Contents

Note: equations referenced using square brackets, *e.g* [3.31], refer to equations in the text of the book; equations referenced using round brackets, *e.g* (1.1), refer to equations in this document.

Chapter 1

1) For a circular orbit of radius r the centrifugal acceleration, $\omega^2 r$ must balance the gravitational acceleration $\frac{GM}{r^2}$,

$$\omega^2 r = \frac{GM}{r^2} \qquad \Rightarrow \qquad \omega = \sqrt{\frac{GM}{r^3}},$$

and $v = \omega r$, so

$$v = \sqrt{\frac{GM}{r}}.$$

2) Non-relativistically the escape velocity can be determined by setting the total energy, the sum of the kinetic and potential energy, to zero. If the total energy is negative the orbit is bounded, if it is positive the orbit is unbounded. Zero energy is the watershed value between these two cases. For a test mass m at the surface of a planet of mass M and radius R, moving vertically upward with speed v, the total energy is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}.$$

The escape velocity is determined by setting $E = 0$, so

$$v^2 = \frac{2GM}{R}.$$

When $R = \frac{2GM}{c^2}$ we have $v = c$.

The fact that the non-relativistic Newtonian escape velocity is equal to c when R is the Schwarzschild radius is just a co-incidence, it uses the non-relativistic kinetic energy which is incorrect when $v = c$. It is an incorrect calculation which gets the correct answer using an erroneous argument, but it can be useful as a mnemonic.

3) a) Since l is constant it can be evaluated for any P. When $P = A$ the length l of the string must be twice the distance from O to A, which is $\Delta + a$, so $l = 2(\Delta + a)$.

b) In Cartesian co-ordinates $x = r\cos\theta$, $y = r\sin\theta$, relative to the origin O, the distance r from O to a point P on the ellipse is $r = \sqrt{x^2 + y^2}$. If $\tilde{r}$ is the distance from $\widetilde{O}$ to P then, by construction,

$$r + \tilde{r} + 2\Delta = l = 2(\Delta + a) \quad \Rightarrow \quad r + \tilde{r} = 2a.$$

c) i) The distance from A to B is $2a$ but this is also the sum of the distance from O to A plus the distance from O to B, which is

$$\frac{r_0}{1-e} + \frac{r_0}{1+e} = \frac{2r_0}{1-e^2},$$

hence

$$a = \frac{r_0}{1-e^2}.$$

ii) Δ is the distance $O'B$ minus the distance OB, or

$$\Delta = a - \frac{r_0}{1+e} = \frac{r_0}{1-e^2} - \frac{r_0}{1+e} = \frac{r_0 e}{1-e^2}.$$

iii) When P is directly above O', $r = \tilde{r}$ we have $r = a$, from (3b), and $OP\widetilde{O}$ is an isosceles triangle, half of which is a right-angled triangle with height b, base Δ and hypotenuse a, so $a^2 = b^2 + \Delta^2$ from Pythagoras' theorem. Hence

$$b^2 = a^2 - \Delta^2 = \frac{r_0^2}{(1-e^2)^2} - \frac{r_0^2 e^2}{(1-e^2)^2} = \frac{r_0^2}{(1-e^2)}$$

$$\Rightarrow \qquad b = \frac{r_0}{\sqrt{1-e^2}}.$$

iv) from questions 3a), 3(c)i) and 3(c)ii)

$$l = \frac{2r_0(e+1)}{(1-e^2)} = \frac{2r_0}{(1-e)}.$$

d) From

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

with $x' = r'\cos\theta'$ and $y' = r'\sin\theta'$, 3(c)i) and 3(c)iii) give

$$
\begin{aligned}
&\frac{r'^2}{r_0^2}\left((1-e^2)^2\cos^2\theta' + (1-e^2)\sin^2\theta'\right) = 1 \\
\Rightarrow\quad & r'^2(1-e^2)\left(1-e^2\cos^2\theta'\right) = r_0^2 \\
\Rightarrow\quad & r'^2 = \frac{r_0^2}{(1-e^2)\left(1-e^2\cos^2\theta'\right)}.
\end{aligned}
\tag{1.1}
$$

e) Again using

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

but now with

$$x' = x + \Delta = r\cos\theta + \frac{r_0 e}{1-e^2}, \qquad y' = y = r\sin\theta$$

we have

$$
\begin{aligned}
&\frac{\left((1-e^2)r\cos\theta + r_0 e\right)^2}{(1-e^2)^2}\frac{(1-e^2)^2}{r_0^2} + r^2\sin^2\theta\frac{(1-e^2)}{r_0^2} = 1 \\
\Rightarrow\quad & \left((1-e^2)r\cos\theta + r_0 e\right)^2 + (1-e^2)r^2\sin^2\theta - r_0^2 = 0 \\
\Rightarrow\quad & r^2(1-e^2\cos^2\theta) + 2rr_0 e\cos\theta = r_0^2 \\
\Rightarrow\quad & r = \frac{-2r_0 e\cos\theta \pm \sqrt{4r_0^2e^2\cos^2\theta + 4r_0^2(1-e^2\cos^2\theta)}}{2(1-e^2\cos^2\theta)} \\
& r = \pm\frac{r_0(1\mp e\cos\theta)}{1-e^2\cos^2\theta} = \pm\frac{r_0}{(1\pm e\cos\theta)}.
\end{aligned}
$$

We take the positive root to ensure that r is positive,

$$r = \frac{r_0}{(1+e\cos\theta)}$$

as required.

f) From 3b), 3(c)i) and 3e)

$$
\begin{aligned}
\tilde{r} = 2a - r &= \frac{2r_0}{1-e^2} - \frac{r_0}{1+e\cos\theta} \\
&= r_0\frac{\left\{2(1+e\cos\theta) - (1-e^2)\right\}}{(1-e^2)(1+e\cos\theta)} \\
&= \frac{(1+2e\cos\theta+e^2)r_0}{(1-e^2)(1+e\cos\theta)}.
\end{aligned}
$$

g) Relative to the origin O the area of a thin triangular wedge associated with an infinitesimal variation $\delta\theta$ of θ is $\frac{1}{2}r^2\delta\theta$, so the area of the whole ellipse is

$$\frac{1}{2}\int_0^{2\pi} r^2(\theta)d\theta = \frac{r_0^2}{2}\int_0^{2\pi}\frac{d\theta}{(1+e\cos\theta)^2} = \frac{\pi r_0^2}{(1-e^2)^{3/2}} = \pi ab.$$

The integral can be evaluated using the calculus of residues, by writing $\cos\theta = \frac{1}{2}(z+z^{-1})$, with $z = e^{i\theta}$, and integrating around the closed contour $z = 1$, on which $d\theta = -i\frac{dz}{z}$. Then

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{(1+e\cos\theta)^2} &= -i\oint\frac{dz}{z\left(1+\frac{e}{2}\left(z+z^{-1}\right)\right)^2}\\ &= -\frac{4i}{e^2}\oint\frac{zdz}{\left(z^2+\frac{2}{e}z+1\right)^2}\\ &= -\frac{4i}{e^2}\oint\frac{zdz}{(z-z_+)^2(z-z_-)^2}\end{aligned}$$

with $z_\pm = \frac{\pm\sqrt{1-e^2}-1}{e}$ and $z_+z_- = 1$. For $0 \le e \le 1$, z_+ is inside the unit circle and z_- is outside it. There is one pole inside the contour, at $z = z_+$, of order 2. Laurent expanding the integrand about z_+, with $z = z_+ + \varepsilon$,

$$\frac{z}{(z-z_+)^2(z-z_-)^2} = \frac{z_+}{(z_+-z_-)}\frac{1}{\varepsilon^2} + \frac{z_+}{(z_+-z_-)^2}\left(\frac{1}{z_+} - \frac{2}{z_+-z_-)}\right)\frac{1}{\epsilon} + \dots,$$

and the residue is

$$a_{-1}(z_+) = \frac{z_+}{(z_+-z_-)^2}\left(\frac{1}{z_+} - \frac{2}{(z_+-z_-)}\right) = -\frac{(z_++z_-)}{(z_+-z_-)^3} = \frac{e^2}{4(1-e^2)^{3/2}}.$$

The calculus of residues then gives

$$\int_0^{2\pi}\frac{d\theta}{(1+e\cos\theta)^2} = 2\pi i\left(-\frac{4i}{e^2}\right)a_{-1}(z_+) = \frac{2\pi}{(1-e^2)^{3/2}}.$$

4) The lunar tides are due to the difference in the gravitational force between the Moon and the Earth on opposite sides of the Earth. Let M be the mas of the Earth, m be the mass of the Moon, D the Earth-Moon distance (between their centres) and R the radius of the Earth. Then the gravitational attraction between the Earth and the Moon is $\frac{GMm}{D^2}$ and acceleration of the Earth is $\frac{Gm}{D^2} = 3.3\times10^{-5}\text{m/s}^2$. In terms of the dimensionless ratios

$$\eta = \frac{R}{D} = 1.66\times10^{-2}, \qquad \mu = \frac{m}{M} = 1.23\times10^{-2}$$

this acceleration is $a = g\mu\eta^2$ where $g = \frac{GM}{R^2}$ is the acceleration due to gravity at the surface of the Earth. The acceleration a does not

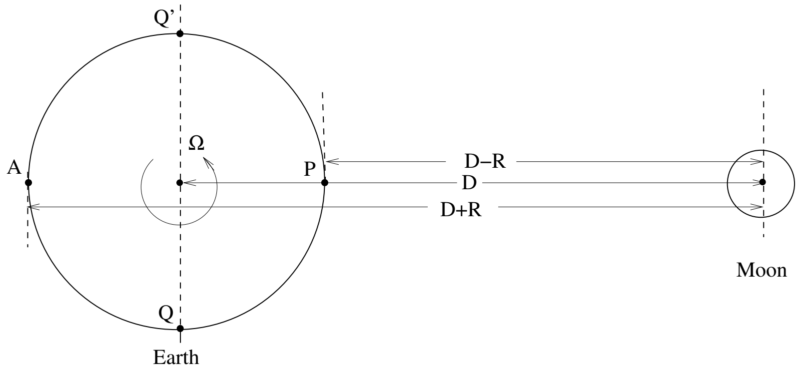

Figure 1.1 Tidal forces due to the change in the Moon's gravitational attraction over the Earth's diameter. This diagram is drawn from above the Moon's orbital plane, looking down on the Earth's equatorial plane.

cause D to decrease however as the Earth and the Moon are rotating about their common centre of gravity and, in this rotating reference frame, there is a compensating centrifugal force that keeps them in orbit around the barycentre.

The calculation of the height of the tides is greatly simplified by the fact that η is small, so we can expand in η. To estimate the magnitude of the lunar tides we compare the magnitude of the gravitational forces exerted by the Moon on a unit mass of water at perilune (the point P in figure 1.1) and at apolune (the point A in figure 1.1) to the lunar force at the centre of the Earth (for a unit mass this the same as calculating an acceleration).

The differences are

$$\frac{Gm}{(D \mp R)^2} - \frac{Gm}{D^2} \approx \pm \frac{2GmR}{D^3} = \pm \frac{2GM}{R^2}\mu\eta^3 = \pm 2g\mu\eta^3 = \pm 1.1\times 10^{-6}\mathrm{m/s}^2.$$

The acceleration due to gravity affecting a mass of water at the two points A and P is the combination of that of the Earth and that of the Moon,

$$a_g = (1 \pm 2\epsilon)g$$

where $\epsilon = \mu\eta^3 = 5.6 \times 10^{-8}$. This should be compared to the vertical acceleration due to gravity at the two points Q (for 'quadrature') and Q', 90° from perilune and apolune in figure 1.1, which is simply g.

The lunar tidal forces in a_g will support a height h of water where

$$a_g = \frac{GM}{(R+h)^2} = \frac{GM}{R^2}\left(1 - \frac{2h}{R} + \cdots\right) = g\left(1 - \frac{2h}{R} + O\left(\frac{h}{R}\right)\right) \tag{1.2}$$

so, to lowest order,

$$h = \pm\epsilon R = \pm\mu\eta^3 R = \pm 0.36\text{m}, \tag{1.3}$$

compared to a point at 90° to perilune and apolune at which $h = 0$.

Repeating the calculation using the figures for the Sun, $\mu = 33.5 \times 10^4$ and $\eta = 4.2 \times 10^{-5}$, gives

$$h = \pm 0.16\text{m},$$

the tidal influence the Sun is about half that of the Moon. Although the Sun is much farther away it is also much more massive than the Moon and that latter property almost compensates for the greater distance. From now on we shall forget about the Sun and just discuss lunar tides, but the same analysis can be applied to solar tides.

This above argument is not the whole story of course, if it were the height of the sea on the side of the Earth farthest away from the Moon would be lowered by 36cm at the same time as the height closest to the Moon is raised by 36cm, and this is not what happens, high tides occur at the same time on opposite sides of the Earth. This is because Earth and the Moon rotate about their common centre of gravity, which results in a significant centrifugal force, and this pushes the water away from the Moon, so that centrifugal force might be expected to lower the ocean surface on the side closest to the Moon and raised it on the other side. The effect is to make the situation symmetric on opposite sides making high tide one half of 36cm, about 18cm on opposite sides of the Earth, though, as we shall see, this is a slight underestimate.

In fact the centre of gravity of the Earth-Moon system lies inside the Earth, under its surface, so one might worry that centrifugal force would raise the tide on the surface closest to the Moon, rather than lowering it, so we now give a more careful analysis.

The Moon raises tides due to the fact that the Earth is a rigid body and the Moon's gravitational field is not uniform. As the Earth and the Moon orbit around their common centre of gravity (their barycentre) the centre of the Earth is in free fall, but because the Earth is a solid body the point on the Earth's surface nearest the

Moon (perilune) is not in free fall, it is moving sightly too slowly to be, and neither is the point on the opposite side of the Earth farthest from the Moon (apolune) in free fall, it is moving slightly too fast to be. If the Earth were made entirely of water, every drop of which was in a stable orbit around the barycentre the water would get stretched and distorted, points at perilune forging ahead of the centre of the Earth and points at apolune lagging behind it in its orbit around the barycentre.

As the earth rotates on its axis the perilune will perform one revolution about the equator every 24 hours, as does apolune (we are ignoring the tilt of the Earth's axis here). This rotation does not affect the tides however: in the reference frame of the rotating Earth it gives rise to a centrifugal acceleration $a_\Omega = \Omega^2 R = 0.034\mathrm{m/s}^2 = 3.4 \times 10^{-3} g$ on the equator, which is far greater than the tidal forces calculated above and makes the ocean at the equator bulge out by some 22km relative to the poles. But it is completely symmetric, all points at the same latitude experience exactly the same height rise due to this centrifugal force and there is no change as the Earth rotates, there is no tidal variation due to this centrifugal force (we are ignoring the tilt of the Earth's axis relative to the plane of the Moon's orbit here).

The estimate above gives the height of the lunar tide as 36cm at perilune, above zero height at Q, but it looks like a low tide of -36cm and apolune, with a total tidal range of 72cm. But this is not correct: high tide at perilune does not coincide with low tide at apolune. The angular velocity of the rotation of the Earth-Moon system is given by $\omega^2 = \frac{G(M+m)}{D^3}$ about the barycentre and results in a centrifugal acceleration in the rotating frame of the Earth-Moon system that modifies the above conclusion. The above calculation ignores the rotation of the Earth about the barycentre. When this is included the exact numbers change slightly, but the order of magnitude, $\mu\eta^3$, is still correct.

