrac{1}{2}e^{-\Gamma t}\Big[\left(1 + \left|\lambda_f\right|^2\right)\cosh(\Delta\Gamma t/2) + \left(1 - \left|\lambda_f\right|^2\right)\cos(\Delta m t) \\ &\quad + 2\Re\left(\lambda_f\right)\sinh(\Delta\Gamma t/2) + 2\Im\left(\lambda_f\right)\sin(\Delta m t)\Big]. \end{aligned}$$

When there are no direct and indirect CP violations, $|A_{\overline{K}^0\to f}/A_{K^0\to f}|^2 = 1$ and $|q/p|^2 = 1$ and thus $|\lambda_f|^2 = 1$. The previous expressions become

$$\begin{aligned} \left|A_{K^0\to f}(t)\right|^2 &= \left|A_{K^0\to f}\right|^2 e^{-\Gamma t}\left[\cosh(\Delta\Gamma t/2) + \Re\left(\lambda_f\right)\sinh(\Delta\Gamma t/2) - \Im\left(\lambda_f\right)\sin(\Delta m t)\right], \\ \left|A_{\overline{K}^0\to f}(t)\right|^2 &= \left|A_{K^0\to f}\right|^2 e^{-\Gamma t}\left[\cosh(\Delta\Gamma t/2) + \Re\left(\lambda_f\right)\sinh(\Delta\Gamma t/2) + \Im\left(\lambda_f\right)\sin(\Delta m t)\right]. \end{aligned}$$

Therefore, the asymmetry is

$$\frac{|A_{\overline{K}^0\to f}(t)|^2 - |A_{K^0\to f}(t)|^2}{|A_{\overline{K}^0\to f}(t)|^2 + |A_{K^0\to f}(t)|^2} = \frac{\Im(\lambda_f)\sin(\Delta m t)}{\cosh(\Delta\Gamma t/2) + \Re(\lambda_f)\sinh(\Delta\Gamma t/2)}.$$

9.7. The vectors $\boldsymbol{AB}$ and $\boldsymbol{AC}$ are defined by

$$\boldsymbol{AB} = \begin{pmatrix}\Re(V_{i1}V_{j1}^*)\\ \Im(V_{i1}V_{j1}^*)\\ 0\end{pmatrix},\quad \boldsymbol{AC} = \begin{pmatrix}\Re(V_{i2}V_{j2}^*)\\ \Im(V_{i2}V_{j2}^*)\\ 0\end{pmatrix},$$

This ABC area, $S_{12}^{ij} = |\boldsymbol{AB}\times\boldsymbol{AC}|/2$, is thus

$$\begin{aligned} S_{12}^{ij} &= \tfrac{1}{2}\left|\Re(V_{i1}V_{j1}^*)\Im(V_{i2}V_{j2}^*) - \Im(V_{i1}V_{j1}^*)\Re(V_{i2}V_{j2}^*)\right| \\ &= \tfrac{1}{2}\left|\Re(V_{i1}V_{j1}^*)\Im(V_{i2}V_{j2}^*) + \Im(V_{i1}^*V_{j1})\Re(V_{i2}V_{j2}^*)\right| \\ &= \tfrac{1}{2}\left|\Im(V_{i1}^*V_{j1}V_{i2}V_{j2}^*)\right|, \end{aligned}$$

the last equality coming from the property $\Im(zz') = \Re(z)\Im(z') + \Re(z')\Im(z)$. We can generalise this result and define $\boldsymbol{AB}$ and $\boldsymbol{AC}$ by

$$\boldsymbol{AB} = \begin{pmatrix} \Re(V_{i\alpha}V^*_{j\alpha}) \\ \Im(V_{i\alpha}V^*_{j\alpha}) \\ 0 \end{pmatrix}, \quad \boldsymbol{AC} = \begin{pmatrix} \Re(V_{i\beta}V^*_{j\beta}) \\ \Im(V_{i\beta}V^*_{j\beta}) \\ 0 \end{pmatrix},$$

and get the area

$$S^{ij}_{\alpha\beta} = \frac{1}{2}\left|\Im(V^*_{i\alpha}V_{j\alpha}V_{i\beta}V^*_{j\beta})\right|,$$

or

$$\boldsymbol{AB} = \begin{pmatrix} \Re(V_{\alpha i}V^*_{\alpha j}) \\ \Im(V_{\alpha i}V^*_{\alpha j}) \\ 0 \end{pmatrix}, \quad \boldsymbol{AC} = \begin{pmatrix} \Re(V_{\beta i}V^*_{\beta j}) \\ \Im(V_{\beta i}V^*_{\beta j}) \\ 0 \end{pmatrix},$$

with the area

$$S^{\alpha\beta}_{ij} = \frac{1}{2}\left|\Im(V^*_{\alpha i}V_{\alpha j}V_{\beta i}V^*_{\beta j})\right|.$$

Now, let us multiply the first equation of Eq. (9.78) by $V^*_{i1}V_{j1}$. This yields

$$\left|V_{i1}V_{j1}\right|^2 + V^*_{i1}V_{j1}V_{i2}V^*_{j2} + V^*_{i1}V_{j1}V_{i3}V^*_{j3} = 0.$$

Taking the imaginary part, it follows that

$$\Im\left(V^*_{i1}V_{j1}V_{i2}V^*_{j2}\right) = -\Im\left(V^*_{i1}V_{j1}V_{i3}V^*_{j3}\right).$$

Similarly, by multiplying the first equation of Eq. (9.78) by $V^*_{i2}V_{j2}$ or $V^*_{i3}V_{j3}$, we would conclude

$$\Im\left(V^*_{i2}V_{j2}V_{i1}V^*_{j1}\right) = -\Im\left(V^*_{i2}V_{j2}V_{i3}V^*_{j3}\right),$$
$$\Im\left(V^*_{i3}V_{j3}V_{i2}V^*_{j2}\right) = -\Im\left(V^*_{i3}V_{j3}V_{i1}V^*_{j1}\right).$$

With the last three equalities, we conclude, given that $\Im(z^*) = -\Im(z)$, that

$$\Im(V^*_{i1}V_{j1}V_{i2}V^*_{j2}) = \begin{cases} -\Im(V^*_{i1}V_{j1}V_{i3}V^*_{j3}) = -\Im(V^*_{i2}V_{j2}V_{i1}V^*_{j1}) = \Im(V^*_{i2}V_{j2}V_{i3}V^*_{j3}), \\ -\Im(V^*_{i3}V_{j3}V_{i2}V^*_{j2}) = \Im(V^*_{i3}V_{j3}V_{i1}V^*_{j1}). \end{cases}$$

Therefore, $S^{ij}_{\alpha\beta}$ are all the same regardless of α and β, provided that $\alpha \neq \beta$ for a given i and j. We can proceed similarly with the second equation of Eq. (9.78). It yields

$$\Im\left(V^*_{1i}V_{1j}V_{2i}V^*_{2j}\right) = -\Im\left(V^*_{1i}V_{1j}V_{3i}V^*_{3j}\right),$$
$$\Im\left(V^*_{2i}V_{2j}V_{1i}V^*_{1j}\right) = -\Im\left(V^*_{2i}V_{2j}V_{3i}V^*_{3j}\right),$$
$$\Im\left(V^*_{3i}V_{3j}V_{2i}V^*_{2j}\right) = -\Im\left(V^*_{3i}V_{3j}V_{1i}V^*_{1j}\right).$$

Therefore, all these quantities are equal up to a sign, leading to conclude that $S^{\alpha\beta}_{ij}$ are all the same regardless of α and β, provided that $\alpha \neq \beta$ for a given i and j. What we found for $S^{ij}_{\alpha\beta}$ and $S^{\alpha\beta}_{ij}$ shows that all (non-trivial) triangles have the same area ($\alpha \neq \beta$, $i \neq j$, otherwise the area is zero).

9.8. The reactions

$$1.\ n \to p + e^+ + \nu_e, \quad 2.\ p + \pi^- \to n + \pi^0, \quad 3.\ p \to n + e^+ + \nu_e,$$
$$4.\ \pi^0 \to \gamma\gamma, \quad 5.\ p \to e^+ + \nu_e, \quad 6.\ \pi^0 \to \gamma.$$

Only reactions 2 and 4 are possible by strong and QED interactions, respectively. Note that the parity conservation in reaction 4 implies that necessarily, the two-photon final state has odd orbital angular momentum (actually, their polarisation vectors are necessarily perpendicular to each other). All reactions involving neutrinos (1, 3, 5) would have required weak interaction, but none is allowed (charge is not conserved for 1, mass is too small for proton decay in 3, and baryon number is violated in 5). Finally, reaction 6 is a 1-to-1 decay which would imply $p_{\pi^0} = p_\gamma$, i.e. $m_{\pi^0} = m_\gamma$, which is not the case.

9.9. Electron-neutrino scattering: (1) $\nu_\mu(k) + e^-(p) \to \nu_e(k') + \mu^-(p')$, (2) $\bar{\nu}_e(k) + e^-(p) \to \bar{\nu}_\mu(k') + \mu^-(p')$.

