

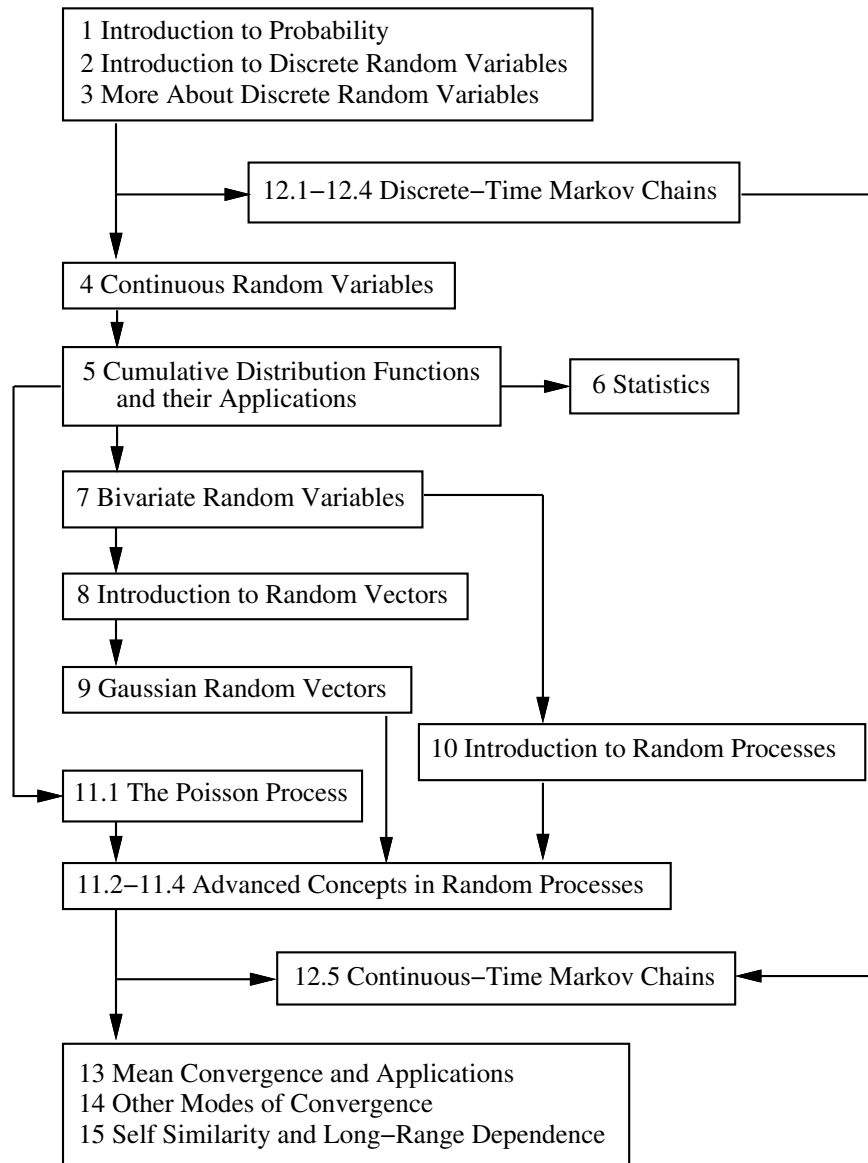
# **Probability and Random Processes for Electrical and Computer Engineers**

**John A. Gubner  
University of Wisconsin–Madison**

---

## Chapter Dependencies

---



---

---

# Preface

---

---

## *Intended Audience*

This book is a primary text for **graduate-level courses** in probability and random processes that are typically offered in electrical and computer engineering departments. The text starts from first principles and contains more than enough material for a two-semester sequence. The **level of the text** varies from advanced undergraduate to graduate as the material progresses. The principal **prerequisite** is the usual undergraduate electrical and computer engineering course on signals and systems, e.g., Haykin and Van Veen [25] or Oppenheim and Willsky [39] (see the Bibliography at the end of the book). However, later chapters that deal with random vectors assume some familiarity with linear algebra; e.g., determinants and matrix inverses.

## *How to Use the Book*

**A First Course.** In a course that assumes at most a modest background in probability, the core of the offering would include Chapters 1–5 and 7. These cover the basics of probability and discrete and continuous random variables. As the Chapter Dependencies graph on the preceding page indicates, there is considerable flexibility in the selection and ordering of additional material as the instructor sees fit.

**A Second Course.** In a course that assumes a solid background in the basics of probability and discrete and continuous random variables, the material in Chapters 1–5 and 7 can be reviewed quickly. In such a review, the instructor may want include **sections and problems marked with a \***, as these indicate more challenging material that might not be appropriate in a first course. Following the review, the core of the offering would include Chapters 8, 9, 10 (Sections 10.1–10.6), and Chapter 11. Additional material from Chapters 12–15 can be included to meet course goals and objectives.

**Level of Course Offerings.** In any course offering, the level can be adapted to the background of the class by omitting or including the **more advanced sections, remarks, and problems that are marked with a \***. In addition, discussions of a highly technical nature are placed in a Notes section at the end of the chapter in which they occur. Pointers to these discussions are indicated by **boldface numerical superscripts** in the text. These notes can be omitted or included as the instructor sees fit.

## *Chapter Features*

- Key equations are boxed:

$$P(A|B) := \frac{P(A \cap B)}{P(B)}.$$

- Important text passages are highlighted:

Two events  $A$  and  $B$  are said to be independent if  $P(A \cap B) = P(A)P(B)$ .

- Tables of discrete random variables and of Fourier transform pairs are found inside the front cover. A table of continuous random variables is found inside the back cover.
- The index was compiled as the book was written. Hence, there are many cross-references to related information. For example, see “chi-squared random variable.”
- When cumulative distribution functions or other functions are encountered that do not have a closed form, MATLAB commands are given for computing them; see “Matlab commands” in the index for a list. The use of many commands is illustrated in the examples and the problems throughout most of the text. Although some commands require the MATLAB Statistics Toolbox, alternative methods are also suggested; e.g., the use of `erf` and `erfinv` for `normcdf` and `norminv`.
- Each chapter contains a **Notes** section. Throughout each chapter, numerical superscripts refer to discussions in the Notes section. These notes are usually rather technical and address subtleties of the theory.
- Each chapter contains a **Problems** section. There are more than 800 problems throughout the book. Problems are grouped according to the section they are based on, and this is clearly indicated. This enables the student to refer to the appropriate part of the text for background relating to particular problems, and it enables the instructor to make up assignments more quickly. In chapters intended for a first course, **the more challenging problems are marked with a \***. Problems requiring MATLAB are indicated by the label **MATLAB**.
- Each chapter contains an **Exam Preparation** section. This serves as a chapter summary, drawing attention to key concepts and formulas.

### **Acknowledgements**

The writing of this book has been greatly improved by the suggestions of many people.

At the University of Wisconsin–Madison, the sharp eyes of the students in my classes on probability and random processes, my research students, and my postdocs have helped me fix countless typos and improve explanations of several topics. My colleagues here have been generous with their comments and suggestions. Professor Rajeev Agrawal, now with Motorola, convinced me to treat discrete random variables before continuous random variables. Discussions with Professor Bob Barmish on robustness of rational transfer functions led to Problems 38–40 in Chapter 5. I am especially grateful to Professors Jim Bucklew, Yu Hen Hu, and Akbar Sayeed, who taught from early, unpolished versions of the manuscript.