Assuming the Earth-Moon system rotates about the centre of the Earth (which is not quite true), the centrifugal acceleration of a point on the Earth's surface would be $a_\omega = \omega^2 R = 4.5 \times 10^{-5}$ m/s^2, which, according to the above logic, would generate an extra height of

$$h = \frac{R}{2}\left(\frac{\omega^2 R}{g}\right)\frac{R\eta^2}{2} = 15m, \tag{1.4}$$

nearly two orders of magnitude greater than that calculated above, but this affects the point Q equally as much as A and P, and this

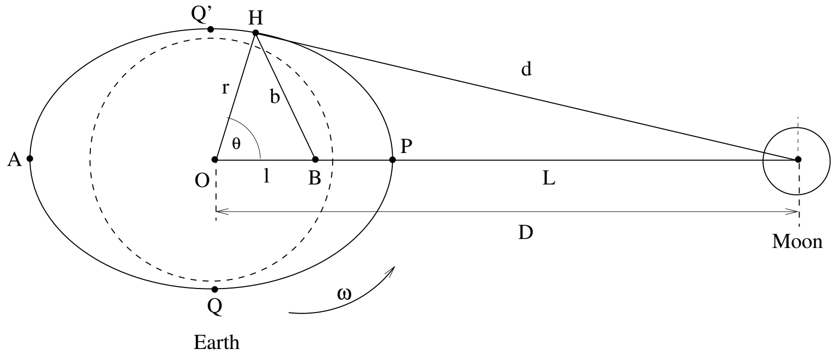

Figure 1.2 Looking down on the Earth-Moon system from above the north pole (ignoring the tilt of the Earth's axis), the diagram shows the Earth's equatorial plane. The dashed circle, radius R, represents the solid surface of the Earth. The ellipse represents the height of the sea level at the equator when tides and rotation and are included (exaggerated for clarity). D is the distance between the centre of the Moon and the centre of the Earth and B is the barycentre. H represents the point on the equator on the Greenwich meridian and θ is the longitude of the perilunal point.

constant shift does not produce tides. To fully understand the tides we need to calculate how the extra height difference generated by this centrifugal acceleration varies for points across the globe from perilune to apolune. This can be done by balancing the forces but a simpler way is to observe that, in equilibrium, the surface of the sea will be an equipotential surface and calculating the potential energy.

For simplicity consider what would happen if the Earth were to rotate in the same plane as the Moon's orbit, as in figure 1.2 below. Of course this is not really the case, the Earth's axis is tilted by $23\frac{1}{2}^\circ$ relative to the ecliptic and the Moon's orbital plane is tilted by 5° relative to the ecliptic, but in a first approximation we shall ignore these complications and assume that Ω and ω are parallel, as in figure 1.2.

Consider the Earth to be a perfect sphere, covered with water to a depth of a few kilometers, with the dashed circle in the figure, of radius $R = 6,370$km, representing the solid surface, assumed spherical, (the figure is not to scale!). The solid ellipse represents the height of the sea level at the equator, with $r(\theta) = R + h(\theta)$.[1] D is the distance

[1] The excess volume comes from water flowing down from higher latitudes, water is incompressible so the total volume of seawater is constant.

between the centre of the Moon and the centre of the Earth and B, the barycentre, is a distance l from the centre of the Earth and a distance L from the centre of the Moon, so $D = L + l$. If H represents the point on the equator on the Greenwich meridian (somewhere just south of Ghana in Africa) then θ is the longitude of the point at perilune.

B is defined by

$$Ml = mL \qquad \Rightarrow \qquad l = \frac{m}{M}L$$

and $l + L = D$ so

$$L = \frac{1}{(1+\mu)}D, \qquad \Rightarrow \qquad l = \frac{\mu}{(1+\mu)}D. \tag{1.5}$$

For the Earth-Moon system $l = 0.74R$ and the barycentre actually lies inside the Earth's surface, about $3/4$ of the way from the centre.

The rotation the Earth about the common centre of gravity of the Earth-Moon system results in a centrifugal force $\omega^2 b$ (in the reference frame of the Earth) on fluid elements of unit mass a distance b from the barycentre. ω is obtained from Kepler's third law

$$\omega^2 l = \frac{Gm}{D^2},$$

where l is given by (1.5), so

$$\omega^2 = \frac{GM(1+\mu)}{D^3} = \frac{GM}{R^3}\eta^3(1+\mu)$$

and $\frac{2\pi}{\omega} = 27$ days, the sidereal period of the Moon's orbit. The centrifugal acceleration (force per unit mass of water) at a point a distance b away from B is $\omega^2 b$ and the potential energy per unit mass that generates this is

$$V_\omega = -\frac{1}{2}\omega^2 b^2.$$

Let $\epsilon = \frac{h}{R} \ll 1$, then $R+h = R(1+\epsilon)$ and, from elementary trigonometry,

$$\begin{aligned} b^2 &= r^2 + l^2 - 2rl\cos\theta \\ &= \frac{R^2}{(1+\mu)^2\eta^2}\{\mu^2 - 2\eta\mu(1+\mu)(1+\epsilon)\cos\theta + \eta^2(1+\mu)^2(1+\epsilon)^2\} \\ &= \frac{R^2}{(1+\mu)^2\eta^2}\{\mu^2 - 2\eta\mu(1+\mu)\cos\theta + \eta^2(1+\mu)^2 + O(\epsilon)\}, \end{aligned}$$

since $l = \frac{D\mu}{1+\mu} = \frac{R\mu}{(1+\mu)\eta}$. So

$$V_\omega = -\frac{1}{2}\frac{GM}{R}\frac{\eta}{(1+\mu)}\{\mu^2 - 2\eta\mu(1+\mu)\cos\theta + \eta^2(1+\mu)^2\} + O(\epsilon\eta). \tag{1.6}$$

The potential energy of a unit mass at the point H due to the Moon's gravitational attraction is

$$V_m = -\frac{Gm}{d} = -\frac{GM\mu}{d}$$

and, again from elementary trigonometry,

$$d^2 = r^2 + D^2 - 2rD\cos\theta = \frac{R^2}{\eta^2}\big(1 - 2\eta(1+\epsilon)\cos\theta + \eta^2(1+\epsilon)^2\big)$$
$$= \frac{R^2}{\eta^2}\big(1 - 2\eta\cos\theta + \eta^2 + O(\epsilon)\big).$$

So

$$V_m = -\frac{GM}{R}\frac{\mu\eta}{\sqrt{1-2\eta\cos\theta+\eta^2}} + O(\epsilon\eta) \tag{1.7}$$

Lastly, the potential energy of a unit mass at the point H due to the Earth's gravitational attraction is

$$V_M = -\frac{GM}{r} = -\frac{GM}{R}\frac{1}{(1+\epsilon)} = -\frac{GM}{R}\big(1 - \epsilon + O(\epsilon^2)\big)$$

Equipotential surfaces are therefore obtained from

$$V = V_M + V_m + V_\omega = const$$
$$= \frac{GM}{R}\Bigg(-1 + \epsilon - \frac{\mu\eta}{\sqrt{1-2\eta\cos\theta+\eta^2}}$$
$$-\frac{1}{2}\frac{\eta}{(1+\mu)}\{\mu^2 - 2\eta\mu(1+\mu)\cos\theta + \eta^2(1+\mu)^2\} + O(\epsilon\eta, \epsilon^2)\Bigg)$$

Expanding in $\eta \ll 1$

$$-1 + \epsilon - \mu\eta\left(1 + \eta\cos\theta - \frac{\eta^2}{2} + \frac{3}{2}\eta^2\cos^2\theta\right)$$
$$-\frac{1}{2}\frac{\eta}{(1+\mu)}\{\mu^2 - 2\eta\mu(1+\mu)\cos\theta\} + O(\eta^3, \epsilon\eta, \epsilon^2) = const.$$
$$\Rightarrow \qquad \epsilon - \frac{3}{2}\mu\eta^3\cos^2\theta + O(\eta^3, \epsilon\eta, \epsilon^2) = const. \tag{1.8}$$

To a very good approximation equipotential surfaces are described by

$$\epsilon(\theta) = \frac{3}{2}\mu\eta^3 \cos^2\theta + const.$$

Sea level is at

$$r(\theta) = R + h(\theta) = R(1+\epsilon(\theta)) = R\left(1 + \frac{3}{2}\mu\eta^3\cos^2\theta + \cdots\right)$$

Referring to the equation for an ellipse centered at O' given in equation (1.1), and dropping the primes,

$$r = \frac{r_0}{\sqrt{(1-e^2)(1-e^2\cos^2\theta)}} = \frac{r_0}{\sqrt{(1-e^2)}}\left(1 + \frac{e^2}{2}\cos^2\theta + O(e^4)\right)$$

and, to a very good approximation, the sea level at the equator forms an ellipse with eccentricity

$$e = \sqrt{3\mu\eta^3} = 4.1 \times 10^{-4}.$$

The full tidal range is

$$\frac{3R}{2}\mu\eta^3 = 0.54\text{m}$$

and high tide is $\frac{3R}{4}\mu\eta^3 = 0.27$m above mean sea level, simultaneously at perilune and apolune (P and A respectively), low tide is 0.27m below mean sea level, at $\theta = \pi/2$ and $3\pi/2$ (Q and it's antipodal point). These values are lower at higher latitudes away from the equator.

Note that the average value of V_ω in (1.6) is

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} V_\omega d\theta &= -\frac{1}{2}\frac{GM}{R}\frac{\eta}{(1+\mu)}\left(\mu^2 - \frac{\eta^2\mu(1+\mu)}{4} + \eta^2(1+\mu)^2\right)\\ &\approx -\frac{1}{2}\frac{GM}{R}\left(\mu^2\eta - \frac{\eta^2\mu}{4} + \eta^3\right)\end{aligned}$$

and this corresponds to a uniform increase of sea level h_0 at the equator given by $h_0 = R\epsilon_0$ where

$$-\frac{GM}{R(1+\epsilon_0)} \approx -\frac{GM}{R}(1-\epsilon_0) = -\frac{GM}{R}\left(1 - \frac{(4\mu^2\eta - \eta^2\mu + 4\eta^3)}{8}\right)$$

so

$$h_0 = \frac{R(4\mu^2\eta + 4\eta^3 - \eta^2\mu)}{8} = 19.8\text{m},$$

somewhat larger than the estimate (1.4), the difference being due to the fact that the barycentre is displaced from O. This is in addition to the 22km due the Earth's daily rotation about O, and does not affect the tides.

The above calculation only assumed that $\eta \ll 1$ and $\epsilon \ll 1$, it did not assume that μ is small, equation (1.8) works just as well for the Sun as for the Moon, giving a solar tidal range of 0.24m, with ellipse eccentricity $e = 2.8 \times 10^{-4}$.

The calculation presented above is very simplistic, assuming that the Earth can be treated as a perfect sphere covered by a film of water, of uniform depth at the equator in the absence of tides. The full story is of course much more complicated than that given above. The Earth is not uniformly covered with an ocean of constant depth, the depth varies and there are continents and indented coastlines which impede the flow of water. This hugely complicates the issue and tidal surges and resonances can lead to differing heights at different places and even more than two tides a day. The highest tidal range on the Earth is actually 16m, in the Bay of Fundy off Novia Scotia.

2

Chapter 2

1) In polar coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$, with $\rho^2 = x^2+y^2$. When $\phi = \pi/2$, $x = 0$ and $y = \rho = b\big(1+\cos(z/a)\big)$ so, rotating around the z-axis, for a general value of ϕ

$$y = b\sin\phi\big(1+\cos(z/a)\big), \qquad x = b\cos\phi\big(1+\cos(z/a)\big).$$

and

$$\rho = b\big(1+\cos(z/a)\big).$$

Infinitesimally

$$d\rho = -\frac{b}{a}\sin(z/a)dz = -\frac{b}{a}\sqrt{1-\cos^2(z/a)}dz = -\frac{b}{a}\sqrt{1-(\rho/b-1)^2}dz$$

and

$$\begin{aligned} d\rho^2 &= \frac{b^2}{a^2}\left(\frac{2\rho}{b} - \frac{\rho^2}{b^2}\right)dz^2 \\ \Rightarrow \qquad dz^2 &= \frac{a^2}{b^2}\frac{d\rho^2}{\left(\frac{2\rho}{b} - \frac{\rho^2}{b^2}\right)} = \frac{a^2}{\rho}\frac{d\rho^2}{(2b-\rho)}, \end{aligned}$$

giving

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \\ &= d\rho^2 + \rho^2 d\phi^2 + \frac{a^2}{\rho}\frac{d\rho^2}{(2b-\rho)} \\ &= \left(\frac{\rho(2b-\rho)+a^2}{\rho(2b-\rho)}\right)d\rho^2 + \rho^2 d\phi^2. \end{aligned}$$

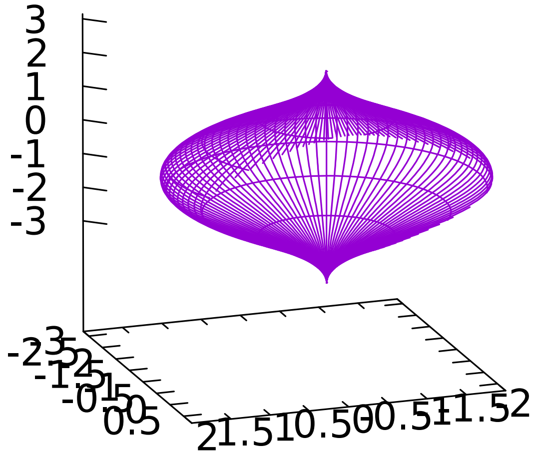

Figure 2.1 Surface with cusps.

The metric is

$$g_{\alpha\beta} = \begin{pmatrix} \frac{\rho(2b-\rho)+a^2}{\rho(2b-\rho)} & 0 \\ 0 & \rho^2 \end{pmatrix}.$$

$g_{\rho\rho}$ diverges as $\rho \to 0$ ($z \to \pm\pi a$) while $g_{\phi\phi}$ vanishes there. The latter is just the usual harmless coordinate singularity in 2-dimensional polar coordinates, ϕ is not a good coordinate on the z-axis, but what about the divergence in $g_{\rho\rho}$?

The surface, for $a = b = 1$ is sketched in figure 2.1:

The shape is that of an old-fashioned wooden top, with sharp cusps at $z = \pm\pi a$. The surface is not differentiable at these sharp points, the curvature is singular there, these are real singularities in the geometry. You can check this by calculating the Ricci scalar, it diverges at these two points.

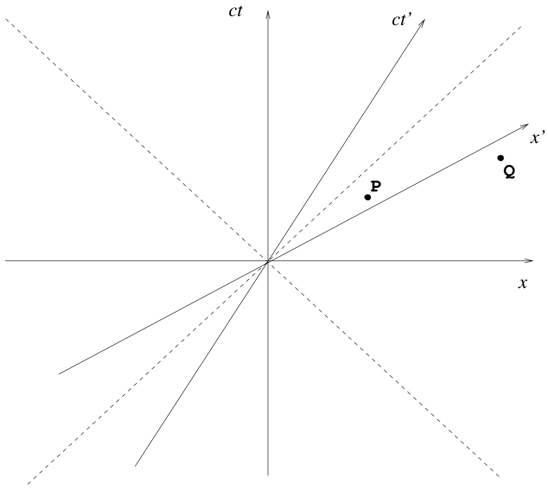

Figure 2.2 Space-like separated events, P and Q.