1. For reaction (1), we have a t-channel diagram with the exchange of a W boson. We denote $q = k - k'$, the 4-momentum of the W boson, with $t = q^2$. The amplitude reads

$$\begin{aligned} i\mathcal{M}_1 &= \bar{u}(p') \cdot \left(-i\frac{g_w}{\sqrt{2}}\gamma^\mu \frac{1-\gamma^5}{2}\right) \cdot u(k) \times i\frac{-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2}}{t - M_W^2} \times \bar{u}(k') \cdot \left(-i\frac{g_w}{\sqrt{2}}\gamma^\nu \frac{1-\gamma^5}{2}\right) \cdot u(p) \\ &= -i\frac{G_F}{\sqrt{2}}\, \bar{u}(p')\, \gamma^\mu (1-\gamma^5)\, u(k) \times \bar{u}(k')\, \gamma_\mu (1-\gamma^5)\, u(p). \end{aligned}$$

Given that we neglect the lepton masses with respect to M_W, the term $q_\mu q_\nu$ in the propagator does not contribute to the amplitude (see the solution of Problem 9.2). We also replaced above the coupling $g_w^2/8M_W^2$ by $G_F/\sqrt{2}$ and considered $t \ll M_W^2$. For reaction (2), we have an s-channel with the exchange of a W boson. With the same assumptions as for (1), we write the amplitude as

$$i\mathcal{M}_2 = -i\frac{G_F}{\sqrt{2}}\, \bar{u}(p')\, \gamma^\mu (1-\gamma^5)\, v(k') \times \bar{v}(k)\, \gamma_\mu (1-\gamma^5)\, u(p).$$

2. We start by

$$\overline{|\mathcal{M}_1|^2} = \frac{1}{2} \sum_{\text{all spins}} \mathcal{M}_1 \mathcal{M}_1^*,$$

where 1/2 stands for the average over the initial polarisation of the electron. Note that we use 1/2 instead of 1/4 because neutrinos only have one helicity (left-handed). We still sum over neutrino polarisations to be able to use the completeness relations

$$\sum_{\text{spins}} \bar{u}(p)u(p) = \not{p} + m \simeq \not{p}, \quad \sum_{\text{spins}} \bar{v}(p)v(p) = \not{p} - m \simeq \not{p}.$$

The calculation of $\mathcal{M}_1\mathcal{M}_1^*$ is very similar to that of Problem 9.2, giving

$$\begin{aligned}\mathcal{M}_1\mathcal{M}_1^* &= \frac{G_F^2}{2}\,\bar{u}(p')\,\gamma^\mu(1-\gamma^5)\,u(k)\times\bar{u}(k')\,\gamma_\mu(1-\gamma^5)\,u(p)\\ &\qquad\times\bar{u}(k)\,\gamma^\nu(1-\gamma^5)\,u(p')\times\bar{u}(p)\,\gamma_\nu(1-\gamma^5)\,u(k')\\ &= \frac{G_F^2}{2}\,\bar{u}(p')\,\gamma^\mu(1-\gamma^5)\,u(k)\times\bar{u}(k)\,\gamma^\nu(1-\gamma^5)\,u(p')\\ &\qquad\times\bar{u}(k')\,\gamma_\mu(1-\gamma^5)\,u(p)\times\bar{u}(p)\,\gamma_\nu(1-\gamma^5)\,u(k').\end{aligned}$$

Using Eqs. (9.8) and (9.9), the sum over the spins yields

$$\begin{aligned}\overline{|\mathcal{M}_1|^2} &= \tfrac{1}{2}\tfrac{G_F^2}{2}\mathrm{Tr}\left[\not{p}\,\gamma^\mu(1-\gamma^5)\,\not{k}'\,\gamma^\nu(1-\gamma^5)\right]\mathrm{Tr}\left[\not{k}\,\gamma_\mu(1-\gamma^5)\,\not{p}'\,\gamma_\nu(1-\gamma^5)\right]\\ &= \tfrac{G_F^2}{4}256(p\cdot k)(k'\cdot p').\end{aligned}$$

Now, as $s = (p+k)^2 = p^2+k^2+2p\cdot k \simeq 2p\cdot k$ (since we consider massless fermions) and $s = (p'+k')^2 \simeq 2p'\cdot k'$, we conclude that

$$\overline{|\mathcal{M}_1|^2} = \frac{G_F^2}{4}\times 256\left(\frac{s}{2}\right)^2 = 64G_F^2\frac{s^2}{4}.$$

The second spin-averaged amplitude squared is obtained by substituting $s \leftrightarrow t$ since the crossing properties show that the s-channel of reaction (1) corresponds to the t-channel of reaction (2). Therefore,

$$\overline{|\mathcal{M}_1|^2} = 64G_F^2\frac{s^2}{4} \quad\text{and}\quad \overline{|\mathcal{M}_2|^2} = 64G_F^2\frac{t^2}{4}.$$

3. We now need to integrate over the angular variables to determine the cross-sections. Using the centre-of-mass frame, we have

$$\sigma = \frac{1}{64\pi s}\frac{|\boldsymbol{p}'^*|}{|\boldsymbol{p}^*|}\int|\mathcal{M}|^2\mathrm{d}\Omega.$$

As fermions masses are neglected, the 4-momenta read $p = (E,\boldsymbol{p}^*)$, $k = (E,-\boldsymbol{p}^*)$ and $k' = (E,\boldsymbol{p}'^*)$, $p' = (E,-\boldsymbol{p}'^*)$, with $E = |\boldsymbol{p}^*| = |\boldsymbol{p}'^*| = \sqrt{s}/2$.
The cross-section ratio can be simply expressed as

$$\frac{\sigma_1}{\sigma_2} = \frac{\int s^2\mathrm{d}\Omega}{\int t^2\mathrm{d}\Omega}. \tag{S9.7}$$

There is no angular dependence in s, so the numerator is simply $4\pi s^2$, while for t,

$$t = (p-k') = -2p\cdot k' = -2(E^2-\boldsymbol{p}.\boldsymbol{k}') = -2E^2(1-\cos\theta) = -\frac{s}{2}(1-\cos\theta),$$

where θ is the angle between $\boldsymbol{p}^*$ and $\boldsymbol{p}'^*$ (i.e. between e^- and $\bar{\nu}_\mu$). The integration $\int t^2\mathrm{d}\Omega$ gives $4\pi s^2/3$, so we conclude that

$$\frac{\sigma(\nu_\mu+e^-\to\nu_e+\mu^-)}{\sigma(\bar{\nu}_e+e^-\to\bar{\nu}_\mu+\mu^-)} = 3.$$

10 Electroweak interaction

10.1. We start with the sum of Eqs. (10.25) and (10.26) and keep only the term proportional to Z^μ. It reads

$$\begin{aligned} j^\mu &= \overline{L}\gamma^\mu\left(g_w\cos\theta_w\tfrac{\sigma_3}{2} - g\sin\theta_w\tfrac{Y}{2}\right)L - g\sin\theta_w\left(\overline{\psi_R}\gamma^\mu\tfrac{Y}{2}\psi_R + \overline{\psi'_R}\gamma^\mu\tfrac{Y}{2}\psi'_R\right) \\ &= \overline{L}\gamma^\mu\left(g_w\cos\theta_w T_3 - g\sin\theta_w(Q-T_3)\right)L - g\sin\theta_w\left(\overline{\psi_R}\gamma^\mu Q\psi_R + \overline{\psi'_R}\gamma^\mu Q\psi'_R\right), \end{aligned}$$

where $Y/2 = Q - T_3$ and $T_3\psi_R = T_3\psi'_R = 0$ for singlets. Now, as

$$\begin{aligned} &\overline{L}\gamma^\mu QL + \overline{\psi_R}\gamma^\mu Q\psi_R + \overline{\psi'_R}\gamma^\mu Q\psi'_R \\ &\quad= \overline{\psi_L}\gamma^\mu Q\psi_L + \overline{\psi'_L}\gamma^\mu Q\psi'_L + \overline{\psi_R}\gamma^\mu Q\psi_R + \overline{\psi'_R}\gamma^\mu Q\psi'_R \\ &\quad= \overline{\psi}\gamma^\mu Q\psi + \overline{\psi'}\gamma^\mu Q\psi', \end{aligned}$$

with $\psi = \psi_L + \psi_R$ and $\psi' = \psi'_L + \psi'_R$, it follows that

$$\begin{aligned} j^\mu &= \overline{L}\gamma^\mu\left(g_w\cos\theta_w + g\sin\theta_w\right)T_3\,L - g\sin\theta_w\left(\overline{\psi}\gamma^\mu Q\psi + \overline{\psi'}\gamma^\mu Q\psi'\right) \\ &= \frac{g_w}{\cos\theta_w}\left[\overline{L}\gamma^\mu T_3\,L - \sin^2\theta_w\left(\overline{\psi}\gamma^\mu Q\psi + \overline{\psi'}\gamma^\mu Q\psi'\right)\right] \end{aligned}$$

since according to Eq. (10.30), $g = g_w\sin\theta_w/\cos\theta_w$. This result is Eq. (10.31).

10.2. The Lagrangian in Eq. (10.52) is

$$\mathcal{L}_{\gamma Z^0W^+W^-} = -eg_w\cos\theta_w\left[2W^+_\mu W^{-\mu}Z_\nu A^\nu - W^+_\mu W^{-\nu}Z_\nu A^\mu - W^+_\mu W^{-\nu}A_\nu Z^\mu\right].$$

Let us rewrite each product of fields with appropriate labels:

$$\begin{aligned} W^+_\mu W^{-\mu}Z_\nu A^\nu &= W^+_\mu W^{-\mu}Z_\alpha A^\alpha = g_{\mu\nu}g_{\alpha\beta}W^{+\mu}W^{-\nu}Z^\beta A^\alpha, \\ W^+_\mu W^{-\nu}Z_\nu A^\nu &= W^{+\mu}W^{-\nu}Z_\nu A_\mu = g_{\nu\beta}g_{\mu\alpha}W^{+\mu}W^{-\nu}Z^\beta A^\alpha, \\ W^+_\mu W^{-\nu}A_\nu Z^\mu &= W^{+\mu}W^{-\nu}A_\nu Z_\mu = g_{\nu\alpha}g_{\mu\beta}W^{+\mu}W^{-\nu}A^\alpha Z^\beta. \end{aligned}$$

Therefore,

$$\mathcal{L}_{\gamma Z^0W^+W^-} = -eg_w\cos\theta_w\left[2g_{\mu\nu}g_{\alpha\beta} - g_{\nu\beta}g_{\mu\alpha} - g_{\nu\alpha}g_{\mu\beta}\right]W^{+\mu}W^{-\nu}Z^\beta A^\alpha.$$

The vertex factor is thus simply i times the terms in front of $W^{+\mu}W^{-\nu}Z^\beta A^\alpha$.