Colleagues at other universities and students in their classes have also been generous with their support. I thank Professors Toby Berger, Edwin Chong, and Dave Neuhoﬀ, who have used recent manuscripts in teaching classes on probability and random processes and have provided me with detailed reviews. Special thanks go to Professor Tom Denney for his multiple careful reviews of each chapter.

Since writing is a solitary process, I am grateful to be surrounded by many supportive family members. I especially thank my wife and son for their endless patience and faith in me and this book, and I thank my parents for their encouragement and help when I was preoccupied with writing.

---

---

# Contents

---

---

## Chapter Dependencies

## Preface

|          |                                                                     |           |
|----------|---------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction to Probability</b>                                  | <b>1</b>  |
|          | Why Do Electrical and Computer Engineers Need to Study Probability? | 1         |
|          | Relative Frequency                                                  | 3         |
|          | What Is Probability Theory?                                         | 5         |
| 1.1      | Sample Spaces, Outcomes, and Events                                 | 6         |
|          | Sample Spaces                                                       | 6         |
|          | Outcomes and Events                                                 | 7         |
| 1.2      | Review of Set Notation                                              | 8         |
|          | Set Operations                                                      | 8         |
|          | Set Identities                                                      | 10        |
|          | Partitions                                                          | 12        |
|          | *Functions                                                          | 13        |
|          | *Countable and Uncountable Sets                                     | 15        |
| 1.3      | Probability Models                                                  | 17        |
| 1.4      | Axioms and Properties of Probability                                | 22        |
|          | Consequences of the Axioms                                          | 23        |
| 1.5      | Conditional Probability                                             | 26        |
|          | The Law of Total Probability and Bayes' Rule                        | 27        |
| 1.6      | Independence                                                        | 30        |
|          | Independence for More Than Two Events                               | 32        |
| 1.7      | Combinatorics and Probability                                       | 34        |
|          | Ordered Sampling with Replacement                                   | 34        |
|          | Ordered Sampling without Replacement                                | 35        |
|          | Unordered Sampling without Replacement                              | 37        |
|          | Unordered Sampling with Replacement                                 | 42        |
|          | Notes                                                               | 43        |
|          | Problems                                                            | 48        |
|          | Exam Preparation                                                    | 62        |
| <b>2</b> | <b>Introduction to Discrete Random Variables</b>                    | <b>63</b> |
| 2.1      | Probabilities Involving Random Variables                            | 63        |
| 2.2      | Discrete Random Variables                                           | 66        |
|          | Integer-Valued Random Variables                                     | 67        |
|          | Probability Mass Functions                                          | 67        |
|          | Uniform Random Variables                                            | 68        |
|          | The Poisson Random Variable                                         | 69        |
| 2.3      | Multiple Random Variables                                           | 70        |
|          | Independence                                                        | 71        |
|          | Max and Min Problems                                                | 73        |
|          | Geometric Random Variables                                          | 74        |

|          |                                                                                                         |            |
|----------|---------------------------------------------------------------------------------------------------------|------------|
|          | Joint Probability Mass Functions                                                                        | 75         |
|          | Computing Probabilities with MATLAB                                                                     | 78         |
| 2.4      | Expectation                                                                                             | 80         |
|          | Expectation of a Function of a Random Variable, or the Law of the Un-<br>conscious Statistician (LOTUS) | 83         |
|          | Linearity of Expectation                                                                                | 84         |
|          | Moments                                                                                                 | 84         |
|          | Indicator Functions                                                                                     | 87         |
|          | The Markov and Chebyshev Inequalities                                                                   | 88         |
|          | Expectations of Products of Functions of Independent Random Variables                                   | 90         |
|          | Correlation and Covariance                                                                              | 91         |
|          | Notes                                                                                                   | 96         |
|          | Problems                                                                                                | 99         |
|          | Exam Preparation                                                                                        | 106        |
| <b>3</b> | <b>More About Discrete Random Variables</b>                                                             | <b>108</b> |
| 3.1      | Probability Generating Functions                                                                        | 108        |
| 3.2      | The Binomial Random Variable                                                                            | 111        |
|          | Poisson Approximation of Binomial Probabilities                                                         | 115        |
| 3.3      | The Weak Law of Large Numbers                                                                           | 115        |
|          | Conditions for the Weak Law                                                                             | 116        |
| 3.4      | Conditional Probability                                                                                 | 117        |
|          | The Law of Total Probability                                                                            | 120        |
|          | The Substitution Law                                                                                    | 123        |
|          | Binary Channel Receiver Design                                                                          | 126        |
| 3.5      | Conditional Expectation                                                                                 | 127        |
|          | Substitution Law for Conditional Expectation                                                            | 128        |
|          | Law of Total Probability for Expectation                                                                | 129        |
|          | Notes                                                                                                   | 130        |
|          | Problems                                                                                                | 132        |
|          | Exam Preparation                                                                                        | 137        |
| <b>4</b> | <b>Continuous Random Variables</b>                                                                      | <b>138</b> |
| 4.1      | Densities and Probabilities                                                                             | 138        |
|          | Introduction                                                                                            | 138        |
|          | Definition                                                                                              | 139        |
|          | Some Common Densities                                                                                   | 139        |
|          | Location and Scale Parameters and the Gamma Densities                                                   | 146        |
|          | The Paradox of Continuous Random Variables                                                              | 149        |
| 4.2      | Expectation of a Single Random Variable                                                                 | 149        |
| 4.3      | Transform Methods                                                                                       | 156        |
|          | Moment Generating Functions                                                                             | 156        |
|          | Characteristic Functions                                                                                | 159        |
|          | Why So Many Transforms?                                                                                 | 161        |
| 4.4      | Expectation of Multiple Random Variables                                                                | 162        |
| 4.5      | *Probability Bounds                                                                                     | 164        |
|          | Notes                                                                                                   | 167        |
|          | Problems                                                                                                | 170        |

|                                                                   |            |
|-------------------------------------------------------------------|------------|
| Exam Preparation                                                  | 183        |
| <b>5 Cumulative Distribution Functions and Their Applications</b> | <b>184</b> |
| 5.1 Continuous Random Variables                                   | 185        |
| Receiver Design for Discrete Signals in Continuous Noise          | 192        |
| Simulation                                                        | 193        |
| 5.2 Discrete Random Variables                                     | 194        |
| Simulation                                                        | 196        |
| 5.3 Mixed Random Variables                                        | 197        |
| 5.4 Functions of Random Variables and Their CDFs                  | 200        |
| 5.5 Properties of CDFs                                            | 205        |
| 5.6 The Central Limit Theorem                                     | 207        |
| Approximation of Densities and PMFs Using the CLT                 | 211        |
| 5.7 Reliability                                                   | 215        |
| Notes                                                             | 219        |
| Problems                                                          | 222        |
| Exam Preparation                                                  | 238        |
| <b>6 Statistics</b>                                               | <b>240</b> |
| 6.1 Parameter Estimators and Their Properties                     | 240        |
| 6.2 Histograms                                                    | 244        |
| *The Chi-Squared Test                                             | 248        |
| 6.3 Confidence Intervals for the Mean – Known Variance            | 250        |
| 6.4 Confidence Intervals for the Mean – Unknown Variance          | 253        |
| Applications                                                      | 254        |
| Sampling with and without Replacement                             | 255        |
| 6.5 Confidence Intervals for Gaussian Data                        | 256        |
| Estimating the Mean                                               | 256        |
| Limiting $t$ Distribution                                         | 258        |
| Estimating the Variance – Known Mean                              | 258        |
| Estimating the Variance – Unknown Mean                            | 260        |
| 6.6 Hypothesis Tests for the Mean                                 | 262        |
| Decision Rules                                                    | 263        |
| Acceptance and Rejection Regions                                  | 264        |
| Types of Errors                                                   | 264        |
| Finding the Critical Value                                        | 264        |
| Small Samples                                                     | 267        |
| 6.7 Regression and Curve Fitting                                  | 267        |
| 6.8 *Monte Carlo Estimation                                       | 271        |
| Notes                                                             | 273        |
| Problems                                                          | 276        |
| Exam Preparation                                                  | 285        |
| <b>7 Bivariate Random Variables</b>                               | <b>287</b> |
| 7.1 Joint and Marginal Probabilities                              | 287        |
| Product Sets and Marginal Probabilities                           | 289        |
| Joint Cumulative Distribution Functions                           | 291        |
| Marginal Cumulative Distribution Functions                        | 292        |
| Independent Random Variables                                      | 294        |