2) For simplicity set $y = z = 0$. Under a Lorentz transformation from Cartesian coordinates (ct, x) in an inertial reference frame S to coordinates (ct', x') in an inertial frame S',

$$ct' = \gamma(v)\left(ct - \frac{vx}{c}\right)$$
$$x' = \gamma(v)(x - vt)$$

where $\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Using this formula we can draw the ct' and x' axes in the ct-x plane, by plotting the lines $x' = 0$ and $ct' = 0$ respectively, as in the figure below, where $v > 0$,

For the two events marked P and Q, P happens before Q in the ct-x coordinates system but after Q in the ct'-x' system. This reversal of temporal order is only possible if P and Q have space-like separation (the dotted lines at 45° represent light-like directions).

3) We have

$$x = \frac{1}{2}(v - u), \qquad ct = \frac{1}{2}(v + u),$$

so

$$dx^2 = \frac{1}{4}(dv^2 - 2\,du\,dv + du^2), \qquad cdt^2 = \frac{1}{4}(dv^2 + 2\,du\,dv + du^2)$$

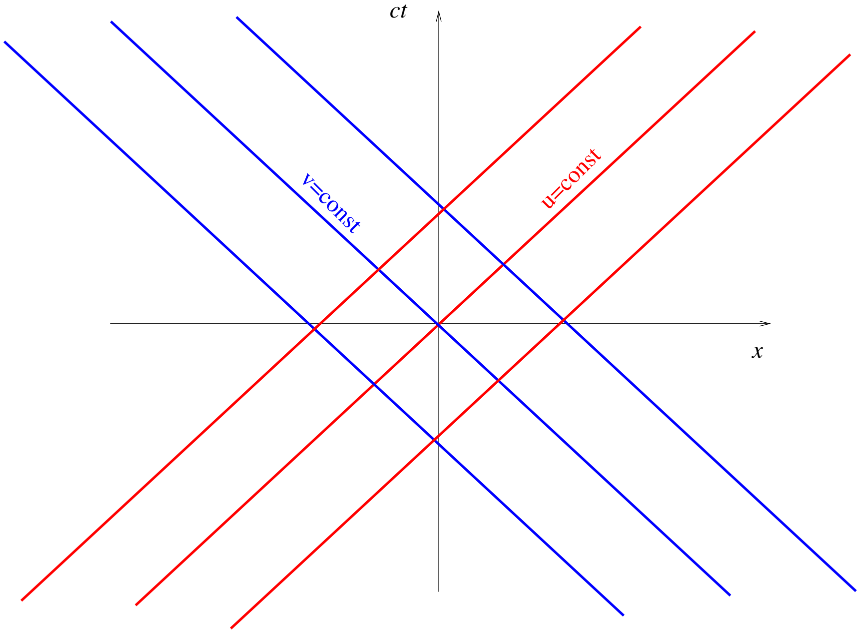

Figure 2.3 Light-like coordinates,

and

$$-c^2dt^2 + dx^2 = -du\,dv = \frac{1}{2}\begin{pmatrix} du & dv \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}.$$

Lines of constant u are at 45° to the x-axis and lines of constant v are at −45°, or 135°, as in the figure below

These are light-like lines, hence the name *light-like coordinates*.

4) Fix X, then

$$x^2 - c^2t^2 = X^2\left(\cosh^2\left(\frac{cT}{L}\right) - \sinh^2\left(\frac{cT}{L}\right)\right) = X^2 > 0$$

is a parabola in the ct-x plane, different values of X give different parabolas.

Fix T then

$$\frac{x}{ct} = \frac{\cosh\left(\frac{cT}{L}\right)}{\sinh\left(\frac{cT}{L}\right)} = \coth\left(\frac{cT}{L}\right)$$

is a straight line with $ct = \tanh\left(\frac{cT}{L}\right)x$, $-1 \le \tanh\left(\frac{cT}{L}\right) \le 1$, and different values of T give different slopes.

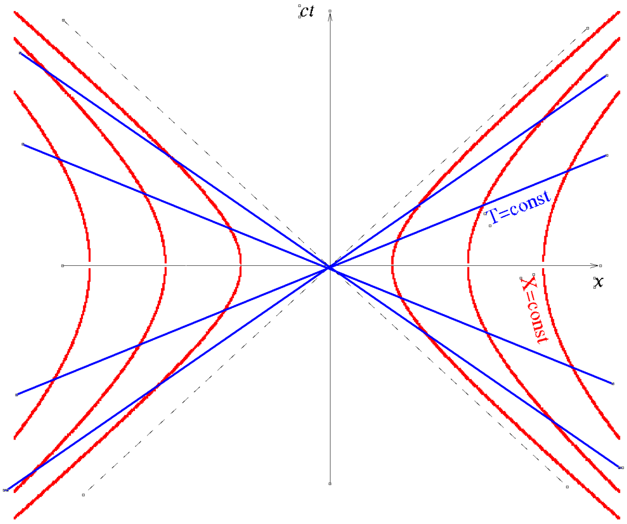

Figure 2.4

5) Fixing

$$\rho^2 - c^2t^2 = L^2 \tag{2.1}$$

$$\begin{aligned}
\Rightarrow \quad & 2\rho d\rho - 2c^2 t dt = 0 \\
\Rightarrow \quad & d\rho^2 = \frac{c^4t^2dt^2}{\rho^2} = \frac{c^4t^2dt^2}{c^2t^2+L^2} \\
\Rightarrow \quad & d\rho^2 + \rho^2 d\phi^2 - c^2dt^2 = \rho^2 d\phi^2 + \left(\frac{c^2t^2}{c^2t^2+L^2} - 1\right) c^2dt^2 \\
& = (L^2 + c^2t^2)d\phi^2 - \left(\frac{L^2}{c^2t^2+L^2}\right) c^2dt^2 .
\end{aligned} \tag{2.2}$$

The surface $c^2t^2 = x^2+y^2-L^2$ is sketched in figure 5, for $L = 1$. It is a hyperboloid of revolution and the metric (2.2) has one time-like and one space-like direction at every point, it is a curved 2-dimensional space-time.

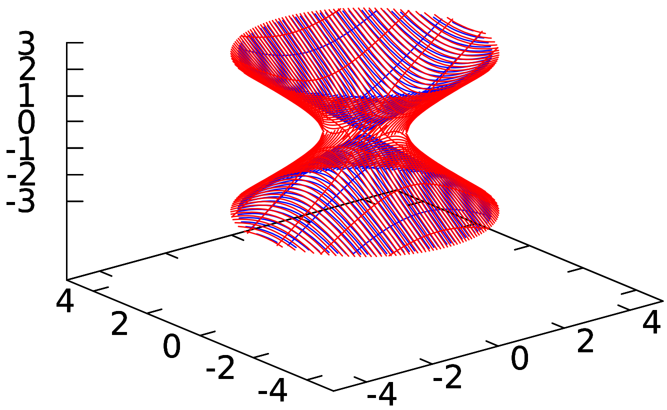

Figure 2.5 2-dimensional de Sitter space-time embedded in 3-dimensional Minkowski space-time. The vertical axis is ct and the surface is the hyperbola $x^2 + y^2 = c^2t^2 + L^2$ (with $L = 1$). The light-cone $x^2 + y^2 = c^2t^2$ sits inside the hyperboloid.

Now

$$ct = L \sinh\left(\frac{cT}{L}\right)$$
$$\Rightarrow \qquad cdt = \cosh\left(\frac{cT}{L}\right) cdT$$
$$\Rightarrow \quad c^2dt^2 = \cosh^2\left(\frac{cT}{L}\right) c^2dT^2$$

and

$$c^2t^2 + L^2 = L^2\left(\sinh^2\left(\frac{cT}{L}\right) + 1\right) = L^2 \cosh^2\left(\frac{cT}{L}\right)$$
$$\Rightarrow \qquad ds^2 = L^2 \cosh^2\left(\frac{cT}{L}\right) d\phi^2 - c^2dT^2$$
$$\Rightarrow \qquad a(T) = L \cosh\left(\frac{cT}{L}\right).$$

This is a 2-dimensional Robertson-Walker space-time, corresponding to an expanding universe when T is positive.

6) The only place on Earth that this could possibly happen is at the south pole, so the bird must be a penguin! [1]

The point is that polar co-ordinates, (latitude and longitude) are not good co-ordinates at the south (or north) pole. Exactly at the poles longitude is simply not defined, every degree of longitude corresponds to exactly the same point. This has no relevance to the photographer of course, she just follows a perfectly regular isosceles triangle which, to all intents and purposes, is on a flat plane. Note that continuing in an easterly direction when you are 10km from the south pole means you travel in a circle of radius 10km — it is not a straight line (in navigation this is called a *rhumb line*), this has nothing to do with curvature of the Earth, it is simply a consequence of using polar co-ordinates around the origin.

[1] Probably an emperor penguin, as these are the only birds that nest deep into the interior of Antarctica, though in reality even an emperor probably wouldn't get as far in as the south pole itself, unless it was very lost!

3

Chapter 3

1) The equations of motion for $\dot{x}$ and $\dot{y}$ are

$$\ddot{\tilde{x}} = \Omega\tilde{y}\ddot{t} + \Omega^2\tilde{x}\dot{t}^2 + 2\Omega\dot{\tilde{y}}\dot{t} \tag{3.1}$$

$$\ddot{\tilde{y}} = -\Omega\tilde{x}\ddot{t} + \Omega^2\tilde{y}\dot{t}^2 - 2\Omega\dot{\tilde{x}}\dot{t} \tag{3.2}$$

and the first integral for t is

$$\dot{t} = \frac{kc^2 + \Omega\big(\tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}}\big)}{c^2 - \Omega^2\big(\tilde{x}^2 + \tilde{y}^2\big)}. \tag{3.3}$$

Combining equations (3.1) and (3.2)

$$\begin{aligned}
\tilde{x}\ddot{\tilde{y}} - \tilde{y}\ddot{\tilde{x}} &= -\Omega\tilde{x}^2\ddot{t} - 2\Omega\tilde{x}\dot{\tilde{x}}\dot{t} - \Omega\tilde{y}^2\ddot{t} - 2\Omega\tilde{y}\dot{\tilde{y}}\dot{t} \\
\Rightarrow\ \frac{d}{dt}\left(\tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}}\right) &= -\frac{d}{dt}\left(\Omega(\tilde{x}^2 + \tilde{y}^2)\dot{t}\right) \\
\Rightarrow\ \tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}} + \Omega(\tilde{x}^2 + \tilde{y}^2)\dot{t} &= A \qquad \text{(a constant)} \\
\Rightarrow\ \left(\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt} + \Omega(\tilde{x}^2 + \tilde{y}^2)\right)\dot{t} &= A, \qquad (3.4)
\end{aligned}$$

where we have used $\dot{\tilde{x}} = \frac{d\tilde{x}}{dt}\dot{t}$ and $\dot{\tilde{y}} = \frac{d\tilde{y}}{dt}\dot{t}$.

Non-relativistically ($c \to \infty$ in (3.3)) $\dot{t} = k$ and

$$\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt} + \Omega(\tilde{x}^2 + \tilde{y}^2) = \frac{A}{k}$$

so, in the question, $l = \frac{A}{k} \ \Rightarrow\ A = lk$.

Now from equation (3.3)

$$\dot{t} = \frac{kc^2 + \Omega\big(\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt}\big)\dot{t}}{c^2 - \Omega^2\big(\tilde{x}^2 + \tilde{y}^2\big)}$$

$$\Rightarrow \qquad \dot{t} = \frac{kc^2}{c^2 - \Omega^2\big(\tilde{x}^2 + \tilde{y}^2\big) - \Omega\big(\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt}\big)}$$

and putting this in (3.4), with $A = lk$, gives

$$\left(\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt} + \Omega(\tilde{x}^2 + \tilde{y}^2)\right) = \left(\frac{c^2 - \Omega^2\big(\tilde{x}^2 + \tilde{y}^2\big) - \Omega\big(\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt}\big)}{c^2}\right) l$$

$$\Rightarrow \left(\tilde{x}\frac{d\tilde{y}}{dt} - \tilde{y}\frac{d\tilde{x}}{dt} + \Omega(\tilde{x}^2 + \tilde{y}^2)\right) = \frac{l}{\big(1 + \frac{l\Omega}{c^2}\big)}.$$

2) The Schwarzschild radius is

$$r_S = \frac{2GM}{c^2}$$

with $G = 6.67 \times 10^{-11}\text{kg}^{-1}\text{m}^3\text{s}^{-2}$ and $c = 3 \times 10^8\text{ms}^{-1}$. Using the naïve volume for a sphere of radius r_S, $\frac{4\pi r_S^3}{3}$, the density is

$$\rho = \frac{3M}{4\pi r_S^3} = \frac{3M}{4\pi}\left(\frac{c^2}{2GM}\right)^3 = \frac{3c^6}{32\pi G^2 M^2}.$$

a) With M=6×10^{24}kg, $r_s = 9$mm and $\rho = 2 \times 10^{30}$kg m^{-3}, this is an unimaginably high density — equivalent to packing the Sun into a metre cubed box..

b) With M=2×10^{30}kg, $r_s = 3$km and $\rho = 1.8 \times 10^{19}$kg m^{-3}, slightly higher than the density of nuclear matter (the density of an atomic nucleus).

c) With M=2×10^{36}kg, $r_s = 3 \times 10^6$km and $\rho = 1.8 \times 10^7$kg m^{-3}, or about 18kg per cc. For a supermassive black hole with a mass of 4 billion solar masses, $\rho = 1.1$ kg m^{-3} $= 1.1 \times 10^{-3}$g/cc, about the density of air.

d) With M=10^{53}kg, $r_s = 1.5 \times 10^{26}$km and $\rho = 7.3 \times 10^{-27}$kg m^{-3}. This is about twice the average density of matter in the Universe.