10.3. The Lagrangian in Eq. (10.53) is

$$\mathcal{L}_{W^+W^-W^+W^-} = \frac{g_w^2}{2}\left[W^+_\mu W^{+\mu}W^-_\nu W^{-\nu} - W^+_\mu W^{-\mu}W^-_\nu W^{+\nu}\right].$$

We first change the labels of the first term:

$$W^+_\mu W^{+\mu}W^-_\nu W^{-\nu} \to W^+_{\mu'}W^{+\mu'}W^-_{\nu'}W^{-\nu'} = g_{\mu'\alpha'}g_{\nu'\beta'}W^{+\alpha'}W^{+\mu'}W^{-\beta'}W^{-\nu'}.$$

In the Feynman diagram, we want the labels to be $W^{+\alpha}$, $W^{+\mu}$, $W^{-\beta}$, $W^{-\nu}$. Four combinations of (α',μ') pairs and (ν',β') pairs lead to this result. They are:

$$\begin{cases}(\alpha',\mu') = (\alpha,\mu)\\ (\nu',\beta') = (\nu,\beta)\end{cases} \quad \begin{cases}(\alpha',\mu') = (\mu,\alpha)\\ (\nu',\beta') = (\nu,\beta)\end{cases} \quad \begin{cases}(\alpha',\mu') = (\mu,\alpha)\\ (\nu',\beta') = (\beta,\nu)\end{cases} \quad \begin{cases}(\alpha',\mu') = (\alpha,\mu)\\ (\nu',\beta') = (\beta,\nu)\end{cases}$$

Therefore, they contribute to a factor proportional to $g_{\alpha\mu}g_{\nu\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\alpha}g_{\beta\nu}+g_{\alpha\mu}g_{\beta\nu} = 4g_{\mu\alpha}g_{\nu\beta}$. Now, the second term in $\mathcal{L}_{W^+W^-W^+W^-}$,

$$W^+_\mu W^{-\mu} W^-_\nu W^{+\nu} \to W^+_{\mu'} W^{-\mu'} W^-_{\nu'} W^{+\nu'} = W^{+\mu'} W^-_{\mu'} W^{-\nu'} W^+_{\nu'} = g_{\mu'\beta'} g_{\nu'\alpha'} W^{+\mu'} W^{-\beta'} W^{-\nu'} W^{+\alpha'}.$$

Here again, four combinations match the desired labels,

$$\begin{cases}(\mu',\beta') = (\mu,\beta)\\(\nu',\alpha') = (\nu,\alpha)\end{cases} \quad \begin{cases}(\mu',\beta') = (\alpha,\beta)\\(\nu',\alpha') = (\nu,\mu)\end{cases} \quad \begin{cases}(\mu',\beta') = (\alpha,\nu)\\(\nu',\alpha') = (\beta,\mu)\end{cases} \quad \begin{cases}(\mu',\beta') = (\mu,\nu)\\(\nu',\alpha') = (\beta,\alpha)\end{cases}$$

giving a factor $g_{\mu\beta}g_{\nu\alpha}+g_{\alpha\beta}g_{\nu\mu}+g_{\alpha\nu}g_{\beta\mu}+g_{\mu\nu}g_{\beta\alpha} = 2g_{\mu\beta}g_{\alpha\nu}+2g_{\mu\nu}g_{\alpha\beta}$. Overall, the vertex factor is thus

$$i \times \frac{g_w^2}{2}\left[4g_{\mu\alpha}g_{\nu\beta} - 2g_{\mu\beta}g_{\alpha\nu} - 2g_{\mu\nu}g_{\alpha\beta}\right].$$

10.4. Determination of the number of light neutrinos.

1. The Z boson can decay in $f\bar{f}$ for all fermions f such that $2m_f \leqslant m_Z$. It includes quarks, charged leptons, and invisible particles, i.e. neutrinos in the context of the Standard Model. Therefore, by definition, we have for the total decay width,

$$\Gamma_Z = \sum_q \Gamma_{Z\to q\bar{q}} + \sum_\ell \Gamma_{Z\to\ell^-\ell^+} + \Gamma_{\text{inv}}.$$

Quarks hadronise, so in the quark sector, we only observe

$$\Gamma_{Z\to\text{had}} \equiv \sum_q \Gamma_{Z\to q\bar{q}}.$$

The lepton universality hypothesis states that weak couplings are the same for all lepton families: the amplitudes $\mathcal{M}_{Z\to\ell^-\ell^+}$ are then all the same. Neglecting lepton masses, the integration over the phase space gives also the same results for all generations so that the partial decay widths $\Gamma_{Z\to\ell^-\ell^+}$ are also all the same. We finally deduce

$$\Gamma_Z = \Gamma_{Z\to\text{had}} + 3\Gamma_{Z\to\ell^-\ell^+} + \Gamma_{\text{inv}}.$$

Using the definition of branching ratios, we conclude

$$\Gamma_{\text{inv}} = \Gamma_z[1 - \text{BR}(h) - 3\text{BR}(\ell)].$$

2. Assuming lepton universality and neglecting neutrino masses, we have, as for charged leptons, an identical value of $\Gamma_{Z\to\nu\bar{\nu}}$ for all neutrinos. Therefore,

$$\Gamma_{\text{inv}} = N_\nu \Gamma_{Z\to\nu\bar{\nu}},$$

where N_ν is the number of neutrinos satisfying the constraint $m_\nu \leq M_Z/2$. The partial decay width $\Gamma_{Z\to\nu\bar{\nu}}$ cannot be measured experimentally, so instead we use the Standard Model prediction Γ_ν^{th} and

$$N_\nu = \frac{\Gamma_{\text{inv}}}{\Gamma_\nu^{th}},$$

which should be

$$N_\nu = \frac{\Gamma_{\text{inv}}}{\Gamma_\ell} \times \frac{\Gamma_\ell^{th}}{\Gamma_\nu^{th}}$$

if the Standard Model is correct for the charged lepton sector.

3. We wish now to calculate Γ_ℓ^{th} and Γ_ν^{th}.
   1. If $q = p_1 + p_2$ denotes the 4-momentum of the Z boson and ϵ_μ its polarisation vector, the amplitude of $Z^0 \to f(p_1)\bar{f}(p_2)$ reads

$$\begin{aligned} i\mathcal{M} &= \bar{u}(p_1) \cdot \left(-i\frac{g_w}{\cos\theta_w}\,\gamma^\mu \frac{1}{2}\left(c_V - c_A\gamma^5\right)\right) \cdot v(p_2) \cdot \epsilon_\mu(q) \\ &= -i\frac{g_w}{2\cos\theta_w}\,\bar{u}(p_1)\gamma^\mu\left(c_V - c_A\gamma^5\right)v(p_2) \cdot \epsilon_\mu(q). \end{aligned}$$

   2. There are three possible polarisations for the Z boson, so the spin-averaged amplitude squared $\overline{|\mathcal{M}|^2}$ reads

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{3}\sum_{\text{spins}} \mathcal{M}\mathcal{M}^* \\ &= \frac{g_w^2}{12\cos^2\theta_w} \times \sum_\lambda \epsilon_\mu(\lambda, q)\epsilon_\nu^*(\lambda, q) \\ &\quad \times \sum_{s_1,s_2}\left[\bar{u}(p_1)\gamma^\mu(c_V - c_A\gamma^5)v(p_2)\right]\left[\bar{u}(p_1)\gamma^\nu(c_V - c_A\gamma^5)v(p_2)\right]^* \\ &= \frac{g_w^2}{12\cos^2\theta_w}\left(-g_{\mu\nu} + \frac{(p_1+p_2)_\mu(p_1+p_2)_\nu}{m_Z^2}\right) \\ &\quad \times \sum_{s_1,s_2}\left[\bar{u}(p_1)\gamma^\mu(c_V - c_A\gamma^5)v(p_2)\right]\left[\bar{u}(p_1)\gamma^\nu(c_V - c_A\gamma^5)v(p_2)\right]^*. \end{aligned}$$

   3. We use the Dirac equation in spinor space with neglected masses, i.e. $\not{p}_2 v \approx 0$ and $\bar{u}\not{p}_1 \approx 0$. Therefore,

$$\begin{aligned} (p_1+p_2)_\mu\left[\bar{u}(p_1)\gamma^\mu(c_V - c_A\gamma^5)v(p_2)\right] &= \bar{u}(p_1)\not{p}_1(c_V - c_A\gamma^5)v(p_2) \\ &\quad + \bar{u}(p_1)(c_V + c_A\gamma^5)\not{p}_2\, v(p_2) \end{aligned}$$

   has both terms vanishing. The spin-averaged amplitude squared simplifies to

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{-g_w^2\, g_{\mu\nu}}{12\cos^2\theta_w} \times \sum_{s_1,s_2}\left[\bar{u}(p_1)\gamma^\mu(c_V - c_A\gamma^5)v(p_2)\right]\left[\bar{u}(p_1)\gamma^\nu(c_V - c_A\gamma^5)v(p_2)\right]^* \\ &= \frac{-g_w^2\, g_{\mu\nu}}{12\cos^2\theta_w} \times \sum_{s_1,s_2}\left[\bar{u}(p_1)\gamma^\mu(c_V - c_A\gamma^5)v(p_2)\right]\left[\bar{v}(p_2)\gamma^\nu(c_V - c_A\gamma^5)u(p_1)\right] \\ &= \frac{-g_w^2\, g_{\mu\nu}}{12\cos^2\theta_w} \times \mathrm{Tr}\left[\not{p}_1\gamma^\mu(c_V - c_A\gamma^5)\not{p}_2\gamma^\nu(c_V - c_A\gamma^5)\right]. \end{aligned}$$