|          |                                                              |            |
|----------|--------------------------------------------------------------|------------|
| 7.2      | Jointly Continuous Random Variables                          | 295        |
|          | Joint and Marginal Densities                                 | 297        |
|          | Independence                                                 | 300        |
|          | Expectation                                                  | 301        |
|          | *Continuous Random Variables That Are Not Jointly Continuous | 302        |
| 7.3      | Conditional Probability and Expectation                      | 302        |
| 7.4      | The Bivariate Normal                                         | 309        |
| 7.5      | Extension to Three or More Random Variables                  | 314        |
|          | The Law of Total Probability                                 | 315        |
|          | Notes                                                        | 317        |
|          | Problems                                                     | 319        |
|          | Exam Preparation                                             | 328        |
| <b>8</b> | <b>Introduction to Random Vectors</b>                        | <b>330</b> |
| 8.1      | Review of Matrix Operations                                  | 330        |
| 8.2      | Random Vectors and Random Matrices                           | 333        |
|          | Expectation                                                  | 333        |
|          | Correlation                                                  | 334        |
|          | Covariance                                                   | 335        |
|          | Cross-Covariance                                             | 336        |
|          | *Characteristic Functions                                    | 337        |
|          | *Decorrelation and the Karhunen–Loève Expansion              | 338        |
| 8.3      | Transformations of Random Vectors                            | 340        |
| 8.4      | *Linear Estimation of Random Vectors (Wiener Filters)        | 344        |
| 8.5      | *Estimation of Covariance Matrices                           | 348        |
| 8.6      | *Nonlinear Estimation of Random Vectors                      | 350        |
|          | Notes                                                        | 354        |
|          | Problems                                                     | 354        |
|          | Exam Preparation                                             | 360        |
| <b>9</b> | <b>Gaussian Random Vectors</b>                               | <b>362</b> |
| 9.1      | Introduction                                                 | 362        |
| 9.2      | Definition of the Multivariate Gaussian                      | 363        |
| 9.3      | Characteristic Function                                      | 365        |
|          | For Gaussian Random Vectors Uncorrelated Implies Independent | 365        |
| 9.4      | Density Function                                             | 367        |
|          | Simulation                                                   | 368        |
|          | Level Sets                                                   | 368        |
| 9.5      | Conditional Expectation and Conditional Probability          | 369        |
| 9.6      | Complex Random Variables and Vectors                         | 371        |
|          | Complex Gaussian Random Vectors                              | 372        |
|          | Notes                                                        | 373        |
|          | Problems                                                     | 375        |
|          | Exam Preparation                                             | 382        |

|                                                       |            |
|-------------------------------------------------------|------------|
| <b>10 Introduction to Random Processes</b>            | <b>383</b> |
| 10.1 Definition and Examples                          | 383        |
| Discrete-Time Processes                               | 383        |
| Continuous-Time Processes                             | 386        |
| 10.2 Characterization of Random Processes             | 388        |
| Mean and Correlation Functions                        | 388        |
| Cross-Correlation Functions                           | 392        |
| 10.3 Strict-Sense and Wide-Sense Stationary Processes | 393        |
| Estimation of Correlation Functions                   | 397        |
| Transforms of Correlation Functions                   | 398        |
| 10.4 WSS Processes through LTI Systems                | 401        |
| Time-Domain Analysis                                  | 401        |
| Frequency-Domain Analysis                             | 402        |
| 10.5 Power Spectral Densities for WSS Processes       | 403        |
| Motivation                                            | 403        |
| Power in a Process                                    | 404        |
| White Noise                                           | 406        |
| 10.6 Characterization of Correlation Functions        | 410        |
| Correlation Functions of Deterministic Signals        | 411        |
| 10.7 The Matched Filter                               | 412        |
| Analysis of the Matched Filter Output                 | 415        |
| 10.8 The Wiener Filter                                | 417        |
| *Causal Wiener Filters                                | 419        |
| 10.9 *The Wiener–Khinchin Theorem                     | 421        |
| 10.10*Mean-Square Ergodic Theorem for WSS Processes   | 423        |
| 10.11*Power Spectral Densities for non-WSS Processes  | 425        |
| Notes                                                 | 427        |
| Problems                                              | 429        |
| Exam Preparation                                      | 440        |
| <b>11 Advanced Concepts in Random Processes</b>       | <b>443</b> |
| 11.1 The Poisson Process                              | 443        |
| *Derivation of the Poisson Probabilities              | 448        |
| Marked Poisson Processes                              | 450        |
| Shot Noise                                            | 450        |
| 11.2 Renewal Processes                                | 452        |
| 11.3 The Wiener Process                               | 453        |
| The Wiener Integral                                   | 456        |
| Random Walk Approximation of the Wiener Process       | 457        |
| 11.4 Specification of Random Processes                | 459        |
| Finitely Many Random Variables                        | 459        |
| Infinite Sequences (Discrete Time)                    | 461        |
| Continuous-Time Random Processes                      | 464        |
| Gaussian Processes                                    | 464        |
| Notes                                                 | 466        |
| Problems                                              | 466        |
| Exam Preparation                                      | 475        |

|                                                               |            |
|---------------------------------------------------------------|------------|
| <b>12 Introduction to Markov Chains</b>                       | <b>476</b> |
| 12.1 Preliminary Results                                      | 476        |
| 12.2 Discrete-Time Markov Chains                              | 477        |
| Conditional Joint PMFs                                        | 478        |
| State Space and Transition Probabilities                      | 480        |
| Examples                                                      | 481        |
| Consequences of Time Homogeneity                              | 483        |
| The Chapman–Kolmogorov Equation                               | 484        |
| Stationary Distributions                                      | 484        |
| 12.3 *Recurrent and Transient States                          | 488        |
| Entrance Times and Intervisit Times                           | 488        |
| Number of Visits to a State and Occupation Time               | 491        |
| 12.4 *Limiting $n$ -Step Transition Probabilities             | 496        |
| Classes of States                                             | 498        |
| 12.5 Continuous-Time Markov Chains                            | 502        |
| Behavior of Continuous-Time Markov Chains                     | 503        |
| Kolmogorov’s Differential Equations                           | 505        |
| Stationary Distributions                                      | 506        |
| Notes                                                         | 507        |
| Problems                                                      | 509        |
| Exam Preparation                                              | 515        |
| <b>13 Mean Convergence and Applications</b>                   | <b>517</b> |
| 13.1 Convergence in Mean of Order $p$                         | 518        |
| Continuity in Mean of Order $p$                               | 521        |
| 13.2 Normed Vector Spaces of Random Variables                 | 522        |
| Mean-Square Integrals                                         | 526        |
| 13.3 The Karhunen–Loève Expansion                             | 527        |
| 13.4 The Wiener Integral (Again)                              | 532        |
| 13.5 Projections, Orthogonality Principle, Projection Theorem | 534        |
| 13.6 Conditional Expectation and Probability                  | 537        |
| The Smoothing Property                                        | 544        |
| 13.7 The Spectral Representation                              | 545        |
| Notes                                                         | 549        |
| Problems                                                      | 550        |
| Exam Preparation                                              | 562        |
| <b>14 Other Modes of Convergence</b>                          | <b>564</b> |
| 14.1 Convergence in Probability                               | 564        |
| 14.2 Convergence in Distribution                              | 566        |
| 14.3 Almost-Sure Convergence                                  | 572        |
| The Skorohod Representation                                   | 577        |
| Notes                                                         | 579        |
| Problems                                                      | 580        |
| Exam Preparation                                              | 589        |