3) For a black hole we use the Schwarzschild metric. When θ and ϕ are constant

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \frac{dr^2}{\big(1 - \frac{2GM}{rc^2}\big)}.$$

With $U^\mu = (c\dot{t}, \dot{r}, 0, 0)$, the Lagrangian is

$$\mathcal{L} = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{rc^2}\right)}.$$

The equation of motion for $t(\tau)$ is

$$\frac{d}{d\tau}\left\{\left(1 - \frac{2GM}{rc^2}\right)\dot{t}\right\} = 0 \quad \Rightarrow \quad \dot{t} = \frac{k}{\left(1 - \frac{2GM}{rc^2}\right)}.$$

If τ is the proper-time then $\mathcal{L} = -c^2$ and

$$-c^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{rc^2}\right)}$$

$$\Rightarrow \quad \dot{r}^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right) + c^2 k^2 = c^2\left(k^2 - 1 + \frac{2GM}{rc^2}\right)$$

The clock is dropped from rest implies that $\dot{r} = \frac{dr}{d\tau} = 0$ when $r = r_0$, so the constant k is given by

$$k^2 = 1 - \frac{2GM}{r_0 c^2}$$

and

$$\dot{r} = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}.$$

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -\left(1 - \frac{2GM}{rc^2}\right)\sqrt{\frac{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}{1 - \frac{2GM}{r_0 c^2}}}.$$

The proper time on the falling clock is

$$\tau_c = \int d\tau = \int_{r_0}^{r_1} \frac{dr}{\dot{r}}$$

which is

$$\tau_c = \frac{1}{\sqrt{2GM}} \int_{r_1}^{r_0} \frac{r^{\frac{1}{2}}}{\sqrt{1 - \frac{r}{r_0}}} dr = \frac{r_0^{3/2}}{\sqrt{2GM}} \int_{\frac{r_1}{r_0}}^{1} \frac{\sqrt{y}}{\sqrt{1-y}} dy.$$

The integral can be done exactly

$$\int \frac{\sqrt{y}}{\sqrt{1-y}} dy = \sin^{-1}(\sqrt{y}) - \sqrt{y(1-y)},$$

but if we only want $r_0 \gg r_1$ then

$$\tau_c \approx \frac{\pi r_0^{3/2}}{2\sqrt{2GM}} = \frac{\pi r_0}{2c}\sqrt{\frac{r_0}{r_S}}.$$

For an observer fixed at $r = r_0$ the proper time on their clock is $d\tau_0 = \sqrt{1 - \frac{2GM}{r_0 c^2}}dt$. Using

$$dt = \frac{dr}{\frac{dr}{dt}} = -\sqrt{1 - \frac{2GM}{r_0 c^2}} \frac{dr}{\left(1 - \frac{2GM}{rc^2}\right)\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}}$$

the time their clock measures for the falling clock to fall from r_0 down to r_1 is

$$\begin{aligned}
\tau_0 = \sqrt{1 - \frac{2GM}{r_0 c^2}} \int_0^t dt &= -\frac{\left(1 - \frac{2GM}{r_0 c^2}\right)}{\sqrt{2GM}} \int_{r_0}^{r_1} \frac{r^{\frac{3}{2}} dr}{\left(1 - \frac{r}{r_0}\right)^{1/2} \left(r - \frac{2GM}{c^2}\right)} \\
&= r_0^{3/2} \frac{\left(1 - \frac{2GM}{r_0 c^2}\right)}{\sqrt{2GM}} \int_{\frac{r_1}{r_0}}^{1} \frac{y^{\frac{3}{2}} dy}{(1-y)^{1/2}\left(y - \frac{r_S}{r_0}\right)} \\
&= \frac{r_0^{3/2}}{\sqrt{2GM}} \left(1 - \frac{2GM}{r_0 c^2}\right) \int_\epsilon^1 \frac{y^{\frac{3}{2}} dy}{(1-y)^{1/2}\left(y - \frac{r_S}{r_1}\epsilon\right)} \\
&\approx \frac{r_0^{3/2}}{\sqrt{2GM}} \int_\epsilon^1 \frac{y}{\left(y - \frac{r_S}{r_1}\epsilon\right)} \frac{y^{\frac{1}{2}} dy}{(1-y)^{1/2}}
\end{aligned}$$

where $y = \frac{r}{r_0}$, $\epsilon = \frac{r_1}{r_0}$ and again the approximation is for $r_0 \gg r_1 > r_S = \frac{2GM}{c^2}$. Since $\frac{y}{y - \frac{r_S}{r_1}\epsilon} > 1$ we see that $\tau_0 > \tau_c$. When r_1 goes all the way down to the event horizon, $r_1 = r_S$, and the integral

$$\int_\epsilon^1 \frac{y}{(y-\epsilon)} \frac{y^{\frac{1}{2}} dy}{(1-y)^{1/2}}$$

diverges logarithmically: while the falling clock takes a finite time to reach the event horizon, according to itself, to a stationary observer outside the event horizon it never reaches the event horizon.

4) In the text we took a short cut and used $\mathcal{L} = -c^2$ when τ is the proper time, here we shall derive equation [3.31] directly. The radial equation [3.25] is

$$\ddot{r} = -\frac{GM}{r^2} \frac{k^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{c^2 r}\right)} \left(\frac{GM}{c^2 r^2}\right) + \left(1 - \frac{2GM}{c^2 r}\right) \frac{l^2}{r^3}.$$

Dividing by $1 - \frac{2GM}{c^2 r}$ and multiplying by $\dot{r}$ leads to a first integral

$$\frac{\dot{r}\ddot{r}}{\left(1 - \frac{2GM}{c^2 r}\right)} - \frac{\dot{r}^3}{\left(1 - \frac{2GM}{c^2 r}\right)^2}\left(\frac{GM}{c^2 r^2}\right) = -\frac{GM}{r^2}\frac{k^2\dot{r}}{\left(1 - \frac{2GM}{c^2 r}\right)^2} + \frac{l^2\dot{r}}{r^3}$$

$$\Rightarrow \qquad \frac{d}{d\tau}\left(\frac{\dot{r}^2}{1 - \frac{2GM}{c^2 r}}\right) = \frac{d}{d\tau}\left(\frac{c^2k^2}{1 - \frac{2GM}{c^2 r}} - \frac{l^2}{r^2}\right)$$

$$\Rightarrow \qquad \frac{\dot{r}^2}{1 - \frac{2GM}{c^2 r}} - \frac{c^2k^2}{1 - \frac{2GM}{c^2 r}} + \frac{l^2}{r^2} = A,$$

where A is a constant.

Now let $u = \frac{r_S}{r}$ and use conservation of angular momentum,

$$l = r^2\dot{\phi} = \frac{r_S^2}{u^2}\dot{\phi} \qquad \Rightarrow \qquad \frac{d}{d\tau} = \frac{lu^2}{r_S^2}\frac{d}{d\phi} \qquad \Rightarrow \qquad \frac{dr}{d\tau} = -\frac{l}{r_S}\frac{du}{d\phi}$$

to get

$$\frac{1}{(1-u)}\frac{l^2}{r_S^2}\left(\frac{du}{d\phi}\right)^2 - \frac{c^2k^2}{1-u} + \frac{l^2u^2}{r_S^2} = A$$

$$\Rightarrow \qquad \left(\frac{du}{d\phi}\right)^2 + u^2(1-u) = \frac{c^2k^2r_S^2}{l^2} + \frac{r_S^2A}{l^2}(1-u).$$

k can now be eliminated by differentiating with respect to ϕ

$$2\frac{du}{d\phi}\left(\frac{d^2u}{d\phi^2}\right) + 2u\frac{du}{d\phi} - 3u^2\frac{du}{d\phi} = -\left(\frac{r_S^2A}{l^2}\right)\frac{du}{d\phi}$$

$$\Rightarrow \qquad \frac{d^2u}{d\phi^2} + u - \frac{3}{2}u^2 = -\frac{r_S^2A}{2l^2}.$$

Choosing $A = -c^2$ is equivalent to requiring that τ is the proper time, though this is not forced on us, it is a choice of parameterisation.

5) Write the equation as

$$\frac{1}{2}\dot{r}^2 + V(r) = E$$

with

$$V(r) = \frac{c^2}{2}\left(-\frac{r_S}{r} + \frac{l^2}{c^2r^2} - \frac{l^2r_S}{c^2r^3}\right)$$

and

$$E = \frac{c^2}{2}(k^2 - 1)$$

and view $V(r)$ as the potential energy for a particle on unit mass moving in one dimension with energy E.

Extrema occur when $\frac{dV}{dr} = 0$,

$$\frac{dV}{dr} = \frac{c^2}{2}\left(\frac{r_S}{r^2} - \frac{2l^2}{c^2r^3} + \frac{3l^2 r_S}{c^2r^4}\right) = 0.$$

There are extrema as $r \to \infty$, where $V \to 0$, and at

$$r = r_\pm = \frac{l^2}{r_S c^2} \pm \sqrt{\frac{l^4}{r_S^2 c^4} - \frac{3l^2}{c^2}}.$$

Three cases to consider are

i) $\boldsymbol{\frac{l^2}{c^2} > 3r_s^2}$: there are two extrema at finite r, r_+ is a minimum and r_- a maximum; there is a circular orbit at r_+ which is stable and one at r_- which is unstable. If $V(r_+) < E < \min\{V(r_-), 0\}$ there are bound elliptical orbits to the right of the peak, with $E = V(r_+)$ being circular. If $E > \min\{V(r_-), 0\}$ there are no bound orbits.

ii) $\boldsymbol{\frac{l^2}{c^2} = 3r_s^2}$: there is an inflexion point at $r = 3r_S$ where there is a marginally stable circular orbit when $E = V(3r_S) = -\frac{c^2}{9}$. Any other value of E is either unbounded or unstable: for $E > 0$ the orbit is unbounded, for $E < 0$ the orbit is inexorably sucked into $r = 0$.

iii) $\boldsymbol{\frac{l^2}{c^2} < 3r_s^2}$: if the angular momentum is too low for a bound orbit, all orbits are either unbounded or unstable, again $E > 0$ orbits are unbounded and $E < 0$ orbits are eventually pulled toward $r = 0$.

The smallest stable circular orbit occurs for $l^2 = 3r_S^2 c^2$ and $r = 3r_S$, so the speed is

$$r\dot{\phi} = r\left(\frac{l}{r^2}\right) = \frac{l}{r} = \frac{c}{\sqrt{3}}.$$

Figure 3.1 is a plot of $\frac{2V}{c^2}$ as a function of $\frac{r}{r_S}$: the closest approach of any stable orbit is when $E = 0$ and $l^2 = 4r_S^2 c^2$, at $r = 2r_S$. So the speed is

$$r\dot{\phi} = r\left(\frac{l}{r^2}\right) = \frac{l}{r} = c,$$

an upper limit to the speed of any massive particle.

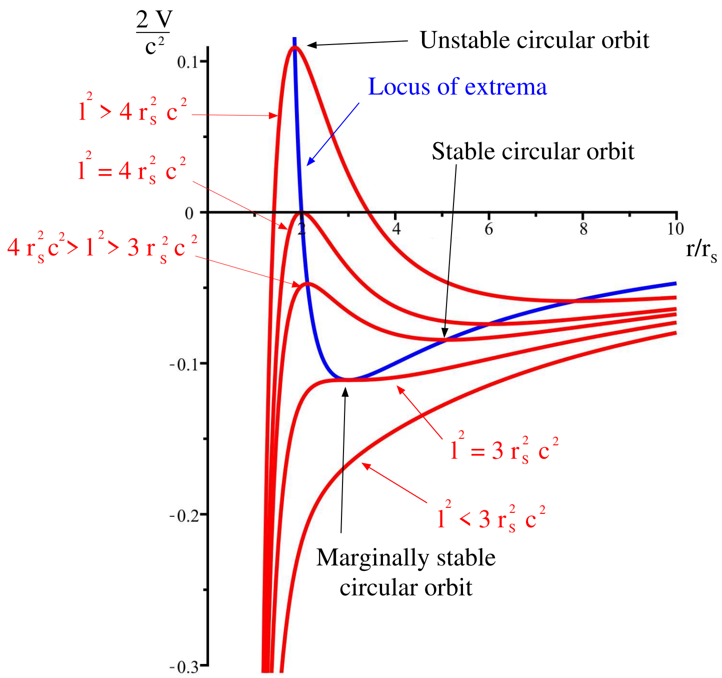

Figure 3.1 Plot of the potential V as a function of $\frac{r}{r_S}$ for various values of the angular momentum l.

6) Since $dr' = \frac{dr}{1-\frac{r_S}{r}}$ the Schwarzschild line element is

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_S}{r}\right)c^2dt^2 + \left(1-\frac{r_S}{r}\right)(dr')^2 + r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &= \left(1-\frac{r_S}{r}\right)\left(-c^2dt^2+(dr')^2\right) + r^2(d\theta^2+\sin^2\theta d\phi^2). \end{aligned}$$

Now $u = ct - r'$ and $v = ct + r'$ implies that

$$du\,dv = (cdt - dr')(cdt + dr') = cdt^2 - (dr')^2$$

hence

$$ds^2 = -\left(1-\frac{r_S}{r}\right)du\,dv + r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{3.5}$$

and

$$\frac{v-u}{2} = r' = r + r_S \ln\left|\frac{r-r_S}{r_S}\right|,$$

which implicitly defines r and t as functions of u and v,

$$r(u,v) + r_S \ln\left|\frac{r(u,v) - r_S}{r_S}\right| = \frac{v-u}{2}, \qquad ct(u,v) = \frac{v+u}{2}.$$

Now let

$$\begin{aligned}
& v' = \exp\left(\frac{v}{2r_S}\right), \qquad && u' = -\exp\left(-\frac{u}{2r_S}\right) \\
\Rightarrow \qquad & dv' = \exp\left(\frac{v}{2r_S}\right)\frac{dv}{2r_S}, \qquad && du' = \exp\left(-\frac{u}{2r_S}\right)\frac{du}{2r_S} \\
\Rightarrow \qquad & dv'du' = \frac{1}{4r_S^2}\exp\left(\frac{v-u}{2r_S}\right) dv\, du
\end{aligned}$$

and

$$\begin{aligned}
& \exp\left(\frac{v-u}{2r_S}\right) = \left|\frac{r-r_S}{r_S}\right| \exp\left(\frac{r}{r_S}\right) \\
\Rightarrow \qquad & du\, dv = 4r_S^2 e^{-r/r_S}\left|\frac{r_S}{r-r_S}\right| du'dv'.
\end{aligned}$$

So now (3.5) can be written as

$$\begin{aligned}
ds^2 &= \mp\frac{4r_S^3}{r}e^{-r/r_S}du'dv' + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
&= \mp F^2(u',v')du'dv' + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}$$

where $F^2 = \frac{4r_S^3}{r}e^{-r/r_S}$ and $r(u',v')$ defined through

$$u'v' = -\exp\left(\frac{v-u}{2r_S}\right) = -\left|\frac{r-r_S}{r_S}\right| e^{r/r_S}$$

(the $\mp$ sign is for $r > r_S$ and $r < r_S$ respectively).

No component of the metric is singular at $r = r_S$, everything is perfectly regular. There is no singularity in the geometry at the event horizon, the apparent singularity in the Schwarzschild line element is just due to the fact that r is not a good coordinate at r_S.