   The trace can be rewritten as

$$\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu(c_V - c_A\gamma^5)^2\right] = (c_V^2 + c_A^2)\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\right] - 2c_Vc_A\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\gamma^5\right].$$

Given that the second term is an antisymmetric tensor under $\mu \leftrightarrow \nu$ (it evaluates to $-4i\, p_{1\alpha} p_{2\beta}\, \varepsilon^{\alpha\mu\beta\nu}$) and that the full trace will be contracted with the symmetric tensor $g_{\mu\nu}$, only the first term has a non-zero contribution. Therefore, we conclude that

$$\overline{|\mathcal{M}|^2} = \frac{g_w^2}{12\cos^2\theta_w}\left(-g_{\mu\nu}\right)(c_V^2 + c_A^2)\mathrm{Tr}\left[\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2\right].$$

4. We need first to determine the trace. We have

$$\begin{aligned}\mathrm{Tr}\left[\not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu\right] &= p_{1\alpha} p_{2\beta} \mathrm{Tr}\left[\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\nu\right] \\ &= 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2) g^{\mu\nu}).\end{aligned}$$

Contracted with $g_{\mu\nu}$, this gives $-8(p_1 \cdot p_2)$. In the Z rest frame, we can write $p_1 = (E, \boldsymbol{p})$ and $p_2 = (E, -\boldsymbol{p})$ with $E = |\boldsymbol{p}| = m_Z/2$, so that

$$p_1 \cdot p_2 = E^2 + |\boldsymbol{p}|^2 = \frac{m_Z^2}{2}.$$

Therefore, the spin-averaged amplitude squared is

$$\overline{|\mathcal{M}|^2} = \frac{g_w^2 m_Z^2}{3\cos^2\theta_w}(c_V^2 + c_A^2),$$

and the partial decay width is

$$\Gamma_{Z\to f\bar{f}} = \frac{|\boldsymbol{p}|}{32\pi^2 m_Z^2}\int \mathrm{d}\Omega\, \overline{|\mathcal{M}|^2} = \frac{g_w^2}{48\pi\cos^2\theta_w} m_Z (c_V^2 + c_A^2).$$

However, if f is a quark, one should also take into account the colour degree of freedom. As the pair $q\bar{q}$ can be produced in three different colour configurations, the partial decay width finally reads

$$\Gamma_{Z\to f\bar{f}} = \frac{|\boldsymbol{p}|}{32\pi^2 m_Z^2}\int \mathrm{d}\Omega\, \overline{|\mathcal{M}|^2} = \frac{g_w^2}{48\pi\cos^2\theta_w} m_Z (c_V^2 + c_A^2)\mathcal{N}_c(f), \qquad \text{(S10.1)}$$

where $\mathcal{N}_c(f)$ is the colour factor: 3 for quarks and 1 for leptons.

4. $\Gamma_\ell^{th}/\Gamma_\nu^{th}$ reduces to the ratio of $c_V^2 + c_A^2$ factors, i.e.

$$\frac{\Gamma_\ell^{th}}{\Gamma_\nu^{th}} = \frac{(c_V^2 + c_A^2)_\ell}{(c_V^2 + c_A^2)_\nu}.$$

According to Table 10.2, for neutrinos $c_V = c_A = 1/2$, so $c_V^2 + c_A^2 = 1/2$; for charged leptons, $c_V = -1/2 + 2\sin^2\theta_w$ and $c_A = -1/2$ so

$$c_V^2 + c_A^2 = 1/2 + 4\sin^4\theta_w - 2\sin^2\theta_w \approx 0.252.$$

As a result, $\Gamma_\ell^{th}/\Gamma_\nu^{th} \approx 0.504$. Using the measured values for $\Gamma_\ell = 83.984$ MeV and $\Gamma_{\mathrm{inv}} = 499.0$ MeV given in the text, we compute the number of neutrinos $N_\nu \approx 2.98$. We thus have three light neutrinos ($m_\nu \leq m_Z/2$).

5. Given our computation of the partial decay width $\Gamma_{Z\to f\bar{f}}$, the total decay width reads

$$\Gamma_Z = \frac{g_w^2}{48\pi\cos^2\theta_w} m_Z \times \sum_f \left[(c_V^f)^2 + (c_A^f)^2\right] \mathcal{N}_c(f),$$

where the sum runs over all fermions such that $m_f \leq m_Z/2$, i.e. all but the top quark. We have:

- 3 neutrinos with $c_V^2 + c_A^2 = 1/2$;
- 3 charged leptons with $c_V^2 + c_A^2 = 1/2 + 4\sin^4\theta_w - 2\sin^2\theta_w \approx 0.252$;
- 3 d-type quarks with $c_V^2 + c_A^2 = 1/2 + 4/9\ \sin^4\theta_w - 2/3\ \sin^2\theta_w \approx 0.370$;
- 2 u-type quarks with $c_V^2 + c_A^2 = 1/2 + 16/9\ \sin^4\theta_w - 4/3\ \sin^2\theta_w \approx 0.287$;

so the numerical factor from the sum amounts to ≈ 7.31. Using $M_Z = 91.19$ GeV and $g_w^2 = 0.426$, we finally find

$$\Gamma_Z^{th} \approx 2.45 \text{ GeV}.$$

It is close to the measured value $\Gamma_Z \approx 2.495$ GeV. The Z lifetime is

$$\tau_Z = \hbar/\Gamma_Z \approx 2.67 \cdot 10^{-25} \text{ s}.$$

10.5. The $W^\pm$ decay width of the W boson.
As in the previous problem, we neglect the masses of the decay products. We can re-use the previous calculations by comparing the coupling of fermions to the W boson with that to the Z boson. The vertex factors are given in sections 10.4.1 and 10.4.3 of the book. They are:

$$W: \ -i\frac{g_w}{\sqrt{2}}\gamma^\mu \frac{1}{2}(1-\gamma^5), \qquad Z: \ -i\frac{g_w}{\cos\theta_w}\gamma^\mu\frac{1}{2}(c_V - c_A\gamma^5).$$

Therefore, we go from Z to W by taking $c_V = c_A = 1$ and changing $g_w/\cos\theta_w$ for $g_w/\sqrt{2}$. The W^+ boson can decay into $f\bar{f}'$ with $f\bar{f}' = \bar{\nu}e^+$, $\nu_\mu\mu^+$, $\nu_\tau\tau^+$, $u\bar{d}$, $u\bar{s}$, $u\bar{b}$, $c\bar{d}$, $c\bar{s}$ and $c\bar{b}$. As before, for quarks, we have to take into account the colour factor. Moreover, the elements of the CKM mixing matrix must also be taken into account in the quark sector since we use mass eigenstates. Starting from Eq. (S10.1), we deduce

$$\Gamma_{W\to f\bar{f}'} = \frac{g_w^2}{96\pi} m_W \times \left[1^2 + 1^2\right] |V_{ff'}|^2 \mathcal{N}_c(f) = \frac{g_w^2}{48\pi} m_W |V_{ff'}|^2 \mathcal{N}_c(f),$$

with $\mathcal{N}_c(f) = 3$ for quarks and 1 for leptons, and V is the CKM matrix for quarks and the identity for leptons. Therefore, the total decay width is

$$\Gamma_W = \frac{g_w^2}{48\pi} m_W \sum_{f,f'} |V_{ff'}|^2 \mathcal{N}_c(f).$$

Note that for quarks,

$$\sum_{f,f'} |V_{ff'}|^2 \mathcal{N}_c(f) = 3\sum_{f,f'} |V_{ff'}|^2 = 3\left(\underbrace{|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2}_{=1} + \underbrace{|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2}_{=1}\right) = 6,$$

the factors 1 coming from the unitarity of the CKM matrix. For leptons,

$$\sum_{(f,f')=(\nu_e,e),(\nu_\mu,\mu),(\nu_\tau,\tau)} |V_{ff'}|^2 \mathcal{N}_c(f) = \sum_{(f,f')} 1 = 3.$$

It follows that

$$\Gamma_W = \frac{9g_w^2}{48\pi} m_W.$$

Using the numerical values, $g_w^2 = 0.426$ and $M_W = 80.38$ GeV, one finds $\Gamma_W = 2.04$ GeV, in good agreement with the experimental value $\Gamma_W = 2.08$ GeV.

11 Electroweak symmetry breaking

11.1. As $\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \mathring{\phi}} \mathring{\phi} + \mathring{\phi}^\dagger \frac{\partial \mathcal{L}}{\partial \mathring{\phi}^\dagger} - \mathcal{L}$, with $\mathcal{L} = \mathring{\phi}^\dagger \mathring{\phi} + \partial_i \phi^\dagger\, \partial^i \phi - V(\phi)$, it follows that

$$\begin{aligned}
\mathcal{H} &= \mathring{\phi}^\dagger \mathring{\phi} + \mathring{\phi}^\dagger \mathring{\phi} - (\mathring{\phi}^\dagger \mathring{\phi} + \partial_i \phi^\dagger\, \partial^i \phi - V(\phi)) \\
&= \mathring{\phi}^\dagger \mathring{\phi} - \partial_i \phi^\dagger\, \partial^i \phi + V(\phi) \\
&= \mathring{\phi}^\dagger \mathring{\phi} + \boldsymbol{\nabla} \phi^\dagger \cdot \boldsymbol{\nabla} \phi + \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \\
&= |\mathring{\phi}|^2 + |\boldsymbol{\nabla} \phi|^2 + \mu^2 |\phi|^2 + \lambda |\phi|^4 .
\end{aligned}$$