|                                                       |            |
|-------------------------------------------------------|------------|
| <b>15 Self Similarity and Long-Range Dependence</b>   | <b>591</b> |
| 15.1 Self Similarity in Continuous Time               | 591        |
| Implications of Self Similarity                       | 592        |
| Stationary Increments                                 | 593        |
| Fractional Brownian Motion                            | 594        |
| 15.2 Self Similarity in Discrete Time                 | 595        |
| Convergence Rates for the Mean-Square Ergodic Theorem | 596        |
| Aggregation                                           | 597        |
| The Power Spectral Density                            | 599        |
| Engineering versus Statistics/Networking Notation     | 601        |
| 15.3 Asymptotic Second-Order Self Similarity          | 601        |
| 15.4 Long-Range Dependence                            | 604        |
| 15.5 ARMA Processes                                   | 606        |
| 15.6 ARIMA Processes                                  | 608        |
| Problems                                              | 610        |
| Exam Preparation                                      | 613        |
| <b>Bibliography</b>                                   | <b>615</b> |

---

---

# Index

---

---

## A

Abel's theorem, 130  
 absolutely continuous random variables, 221, 318  
 absorbing state, 482  
 acceptance region, 264  
 accessible state, 499  
 affine function, 186, 344  
 aggregated process, 598  
 almost-sure convergence, 572  
 almost-sure event, 23  
 alternative hypothesis, 263  
 analog-to-digital converter, 150  
 Anderson–Darling test, 248  
 angle  
     of a point in the plane, 354  
 AR process, *see* autoregressive process  
 arcsine random variable, 233, 302, 538  
     relation to beta, 233  
 ARIMA, *see* autoregressive integrated moving average process  
 ARMA process, *see* autoregressive moving average process  
 arrival times, 446  
 associative laws, 10  
 asymptotic second-order self similarity, 602, 611  
 asymptotically unbiased estimator, 243  
 atomic weight, 115  
 auto-correlation function, 392  
 autoregressive integrated moving average, 608  
 autoregressive integrated moving average process, 602  
     fractional, 608  
 autoregressive moving average process, 606  
 autoregressive process, 606  
 Avogadro's number, 115

## B

Banach space, 524  
 bandlimited white noise, 406  
 bandwidth, 408  
 Bayes' rule, 28, 29  
 Bernoulli random variable, 71  
     mean, 81  
     second moment and variance, 86  
     simulation, 196  
 Bernoulli trials, 3  
 Bernoulli, Jacob, 117  
 Bessel function, 227  
     properties, 229  
 beta function, 175, 176  
 beta random variable, 175  
     relation to arcsine random variable, 233

    relation to gamma and chi-squared, 325  
 betting on fair games, 104  
 biased estimator, 243  
 binary symmetric channel, 58  
 binomial approximation  
     by normal, 213  
     by Poisson, 115, 584  
 binomial coefficient, 38, 114  
 binomial random variable, 113  
     mean, variance, and pgf, 133  
     simulation, 197  
 binomial theorem, 38, 113, 133, 603  
 birth process, *see* Markov chain  
 birth–death process, *see* Markov chain  
 birthday problem, 36  
 bivariate characteristic function, 301  
 bivariate Gaussian random variables, 309  
 block matrices, 332  
 Borel–Cantelli lemma  
     first, 54, 575  
     second, 60  
 Borel set, 56, 96  
 Borel sets of  $\mathbb{R}^2$ , 317  
 Borel  $\sigma$ -field, 56, 96  
 Brown, Robert, 387  
 Brownian motion, 387  
     fractional, *see* fractional Brownian motion  
     ordinary, *see* Wiener process

## C

Campbell's theorem, 452  
 Cantor set, 55  
 cardinality, 15, 18  
 Cartesian product, 289  
 Cauchy random variable, 144  
     as quotient of Gaussians, 323  
     as tangent of uniform, 194  
     cdf, 186  
     characteristic function, 180  
     nonexistence of mean, 154  
     simulation, 194  
     special case of Student's  $t$ , 176  
 Cauchy–Schwarz inequality  
     for column vectors, 331, 355  
     for random variables, 92, 524  
     for time functions, 429  
 Cauchy sequence  
     of  $L^p$  random variables, 524  
     of real numbers, 524  
 causal Wiener filter, 419  
     prediction, 439  
     smoothing, 439

- cdf, *see* cumulative distribution function
- central chi-squared random variable, *see* chi-squared random variable
- central limit theorem, 6, 185, 208, 252, 458, 570
  - compared with weak law of large numbers, 571
- central moment, 86
- certain event, 23
- chain rule, 190
  - of calculus, 319
  - of conditional probability, 58, 510
- change of variable (multivariate), 341
- Chapman–Kolmogorov equation
  - continuous time, 503
  - derivation via smoothing property, 544
  - discrete time, 484
  - for Markov processes, 515
- characteristic function
  - bivariate, 301
  - compared with pgf and mgf, 161
  - multivariate (joint), 337
  - univariate, 159
- Chebyshev inequality, 89, 164, 165, 182
  - used to derive the weak law, 116
- Chernoff bound, 164, 165, 182
- Chevalier de Mere, 3
- chi-squared random variable, 148, 174
  - as squared zero-mean Gaussian, 179, 192, 222
  - cdf – special case of gamma, 225
  - characteristic function, 179
  - moment generating function, 179
  - parameter estimation, 276
  - relation to  $F$  random variable, 325
  - relation to beta random variable, 325
  - relation to generalized gamma, 224
  - see also* noncentral chi-squared, 180
  - simulation, 276
  - square root of = Rayleigh, 223
- chi-squared test, 248
- circularly symmetric complex Gaussian, 373
- closed set, 535
- CLT, *see* central limit theorem
- co-domain of a function, 13
- combinatorics, 34
- communicating states, 499
- commutative laws, 10
- complement of a set, 8
- complementary cdf
  - Gaussian, 187, 225
- complementary error function, 219
- complete orthonormal set, 530
- completeness
  - of the  $L^p$  spaces, 524
  - of the real numbers, 524
- complex conjugate, 371
- complex Gaussian random vector, 372
- complex random variable, 371
- complex random vector, 372
- conditional cdf, 192, 303
- conditional density, 192, 303
- conditional expectation
  - abstract definition, 538
  - for discrete random variables, 127
  - for jointly continuous random variables, 302
  - linearity, 542
  - smoothing property, 544, 557
- conditional independence, 60, 476
- conditional probability, 27
  - for jointly continuous random variables, 303
- conditional probability mass functions, 118
- confidence interval, 250
- confidence level, 250
- conservative Markov chain, 504, 508
- consistency condition
  - continuous-time processes, 464
  - discrete-time processes, 461
- continuity in mean of order  $p$ , 521
- continuous random variable, 139
  - arcsine, 233
  - beta, 175
  - Cauchy, 144
  - chi-squared, 174
  - Erlang, 174
  - exponential, 141
  - $F$ , 325
  - gamma, 173
  - Gaussian = normal, 145
  - generalized gamma, 224
  - Laplace, 143
  - lognormal, 190
  - Maxwell, 222
  - multivariate Gaussian, 363
  - Nakagami, 224
  - noncentral chi-squared, 182
  - noncentral Rayleigh, 227
  - Pareto, 237
  - Rayleigh, 177
  - Rice, 227
  - Student's  $t$ , 176
  - uniform, 140
  - Weibull, 171
- continuous sample paths, 455
- convergence
  - almost-sure (a.s.), 572
  - in distribution, 566
  - in mean of order  $p$ , 518
  - in mean square, 518
  - in probability, 564
  - in quadratic mean, 518
  - of real numbers, 573
  - sure, 573
  - weak, 566
- convex function, 105
- convolution
  - of densities, 163
  - of probability mass functions, 125
- correlation, 91