4

Chapter 4

1) Starting from the definition of the Christoffel symbols

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(g_{\nu\lambda,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda})$$

use

$$g_{\mu'\lambda'} = \frac{\partial x^{\tau}}{\partial x^{\mu'}} \frac{\partial x^{\zeta}}{\partial x^{\lambda'}} g_{\tau\zeta} \qquad \text{and} \qquad g^{\mu'\lambda'} = \frac{\partial x^{\mu'}}{\partial x^{\eta}} \frac{\partial x^{\lambda'}}{\partial x^{\sigma}} g^{\eta\sigma},$$

so

$$\begin{aligned} \Gamma^{\mu'}_{\nu'\rho'} &= \frac{1}{2} g^{\mu'\lambda'}(g_{\nu'\lambda',\rho'} + g_{\lambda'\rho',\nu'} - g_{\nu'\rho',\lambda'}) \\ &= \frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}} \frac{\partial x^{\lambda'}}{\partial x^{\sigma}} g^{\eta\sigma}\right)\left\{\partial_{\rho'}\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}} \frac{\partial x^{\zeta}}{\partial x^{\lambda'}} g_{\tau\zeta}\right) + \partial_{\nu'}\left(\frac{\partial x^{\tau}}{\partial x^{\lambda'}} \frac{\partial x^{\zeta}}{\partial x^{\rho'}} g_{\tau\zeta}\right) - \partial_{\lambda'}\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}} \frac{\partial x^{\zeta}}{\partial x^{\rho'}} g_{\tau\zeta}\right)\right\} \\ &= \frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}} \frac{\partial x^{\lambda'}}{\partial x^{\sigma}} g^{\eta\sigma}\right)\left\{\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}} \frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right)\partial_{\rho'} g_{\tau\zeta} + \left(\frac{\partial x^{\tau}}{\partial x^{\lambda'}} \frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\partial_{\nu'} g_{\tau\zeta} - \left(\frac{\partial x^{\tau}}{\partial x^{\nu'}} \frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\partial_{\lambda'} g_{\tau\zeta}\right. \\ &\qquad \left. + \partial_{\rho'}\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}} \frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right) g_{\tau\zeta} + \partial_{\nu'}\left(\frac{\partial x^{\tau}}{\partial x^{\lambda'}} \frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right) g_{\tau\zeta} - \partial_{\lambda'}\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}} \frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right) g_{\tau\zeta}\right\}. \end{aligned} \tag{4.1}$$

Break this into two parts and consider them separately: the first three terms on the right hand side are

$$\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}g^{\eta\sigma}\right)\left\{\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right)\partial_{\rho'}g_{\tau\zeta}+\left(\frac{\partial x^{\tau}}{\partial x^{\lambda'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\partial_{\nu'}g_{\tau\zeta}\right.$$
$$\left.-\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\partial_{\lambda'}g_{\tau\zeta}\right\}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}g^{\eta\sigma}\right)\times$$
$$\left\{\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)\partial_{\omega}g_{\tau\zeta}+\left(\frac{\partial x^{\tau}}{\partial x^{\lambda'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\nu'}}\right)\partial_{\omega}g_{\tau\zeta}\right.$$
$$\left.-\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\lambda'}}\right)\partial_{\omega}g_{\tau\zeta}\right\}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right)\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\zeta}$$
$$+\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}\frac{\partial x^{\tau}}{\partial x^{\lambda'}}\right)\left(\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\nu'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\zeta}$$
$$-\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}\frac{\partial x^{\omega}}{\partial x^{\lambda'}}\right)\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\zeta}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\delta^{\zeta}_{\sigma}\right)\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\zeta}+\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\delta^{\tau}_{\sigma}\right)\left(\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\nu'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\zeta}$$
$$-\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\delta^{\omega}_{\sigma}\right)\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\zeta}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\right)\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\sigma}+\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\right)\left(\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\left(\frac{\partial x^{\omega}}{\partial x^{\nu'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\sigma\zeta}$$
$$-\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\right)\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\sigma}g_{\tau\zeta}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\tau\sigma}+\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\tau}}{\partial x^{\rho'}}\frac{\partial x^{\omega}}{\partial x^{\nu'}}\right)g^{\eta\sigma}\partial_{\omega}g_{\sigma\tau}$$
$$-\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\partial_{\sigma}g_{\tau\omega}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)g^{\eta\sigma}\left(\partial_{\omega}g_{\tau\sigma}+\partial_{\tau}g_{\sigma\omega}-\partial_{\sigma}g_{\tau\omega}\right)$$

$$=\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\omega}}{\partial x^{\rho'}}\right)\Gamma^{\eta}_{\omega\tau};$$

the second three terms on the right hand side of (4.1) are

$$\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}g^{\eta\sigma}\right)\left(\partial_{\rho'}\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right)+\partial_{\nu'}\left(\frac{\partial x^{\tau}}{\partial x^{\lambda'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)-\partial_{\lambda'}\left(\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right)\right)g_{\tau\zeta}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}g^{\eta\sigma}\right)\left(\frac{\partial^2 x^{\tau}}{\partial x^{\rho'}\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}+\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial^2 x^{\zeta}}{\partial x^{\rho'}\partial x^{\lambda'}}+\frac{\partial^2 x^{\tau}}{\partial x^{\nu'}\partial x^{\lambda'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}\right.$$

$$\left.+\frac{\partial x^{\tau}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\zeta}}{\partial x^{\nu'}\partial x^{\rho'}}-\frac{\partial^2 x^{\tau}}{\partial x^{\lambda'}\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\rho'}}-\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial^2 x^{\zeta}}{\partial x^{\lambda'}\partial x^{\rho'}}\right)g_{\tau\zeta}$$

$$=\frac{1}{2}\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}\right)\left(\frac{\partial^2 x^{\tau}}{\partial x^{\rho'}\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}+\frac{\partial x^{\tau}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\zeta}}{\partial x^{\rho'}\partial x^{\nu'}}\right)g^{\eta\sigma}g_{\tau\zeta}$$

$$=\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\lambda'}}{\partial x^{\sigma}}\right)\left(\frac{\partial^2 x^{\tau}}{\partial x^{\rho'}\partial x^{\nu'}}\frac{\partial x^{\zeta}}{\partial x^{\lambda'}}\right)g^{\eta\sigma}g_{\tau\zeta}$$

$$=\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\right)\left(\frac{\partial^2 x^{\tau}}{\partial x^{\rho'}\partial x^{\nu'}}\right)\delta^{\zeta}_{\sigma}g^{\eta\sigma}g_{\tau\zeta}=\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\right)\left(\frac{\partial^2 x^{\tau}}{\partial x^{\rho'}\partial x^{\nu'}}\right)\delta^{\eta}_{\tau}$$

$$=\left(\frac{\partial x^{\mu'}}{\partial x^{\eta}}\right)\left(\frac{\partial^2 x^{\eta}}{\partial x^{\rho'}\partial x^{\nu'}}\right).$$

Putting these together

$$\Gamma^{\mu'}_{\nu'\rho'}=\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial x^{\tau}}{\partial x^{\nu'}}\frac{\partial x^{\omega}}{\partial x^{\rho'}}\Gamma^{\eta}_{\omega\tau}+\frac{\partial x^{\mu'}}{\partial x^{\eta}}\frac{\partial^2 x^{\eta}}{\partial x^{\rho'}\partial x^{\nu'}}, \tag{4.2}$$

as claimed.

2) From [E.7] the non-zero components of the Riemann tensor for a general Robertson-Walker metric with cosmological scale $a(t)$ are

$$R_{0\alpha0\beta}=-\frac{1}{c^2}a\ddot{a}\tilde{g}_{\alpha\beta},\qquad R_{\alpha\beta\gamma\delta}=\frac{1}{c^2}(\dot{a}^2+Kc^2)a^2\big(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta}-\tilde{g}_{\gamma\delta}\tilde{g}_{\beta k}\big),$$

all other components vanish. Setting $a=ct$ and $K=-1$, $\ddot{a}=0$ and $\dot{a}^2+c^2K=0$ hence

$$R_{0\alpha0\beta}=R_{\alpha\beta\gamma\delta}=0,$$

the Riemann tensor vanishes identically so the space-time is necessarily flat and is Minkowski space-time.

3) With

$$\vec{\boldsymbol{n}}=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)$$

we have

$$\vec{\boldsymbol{u}}_\theta = \partial_\theta \vec{\boldsymbol{n}} = (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta)$$
$$\vec{\boldsymbol{u}}_\phi = \partial_\phi \vec{\boldsymbol{n}} = (-\sin\phi\sin\theta, \cos\phi\sin\theta, 0).$$

These vectors are tangent to the sphere with constant radius r, since

$$\vec{\boldsymbol{u}}_\theta.\vec{\boldsymbol{n}} = \vec{\boldsymbol{u}}_\phi.\vec{\boldsymbol{n}} = 0.$$

The second derivatives

$$\partial_\phi \vec{\boldsymbol{u}}_\theta = \partial_\theta \vec{\boldsymbol{u}}_\phi = (-\sin\phi\cos\theta, \cos\phi\cos\theta, 0) = \cot\theta\, \vec{\boldsymbol{u}}_\phi$$

are also tangent to the sphere, but

$$\partial_\theta \vec{\boldsymbol{u}}_\theta = -(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) = -\hat{\boldsymbol{n}}$$
$$\partial_\phi \vec{\boldsymbol{u}}_\phi = -(\cos\phi\sin\theta, \sin\phi\sin\theta, 0) = -\sin^2\theta\, \hat{\boldsymbol{n}} - \sin\theta\cos\theta\, \vec{\boldsymbol{u}}_\theta$$

are not. But if we now project these vectors back on to the sphere we get

$$\begin{aligned}
\partial_\theta \vec{\boldsymbol{u}}_\theta &\rightarrow 0, \\
\partial_\phi \vec{\boldsymbol{u}}_\phi &\rightarrow -\sin\theta\cos\theta\, \vec{\boldsymbol{u}}_\theta \\
\partial_\theta \vec{\boldsymbol{u}}_\phi &\rightarrow \cot\theta \vec{\boldsymbol{u}}_\phi, \\
\partial_\phi \vec{\boldsymbol{u}}_\theta &\rightarrow \cot\theta \vec{\boldsymbol{u}}_\phi.
\end{aligned}$$

4) Hopefully there are enough pointers given in the appendix for the student to fill in the gaps.

5) a) From the definition

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma\rho,\nu} + g_{\nu\sigma,\rho} - g_{\nu\rho,\sigma})$$

so

$$\delta\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}(\delta g^{\mu\sigma})(g_{\sigma\rho,\nu} + g_{\nu\sigma,\rho} - g_{\nu\rho,\sigma}) + \frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\rho})_{,\nu} + \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\sigma})_{,\rho} - \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\rho})_{,\sigma}$$

$$= -\frac{1}{2}g^{\mu\lambda}\big(\delta g_{\lambda\tau})g^{\tau\sigma}(g_{\sigma\rho,\nu} + g_{\nu\sigma,\rho} - g_{\nu\rho,\sigma}\big) + \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\sigma\rho})_{;\nu} + \Gamma^{\lambda}_{\nu\sigma}(\delta g_{\lambda\rho}) + \Gamma^{\lambda}_{\nu\rho}(\delta g_{\lambda\sigma})\big)$$
$$+ \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\nu\sigma})_{;\rho} + \Gamma^{\lambda}_{\rho\nu}(\delta g_{\lambda\sigma}) + \Gamma^{\lambda}_{\rho\sigma}(\delta g_{\lambda\nu})\big) - \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\nu\rho})_{;\sigma} + \Gamma^{\lambda}_{\sigma\nu}(\delta g_{\lambda\rho}) + \Gamma^{\lambda}_{\sigma\rho}(\delta g_{\lambda\nu})\big)$$

$$= -g^{\mu\lambda}\big(\delta g_{\lambda\tau})\Gamma^{\tau}_{\nu\rho} + \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\sigma\rho})_{;\nu} + \Gamma^{\lambda}_{\nu\sigma}(\delta g_{\lambda\rho}) + \Gamma^{\lambda}_{\nu\rho}(\delta g_{\lambda\sigma})\big)$$
$$+ \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\nu\sigma})_{;\rho} + \Gamma^{\lambda}_{\rho\nu}(\delta g_{\lambda\sigma}) + \Gamma^{\lambda}_{\rho\sigma}(\delta g_{\lambda\nu})\big) - \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\nu\rho})_{;\sigma} + \Gamma^{\lambda}_{\sigma\nu}(\delta g_{\lambda\rho}) + \Gamma^{\lambda}_{\sigma\rho}(\delta g_{\lambda\nu})\big)$$

$$= \frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\rho})_{;\nu} + \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\sigma})_{;\rho} - \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\rho})_{;\sigma} - g^{\mu\lambda}\big(\delta g_{\lambda\sigma})\Gamma^{\sigma}_{\nu\rho}$$
$$+ \underline{\frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\nu\sigma}(\delta g_{\lambda\rho})} + \frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\nu\rho}(\delta g_{\lambda\sigma}) + \frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\rho\nu}(\delta g_{\lambda\sigma}) + \underline{\frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\rho\sigma}(\delta g_{\lambda\nu})}$$
$$- \underline{\frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\sigma\nu}(\delta g_{\lambda\rho})} - \underline{\frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\sigma\rho}(\delta g_{\lambda\nu})} \qquad \text{(underlined terms cancel)}$$

$$= \frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\rho})_{;\nu} + \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\sigma})_{;\rho} - \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\rho})_{;\sigma} - g^{\mu\lambda}\big(\delta g_{\lambda\sigma})\Gamma^{\sigma}_{\nu\rho}$$
$$+ \frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\nu\rho}(\delta g_{\lambda\sigma}) + \frac{1}{2}g^{\mu\sigma}\Gamma^{\lambda}_{\rho\nu}(\delta g_{\lambda\sigma})$$

$$= \frac{1}{2}g^{\mu\sigma}(\delta g_{\sigma\rho})_{;\nu} + \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\sigma})_{;\rho} - \frac{1}{2}g^{\mu\sigma}(\delta g_{\nu\rho})_{;\sigma} - g^{\mu\sigma}\big(\delta g_{\sigma\lambda})\Gamma^{\lambda}_{\nu\rho}$$
$$+ g^{\mu\sigma}\Gamma^{\lambda}_{\nu\rho}(\delta g_{\lambda\sigma})$$

$$= \frac{1}{2}g^{\mu\sigma}\big((\delta g_{\sigma\rho})_{;\nu} + (\delta g_{\nu\sigma})_{;\rho} - (\delta g_{\nu\rho})_{;\sigma}\big).$$

b) From the definition

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}{}_{\nu\rho}$$

we have

$$\delta R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho(\delta\Gamma^{\mu}_{\nu\sigma}) - \partial_\sigma(\delta\Gamma^{\mu}_{\nu\rho}) + (\delta\Gamma^{\mu}_{\rho\lambda})\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\mu}_{\rho\lambda}(\delta\Gamma^{\lambda}_{\nu\sigma}) - (\delta\Gamma^{\mu}_{\sigma\lambda})\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\mu}_{\sigma\lambda}(\delta\Gamma^{\lambda}_{\nu\rho})$$

$$\begin{aligned}
&= (\delta\Gamma^{\mu}_{\nu\sigma})_{;\rho} - \underline{\Gamma^{\mu}_{\lambda\rho}(\delta\Gamma^{\lambda}_{\nu\sigma})} + \Gamma^{\lambda}_{\nu\rho}(\delta\Gamma^{\mu}_{\lambda\sigma}) + \underline{\Gamma^{\lambda}_{\sigma\rho}(\delta\Gamma^{\mu}_{\lambda\nu})} \\
&\quad - (\delta\Gamma^{\mu}_{\nu\rho})_{;\sigma} + \Gamma^{\mu}_{\lambda\sigma}(\delta\Gamma^{\lambda}_{\nu\rho}) - \Gamma^{\lambda}_{\nu\sigma}(\delta\Gamma^{\mu}_{\lambda\rho}) - \underline{\Gamma^{\lambda}_{\rho\sigma}(\delta\Gamma^{\mu}_{\nu\lambda})} \\
&\qquad + (\delta\Gamma^{\mu}_{\rho\lambda})\Gamma^{\lambda}_{\nu\sigma} + \underline{\Gamma^{\mu}_{\rho\lambda}(\delta\Gamma^{\lambda}_{\nu\sigma})} - (\delta\Gamma^{\mu}_{\sigma\lambda})\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\mu}_{\sigma\lambda}(\delta\Gamma^{\lambda}_{\nu\rho})
\end{aligned}$$

$$\begin{aligned}
&= (\delta\Gamma^{\mu}_{\nu\sigma})_{;\rho} - (\delta\Gamma^{\mu}_{\nu\rho})_{;\sigma} + \underline{\Gamma^{\lambda}_{\nu\rho}(\delta\Gamma^{\mu}_{\lambda\sigma})} \\
&\quad + \Gamma^{\mu}_{\lambda\sigma}(\delta\Gamma^{\lambda}_{\nu\rho}) - \underline{\Gamma^{\lambda}_{\nu\sigma}(\delta\Gamma^{\mu}_{\lambda\rho})} + \underline{(\delta\Gamma^{\mu}_{\rho\lambda})\Gamma^{\lambda}_{\nu\sigma}} - \underline{(\delta\Gamma^{\mu}_{\sigma\lambda})\Gamma^{\lambda}_{\nu\rho}} - \Gamma^{\mu}_{\sigma\lambda}(\delta\Gamma^{\lambda}_{\nu\rho})
\end{aligned}$$