11.2. Let us insert $\phi(x) = \frac{1}{\sqrt{2}} [v + h(x)]\, e^{i\theta(x)}$ into $V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4$. As

$$\phi^\dagger(x) = \frac{1}{\sqrt{2}} [v + h(x)]\, e^{-i\theta(x)}$$

(remember, h and θ are real scalar fields), $\phi^\dagger \phi = (v + h)^2/2$. Therefore, given that $\mu^2 = -\lambda v^2$,

$$\begin{aligned}
V(\phi) &= (v+h)^2 \left(\tfrac{\mu^2}{2} + \tfrac{\lambda}{4} (v+h)^2 \right) \\
&= (v+h)^2 \left(-\tfrac{\lambda v^2}{2} + \tfrac{\lambda v^2}{4} + \tfrac{\lambda v h}{2} + \tfrac{\lambda h^2}{4} \right) \\
&= \left(v^2 + 2vh + h^2 \right) \left(-\tfrac{\lambda v^2}{4} + \tfrac{\lambda v h}{2} + \tfrac{\lambda h^2}{4} \right) \\
&= -\tfrac{\lambda}{4} v^4 + \lambda v^2 h^2 + \lambda v h^3 + \tfrac{\lambda}{4} h^4 .
\end{aligned}$$

The derivatives in the Lagrangian (11.1) read

$$\partial^\mu \phi^\dagger = \frac{1}{\sqrt{2}} \left[(\partial^\mu h) e^{-i\theta} - i(\partial^\mu \theta)(v+h) e^{-i\theta} \right], \quad \partial_\mu \phi = \frac{1}{\sqrt{2}} \left[(\partial_\mu h) e^{i\theta} + i(\partial_\mu \theta)(v+h) e^{i\theta} \right],$$

yielding for the Lagrangian,

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2} \partial^\mu h\, \partial_\mu h + \tfrac{1}{2} (v+h)^2 \partial^\mu \theta\, \partial_\mu \theta + \tfrac{\lambda v^4}{4} - \lambda v^2 h^2 - \lambda v h^3 - \tfrac{\lambda h^4}{4} \\
&= \tfrac{1}{2} \left(\partial_\mu h \partial^\mu h - \left[\sqrt{2} |\mu| \right]^2 h^2 \right) + \tfrac{1}{2} \partial_\mu \tilde{\theta}\, \partial^\mu \tilde{\theta} + \lambda \tfrac{v^4}{4} + \ldots ,
\end{aligned}$$

where $-\lambda v^2 = \mu^2 = -|\mu|^2$ and $\tilde{\theta} = v\theta$ have been used.

11.3. Conditions to ensure that photons stay massless.

1. Checking that $(\mathrm{D}_\mu \Phi)^\dagger = \partial_\mu \Phi^\dagger - i g_w \Phi^\dagger \frac{\sigma_i}{2} W^i_\mu - i \frac{g}{2} B_\mu \Phi^\dagger$ is straightforward. We just need to set $Y = 1$ as Φ has the hypercharge eigenvalue 1 and to remember that Φ has a doublet structure of isospin so that $(\sigma_i \Phi)^\dagger = \Phi^\dagger \sigma_i$, the Pauli matrices being hermitian. Note also that W^i_μ, $i = 1, 2, 3$ are also hermitian, whereas $W^\pm_\mu$ are not.
2. $(\mathrm{D}_\mu \Phi)^\dagger (\mathrm{D}^\mu \Phi)$ reads

$$(\mathrm{D}_\mu \Phi)^\dagger (\mathrm{D}^\mu \Phi) = \left(\partial_\mu \Phi^\dagger - i g_w \Phi^\dagger \frac{\sigma_i}{2} W^i_\mu - i \frac{g}{2} B_\mu \Phi^\dagger \right) \left(\partial^\mu \Phi + i g_w \frac{\sigma_i}{2} W^{\mu i} \Phi + i \frac{g}{2} B^\mu \Phi \right).$$

We need to find out where the $A_\mu A^\mu$ terms come from. Because of the mixing due to the Weinberg angle in Eq. (10.24), W^3_μcontributes to a $\sin\theta_w A_\mu$ term and B_μ to $\cos\theta_w A_\mu$. As a result, given that after spontaneous symmetry breaking, $\Phi = \frac{1}{\sqrt{2}}\binom{0}{v}$ implies that $\Phi^\dagger\Phi = v^2/2$, and $\Phi^\dagger\sigma_3\Phi = -v^2/2$, the terms depending on A^μ are

$$\begin{aligned}
-ig_w\Phi^\dagger \tfrac{\sigma_3}{2} W^3_\mu \times ig_w \tfrac{\sigma_3}{2} W^{3\mu}\Phi &\longrightarrow \tfrac{g_w^2}{4}\ \Phi^\dagger(\sigma_3)^2\Phi\ \sin^2\theta_w A_\mu A^\mu = \\
&\ \tfrac{g_w^2 v^2}{8}\ \sin^2\theta_w A_\mu A^\mu, \\
-i\tfrac{g}{2} B_\mu \Phi^\dagger \times i\tfrac{g}{2} B^\mu\Phi &\longrightarrow \tfrac{g^2}{4}\Phi^\dagger\Phi\cos^2\theta_w A_\mu A^\mu = \\
&\ \tfrac{g^2 v^2}{8}\ \cos^2\theta_w A_\mu A^\mu, \\
-ig_w\Phi^\dagger \tfrac{\sigma_3}{2} W^3_\mu \times i\tfrac{g}{2} B^\mu\Phi &\longrightarrow \tfrac{g_w g}{4}\ \Phi^\dagger\sigma_3\Phi\ \sin\theta_w\cos\theta_w A_\mu A^\mu = \\
&\ -\tfrac{g_w g v^2}{8}\ \sin\theta_w\cos\theta_w A_\mu A^\mu, \\
-i\tfrac{g}{2} B_\mu\Phi^\dagger \times ig_w\tfrac{\sigma_i}{2} W^{\mu i}\Phi &\longrightarrow -\tfrac{g_w g v^2}{8}\ \sin\theta_w\cos\theta_w A_\mu A^\mu,
\end{aligned}$$

so that we obtain globally

$$\frac{v^2}{8}\left(g^2\cos^2\theta_w + g_w^2\sin^2\theta_w - 2g_w g\sin\theta_w\cos\theta_w\right)\ A_\mu A^\mu.$$

The numerical factor in front of $A_\mu A^\mu$ is interpreted as $m_\gamma^2/2$. Using the relations $e = g\cos\theta_w = g_w\sin\theta_w$ we conclude that $m_\gamma^2 = 0$.

3. We have previously set $Y = 1$, obtaining a term in $\mathrm{D}^\mu\Phi$ of the form $i\frac{g}{2}B^\mu\Phi$. But the generic expression of $\mathrm{D}^\mu\Phi$ includes a term $i\frac{g}{2}YB^\mu\Phi$. Therefore, by changing $g \to gY$, we deduce the photon mass

$$\begin{aligned}
m_\gamma^2 &= \frac{v^2}{4}\left(g^2Y^2\cos^2\theta_w + g_w^2\sin^2\theta_w - 2g_w gY\sin\theta_w\cos\theta_w\right) \\
&= \frac{v^2e^2}{4}\left(Y^2 + 1 - 2Y\right) \\
&= \frac{v^2e^2}{4}(Y-1)^2.
\end{aligned}$$

The photon is massless if and only if the hypercharge of the Higgs doublet is 1. Recall that before spontaneous symmetry breaking, Φ writes in full generality

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},$$

ϕ_1 and ϕ_2 being two complex scalar fields. The Gell-Mann–Nishijima relation relates the charge of these fields to the hypercharge of the doublet:

$$Q(\phi_1) = (Y+1)/2 \quad \text{and} \quad Q(\phi_2) = (Y-1)/2.$$

With our spontaneous symmetry breaking prescription, the non-zero v.e.v. is carried by ϕ_2 so that the photon mass writes $m_\gamma = ev\,|Q(\phi_2)|$: the photon is massless if and only if the vacuum expectation value is carried by the neutral component. Since the Higgs boson is a radial excitation around the vacuum expectation value, it carries the same charge as the initial field. Therefore, the photon is massless if and only if the Higgs is neutral.