correlation coefficient, 92, 104, 311  
 correlation function, 392  
   engineering definition, 601  
   of a deterministic signal, 411  
   of a random process, 389  
   properties, 391  
   statistics/networking definition, 601  
   unbiased estimator of, 397  
   univariate, for WSS processes, 395  
 correlation matrix, 334  
 countable additivity, 23  
 countable set, 15, 462  
 countable subadditivity, 26  
 countably infinite set, 15  
 counting process, 443  
 covariance, 94  
   distinction between scalar and matrix, 335  
   function, 392  
   matrix, 335  
 covering of intervals, 221  
 Craig's formula, 322  
 critical region, 264  
 critical value, 249, 264  
 cross power spectral density, 406  
 cross-correlation  
   function, 392  
   univariate, for WSS processes, 402  
   matrix, 337  
 cross-covariance  
   function, 392  
   matrix, 336  
 crossover probabilities, 58, 121  
 cumulative distribution function (cdf), 184  
   continuous random variable, 185  
   discrete random variable, 194  
   joint, 291  
   multivariate, 351  
   properties, 205  
 curve fitting, *see* regression  
 cyclostationary process, 425

## D

dB, *see* decibel  
 de Moivre, Abraham, 208  
 de Moivre–Laplace theorem, 255  
 De Morgan's laws, 10  
   generalized, 12  
 decibel, 188, 437  
 decorrelating transformation, 338  
   applied to a Gaussian vector, 366  
 delta function, 406  
   Dirac, 199, 406  
   Kronecker, 397, 483  
 diagonal argument, 17  
 difference of sets, 9  
 differencing filter, 608  
 differential entropy, 178  
 Dirac delta function, 199

discrete random variable, 66  
   Bernoulli, 71  
   binomial, 113  
   geometric, 74  
   hypergeometric, 256  
   negative binomial = Pascal, 133  
   Poisson, 69  
   uniform, 68  
   zeta = Zipf, 105  
 discrete-time Fourier transform, 400  
 disjoint sets, 9  
 distribution, 97  
 distributive laws, 10  
   generalized, 12  
 domain of a function, 13  
 dominated convergence theorem, 424, 508, 549  
 Doob decomposition, 559  
 dot product, *see* inner product  
 double factorial, 153  
 double-sided exponential = Laplace, 143

## E

eigenvalue, 485, 528  
 eigenvector, 485  
 ellipsoids, 368  
 embedded chain, 504  
 empty set, 8  
 energy spectral density, 412  
 ensemble mean, 241  
 ensemble variance, 241  
 entropy, 105  
   differential, 178  
 equilibrium distribution, 485  
 equivalence classes, 500, 513  
 equivalence relation, 500  
 ergodic theorem, 397  
   for Markov chains, 495  
   mean-square  
     for WSS processes, 424  
     for WSS sequences, 519  
 Erlang random variable, 148, 174  
   as  $n$ th arrival time of Poisson process, 446  
   as sum of i.i.d. exponentials, 181  
   cdf – special case of gamma, 225  
   cumulative distribution function, 174  
   moment generating function, 179  
   relation to generalized gamma, 224  
   simulation, 277  
 error function, 188, 219  
   complementary, 219  
 estimation of nonrandom parameters  
   covariance matrices, 348  
 estimation of random vectors  
   linear MMSE, 344  
   maximum likelihood (ML), 350  
   MMSE, 350  
 estimator  
   asymptotically unbiased, 243

- biased, 243
- unbiased, 241
- event, 7, 43, 580
- expectation
  - additive operator, 84
  - homogeneous operator, 83
  - linearity for arbitrary random variables, 163
  - linearity for discrete random variables, 84
  - monotonicity for arbitrary random variables, 163
  - monotonicity for discrete random variables, 106
  - of a discrete random variable, 80
  - of an arbitrary random variable, 155
  - when it is undefined, 82, 154
- expected average power, 404
  - and the Wiener–Khinchin theorem, 421
- expected instantaneous power, 404
- exponential random variable, 141
  - difference of = Laplace, 180
  - double sided, *see* Laplace random variable
  - memoryless property, 171
  - moment generating function, 158
  - moments, 158
  - relation to generalized gamma, 224

## F

- $F$  random variable, 325
  - relation to chi-squared, 325
- factorial
  - double, 153
- factorial function, 173
- factorial moment, 111
- factorization property, 109
- fading
  - channel, 223
  - Rayleigh, 324
- failure rate, 216
  - constant, 218
  - Erlang, 237
  - Pareto, 237
  - Weibull, 237
- FARIMA, *see* fractional ARIMA
- filtered Poisson process, 451
- first entrance time, 488
- first passage time, 488
- Fourier series, 400
  - as characteristic function, 161
- Fourier transform, 398
  - as bivariate characteristic function, 301
  - as multivariate characteristic function, 337
  - as univariate characteristic function, 160
  - discrete time, 400, 432
  - inversion formula, 398
- fractional ARIMA process, 608
- fractional Brownian motion, 594
- fractional Gaussian noise, 596
- fractional integrating filter, 608
- function
  - co-domain, 13

- definition, 13
- domain, 13
- inverse image, 14
- invertible, 14
- one-to-one, 14
- onto, 14
- probability measure as a function, 23
- range, 13