$$= (\delta\Gamma^{\mu}_{\nu\sigma})_{;\rho} - (\delta\Gamma^{\mu}_{\nu\rho})_{;\sigma} + \Gamma^{\mu}_{\lambda\sigma}(\delta\Gamma^{\lambda}_{\nu\rho}) - \Gamma^{\mu}_{\sigma\lambda}(\delta\Gamma^{\lambda}_{\nu\rho})$$

$$= (\delta\Gamma^{\mu}_{\nu\sigma})_{;\rho} - (\delta\Gamma^{\mu}_{\nu\rho})_{;\sigma}.$$

c) The first line is immediate from 5b) with $\mu = \rho$ and then sending $\nu \to \mu$, $\sigma \to \nu$. Then from 5a)

$$\begin{aligned}
\left(\delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda} - \left(\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} &= \frac{1}{2}g^{\lambda\sigma}\big((\delta g_{\sigma\nu})_{;\mu} + (\delta g_{\mu\sigma})_{;\nu} - (\delta g_{\mu\nu})_{;\sigma}\big)_{;\lambda} \\
&\quad - \frac{1}{2}g^{\lambda\sigma}\big((\delta g_{\sigma\lambda})_{;\mu} + (\delta g_{\mu\sigma})_{;\lambda} - (\delta g_{\mu\lambda})_{;\sigma}\big)_{;\nu}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}g^{\lambda\sigma}\big((\delta g_{\sigma\nu})_{;\mu;\lambda} + (\delta g_{\mu\sigma})_{;\nu;\lambda} - (\delta g_{\mu\nu})_{;\sigma;\lambda} - (\delta g_{\sigma\lambda})_{;\mu;\nu} \\
&\qquad - \underline{(\delta g_{\mu\sigma})_{;\lambda;\nu}} + \underline{(\delta g_{\mu\lambda})_{;\sigma;\nu}}\big)
\end{aligned}$$

$$= \frac{1}{2}g^{\lambda\sigma}\big((\delta g_{\sigma\nu})_{;\mu;\lambda} + (\delta g_{\mu\sigma})_{;\nu;\lambda} - (\delta g_{\mu\nu})_{;\sigma;\lambda} - (\delta g_{\sigma\lambda})_{;\mu;\nu}\big)$$

d)

$$\begin{aligned}
\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) &= (\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}(\delta R_{\mu\nu}) \\
&= -g^{\mu\lambda}(\delta g_{\lambda\sigma})g^{\sigma\nu}R_{\mu\nu} + g^{\mu\nu}(\delta R_{\mu\nu}) \\
&= -(\delta g_{\lambda\sigma})R^{\lambda\sigma} + \frac{1}{2}g^{\mu\nu}g^{\lambda\sigma}\big((\delta g_{\sigma\nu})_{;\mu;\lambda} + (\delta g_{\mu\sigma})_{;\nu;\lambda} - (\delta g_{\mu\nu})_{;\sigma;\lambda} - (\delta g_{\sigma\lambda})_{;\mu;\nu}\big) \\
&= -(\delta g_{\lambda\sigma})R^{\lambda\sigma} + g^{\mu\nu}g^{\lambda\sigma}\big((\delta g_{\sigma\nu})_{;\mu;\lambda} - (\delta g_{\mu\nu})_{;\sigma;\lambda}\big) \\
&= -(\delta g_{\lambda\sigma})R^{\lambda\sigma} + \big\{g^{\mu\nu}g^{\lambda\sigma}\big((\delta g_{\sigma\nu})_{;\mu} - (\delta g_{\mu\nu})_{;\sigma}\big)\big\}_{;\lambda}\,.
\end{aligned}$$

5

Chapter 5

1) From the definition of the Einstein tensor[1]

$$
\begin{aligned}
G^{\mu}{}_{\nu;\mu} &= R^{\mu}{}_{\nu;\mu} - \frac{1}{2}R_{;\nu} \\
&= R^{\mu\rho}{}_{\nu\rho;\mu} - \frac{1}{2}R^{\mu\rho}{}_{\mu\rho;\nu} \\
&= \frac{1}{2}R^{\mu\rho}{}_{\nu\rho;\mu} - \frac{1}{2}R^{\mu\rho}{}_{\nu\mu;\rho} - \frac{1}{2}R^{\mu\rho}{}_{\mu\rho;\nu} \qquad (\text{since } \; R^{\mu\rho}{}_{\nu\rho;\mu} = -R^{\rho\mu}{}_{\nu\rho;\mu} = -R^{\mu\rho}{}_{\nu\mu;\rho}) \\
&= -\frac{1}{2}R^{\mu\rho}{}_{\rho\nu;\mu} - \frac{1}{2}R^{\mu\rho}{}_{\nu\mu;\rho} - \frac{1}{2}R^{\mu\rho}{}_{\mu\rho;\nu} \qquad (\text{since } \; R^{\mu\rho}{}_{\nu\rho} = -R^{\mu\rho}{}_{\rho\nu}) \\
&= -\frac{1}{2}R^{\mu\rho}{}_{[\rho\nu;\mu]} \\
&= 0,
\end{aligned}
$$

from the second Bianchi identity.

2) i) A Maple™ script that achieves this (with Maple v2020.2) is:

```
> with(Physics)
> Setup(mathematicalnotation = true)
> Setup(coordinatesystems = cartesian)
> ds2 := ((dx)^2 + (dy)^2 + (dz)^2)/(1 + 2*Phi(x,y,z)/c^2) - (c^2 + 2*Phi(x,y,z))*(dt)^2
> Setup(metric = ds2)
> CompactDisplay()
> G_44 := simplify(Einstein[4, 4])
> G_11 := simplify(Einstein[1, 1])
> G_22 := simplify(Einstein[2, 2])
> G_33 := simplify(Einstein[3, 3])
> G_14 := simplify(Einstein[1, 4])
> G_24 := simplify(Einstein[2, 4])
> G_34 := simplify(Einstein[3, 4])
> G_23 := simplify(Einstein[2, 3])
> G_13 := simplify(Einstein[1, 3])
> G_23 := simplify(Einstein[2, 3])
```

[1] For a scalar, such as the Ricci scalar, a co-variant derivative is just a partial derivative (there are no connection terms), so a semi-colon just means a partial derivative on R.

though note that G_{44} is G_{tt} and not G_{00}, with $x^0 = ct$, so $G_{44} = c^2 G_{00}$.

ii) A Mathematica™ script (with Mathematica 12.1.1.0) is

```
In[1] Mathematica/EinsteinTensor.m
In[2]:= X = {t,x,y,z}
(* Cartesian coordinates: *)
In[3]:= (metric = DiagonalMatrix[{-(c^2+2*Phi[x,y,z]), 1/(1+2*Phi[x,y,z]/c^2),
1/(1+2*Phi[x,y,z]/c^2), 1/(1+2*Phi[x,y,z]/c^2)}] ) //MatrixForm
(* The line element: *)
In[4]:= (Einstein = Simplify[EinsteinTensor[metric,X]])
(*Calculate the Einstein tensor.*)
```

3) For a relativistic fluid with

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu + g^{\mu\nu} P, \tag{5.1}$$

and $U^\mu = \gamma(v)(c, v^1, v^2, v^3)$,

$$\begin{aligned} T^{00} &= \gamma^2(v)\rho c^2 + \big(g^{00} + \gamma^2(v)\big)P, \\ T^{\alpha 0} &= \gamma^2(v)\rho c v^\alpha + \left(g^{\alpha 0} + \gamma^2(v)\frac{v^\alpha}{c}\right) P \\ T^{\alpha\beta} &= \gamma^2(v)\rho v^\alpha v^\beta + \left(g^{\alpha\beta} + \gamma^2(v)\frac{v^\alpha v^\beta}{c^2}\right) P. \end{aligned}$$

With $g^{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + +O\left(\frac{v^4}{c^4}\right),$$

a large c expansion is

$$\begin{aligned} T^{00} &= \gamma^2(v)\rho c^2 + \big(\gamma^2(v) - 1\big)P = \rho_0 c^2 + \frac{1}{2}\rho_0 v^2 + \cdots \\ T^{\alpha 0} &= \gamma^2(v)\rho c v^\alpha + \gamma^2(v)\frac{v^\alpha}{c} P = \rho_0 c v^\alpha + \frac{v^\alpha}{c}\left(\frac{1}{2}\rho_0 v^2 + P\right) + \cdots \\ T^{\alpha\beta} &= \gamma^2(v)\rho v^\alpha v^\beta + \left(\delta^{\alpha\beta} + \gamma^2(v)\frac{v^\alpha v^\beta}{c^2}\right) P \\ &= \big(P\delta^{\alpha\beta} + \rho_0 v^\alpha v^\beta\big) + \frac{v^\alpha v^\beta}{c^2}\left(\frac{1}{2}\rho_0 v^2 + P\right) + \cdots, \end{aligned}$$

where $\rho_0 = \gamma(v)\rho$. In T^{00}, $\rho_0 c^2$ is the rest energy per unit volume associated with mass density ρ_0, while $\frac{1}{2}\rho_0 v^2$ is the non-relativistic kinetic energy per unit volume. In $T^{0\alpha}$, $\rho_0 v^\alpha$ is the momentum per unit volume associated with mass density ρ_0 moving with velocity. In $T^{\alpha\beta}$, $P\delta^{\alpha\beta}$ is the usual isotropic pressure of a fluid while $\rho_0 v^\alpha v^\beta$ is the extra contribution to the pressure due to the movement of the fluid.

4) The enthalpy density h is

$$h = P + \frac{U}{V}.$$

Identifying U/V with the energy density at rest, the mass density times c^2, ρc^2, the enthalpy density is

$$h = \rho c^2 + P.$$

So if the enthalpy is zero then $P = -\rho c^2$. This in turn implies, from (5.1) above, that

$$T^{\mu\nu} = P g^{\mu\nu},$$

even when the fluid is in motion. Einstein's equations are then

$$G^{\mu\nu} = \frac{8\pi G}{c^4} P g^{\mu\nu} = -\Lambda g^{\mu\nu}$$

and a cosmological constant is equivalent to a fluid with vanishing enthalpy and $P = -\frac{\Lambda c^4}{8\pi G}$.

5) From

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu + g^{\mu\nu} P$$

we have, since $g_{\mu\nu} U^\mu U^\nu = -c^2$,

$$T^\mu{}_\mu = g_{\mu\nu} T^{\mu\nu} = -(\rho c^2 + P) + 4P = -(\rho c^2 - 3P).$$

For photons the radiation pressure is $P = \frac{\rho c^2}{3}$ and $T^\mu{}_\mu = 0$ is traceless.

6

Chapter 6

1) The Einstein tensor, indeed the whole Riemann tensor, vanishes. This can be verified explicitly but quicker is to observe that changing the azimuthal co-ordinate from ϕ to $\phi' = \phi - \omega t$ renders the line element in the form

$$ds^2 = -c^2dt^2 + dr^2 + (d\theta^2 + \sin^2\theta d\phi'^2)$$

which is just flat space-time in spherical polar co-ordinates. This is just a co-ordinate transformation of the Minkowski space-time line element.

Even if $\omega(t)$ is a function of time $\phi' = \phi - \int^t \omega(t)dt$ is a perfectly good co-ordinate (provided the integral is not singular) and the space-time is still flat.

2) With the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and $h_{\mu\nu} \ll 1$,

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\rho}h_{\rho\sigma}\eta^{\sigma\nu} + O(h^2) = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2).$$

The Christoffel symbols are

$$\begin{aligned}\Gamma^{\mu}_{\nu\rho} &= \frac{1}{2}g^{\mu\tau}(g_{\tau\rho,\nu} + g_{\nu\tau,\rho} - g_{\nu\rho,\tau}) \\ &= \frac{1}{2}\eta^{\mu\tau}(h_{\tau\rho,\nu} + h_{\nu\tau,\rho} - h_{\nu\rho,\tau}) + O(h^2)\end{aligned}$$

and the Riemann tensor

$$\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + O(h^2) \\
&= \frac{1}{2}\eta^{\mu\tau}(h_{\tau\sigma,\nu,\rho} + h_{\nu\tau,\sigma,\rho} - h_{\nu\sigma,\tau,\rho}) - \frac{1}{2}\eta^{\mu\tau}(h_{\tau\rho,\nu,\sigma} + h_{\nu\tau,\rho,\sigma} - h_{\nu\rho,\tau,\sigma}) + O(h^2) \\
&= \frac{1}{2}\eta^{\mu\tau}(h_{\tau\sigma,\nu,\rho} - h_{\nu\sigma,\tau,\rho} - h_{\tau\rho,\nu,\sigma} + h_{\nu\rho,\tau,\sigma}) + O(h^2),
\end{aligned}$$

since $h_{\nu\tau,\sigma,\rho} = h_{\nu\tau,\rho,\sigma}$ (partial derivatives commute).

Using the gravitational wave form, the real part of [6.12], the Riemann tensor is

$$\begin{aligned}
R^{\mu}{}_{\nu\rho\sigma} &= \frac{1}{2}\eta^{\mu\tau} Re\big\{\big((ik_\nu)(ik_\rho)P_{\tau\sigma} - (ik_\tau)(ik_\rho)P_{\nu\sigma} - (ik_\nu)(ik_\sigma)P_{\tau\rho} + (ik_\tau)(ik_\sigma)P_{\nu\rho}\big)e^{ik.x}\big\} \\
&\qquad + O(h^2) \\
\Rightarrow \qquad R_{\mu\nu\rho\sigma} &= \frac{1}{2}(-k_\nu k_\rho P_{\mu\sigma} + k_\mu k_\rho P_{\nu\sigma} + k_\nu k_\sigma P_{\mu\rho} - k_\mu k_\sigma P_{\nu\rho})\cos(k.x) + O(h^2).
\end{aligned}$$

For example, with $k^\mu = (\frac{\omega}{c}, k, 0, 0)$, $\omega = ck$ and $P_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P_+ & P_\times \\ 0 & 0 & P_\times & -P_+ \end{pmatrix}$,

$$R_{ct\,y\,x\,z} = \frac{1}{2}k^2 P_\times \cos\big(k(x - ct)\big).$$

This is non-zero if $P_\times$ and k are non-zero, so the space-time is not flat, though $R_{\mu\nu} = 0$.