11.4. Higgs boson decay $H \to f(p_1) + \bar{f}(p_2)$.

1. The diagram corresponds to a simple two-body decay. Using the Feynman rules, the amplitude reads
$$i\mathcal{M} = -i\left(\sqrt{2}G_F\right)^{1/2} m_f\, \bar{u}(p_1)\cdot v(p_2).$$
2. The squared amplitude reads
$$|\mathcal{M}|^2 = \sqrt{2}G_F\, m_f^2\, \bar{u}(p_1)v(p_2)\, \bar{v}(p_2)u(p_1).$$
The spin/colour-averaged amplitude squared is then
$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \sum_{\text{pol.}H}\sum_{\text{pol.}f,\bar{f}}\sum_{\text{colors}} |\mathcal{M}|^2\\ &= \mathcal{N}_c(f)\sum_{\text{pol.}f,\bar{f}} \sqrt{2}G_F\, m_f^2\, \bar{u}(p_1)v(p_2)\, \bar{v}(p_2)u(p_1),\end{aligned}$$
given that there is only one polarisation for the Higgs. Of course, the colour factor is 1 for leptons. Concerning quarks, there are three colours and three anti-colours, so nine combinations. However, only three ($r\bar{r}$, $g\bar{g}$ and $b\bar{b}$) are possible since the initial state is colourless. As a result, $\mathcal{N}_c(q) = 3$. The sum over the fermion polarisations yields a trace [see Eq. (6.84) with $\Gamma_1 = \Gamma_2 = \mathbb{1}$],
$$\begin{aligned}\textstyle\sum_{\text{pol.}f,\bar{f}} \bar{u}(p_1)v(p_2)\, \bar{v}(p_2)u(p_1) &= \text{Tr}\left[\left(\not{p}_1 + m_f\right)\left(\not{p}_2 - m_f\right)\right]\\ &= \text{Tr}\left[\not{p}_1\not{p}_2 + m_f\left(\not{p}_1 - \not{p}_2\right) - m_f^2\mathbb{1}\right].\end{aligned}$$
As $\text{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $\text{Tr}(\gamma^\mu) = 0$, we conclude that
$$\text{Tr}\left[\not{p}_1\not{p}_2 + m_f\left(\not{p}_1 - \not{p}_2\right) - m_f^2\mathbb{1}\right] = 4p_1\cdot p_2 - 4m_f^2,$$
yielding
$$\overline{|\mathcal{M}|^2} = 4\sqrt{2}m_f^2 G_F(p_1\cdot p_2 - m_f^2)\mathcal{N}_c(f).$$
3. In the Higgs rest frame, the two fermions are emitted back-to-back, so the expression of the momenta are $p_1 = (E^*, \boldsymbol{p}^*)$ and $p_2 = (E^*, -\boldsymbol{p}^*)$, with $E^* = m_H/2$ and $|\boldsymbol{p}^*|^2 = E^{*2} - m_f^2 = m_H^2/4 - m_f^2$. Therefore, $p_1\cdot p_2 = m_H^2/2 - m_f^2$ and
$$\overline{|\mathcal{M}|^2} = 4\sqrt{2}m_f^2 G_F\left(\frac{m_H^2}{2} - 2m_f^2\right)\cdot\mathcal{N}_c(f) = 2\sqrt{2}m_f^2 m_H^2 G_F\left(1 - \frac{4m_f^2}{m_H^2}\right)\cdot\mathcal{N}_c(f).$$
The decay partial width reads
$$\Gamma_{H\to f\bar{f}} = \frac{|\boldsymbol{p}^*|}{32\pi^2 s}\int \mathrm{d}\Omega\, \overline{|\mathcal{M}|^2},$$
which leads to
$$\Gamma(H\to f\bar{f}) = \frac{G_F}{4\pi\sqrt{2}}M_H\, m_f^2\left(1 - \frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}},$$
given that $s = m_H^2$ and there is no angular dependence in $\overline{|\mathcal{M}|^2}$.

12 The Standard Model and beyond

12.1. Assuming the lepton universality, i.e. $\Gamma_{\ell\ell} = \Gamma_{ee} = \Gamma_{\mu\mu}$ and $\Gamma_{\tau\tau} = \Gamma_{\ell\ell}(1+\delta_\tau)$ to take into account the difference of phase space due to the large mass of τ's, the total decay width is by definition

$$\Gamma_{\text{tot}} = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\text{had}} + \Gamma_{\text{inv}} = 3\Gamma_{\ell\ell} + \Gamma_{\ell\ell}\delta_\tau + \Gamma_{\tau\tau} + \Gamma_{\text{had}},$$

implying

$$\frac{\Gamma_{\text{inv}}}{\Gamma_{\ell\ell}} = \frac{\Gamma_{\text{tot}}}{\Gamma_{\ell\ell}} - \frac{\Gamma_{\text{had}}}{\Gamma_{\ell\ell}} - 3 - \delta_\tau.$$

Now, from Eq. (12.12),

$$\Gamma_{\text{tot}}^2 = \frac{12\pi}{m_Z^2}\frac{\Gamma_{ee}\Gamma_{\text{had}}}{\sigma^0_{\text{had}}} = \frac{12\pi}{m_Z^2}\frac{\Gamma_{\ell\ell}\Gamma_{\text{had}}}{\sigma^0_{\text{had}}},$$

so that

$$\frac{\Gamma_{\text{inv}}}{\Gamma_{\nu\nu}} = \sqrt{\frac{12\pi}{m_Z^2\sigma^0_{\text{had}}}\frac{\Gamma_{\text{had}}}{\Gamma_{\ell\ell}}} - \frac{\Gamma_{\text{had}}}{\Gamma_{\ell\ell}} - 3 - \delta_\tau.$$

12.2. The reaction $e^-(k) + e^+(k') \to \mu^-(p) + \mu^+(p')$.

1. The γ or Z^0 exchange proceeds via s-channel diagrams. The amplitudes are

$$\begin{aligned} i\mathcal{M}_\gamma &= [\bar u(p)\,(ie\gamma^\mu)\,v(p')]\left(-i\tfrac{g_{\mu\nu}}{q^2}\right)[\bar v(k')\,(ie\gamma^\nu)\,u(k)]\,,\\ i\mathcal{M}_Z &= \left[\bar u(p)\left(-i\tfrac{g_w}{\cos\theta_w}\gamma^\mu\tfrac{c_V^\mu - c_A^\mu\gamma^5}{2}\right)v(p')\right]\left(i\tfrac{-g_{\mu\nu}+q_\mu q_\nu/m_Z^2}{q^2-m_z^2}\right)\left[\bar v(k')\left(-i\tfrac{g_w}{\cos\theta_w}\gamma^\nu\tfrac{c_V^e - c_A^e\gamma^5}{2}\right)u(k)\right], \end{aligned}$$

where $q = p + p'$. After rearrangement, they read

$$\begin{aligned} \mathcal{M}_\gamma &= \tfrac{e^2}{q^2}\,[\bar u(p)\gamma^\mu v(p')]\left[\bar v(k')\gamma_\mu u(k)\right],\\ \mathcal{M}_Z &= \tfrac{g_w^2}{4\cos^2\theta_w}\left[\bar u(p)\gamma^\mu(c_V^\mu - c_A^\mu\gamma^5)v(p')\right]\left(\tfrac{g_{\mu\nu}-q_\mu q_\nu/m_Z^2}{q^2-m_Z^2+im_Z\Gamma_Z}\right)\left[\bar v(k')\gamma^\nu(c_V^e - c_A^e\gamma^5)u(k)\right]. \end{aligned}$$

2. Since we neglect the lepton masses, the Dirac equation implies $\bar u(p)\not{p} = \not{p}'v(p') = \not{k}u(k) = \bar v(k')\not{k}' = 0$. We have already seen in previous problems that this implies that the term $q^\mu q^\nu/m_Z^2$ from the Z^0 propagator does not contribute to the amplitude. (See, for example, Problem 9.2). It follows that

$$\mathcal{M}_Z = \frac{g_w^2}{4\cos^2\theta_w(q^2-m_Z^2+im_Z\Gamma_Z)}\left[\bar u(p)\gamma^\mu(c_V^\mu - c_A^\mu\gamma^5)v(p')\right]\left[\bar v(k')\gamma_\mu(c_V^e - c_A^e\gamma^5)u(k)\right].$$

3. Masses being neglected, the helicity eigenstates match the chirality eigenstates. In this context, we learned in Chapter 6 that necessarily the helicity of the anti-fermion

is opposite to that of the fermion. Therefore, introducing $\alpha = e^2/(4\pi)$ and the Mandelstam variable $s = q^2$, $\mathcal{M}_\gamma$ reads

$$\begin{aligned}\mathcal{M}_\gamma &= \tfrac{4\pi\alpha}{s}\left[\overline{u_L}(p)\gamma^\mu v_R(p') + \overline{u_R}(p)\gamma^\mu v_L(p')\right]\left[\overline{v_L}(k')\gamma_\mu u_R(k) + \overline{v_R}(k')\gamma_\mu u_L(k)\right] \\ &= \tfrac{4\pi\alpha}{s}\left(A_{LL} + A_{LR} + A_{RL} + A_{RR}\right),\end{aligned} \tag{S12.1}$$

with the variables $A_{LL}, A_{LR}, A_{RL}, A_{RR}$ given in the text. Concerning $\mathcal{M}_Z$, with the definition of c_R and c_L, $c_V = (c_L + c_R)/2$ and $c_A = (c_L - c_R)/2$. Therefore,

$$c_V - c_A\gamma^5 = c_L\frac{1-\gamma^5}{2} + c_R\frac{1+\gamma^5}{2} = c_L P_L + c_R P_R,$$

where P_L and P_R are the usual chirality projectors. Moreover, using the definition of G_F and the relation coming from the electroweak unification,

$$\left.\begin{aligned}\tfrac{G_F}{\sqrt{2}} &= \tfrac{g_w^2}{8m_W^2} \\ m_Z &= \tfrac{m_W}{\cos\theta_w}\end{aligned}\right\} \Rightarrow \frac{g_w^2}{\cos^2\theta_w} = \frac{8G_F}{\sqrt{2}}m_z^2,$$

it follows that

$$\begin{aligned}\mathcal{M}_Z &= \tfrac{\sqrt{2}G_F m_Z^2}{s - m_Z^2 + i m_Z\Gamma_Z}\left[\bar{u}(p)\gamma^\mu(c_L^\mu P_L + c_R^\mu P_R)v(p')\right]\left[\bar{v}(k')\gamma_\mu(c_L^e P_L + c_R^e P_R)u(k)\right] \\ &= \tfrac{4\pi\alpha}{s}\chi\left[\bar{u}(p)\gamma^\mu(c_L^\mu P_L + c_R^\mu P_R)v(p')\right]\left[\bar{v}(k')\gamma_\mu(c_L^e P_L + c_R^e P_R)u(k)\right],\end{aligned}$$

where χ is the variable defined in the text. Now, as $\gamma^\mu P_{L/R} = P_{R/L}\gamma^\mu$, note that

$$\begin{aligned}\bar{u}(p)\gamma^\mu(c_L^\mu P_L + c_R^\mu P_R)v(p') &= \bar{u}(p)\gamma^\mu(c_L^\mu P_L P_L + c_R^\mu P_R P_R)v(p') \\ &= c_L^\mu\bar{u}(p)P_R\gamma^\mu v_R(p') + c_R^\mu\bar{u}(p)P_L\gamma^\mu v_L(p').\end{aligned}$$