## G

- gambler's ruin, 482
- gamma function, 79, 148, 173
  - incomplete, 225
- gamma random variable, 147, 173
  - cdf, 225
  - characteristic function, 160, 179, 180
  - generalized, 224, 325
  - moment generating function, 179
  - moments, 177
  - parameter estimation, 276, 277
  - relation to beta random variable, 325
  - with scale parameter, 174
- Gaussian pulse, 160
- Gaussian random process, 464
  - fractional, 595
  - Karhunen–Loève expansion, 584
- Gaussian random variable, 145
  - ccdf
    - approximation, 225
    - Craig's formula, 322
    - definition, 187
    - table, 189
  - cdf, 187
    - related to error function, 219
    - table, 189
  - characteristic function, 160, 180
  - complex, 372
  - complex circularly symmetric, 373
  - moment generating function, 157, 159
  - moments, 152
  - quotient of = Cauchy, 323
  - simulation, 194, 278
- Gaussian random vector, 363
  - characteristic function, 365
  - complex circularly symmetric, 373
  - joint density, 367
  - multivariate moments, Wick's theorem, 377
  - proper, 373
  - simulation, 368
- generalized density, 199
- generalized gamma random variable, 224, 325
  - relation to Rayleigh, Maxwell, Weibull, 224
- generator matrix, 506
- geometric random variable, 74
  - mean, variance, and pgf, 132
  - memoryless property, 101
- geometric series, 52
- goodness-of-fit tests, 248

greatest common divisor, 500

## H

$H$ -sssi, 593

Herglotz's theorem, 545

Hilbert space, 524

histogram, 244

Hölder's inequality, 550

holding time, 504, 514

Hurst parameter, 591

hypergeometric random variable, 256  
derivation, 274

hypothesis, 248

hypothesis testing, 262, 263

## I

i.i.d., *see* independent identically distributed

i.o., *see* infinitely often

ideal gas, 224

identically distributed random variables, 72

importance sampling, 272

impossible event, 22

impulse function, 199

impulse response, 390

impulsive, 199

inclusion–exclusion formula, 24

incomplete gamma function, 225

increment process, 593

increments, 390

increments of a random process, 444

independent events

more than two events, 32

pairwise, 32, 46

two events, 30

independent identically distributed (i.i.d.), 72

independent increments, 444

independent random variables, 71

cdf characterization, 295

ch. fcn. characterization, 302, 338

jointly continuous, 300

multiple, 72

pdf characterization, 301

pmf characterization, 76

uncorrelated does not imply independent, 104, 322,  
327

indicator function, 87

infinitely often (i.o.), 492

inner product

of column vectors, 331

of matrices, 355

of random variables, 524

inner-product space, 524

integrating filter, 608

integration by parts formula, 168

intensity of a Poisson process, 444

interarrival times, 446

intersection of sets, 9

intervisit times, 490

inverse image, 14

inverse tangent

principal, 354

irreducible Markov chain, 488, 499

Itô correction term, 457

Itô integral, 457

Itô rule, 457

## J

J-WSS, *see* jointly wide-sense stationary

Jacobian, 341

formulas, 341

Jensen's inequality, 105

joint characteristic function, 337

joint cumulative distribution function, 291

joint density, 295

joint probability mass function, 75

joint wide-sense stationarity, 402

for discrete-time processes, 434

jointly continuous random variables

bivariate, 295

jointly Gaussian random variables, 363

jointly normal random variables, 363

jump chain, 504

jump times

of a Poisson process, 445

## K

Karhunen–Loève expansion, 527

finite-dimensional, 338

Gaussian process, 584

Ornstein–Uhlenbeck process, 554

signal detection, 530

white noise, 531

Wiener process, 531

Kolmogorov

and axiomatic theory of probability, 5, 517

backward equation, 506

characterization of random processes, 388

consistency/extension theorem, 462

forward equation, 505

Kolmogorov–Smirnov test, 248

Kronecker delta, 397, 483

Kronecker product, 103, 447

kurtosis, 86

## L

Laplace random variable, 143

as difference of exponentials, 180

parameter estimation, 277

quotient of, 324

simulation, 277

variance and moment generating function, 179

Laplace transform, 158

Laplace, Pierre-Simon, 208

law of large numbers

convergence rates, 596

- mean square, for second-order self-similar sequences, 596
  - mean square, uncorrelated, 519
  - mean square, WSS sequences, 519
  - strong, 273, 576
  - weak, for independent random variables, 576
  - weak, for uncorrelated random variables, 116, 565
  - law of the unconscious statistician, 83, 149
  - law of total probability, 27, 29
    - discrete conditioned on continuous, 472
    - for conditional expectation, 544
    - for conditional probability, 503, 515, 544
    - for continuous random variables, 304
    - for expectation (continuous random variables), 308, 315
    - for expectation (discrete random variables), 129
    - unified formula, 540
  - Lebesgue
    - dominated convergence theorem, 424, 549
    - measure, 45, 57
    - monotone convergence theorem, 169, 549
  - Leibniz' rule, 191, 307
    - derivation, 318
  - level curves, 310
  - level sets, 368
  - likelihood, 127, 192
  - likelihood ratio
    - continuous random variables, 193
    - discrete random variables, 127
    - martingale, 559
  - likelihood-ratio test, 127, 136, 193, 223
  - limit inferior, 567, 579
  - limit properties of  $P$ , 25
  - limit superior, 567, 579
  - Lindeberg–Lévy theorem, 208
  - linear estimators, 535
  - linear MMSE estimator, 344
  - linear time-invariant system, 390
  - location parameter, 146
  - lognormal random variable
    - definition, 190
    - moments, 222
  - long-range dependence, 604
  - LOTUS, *see* law of the unconscious statistician
  - LRD, *see* long-range dependence
  - LTI, *see* linear time-invariant (system)
  - Lyapunov's inequality, 576
    - derived from Hölder's inequality, 551
    - derived from Jensen's inequality, 105
- M**
- MA process, *see* moving average process
  - MAP, *see* maximum a posteriori probability
  - Marcum  $Q$  function, 228, 322
  - marginal cumulative distributions, 292
  - marginal density, 299
  - marginal probability, 290
  - marginal probability mass functions, 75
  - Markov chain, 544
    - absorbing barrier, 482
    - accessible state, 499
    - aperiodic state, 500
    - birth–death process, 482
    - Chapman–Kolmogorov equation, 484
    - communicating states, 499
    - conservative, 504, 508
    - continuous time, 502
    - discrete time, 477
    - embedded chain, 504
    - equilibrium distribution, 485
    - ergodic theorem, 495
    - first entrance time, 488
    - first passage time, 488
    - gambler's ruin, 482
    - generator matrix, 506
    - holding time, 504, 514
    - intervisit times, 490
    - irreducible, 488, 499
    - jump chain, 504
    - Kolmogorov's backward equation, 506
    - Kolmogorov's forward equation, 505
    - $m$ -step transition probabilities, 483
    - model for queue
      - with finite buffer, 482
      - with infinite buffer, 482, 513
    - $n$ th entrance time, 489
    - null recurrent, 489
    - occupation time, 491
      - average, 491
      - convergence, 495
      - total, 492, 512
    - period of a state, 500
    - periodic state, 500
    - positive recurrent, 489
    - pure birth process, 482
    - random walk
      - construction, 477
      - continuous time, 513
      - definition, 481
      - symmetric, 478
    - rate matrix, 506
    - reachable state, 499
    - recurrent state, 488
    - reflecting barrier, 482
    - sojourn time, 504, 514
    - state space, 480
    - state transition diagram, 480
    - stationary distribution, 485
    - time homogeneous
      - continuous time, 503
      - discrete time, 480
    - transient state, 488
    - transition probabilities
      - continuous time, 502
      - discrete time, 480
    - transition probability matrix, 480