3) From appendix D of the text

$$R_{0r0r} = fg\left(\frac{f'}{g}\right)', \tag{6.1}$$

$$R_{0\theta 0\theta} = \frac{rff'}{g^2}, \qquad R_{0\phi 0\phi} = \frac{rff'}{g^2}\sin^2\theta, \tag{6.2}$$

$$R_{r\theta r\theta} = \frac{rg'}{g}, \qquad R_{r\phi r\phi} = \frac{rg'}{g}\sin^2\theta, \tag{6.3}$$

$$R_{\theta\phi\theta\phi} = \frac{(g^2-1)}{g^2}r^2\sin^2\theta. \tag{6.4}$$

Setting $g = \frac{1}{f}$ these are

$$R_{0r0r} = \frac{1}{2}(f^2)'', \tag{6.5}$$

$$R_{0\theta0\theta} = \frac{1}{2}rf^2(f^2)', \qquad R_{0\phi0\phi} = \frac{1}{2}rf^2(f^2)'\sin^2\theta, \tag{6.6}$$

$$R_{r\theta r\theta} = -\frac{rf'}{f}, \qquad R_{r\phi r\phi} = -\frac{rf'}{f}\sin^2\theta, \tag{6.7}$$

$$R_{\theta\phi\theta\phi} = (1-f^2)r^2\sin^2\theta. \tag{6.8}$$

Furthermore $f(r) = \sqrt{1 - \frac{2GM}{rc^2}}$ gives

$$\begin{aligned}
R_{0r0r} &= -\frac{2GM}{r^3c^2},\\
R_{0\theta0\theta} &= \frac{GM}{rc^2}\left(1-\frac{2GM}{rc^2}\right), \qquad R_{0\phi0\phi} = \frac{GM}{rc^2}\left(1-\frac{2GM}{rc^2}\right)\sin^2\theta,\\
R_{r\theta r\theta} &= -\frac{GM}{rc^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}, \qquad R_{r\phi r\phi} = -\frac{GM}{rc^2}\left(1-\frac{2GM}{rc^2}\right)^{-1}\sin^2\theta,\\
R_{\theta\phi\theta\phi} &= \frac{2GMr}{c^2}\sin^2\theta.
\end{aligned}$$

With this form for the Riemann tensor the off-diagonal components of the Ricci tensor vanish identically, because the metric is diagonal, and we only need check the diagonal components:

$$\begin{aligned}
R_{00} &= g^{rr}R_{0r0r} + g^{\theta\theta}R_{0\theta0\theta} + g^{\phi\phi}R_{0\phi0\phi}\\
&= -\left(1-\tfrac{2GM}{rc^2}\right)\tfrac{2GM}{r^3c^2} + \tfrac{1}{r^2}\left(\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right) + \tfrac{1}{r^2\sin^2\theta}\left(\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right)\sin^2\theta\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
R_{rr} &= g^{00}R_{0r0r} + g^{\theta\theta}R_{r\theta r\theta} + g^{\phi\phi}R_{r\phi r\phi}\\
&= -\left(1-\tfrac{2GM}{rc^2}\right)^{-1}\left(-\tfrac{2GM}{r^3c^2}\right) + \tfrac{1}{r^2}\left(-\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right)^{-1} + \tfrac{1}{r^2\sin^2\theta}\left(-\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right)^{-1}\sin^2\theta\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
R_{\theta\theta} &= g^{00}R_{0\theta0\theta} + g^{rr}R_{r\theta r\theta} + g^{\phi\phi}R_{\theta\phi\theta\phi}\\
&= -\left(1-\tfrac{2GM}{rc^2}\right)^{-1}\left(\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right) + \left(1-\tfrac{2GM}{rc^2}\right)\left(-\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right)^{-1} + \tfrac{1}{r^2\sin^2\theta}\left(\tfrac{2GMr}{c^2}\right)\sin^2\theta\\
&= 0,\\
R_{\phi\phi} &= g^{00}R_{0\phi0\phi} + g^{rr}R_{r\phi r\phi} + g^{\theta\theta}R_{\theta\phi\theta\phi}\\
&= -\left(1-\tfrac{2GM}{rc^2}\right)^{-1}\left(\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right)\sin^2\theta + \left(1-\tfrac{2GM}{rc^2}\right)\left(-\tfrac{GM}{rc^2}\right)\left(1-\tfrac{2GM}{rc^2}\right)^{-1}\sin^2\theta + \tfrac{1}{r^2}\left(\tfrac{2GMr}{c^2}\right)\sin^2\theta\\
&= 0.
\end{aligned}$$

4) One needs access to either Maple™ or Mathematica™ to do this problem.

5) a) From (6.1)-(6.4) above the Ricci tensor is

$$\begin{aligned} R_{00} &= g^{rr}R_{0r0r} + g^{\theta\theta}R_{0\theta0\theta} + g^{\phi\phi}R_{0\phi0\phi} \\ &= \frac{1}{g^2}\left(fg\left(\frac{f'}{g}\right)'\right) + \frac{1}{r^2}\left(\frac{rff'}{g^2}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{rff'}{g^2}\sin^2\theta\right) \\ &= \frac{f}{g}\left(\frac{f'}{g}\right)' + \frac{2ff'}{rg^2}, \end{aligned}$$

$$\begin{aligned} R_{rr} &= g^{00}R_{0r0r} + g^{\theta\theta}R_{r\theta r\theta} + g^{\phi\phi}R_{r\phi r\phi} \\ &= -\frac{1}{f^2}\left(fg\left(\frac{f'}{g}\right)'\right) + \frac{1}{r^2}\left(\frac{rg'}{g}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{rg'}{g}\right)\sin^2\theta \\ &= -\frac{g}{f}\left(\frac{f'}{g}\right)' + \frac{2g'}{rg}, \end{aligned}$$

$$\begin{aligned} R_{\theta\theta} &= g^{00}R_{0\theta0\theta} + g^{rr}R_{r\theta r\theta} + g^{\phi\phi}R_{\theta\phi\theta\phi} \\ &= -\frac{1}{f^2}\left(\frac{rff'}{g^2}\right) + \frac{1}{g^2}\left(\frac{rg'}{g}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{g^2-1}{g^2}\right)r^2\sin^2\theta \\ &= -\frac{rf'}{fg^2} + \frac{rg'}{g^3} + \left(\frac{g^2-1}{g^2}\right) \end{aligned}$$

$$\begin{aligned} R_{\phi\phi} &= g^{00}R_{0\phi0\phi} + g^{rr}R_{r\phi r\phi} + g^{\theta\theta}R_{\theta\phi\theta\phi} \\ &= -\frac{1}{f^2}\frac{rff'}{g^2}\sin^2\theta + \frac{1}{g^2}\left(\frac{rg'}{g}\right)\sin^2\theta + \frac{1}{r^2}\left(\frac{g^2-1}{g^2}\right)r^2\sin^2\theta \\ &= \left\{-\frac{rf'}{fg^2} + \frac{rg'}{g^3} + \left(\frac{g^2-1}{g^2}\right)\right\}\sin^2\theta. \end{aligned}$$

The Ricci scalar is

$$\begin{aligned} R &= g^{00}R_{00} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= -\frac{1}{f^2}\left\{\frac{f}{g}\left(\frac{f'}{g}\right)' + \frac{2ff'}{rg^2},\right\} + \frac{1}{g^2}\left\{-\frac{g}{f}\left(\frac{f'}{g}\right)' + \frac{2g'}{rg}\right\} \\ &\qquad + \frac{2}{r^2}\left\{-\frac{rf'}{fg^2} + \frac{rg'}{g^3} + \left(\frac{g^2-1}{g^2}\right)\right\}. \end{aligned}$$

The Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, then works out to be

$$\begin{aligned}
G_{00} &= \frac{f^2\big(g^3 - g + 2rg'\big)}{r^2 g^3},\\
G_{rr} &= \frac{f - fg^2 + 2rf'}{r^2 f} = \frac{1-g^2}{r^2} + \frac{2f'}{rf},\\
G_{\theta\theta} &= \frac{r(gf' + rgf'' - rf'g' - fg')}{fg^3} = \frac{r\big(g(rf')' - (rf)'g'\big)}{fg^3},\\
G_{\phi\phi} &= G_{\theta\theta}\sin^2\theta
\end{aligned} \tag{6.9}$$

(note $x^0 = ct$ here, $G_{tt} = c^2 G_{00}$). Then Einstein's equations require

$$G_{\theta\theta} = \frac{8\pi G}{c^4} g_{\theta\theta} T^{\theta}{}_{\theta} = \frac{8\pi G}{c^4} r^2 P \quad \Rightarrow \quad \frac{g(rf')' - (rf)'g'}{rfg^3} = \frac{8\pi GP}{c^4} \tag{6.10}$$

and $G_{\phi\phi}$ gives the same equation.

b) With the form of $T^{\mu}{}_{\nu}$ specified in the question, $T_{\mu\nu} = g_{\mu\sigma}T^{\sigma}{}_{\nu}$ the 00 component of Einstein's equations is

$$G_{00} = \frac{f^2\big(g^3 - g + 2rg'\big)}{r^2 g^3} = \frac{8\pi\rho f^2}{c^2}. \tag{6.11}$$

With $g = \frac{1}{\sqrt{1 - \frac{2Gm(r)}{rc^2}}}$

$$\begin{aligned}
g' &= -\frac{\left(\frac{Gm}{r^2c^2} - \frac{Gm'}{rc^2}\right)}{\left(1 - \frac{2Gm}{rc^2}\right)^{3/2}}\\
\Rightarrow \qquad \frac{g^3 - g + 2rg'}{r^2 g^3} &= \frac{2Gm'}{r^2c^2}.
\end{aligned}$$

Equation (6.11 is satisfied if

$$m(r) = 4\pi \int^r \rho(r) r^2 dr \qquad \Rightarrow \qquad \rho = \frac{m'}{4\pi r^2}.$$

c) A differential equation for f can be obtained from (6.9),

$$\begin{aligned}
G_{rr} &= \frac{8\pi G}{c^4} g_{rr} T^{r}{}_{r} = \frac{8\pi G}{c^4} g^2 T^{r}{}_{r} = \frac{8\pi GP}{c^4\left(1 - \frac{2Gm}{rc^2}\right)}\\
\Rightarrow \qquad \frac{(1-g^2)}{r^2} + \frac{2f'}{rf} &= \frac{8\pi GP}{c^4\left(1 - \frac{2Gm}{rc^2}\right)}\\
\Rightarrow \qquad \frac{f'}{f} &= \frac{1}{\left(1 - \frac{2Gm}{rc^2}\right)}\left(\frac{Gm}{r^2c^2} + \frac{4\pi GrP}{c^4}\right).
\end{aligned} \tag{6.12}$$

d) Equation (6.10) gives another relation between f and P but a simpler equation for $\frac{f'}{f}$ follows from conservation of energy-momentum

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\rho} T^\rho{}_\nu - \Gamma^\rho_{\mu\nu} T^\mu{}_\rho = 0$$

with $\nu = r$ (via Einstein's equations this follows from the second Bianchi identity). Since $T^\mu{}_\nu$ is diagonal this is

$$\begin{aligned} \frac{dT^r{}_r}{dr} + \Gamma^\mu_{\mu r} T^r{}_r - \Gamma^\rho_{\mu r} T^\mu{}_\rho &= 0, \\ \Rightarrow \quad P' + \Gamma^\mu_{\mu r} P - \Gamma^0_{0r}(-\rho c^2) - \left(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}\right) P &= 0, \\ \Rightarrow \quad P' + \Gamma^0_{0r}(\rho c^2 + P) &= 0, \end{aligned}$$

and we only need

$$\Gamma^0_{0r} = \frac{1}{2} g^{00}\left(g_{00,r} + g_{0r,0} - g_{0r,0}\right) = \frac{1}{2}\frac{(f^2)'}{f^2}$$

to evaluate the left-hand side and the radial component of the energy-momentum conservation equation is

$$P' + \frac{1}{2}\frac{(f^2)'}{f^2}(\rho c^2 + P) = 0 \quad \Rightarrow \quad \frac{f'}{f} = -\frac{P'}{(\rho c^2 + P)}. \qquad (6.13)$$

Combining this with (6.12)

$$\begin{aligned} -\frac{P'}{(\rho c^2 + P)} &= \frac{1}{\left(1 - \frac{2Gm}{rc^2}\right)}\left(\frac{Gm}{r^2 c^2} + \frac{4\pi r G P}{c^4}\right) \\ \Rightarrow \quad P' &= -\frac{G\left(\rho + \frac{P}{c^2}\right)}{r^2\left(1 - \frac{2Gm}{rc^2}\right)}\left(m + \frac{4\pi r^3 P}{c^2}\right). \qquad (6.14) \end{aligned}$$

e) i) For constant $\rho = \rho_0$, $m(r) = \frac{4\pi}{3}\rho_0 r^3$ and

$$P' = -\frac{4\pi r G}{3c^4}\frac{\left(\rho_0 c^2 + P\right)}{\left(1 - \frac{8\pi G \rho_0 r^2}{3c^2}\right)}\left(\rho_0 c^2 + 3P\right).$$

The total mass of the star is $M = \frac{4\pi}{3}\rho_0 R^3$ and $r_S = \frac{2GM}{c^2} = \frac{8\pi G \rho_0 R^3}{3c^2}$ so

$$\left(1 - \frac{r_S r^2}{R^3}\right)\frac{dP}{rdr} = -\frac{4\pi G}{3c^4}\left(\rho_0 c^2 + P\right)\left(\rho_0 c^2 + 3P\right).$$

Define the dimensionless ratios $\epsilon = \frac{r_S}{R}$, $\widetilde{P} = \frac{P}{\rho_0 c^2}$ and $x = \frac{r^2}{R^2}$, so $x = 1$ is the surface of the star, then this is

$$(1-\epsilon x)\frac{d\widetilde{P}}{dx} = -\frac{2\pi G \rho_0 R^2}{3c^2}(1+\widetilde{P})(1+3\widetilde{P}) = -\frac{\epsilon}{4}(1+\widetilde{P})(1+3\widetilde{P}).$$

With $y = \ln(1-\epsilon x)$, $dy = -\epsilon \frac{dx}{1-\epsilon x}$ and

$$\begin{aligned} \frac{d\widetilde{P}}{dy} &= \frac{1}{4}(1+\widetilde{P})(1+\tilde{3}P) \\ \Rightarrow \qquad \frac{dy}{4} &= \frac{d\widetilde{P}}{(1+\widetilde{P})(1+\tilde{3}P)} = \frac{3}{2}\frac{d\widetilde{P}}{(1+3\widetilde{P})} - \frac{1}{2}\frac{d\widetilde{P}}{(1+\widetilde{P})} \end{aligned}$$

which integrates to

$$\ln(1-\epsilon x) = 2\ln\left(\frac{1+3\widetilde{P}}{1+\widetilde{P}}\right) + const.$$

With $\widetilde{P} = 0$ at the surface $x = 1$, the integration constant is $\ln(1-\epsilon)$ and

$$\begin{aligned} \ln\left(\frac{1-\epsilon x}{1-\epsilon}\right) &= 2\ln\left(\frac{1+3\widetilde{P}}{1+\widetilde{P}}\right) \\ \Rightarrow \qquad \frac{1+3\widetilde{P}}{1+\widetilde{P}} &= \sqrt{\frac{1-\epsilon x}{1-\epsilon}} \\ \Rightarrow \qquad \widetilde{P} &= \frac{\sqrt{1-\epsilon x} - \sqrt{1-\epsilon}}{3\sqrt{1-\epsilon} - \sqrt{1-\epsilon x}} \\ \Rightarrow \qquad P &= \rho_0 c^2 \left(\frac{\sqrt{1-\frac{r^2 r_S}{R^3}} - \sqrt{1-\frac{r_S}{R}}}{3\sqrt{1-\frac{r_S}{R}} - \sqrt{1-\frac{r^2 r_S}{R^3}}}\right) \end{aligned}$$

as claimed. The pressure at the centre is

$$P_c = \rho_0 c^2 \left(\frac{1-\sqrt{1-\frac{r_S}{R}}}{3\sqrt{1-\frac{r_S}{R}} - 1}\right)$$

which is positive only if $R > \frac{9}{8}r_S$. If $R \leq \frac{9}{8}r_S$ the pressure diverges at $r^2 = R^2\left(9 - \frac{8R}{r_S}\right)$ and the model is not valid all the way down to $r = 0$ when $R \leq \frac{9}{8}r_S$.

ii) Firstly

$$g(r) = \frac{1}{\sqrt{1-\frac{2Gm(r)}{rc^2}}} = \frac{1}{\sqrt{1-\frac{8\pi G\rho_0 r^2}{3c^2}}} = \frac{1}{\sqrt{1-\frac{r^2 r_S}{R^3}}}.$$

Next, from (6.13),

$$\ln f = -\ln(\rho_0 c^2 + P) + const. \qquad \Rightarrow \qquad f = \frac{const}{\rho_0 c^2 + P}.$$

The constant of integration can be determined by demanding that the metric must be Schwarzschild just above and at the surface of the star, where $P = 0$, so

$$f^2(R) = 1 - \frac{2GM}{Rc^2} = 1 - \frac{r_S}{R}$$
$$\Rightarrow \qquad f^2(r) = \left(1 - \frac{r_S}{R}\right)\left(\frac{\rho_o c^2}{\rho_0 c^2 + P}\right)^2$$
$$= \frac{1}{4}\left(1 - \frac{r_S}{R}\right)\left\{3\sqrt{1 - \frac{r_S}{R}} - \sqrt{1 - \frac{r^2 r_S}{R^3}}\right\}^2.$$

You should convince yourself that (6.10) is also satisfied by this solution. This is not a co-incidence, Einstein's equations give 3 independent equations, (6.9), and conservation of energy-momentum, which is equivalent to the second Bianchi identity when Einstein's equations are satisfied, is not an independent equation, it is a consequence of the original 3 equations.