Given that

$$\bar{u}(p)P_R = u^\dagger(p)\gamma^0 P_R = u^\dagger(p)P_L\gamma^0 = (P_L u(p))^\dagger\gamma^0 = \overline{u_L}(p)$$

and similarly, $\bar{u}(p)P_L = \overline{u_R}(p)$, $\bar{v}(k')P_L = \overline{v_L}(k')$, $\bar{v}(k')P_R = \overline{v_R}(k')$, we deduce

$$\begin{aligned}\bar{u}(p)\gamma^\mu(c_L^\mu P_L + c_R^\mu P_R)v(p') &= c_L^\mu\overline{u_L}(p)\gamma^\mu v_R(p') + c_R^\mu\overline{u_R}(p)\gamma^\mu v_L(p'), \\ \bar{v}(k')\gamma_\mu(c_L^e P_L + c_R^e P_R)u(k) &= c_L^e\overline{v_R}(k')\gamma_\mu u_L(k) + c_R^e\overline{v_L}(k')\gamma_\mu u_R(k).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{M}_Z &= \tfrac{4\pi\alpha}{s}\chi\left[c_L^\mu\overline{u_L}(p)\gamma^\mu v_R(p') + c_R^\mu\overline{u_R}(p)\gamma^\mu v_L(p')\right]\times \\ &\qquad\left[c_L^e\overline{v_R}(k')\gamma_\mu u_L(k) + c_R^e\overline{v_L}(k')\gamma_\mu u_R(k)\right] \\ &= \tfrac{4\pi\alpha}{s}\chi\left[c_L^e c_L^\mu A_{LL} + c_R^e c_L^\mu A_{RL} + c_L^e c_R^\mu A_{LR} + c_R^e c_R^\mu A_{RR}\right].\end{aligned} \tag{S12.2}$$

4. Most of the calculation has already been done in Problem 6.11, where we showed in Eqs. (S6.4) and (S6.6) that

$$\begin{aligned}\tfrac{e^2}{s}\left[\overline{v_R}_{\mathrm{e}}(k)\gamma^\mu u_{L_{\mathrm{e}}}(p)\right]\left[\bar{u}_{L_{\mathrm{mu}}}(p')\gamma_\mu v_{R_{\mathrm{mu}}}(k')\right] &= -e^2(1+\cos\theta), \\ \tfrac{e^2}{s}\left[\overline{v_R}_{\mathrm{e}}(k)\gamma^\mu u_{L_{\mathrm{e}}}(p)\right]\left[\bar{u}_{R_{\mathrm{mu}}}(p')\gamma_\mu v_{L_{\mathrm{mu}}}(k')\right] &= e^2(1-\cos\theta).\end{aligned}$$

The momentum labels in Problem 6.11 were $e^-(p) + e^+(k) \to \mu^-(p') + \mu^+(k')$ while in this problem they are $e^-(k) + e^+(k') \to \mu^-(p) + \mu^+(p')$. Therefore, we conclude

$$\begin{aligned}\left[\overline{v_R}(k')\gamma^\mu u_L(k)\right]\left[\bar{u}_L(p)\gamma_\mu v_R(p')\right] &= A_{LL} = -s(1+\cos\theta), \\ \left[\overline{v_R}(k')\gamma^\mu u_L(k)\right]\left[\bar{u}_R(p)\gamma_\mu v_L(p')\right] &= A_{LR} = s(1-\cos\theta).\end{aligned} \tag{S12.3}$$

Problem 6.11 also showed that $A_{RR} = A_{LL}$ and $A_{RL} = A_{LR}$.

5. Masses being neglected, in the centre-of-mass frame,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2.$$

As the quantities A_{LL} or A_{LR} do not depend on the azimutal angle,

$$\frac{d\sigma}{d(\cos\theta)} = \frac{1}{32\pi s}|\mathcal{M}|^2.$$

The reaction $e_L^- + e_R^+ \to \mu_L^- + \mu_R^+$ involves the term with A_{LL} in Eq. (S12.2). Therefore,

$$\mathcal{M} = \mathcal{M}_\gamma + \mathcal{M}_Z = \frac{4\pi\alpha}{s}A_{LL} + \frac{4\pi\alpha}{s}\chi c_L^e c_L^\mu A_{LL} = \frac{4\pi\alpha}{s}A_{LL}\left(1 + \chi c_L^e c_L^\mu\right),$$

yielding

$$\frac{d\sigma}{d(\cos\theta)}(e_L^- e_R^+ \to \mu_L^- \mu_R^+) = \frac{\pi\alpha^2}{2s^3}|A_{LL}|^2 \left|1 + \chi c_L^e c_L^\mu\right|^2 = \frac{\pi\alpha^2}{2s}(1 + \cos\theta)^2 \left|1 + \chi c_L^e c_L^\mu\right|^2.$$

For the pure QED process, we can simply set $c_L^e c_L^\mu = 0$. Therefore, the ratio of differential cross-sections is

$$r \equiv \frac{\frac{d\sigma}{d(\cos\theta)}}{\frac{d\sigma^{\text{QED}}}{d(\cos\theta)}} = \left|1 + \chi c_L^e c_L^\mu\right|^2.$$

If we neglect Γ_Z, χ is real and the constraint $r = 2$ implies

$$\chi \simeq \frac{\sqrt{2}G_F m_Z^2}{s - m_Z^2}\frac{s}{4\pi\alpha} = \frac{\pm\sqrt{2} - 1}{c_L^e c_L^\mu}.$$

Table 10.2 tells us that $c_L^e = c_L^\mu = c_V + c_A \simeq -0.54$, implying $\chi = 1.43$ or $\chi = -8.36$. Only the latter value leads to a positive Mandelstam constant,

$$s = \frac{4\pi\alpha m_Z^2 \chi}{4\pi\alpha\chi - \sqrt{2}G_F m_Z^2} \simeq 7053\ \text{GeV}^2,$$

leading to $E = \sqrt{s} = 84$ GeV. At this value, the weak interaction (Z exchange) and the QED interaction (γ exchange) contribute equally to the cross-section.

6. For the reaction $e_L^- + e_R^+ \to \mu_R^- + \mu_L^+$, only the term with A_{LR} in Eq. (S12.2) matters. As in the previous question, the differential cross-section is then

$$\frac{d\sigma}{d(\cos\theta)}(e_L^- e_R^+ \to \mu_R^- \mu_L^+) = \frac{\pi\alpha^2}{2s^3}|A_{LR}|^2 \left|1 + \chi c_L^e c_R^\mu\right|^2 = \frac{\pi\alpha^2}{2s}(1 - \cos\theta)^2 \left|1 + \chi c_L^e c_R^\mu\right|^2.$$

The cross-section vanishes when $\theta = 0$. In the reaction $e_L^- + e_R^+ \to \mu_R^- + \mu_L^+$, if the z-axis is along the electron direction, both e^- and e^+ have a spin projection $s_z = -1/2$, so a total spin projection $s_z = -1$. On the other hand, at $\theta = 0$, the final state $\mu_R^- \mu_L^+$ would have $s_z = +1$, which is forbidden by angular momentum conservation.

7. For the reaction $e_L^- + e_R^+ \to \mu^- + \mu^+$, the muon can be left or right-handed. Therefore, both terms with A_{LL} and A_{LR} must be taken into account, such that

$$\begin{aligned}|\mathcal{M}|^2 &= |\mathcal{M}_{LL}|^2 + |\mathcal{M}_{LR}|^2 \\ &= \left(\tfrac{4\pi\alpha}{s}\right)^2 \left[\left(A_{LL} + \chi c_L^e c_L^\mu A_{LL}\right)^2 + \left(A_{LR} + \chi c_L^e c_R^\mu A_{LR}\right)^2\right].\end{aligned}$$

This leads to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}(e_L^- e_R^+ \to \mu^-\mu^+) = \frac{\pi\alpha^2}{2s}\left[(1+\cos\theta)^2\left|1+\chi c_L^e c_L^\mu\right|^2 + (1-\cos\theta)^2\left|1+\chi c_L^e c_R^\mu\right|^2\right].$$

Similarly,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}(e_R^- e_L^+ \to \mu^-\mu^+) = \frac{\pi\alpha^2}{2s}\left[(1+\cos\theta)^2\left|1+\chi c_R^e c_R^\mu\right|^2 + (1-\cos\theta)^2\left|1+\chi c_R^e c_L^\mu\right|^2\right].$$

12.3. In this problem, the amplitude is reduced to Eq. (S12.2) since the photon exchange is neglected.

This problem uses notations and results from Problem 12.2. We are now interested in the reaction $e^- + e^+ \to \mu^- + \mu^+$ for $\sqrt{s} \simeq m_Z$. Therefore, the γ exchange contribution can be safely neglected.

1. The polarised amplitudes squared are thus given by

$$\begin{aligned}|\mathcal{M}_{LL}|^2 &= \left(\tfrac{4\pi\alpha}{s}\right)^2 |\chi|^2 \left(c_L^e c_L^\mu\right)^2 |A_{LL}|^2, \quad & |\mathcal{M}_{RR}|^2 &= \left(\tfrac{4\pi\alpha}{s}\right)^2 |\chi|^2 \left(c_R^e c_R^\mu\right)^2 |A_{RR}|^2, \\ |\mathcal{M}_{LR}|^2 &= \left(\tfrac{4\pi\alpha}{s}\right)^2 |\chi|^2 \left(c_L^e c_R^\mu\right)^2 |A_{LR}|^2, \quad & |\mathcal{M}_{RL}|^2 &= \left(\tfrac{4\pi\alpha}{s}\right)^2 |\chi|^2 \left(c_R^e c_L^\mu\right)^2 |A_{RL}|^2.\end{aligned}$$

The quantities $|A_{LL}|^2 = |A_{RR}|^2 = s^2(1+\cos\theta)^2$ and $|A_{LR}|^2 = |A_{RL}|^2 = s^2(1-\cos\theta)^2$ are given in Eq. (S12.3). Now,

$$\left(\frac{4\pi\alpha}{s}\right)^2 |\chi|^2 = \left(\frac{4\pi\alpha}{s}\right)^2 \left|\frac{\sqrt{2}G_F m_Z^2}{s - m_Z^2 + i m_Z \Gamma_Z}\frac{s}{4\pi\alpha}\right|^2 = \left|\frac{\sqrt{2}G_F m_Z^2}{s - m_Z^2 + i m_Z\Gamma_Z}\right|^2 = |C|^2,$$

where C is given in Eq. (12.17). Hence, the polarised amplitudes squared correspond to Eq. (12.16).