- transition rates, 503
- Markov inequality, 88, 164, 182
- Markov process, 515
- Markov property, 477
- martingale, 558
  - likelihood ratio, 559
- Matlab commands
  - `./`, 79
  - `.^`, 78
  - axis, 282
  - bar, 245
  - besseli, 227
  - chi2cdf, 227
  - chi2inv, 259
  - diag, 340
  - eig, 340
  - erf, 219
  - erfc, 219
  - erfinv, 252
  - eye, 346
  - factorial, 134
  - fft, 432
  - fftshift, 433
  - find, 79, 197
  - for, 78
  - format rat, 80
  - gamcdf, 225
  - gamma, 173
  - gammainc, 225
  - gammaln, 231
  - geopdf, 79
  - histc, 244
  - hold off, 246
  - hold on, 245
  - kron, 447
  - linspace, 247
  - max, 244
  - mean, 241
  - mean (to compute mean vectors), 350
  - min, 244
  - nchoosek, 38
  - ncx2cdf, 227
  - normcdf, 187
  - norminv, 252
  - ones, 197
  - plot, 247
  - poisspdf, 79
  - polyfit, 270
  - polyval, 270
  - rand, 194
  - randn, 194
  - semilogy, 183
  - size, 197
  - sqrt, 276
  - std, 241
  - stem, 231
  - subplot, 282
  - sum, 78
  - sum (of matrix), 80
  - tan, 194
  - tinv, 257
  - trace, 331
  - var, 241
  - zeros, 197
- Matlab M-files
  - allpairs, 102
  - bernrnd, 197
  - binpmf, 231
- matrix exponential, 506
- matrix inverse formula, 358, 381
- maximum a posteriori probability
  - estimator, 350, 360
- maximum a posteriori probability rule
  - continuous observations, 193
  - derivation, 131
  - discrete observations, 126
- maximum-likelihood estimator, 350
- maximum-likelihood rule, 127, 193
- Maxwell random variable, 343
  - as square root of chi-squared, 223
  - cdf, 222
  - relation to generalized gamma, 225
  - speed of particle in ideal gas, 224
- mean, *see* expectation
- mean function, 388
- mean matrix, 333
- mean time to failure, 216
- mean vector, 333
- mean-square convergence, 518
- mean-square ergodic theorem
  - for WSS processes, 423
  - for WSS sequences, 519
- mean-square law of large numbers
  - for uncorrelated random variables, 519
  - for WSS processes, 424
- mean-square periodicity, 551
- mean-squared error, 103, 104, 344, 417
- measure, 45
- median, 170
- memoryless property
  - exponential random variable, 171
  - geometric random variable, 101
- Mercer's theorem, 529
- mgf, *see* moment generating function
- minimum mean squared error, 535
- Minkowski's inequality, 523, 551
- mixed random variable, 199
- mixture density, 172
  - noncentral chi-squared, 182
- ML, *see* maximum likelihood
- MMSE, *see* minimum mean-squared error
- modified Bessel function of the first kind, 227
  - properties, 229
- moment, 84
  - central, 86
  - factorial, 111

moment generating function (mgf), 156  
     compared with pgf and char. fcn., 162  
 monotone convergence theorem, 169, 549  
 monotonic sequence property, 549  
 monotonicity  
     of  $E$ , 106, 163  
     of  $P$ , 24  
 Monte Carlo estimation, 271  
 Mother Nature, 23  
 moving average process, 606  
 MSE, *see* mean-squared error  
 MTTF, *see* mean time to failure  
 multinomial coefficient, 42  
 multivariate change of variable, 374  
 mutually exclusive sets, 9  
 mutually independent events, 32

## N

Nakagami random variable, 224, 381  
     as square root of chi-squared, 224  
 negative binomial random variable, 133  
 noiseless detection, 554  
     discrete time, 339  
 noncentral chi-squared random variable  
     as squared non-zero-mean Gaussian, 180, 192, 223  
     cdf (series form), 227  
     density (closed form using Bessel function), 228  
     density (mixture form), 182  
     moment generating function, 180, 182  
     noncentrality parameter, 180  
     parameter estimation, 276  
     simulation, 277  
     square root of = Rice, 227  
 noncentral Rayleigh random variable, 227  
     square of = noncentral chi-squared, 227  
 noncentrality parameter, 180, 182  
 norm  
      $L^p$  random variables, 523  
     matrix, 355  
     vector, 331  
 norm preserving, 547  
 normal approximation of the binomial, 213  
 normal random variable, *see* Gaussian  
 $n$ th entrance time, 489  
 null hypothesis, 263  
 null recurrent, 489  
 null set, 8

## O

occupation time, 491  
     average, 491  
     convergence, 495  
     total, 492, 512  
 occurrence times, 446  
 odds, 104  
 one-sided test, 266  
 one-tailed test, 266

one-to-one, 14  
 onto, 14  
 open set, 57  
 Ornstein–Uhlenbeck process, 456, 470  
     Karhunen–Loève expansion, 554  
 orthogonal increments, 561  
 orthogonality principle  
     for regression, 269  
     general statement, 534  
     in the derivation of linear estimators, 347  
     in the derivation of the Wiener filter, 417  
 orthonormal, 528  
 outcomes, 7  
 outer product, 331  
 overshoot, 234

## P

pairwise disjoint sets, 12  
 pairwise independent events, 32  
 Paley–Wiener condition, 420  
 paradox of continuous random variables, 149  
 parallelogram law, 524, 552  
 Pareto failure rate, 237  
 Pareto random variable, 154, 170, 177, 179, 182, 237, 588  
 partition, 12  
 Pascal, 3  
 Pascal random variable = negative binomial, 133  
 Pascal’s triangle, 114  
 pdf, *see* probability density function  
 period, 500  
 periodic state, 500  
 permanence of form argument, 169, 612  
 permutation, 37  
 pgf, *see* probability generating function  
 $\pi$ - $\lambda$  theorem, 221  
 pmf, *see* probability mass function  
 Poisson approximation of binomial, 115, 584  
 Poisson process, 444  
     arrival times, 446  
     as a Markov chain, 502  
     filtered, 451  
     independent increments, 444  
     intensity, 444  
     interarrival times, 446  
     marked, 450  
     occurrence times, 446  
     rate, 444  
     shot noise, 451  
     thinned, 467  
 Poisson random variable, 69  
     mean, 81  
     mean, variance, and pgf, 111  
     probability generating function, 108  
     second moment and variance, 86  
 population mean, 241  
 population variance, 241  
 positive definite matrix, 336

positive recurrent, 489  
 positive semidefinite  
   function, 429  
   matrix, 336  
 posterior probability, 30, 126  
 power  
   expected average, 404  
   expected instantaneous, 404  
 power set, 44  
 power spectral density, 403, 405  
   nonnegativity, 422  
 predictable process, 559  
 prediction  
   using the Wiener filter, 439  
 principal  
   angle, 354  
   inverse tangent, 354  
 principal inverse tangent, 354  
 prior probabilities, 30, 127  
 probability  
   written as an expectation, 87  
 probability density function (pdf), 139  
 probability generating function (pgf), 108  
   compared with mgf and char. fcn., 161  
   related to  $z$  transform, 108  
 probability mass function (pmf), 67  
 probability measure, 22, 460  
 probability space, 43  
 projection, 534  
   in linear estimation, 347  
   onto the unit ball, 534  
   theorem, 535, 536  
 proper subset, 8

## Q

$Q$  function  
   Gaussian, 225, 226  
   Marcum, 228  
 quadratic-mean convergence, 518  
 quantizer, 150  
 queue, *see* Markov chain