2. The muon polarisations being ignored (i.e. summed over), if the initial electron is left-handed, the corresponding amplitude squared reads

$$\begin{aligned}|\mathcal{M}_L|^2 &= |\mathcal{M}_{LL}|^2 + |\mathcal{M}_{LR}|^2 \\ &= s^2|C|^2\left[(c_L^e c_L^\mu)^2(1+\cos\theta)^2 + (c_L^e c_R^\mu)^2(1-\cos\theta)^2\right] \\ &= s^2|C|^2(c_L^e)^2\left[(c_L^\mu)^2 + (c_R^\mu)^2 + \left((c_L^\mu)^2 + (c_R^\mu)^2\right)\cos^2\theta + 2\left((c_L^\mu)^2 - (c_R^\mu)^2\right)\cos\theta\right] \\ &= s^2|C|^2(c_L^e)^2\left((c_L^\mu)^2 + (c_R^\mu)^2\right)\left[1+\cos^2\theta + 2\mathcal{A}_\mu\cos\theta\right],\end{aligned}$$

where $\mathcal{A}_\mu$ is defined in Eq. (12.19). Similarly, for a right-handed electron,

$$\begin{aligned}|\mathcal{M}_R|^2 &= |\mathcal{M}_{RR}|^2 + |\mathcal{M}_{RL}|^2 \\ &= s^2|C|^2\left[(c_R^e c_R^\mu)^2(1+\cos\theta)^2 + (c_R^e c_L^\mu)^2(1-\cos\theta)^2\right] \\ &= s^2|C|^2(c_R^e)^2\left[(c_L^\mu)^2 + (c_R^\mu)^2 + \left((c_L^\mu)^2 + (c_R^\mu)^2\right)\cos^2\theta - 2\left((c_L^\mu)^2 - (c_R^\mu)^2\right)\cos\theta\right] \\ &= s^2|C|^2(c_R^e)^2\left((c_L^\mu)^2 + (c_R^\mu)^2\right)\left[1+\cos^2\theta - 2\mathcal{A}_\mu\cos\theta\right].\end{aligned}$$

The positron is unpolarised. Therefore, we have to average the amplitude squared.

On the other hand, the electron is polarised, such as $P_e = 1$ if the electron beam is 100% right-handed and $P_e = -1$ if it is left-handed. The spin-averaged amplitude squared for a given polarisation p_e of the electron is thus

$$|\overline{\mathcal{M}}(P_e)|^2 = \frac{1}{2}\left(\frac{1+P_e}{2}|\mathcal{M}_R|^2 + \frac{1-P_e}{2}|\mathcal{M}_L|^2\right).$$

Note that if $P_e = 0$, the electron beam is not polarised, and we recover the usual formula $|\overline{\mathcal{M}}|^2 = \frac{1}{2}\frac{1}{2}(|\mathcal{M}_R|^2 + |\mathcal{M}_L|^2)$. Inserting the expressions of $|\mathcal{M}_R|^2$ and $|\mathcal{M}_L|^2$ determined above, it follows that

$$\begin{aligned}|\overline{\mathcal{M}}(Pe)|^2 = \tfrac{s^2|C|^2}{4}\left((c_L^\mu)^2 + (c_R^\mu)^2\right)\Big\{ &\left[(c_L^e)^2 + (c_R^e)^2 - P_e\left((c_L^e)^2 - (c_R^e)^2\right)\right](1+\cos^2\theta) \\ &+2\left[(c_L^e)^2 - (c_R^e)^2 - P_e\left((c_L^e)^2 + (c_R^e)^2\right)\right]\mathcal{A}_\mu\cos\theta\Big\},\end{aligned}$$

i.e.

$$\begin{aligned}|\overline{\mathcal{M}}(Pe)|^2 = \tfrac{s^2|C|^2}{4}\; &\left((c_L^\mu)^2 + (c_R^\mu)^2\right)\left((c_L^e)^2 + (c_R^e)^2\right)\times \\ &\left\{(1-P_e\mathcal{A}_e)(1+\cos^2\theta) + 2(\mathcal{A}_e - P_e)\mathcal{A}_\mu\cos\theta\right\}.\end{aligned}$$

3. The expression of the cross-section is thus

$$\begin{aligned}\tfrac{d\sigma}{d(\cos\theta)} = \tfrac{1}{32\pi s}|\overline{\mathcal{M}}(Pe)|^2 = \tfrac{s|C|^2}{128\pi}\; &\left((c_L^\mu)^2 + (c_R^\mu)^2\right)\left((c_L^e)^2 + (c_R^e)^2\right)\times \\ &\left\{(1-P_e\mathcal{A}_e)(1+\cos^2\theta) + 2(\mathcal{A}_e - P_e)\mathcal{A}_\mu\cos\theta\right\}.\end{aligned} \tag{S12.4}$$

By definition,

$$\begin{aligned}\sigma_{\text{tot}} = \int_{-1}^{1} d(\cos\theta)\,\tfrac{d\sigma}{d(\cos\theta)}\Big|_{P_e=0} = \tfrac{s|C|^2}{128\pi}\; &\left((c_L^\mu)^2 + (c_R^\mu)^2\right)\left((c_L^e)^2 + (c_R^e)^2\right)\times \\ &\int_{-1}^{1}\left(1+\cos^2\theta + 2\mathcal{A}_e\mathcal{A}_\mu\cos\theta\right)d(\cos\theta),\end{aligned}$$

where the integral is just $\left[x + \frac{x^3}{3} + \mathcal{A}_e\mathcal{A}_\mu x^2\right]_{x=-1}^{x=1} = 8/3$. Therefore,

$$\frac{d\sigma}{d\cos\theta} = \frac{3}{8}\sigma_{\text{tot}}\left[(1-P_e\mathcal{A}_e)(1+\cos^2\theta) + 2(\mathcal{A}_e - P_e)\mathcal{A}_\mu\cos\theta\right].$$

12.4. According to Eq. (10.38), the $T_3 = 1/2$ component of the weak isospin doublet satisfies

$$c_V^f/c_A^f = 1 - 4\sin^2\theta_w Q,$$

while the $T_3 = -1/2$ component satisfies

$$c_V^{f'}/c_A^{f'} = 1 + 4\sin^2\theta_w Q.$$

Since for charged fermions, the up component of the isodoublet always has a positive electric charge and the down component a negative charge, both equalities above can be summarised as

$$c_V^f/c_A^f = 1 - 4\sin^2\theta_w|Q|,$$

which implies Eq. (12.20).

12.5. The reaction $e^- + e^+ \to f + \bar{f}$. With the results of the previous problem, we just have to change μ for f in Eq. (S12.4). Therefore, given the calculation of $\sigma_{\rm tot}$, checking Eq. (12.21) for σ_F and σ_B is straightforward since

$$\sigma_F = \int_0^1 \mathrm{d}(\cos\theta)\ \frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\Big|_{P_e=0} = \frac{3}{8}\sigma_{\rm tot}\left[x + \frac{x^3}{3} + \mathcal{A}_e\mathcal{A}_f x^2\right]_{x=0}^{x=1} = \frac{3}{8}\sigma_{\rm tot}\left(\frac{4}{3} + \mathcal{A}_e\mathcal{A}_f\right)$$

and

$$\sigma_B = \int_{-1}^0 \mathrm{d}(\cos\theta)\ \frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\Big|_{P_e=0} = \frac{3}{8}\sigma_{\rm tot}\left[x + \frac{x^3}{3} + \mathcal{A}_e\mathcal{A}_f x^2\right]_{x=-1}^{x=0} = \frac{3}{8}\sigma_{\rm tot}\left(\frac{4}{3} - \mathcal{A}_e\mathcal{A}_\mu\right).$$

Therefore,

$$A_{FB}^f = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_\mu.$$

Concerning, σ_R and σ_L, they are defined by $\sigma_{R/L} = \int_{-1}^1 \mathrm{d}(\cos\theta)\ \frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)}\Big|_{P_e=\pm1}$. Using Eq. (S12.4), this yields

$$\begin{aligned}\sigma_R &= \textstyle\int_{-1}^1 \frac{3}{8}\sigma_{\rm tot}\left[(1-\mathcal{A}_e)(1+\cos^2\theta) + 2(\mathcal{A}_e - 1)\mathcal{A}_f\cos\theta\right]\mathrm{d}(\cos\theta)\\ &= \textstyle\frac{3}{8}\sigma_{\rm tot}\left[(1-\mathcal{A}_e)(x + \frac{x^3}{3}) + (\mathcal{A}_e - 1)\mathcal{A}_f x^2\right]_{x=-1}^{x=1}\\ &= \sigma_{\rm tot}(1-\mathcal{A}_e).\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_L &= \textstyle\int_{-1}^1 \frac{3}{8}\sigma_{\rm tot}\left[(1+\mathcal{A}_e)(1+\cos^2\theta) + 2(\mathcal{A}_e + 1)\mathcal{A}_f\cos\theta\right]\mathrm{d}(\cos\theta)\\ &= \textstyle\frac{3}{8}\sigma_{\rm tot}\left[(1+\mathcal{A}_e)(x + \frac{x^3}{3}) + (\mathcal{A}_e + 1)\mathcal{A}_f x^2\right]_{x=-1}^{x=1}\\ &= \sigma_{\rm tot}(1+\mathcal{A}_e).\end{aligned}$$

Consequently,

$$A_{LR}^f = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \mathcal{A}_e.$$