## R

$\mathbb{R} := (-\infty, \infty)$ , the real numbers, 11  
 random matrix, 333  
 random points on the unit sphere, 325  
 random process, 383  
   continuous-time, 386  
   discrete-time, 383  
 random sum, 316  
 random variable  
   absolutely continuous, 221  
   complex-valued, 371  
   continuous, 139  
   definition, 63  
   discrete, 66  
   integer-valued, 67  
   precise definition, 96

  singular, 221  
   traditional interpretation, 63  
 random variables  
   identically distributed, 72  
   independent, 71  
 random vector, 333  
 random walk  
   approximation of the Wiener process, 457  
   construction, 477  
   definition, 481  
   symmetric, 478  
   with a barrier at the origin, 481  
 range of a function, 13  
 rate matrix, 506  
 rate of a Poisson process, 444  
 Rayleigh random variable  
   as square root of chi-squared, 223  
   cdf, 222  
   distance from origin, 141, 224  
   generalized, 223  
   moments, 177  
   parameter estimation, 276  
   quotient of, 324  
   relation to generalized gamma, 225  
   simulation, 276  
   square of = chi-squared, 223  
 reachable state, 499  
 real numbers,  $\mathbb{R} := (-\infty, \infty)$ , 11  
 realization, 383  
 rectangle formula, 291  
 recurrent state, 488  
 reflecting state, 482  
 reflexive  
   property of an equivalence relation, 499  
 regression, 267  
   curve, 267  
   relation to conditional expectation, 282  
 rejection region, 264  
 relative frequency, 3  
 reliability function, 215  
 renewal equation, 453  
   derivation, 468  
 renewal function, 453  
 renewal process, 452, 588  
 resonant frequency, 233  
 Rice random variable, 227, 380  
   square of = noncentral chi-squared, 227  
 Riemann sum, 391, 431, 439, 526  
 Riesz–Fischer theorem, 524, 536, 538

## S

sample, 240  
   mean, 115, 240  
   standard deviation, 241  
   variance, 241  
 sample function, 383  
 sample path, 383  
 sample space, 6, 22

- sampling
    - with replacement, 255
    - without replacement, 255
  - sampling without replacement, 274
  - scale parameter, 146, 174, 224
  - scatter plot, 268
  - second-order process, 392
  - second-order self similarity, 596
  - self similarity, 591
  - sequential continuity, 26
  - set difference, 9
  - shot noise, 451
  - $\sigma$ -algebra, 43
  - $\sigma$ -field, 43, 96, 317, 466
  - signal-to-noise ratio, 94, 188, 413
  - significance level, 248, 264
  - Simon's formula, 327
  - simulation, 271
    - confidence intervals, 271
    - continuous random variables, 193
    - discrete random variables, 196
    - Gaussian random vectors, 368
    - importance sampling, 272
  - sinc function, 400
  - singular random variable, 221
  - skewness, 86
  - Skorohod representation, 577
    - derivation, 578
  - SLLN, *see* strong law of large numbers
  - slowly varying function, 605
  - Slutsky's theorem, 274, 571
  - smoothing
    - using the Wiener filter, 439
  - smoothing property, 544, 557
  - SNR, *see* signal-to-noise ratio
  - sojourn time, 504, 514
  - spectral distribution, 545
  - spectral factorization, 420
  - spectral process, 545
  - spectral representation, 549
  - spontaneous generation, 482
  - square root of a nonnegative definite matrix, 375
  - standard deviation, 85
  - standard normal density, 145
  - state space of a Markov chain, 480
  - state transition diagram, *see* Markov chain
  - stationary distribution, 485
  - stationary increments, 593
  - stationary process, 394
    - i.i.d. example, 394
    - of order  $n$ , 393
  - stationary random process
    - Markov chain example, 474
  - statistic, 240
  - statistical independence, 30
  - statistical regularity, 4
  - Stirling's formula, 176, 584, 609, 612
    - derivation using exponential, 212
    - derivation using Poisson, 236
    - more precise version, 212
  - stochastic process, 383
  - strictly stationary process, 394
    - Markov chain example, 474
    - of order  $n$ , 393
  - strong law of large numbers, 6, 273, 576
  - Student's  $t$ , 176, 325
    - cdf converges to normal cdf, 258
    - density converges to normal density, 176
    - generalization of Cauchy, 176
    - moments, 177, 178
  - submartingale, 558
  - subset, 8
    - proper, 8
  - substitution law, 304
    - continuous random variables, 308, 315
    - discrete random variables, 124, 129
    - general case, 542
  - sum of squared errors, 268
  - supermartingale, 558
  - sure event, 23
  - symmetric
    - function, 391
    - matrix, 334, 335, 374
    - property of an equivalence relation, 499
    - random walk, 478
- T**
- $t$ , *see* Student's  $t$
  - thinned Poisson process, 467
  - tilted density, 273
  - time constant, 407
  - time-homogeneity, *see* Markov chain
  - trace, 331
  - transfer function, 402
  - transient state, 488
  - transition matrix, *see* Markov chain
  - transition probability, *see* Markov chain
  - transition rates, 503
  - transitive
    - property of an equivalence relation, 499
  - transpose of a matrix, 330
  - trial, 3
  - triangle inequality
    - for  $L^p$  random variables, 523
    - for numbers, 522
  - trigonometric identity, 389
  - twisted density, 273
  - two-sided test, 265
  - two-tailed test, 265
  - Type I error, 264
  - Type II error, 264
- U**
- unbiased estimator, 241
    - of a correlation function, 397
  - uncorrelated random variables, 93

example that are not independent, 104, 322, 327  
 uncountable set, 16  
 uniform random variable (continuous), 140  
   cdf, 186  
   simulation, 194  
   tangent of = Cauchy, 194  
 uniform random variable (discrete), 68  
 union bound, 26  
   derivation, 54  
 union of sets, 9  
 unit impulse, 199  
 unit-step function, 87, 421

## V

variance, 84  
 variance formula, 85  
 Venn diagrams, 8

## W

weak law of large numbers, 6, 116, 423, 565, 576  
   compared with the central limit theorem, 571  
 Weibull failure rate, 237  
 Weibull random variable, 171, 222  
   moments, 178  
   relation to generalized gamma, 225  
 white noise, 406  
   bandlimited, 406  
   infinite average power, 406  
   Karhunen–Loève expansion, 531  
 whitening filter, 419  
 Wick's theorem, 377  
 wide-sense stationarity  
   continuous time, 395  
   discrete time, 431, 432  
 Wiener filter, 419, 535  
   causal, 419  
   for random vectors, 344  
   prediction, 439  
   smoothing, 439  
 Wiener integral, 456, 532  
   normality, 584  
 Wiener process, 388, 454  
   approximation using random walk, 457  
   as a Markov process, 515  
   defined for negative and positive time, 471  
   independent increments, 455  
   Karhunen–Loève expansion, 531  
   normality, 474  
   relation to Ornstein–Uhlenbeck process, 470  
   self similarity, 592, 610  
   standard, 455  
   stationarity of its increments, 610  
 Wiener, Norbert, 388  
 Wiener–Hopf equation, 419  
 Wiener–Khinchin theorem, 422  
   alternative derivation, 427  
 WLLN, *see* weak law of large numbers  
 WSS, *see* wide-sense stationary

## Z

$z$  transform, 606  
   related to pgf, 108  
 Zener diode, 266  
 zero random variable, 579  
 zeta random variable, 105  
 Zipf random variable = zeta, 82, 105