6) We can use the Riemann tensor in equations (6.5)-(6.8). with

$$f^2 = \left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right).$$

The result is

$$R_{0r0r} = -\frac{2GM}{r^3c^2} - \frac{1}{L^2},$$
$$R_{0\theta 0\theta} = \left(\frac{GM}{rc^2} - \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right),$$
$$R_{0\phi 0\phi} = \left(\frac{GM}{rc^2} - \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)\sin^2\theta,$$
$$R_{r\theta r\theta} = \left(-\frac{GM}{rc^2} + \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1},$$
$$R_{r\phi r\phi} = \left(-\frac{GM}{rc^2} + \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1}\sin^2\theta,$$
$$R_{\theta\phi\theta\phi} = \left(\frac{2GM}{rc^2} + \frac{r^2}{L^2}\right) r^2 \sin^2\theta.$$

The Ricci tensor is

$$
\begin{aligned}
R_{00} &= g^{rr}R_{0r0r} + g^{\theta\theta}R_{0\theta 0\theta} + g^{\phi\phi}R_{0\phi 0\phi} \\
&= -\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)\left(\frac{2GM}{r^3c^2} + \frac{1}{L^2}\right) + \frac{1}{r^2}\left(\frac{GM}{rc^2} - \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right) \\
&\qquad + \frac{1}{r^2\sin^2\theta}\left(\frac{GM}{rc^2} - \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)\sin^2\theta \\
&= -\frac{3}{L^2}\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
R_{rr} &= g^{00}R_{0r0r} + g^{\theta\theta}R_{r\theta r\theta} + g^{\phi\phi}R_{r\phi r\phi} \\
&= -\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1}\left(-\frac{2GM}{r^3c^2} - \frac{1}{L^2}\right) + \frac{1}{r^2}\left(-\frac{GM}{rc^2} + \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1} \\
&\qquad + \frac{1}{r^2\sin^2\theta}\left(-\frac{GM}{rc^2} + \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1}\sin^2\theta \\
&= \frac{3}{L^2}\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
R_{\theta\theta} &= g^{00}R_{0\theta 0\theta} + g^{rr}R_{r\theta r\theta} + g^{\phi\phi}R_{\theta\phi\theta\phi} \\
&= -\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1}\left(\frac{GM}{rc^2} - \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right) \\
&+ \left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)\left(-\frac{GM}{rc^2} + \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1} + \frac{1}{r^2\sin^2\theta}\left(\frac{2GM}{rc^2} + \frac{r^2}{L^2}\right)r^2\sin^2\theta \\
&= \frac{3r^2}{L^2},
\end{aligned}
$$

$$
\begin{aligned}
R_{\phi\phi} &= g^{00}R_{0\phi 0\phi} + g^{rr}R_{r\phi r\phi} + g^{\theta\theta}R_{\theta\phi\theta\phi} \\
&= -\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1}\left(\frac{GM}{rc^2} - \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)\sin^2\theta \\
&+ \left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)\left(-\frac{GM}{rc^2} + \frac{r^2}{L^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{r^2}{L^2}\right)^{-1}\sin^2\theta + \frac{1}{r^2}\left(\frac{2GM}{rc^2} + \frac{r^2}{L^2}\right)r^2\sin^2\theta \\
&= \frac{3r^2}{L^2}\sin^2\theta.
\end{aligned}
$$

This can be neatly summarised as

$$
R_{\mu\nu} = \frac{3}{L^2}g_{\mu\nu},
$$

so the Ricci scalar is

$$R = \frac{12}{L^2}$$

and the Einstein tensor

$$G_{\mu\nu} = -\frac{3}{L^2} g_{\mu\nu}.$$

This corresponds to Einstein's equations with a cosmological constant $\Lambda = \frac{3}{L^2}$ and $T_{\mu\nu} = 0$.

7

Chapter 7

1) This is a matter of working through the details of equations [E.2]-[E.6] using the definitions

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(g_{\nu\lambda,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda}),$$

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\lambda\nu} - \partial_{\nu}\Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\lambda\mu},$$

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu},$$

$$R = R^{\mu}{}_{\mu}$$

and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$

It is convenient to decompose 4-dimensional indices $\mu, \nu, \ldots$ into time 0, with $x^0 = ct$, and space $\alpha, \beta, \ldots$. The Einstein tensor is

$$G_{00} = \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2, \qquad G_{\alpha\beta} = -\frac{1}{c^2}(2\ddot{a}a + \dot{a}^2)\delta_{\alpha\beta}$$

and Einstein's equations are

$$G_{00} = \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{c^4} T_{00} = \frac{8\pi G\rho}{c^2},$$

$$G_{\alpha\beta} = -\frac{1}{c^2}(2\ddot{a}a + \dot{a}^2)\delta_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta} = \frac{8\pi G a^2 P}{c^4}\delta_{\alpha\beta},$$

$$\Rightarrow \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3},$$

$$\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G P}{c^2}.$$

2) The Robertson line element for $K = 0$ [7.7], with θ and ϕ constant, is

$$ds^2 = -c^2 dt^2 + a^2(t) dr^2$$

from which the Lagrangian governing geodesic motion can immediately be written down

$$\mathcal{L}(t, r; \dot{t}, \dot{r}) = -c^2 \dot{t}^2 + a^2(t) \dot{r}^2 \tag{7.1}$$

and the equations of motion are

$$c^2 \ddot{t} = -a \frac{da}{dt} \dot{r}^2 \tag{7.2}$$

$$\frac{d}{d\tau}(a^2 \dot{r}) = 0, \tag{7.3}$$

and $a^2 \dot{r} = A$ is constant. We can now use this in (7.2) to write

$$c^2 \ddot{t} = -a \frac{da}{dt} \frac{A^2}{a^4} = -\frac{\dot{a}}{\dot{t}} \frac{A^2}{a^3} \tag{7.4}$$

$$\Rightarrow \qquad c^2 \dot{t}\, \ddot{t} = -A^2 \frac{\dot{a}}{a^3}$$

$$\Rightarrow \qquad \frac{1}{2} c^2 \frac{d(\dot{t}^2)}{d\tau} = \frac{A^2}{2} \frac{d}{d\tau} \left(\frac{1}{a^2} \right)$$

$$\Rightarrow \qquad c^2 \dot{t}^2 - \frac{A^2}{a^2} = B$$

with B a constant. Note that the left-hand side of this equation is minus the Lagrangian (7.1) so $B = 0$ is a light-like trajectory, along which $ds^2 = 0$, while for a time-like trajectory choosing $B = c^2$ makes τ the proper time. The solution with $A = 0$, $B = c^2$ is a geodesic with constant r.

3) The Robertson line element for general K, with θ and ϕ constant, is

$$ds^2 = -c^2 dt^2 + \frac{a^2(t) dr^2}{1 - Kr^2}$$

from which the Lagrangian governing geodesic motion can immediately be written down

$$\mathcal{L}(t, r; \dot{t}, \dot{r}) = -c^2 \dot{t}^2 + \frac{a^2(t) \dot{r}^2}{1 - Kr^2} \tag{7.5}$$

and the equations of motion are

$$c^2\ddot{t} = -a\frac{da}{dt}\left(\frac{\dot{r}^2}{1-Kr^2}\right) \tag{7.6}$$

$$\frac{d}{d\tau}\left(\frac{a^2\dot{r}}{1-Kr^2}\right) = \frac{Ka^2r\dot{r}^2}{(1-Kr^2)^2}. \tag{7.7}$$

Tackling equation (7.7) first

$$\begin{aligned} \frac{1}{(1-Kr^2)}\frac{d}{d\tau}\left(a^2\dot{r}\right) + \frac{2Ka^2r\dot{r}^2}{(1-Kr^2)^2} &= \frac{Ka^2r\dot{r}^2}{(1-Kr^2)^2} \\ \Rightarrow \quad \frac{1}{(1-Kr^2)}\frac{d}{d\tau}\left(a^2\dot{r}\right) + \frac{Ka^2r\dot{r}^2}{(1-Kr^2)^2} &= 0 \\ \Rightarrow \quad \frac{d}{d\tau}\left(\frac{a^2\dot{r}}{\sqrt{1-Kr^2}}\right) &= 0, \end{aligned}$$

so

$$\frac{a^2\dot{r}}{\sqrt{1-Kr^2}} = A$$

is a constant and

$$\dot{r}^2 = \frac{A^2}{a^4}(1-Kr^2).$$

With this expression for $\dot{r}$ the t equation of motion is the same as 7.4), so

$$c^2\dot{t}^2 - \frac{A^2}{a^2} = B \tag{7.8}$$

with B constant. For $A = 0$, $r = const$ is a solution, with t linear in τ, for any $a(t)$.

When $A \neq 0$ let $B = c^2$, then τ is the proper time and we can interpret $\dot{t}$ as the Lorentz γ-factor for speed v, $\dot{t} = \gamma(v)$. Then (7.8) is[1]

$$\begin{aligned} \gamma^2(v) - \frac{A^2}{a^2c^2} &= 1 \\ \Rightarrow \quad \frac{A^2}{a^2c^2} &= \frac{1}{1-\frac{v^2}{c^2}} - 1 = \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} \\ \Rightarrow \quad \frac{v^2}{c^2} &= \frac{A^2}{A^2+a^2c^2} = \frac{a^2\dot{r}^2}{a^2\dot{r}^2 + c^2(1-Kr^2)}. \end{aligned}$$

[1] In chapter 2.6, equation [2.23], we took $\boldsymbol{v} = a\frac{d\boldsymbol{x}}{dt}$ for $K = 0$, but we see here that this is only valid non-relativistically, at large c

When $A \neq 0$, $v(\tau)$ is a function of τ and

$$v \to \begin{cases} c & \text{for } K = 1,\ r \to 1; \\ 0 & \text{for } K = -1,\ r \to \infty. \end{cases}$$

4) Write

$$\begin{aligned} ds^2 &= -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \\ &= a^2\big(t\big)\left(-\frac{c^2}{a^2(t)}dt^2 + (1-Kr^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \end{aligned}$$

and define t' as

$$t' = \int \frac{dt}{a(t)},$$

then

$$ds^2 = a^2\big(t(t')\big)\left(-c^2 dt'^2 + (1-Kr^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

5) First check that the definitions satisfy the constraint

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = L^2.$$

a) This is automatic since

$$z_4^2 - z_0^2 = L^2 - \big(\sinh(ct/L) + \cosh(ct/L)\big)e^{ct/L}(x_1^2 + x_2^2 + x_3^2) = L^2 - e^{2ct/L}(x_1^2 + x_2^2 + x_3^2)$$

while

$$z_1^2 + z_2^2 + z_3^2 = e^{2ct/L}(x_1^2 + x_2^2 + x_3^2).$$

Now evaluate the 5-dimensional Minkowski line element with this constraint:

$$dz_0 = \cosh(ct/L)cdt + \frac{1}{2L^2}e^{ct/L}(x_1^2 + x_2^2 + x_3^2)cdt + \frac{1}{L}e^{ct/L}(x_1dx_1 + x_2dx_2 + x_3dx_3),$$

$$dz_4 = \sinh(ct/L)cdt - \frac{1}{2L^2}e^{ct/L}(x_1^2 + x_2^2 + x_3^2)cdt - \frac{1}{L}e^{ct/L}(x_1dx_1 + x_2dx_2 + x_3dx_3),$$

$$dz_\alpha = \frac{1}{L}e^{ct/L}x_\alpha cdt + e^{ct/L}dx_\alpha,$$

$$\begin{aligned}\Rightarrow \quad -dz_0^2 + dz_4^2 &= -c^2dt^2 \\ &- \cosh(ct/L)cdt\left\{\frac{1}{L^2}e^{ct/L}(x_1^2 + x_2^2 + x_3^2)cdt - \frac{2}{L}e^{ct/L}(x_1dx_1 + x_2dx_2 + x_3dx_3)\right\} \\ &+ \frac{1}{L^3}e^{2ct/L}(x_1^2 + x_2^2 + x_3^2)cdt(x_1dx_1 + x_2dx_2 + x_3dx_3) \\ &- \sinh(ct/L)cdt\left\{\frac{1}{L^2}e^{ct/L}(x_1^2 + x_2^2 + x_3^2)cdt + \frac{2}{L}e^{ct/L}(x_1dx_1 + x_2dx_2 + x_3dx_3)\right\} \\ &- \frac{1}{L^3}e^{2ct/L}(x_1^2 + x_2^2 + x_3^2)cdt(x_1dx_1 + x_2dx_2 + x_3dx_3) \\ &= -c^2dt^2 - \frac{1}{L^2}e^{2ct/L}(x_1^2 + x_2^2 + x_3^2)c^2dt^2 - \frac{2}{L}e^{2ct/L}cdt(x_1dx_1 + x_2dx_2 + x_3dx_3)\end{aligned}$$

and

$$dz_1^2 + dz_2^2 + dz_3^2 = \frac{1}{L^2}e^{2ct/L}(x_1^2 + x_2^2 + x_3^2)c^2dt^2 + e^{2ct/L}(dx_1^2 + dx_2^2 + dx_3^2) + \frac{2}{L}e^{2ct/L}cdt(x_1dx_1 + x_2dx_2 + x_3dx_3),$$

hence

$$-dz_0^2 + dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 = -c^2dt^2 + e^{2ct/L}(dx_1^2 + dx_2^2 + dx_3^2),$$

as required.

b) In this case

$$-z_0^2 + z_1^2 = a^2 - r^2$$

and

$$z_2^2 + z_3^2 + z_4^2 = r^2$$

so

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = L^2$$

is immediate.

Now

$$\begin{aligned} dz_0 &= -\frac{\sinh(ct'/L)}{\sqrt{L^2-r^2}} r dr + \frac{\sqrt{L^2-r^2}}{L}\cosh(ct'/L)cdt', \\ dz_1 &= -\frac{\cosh(ct'/L)}{\sqrt{L^2-r^2}} r dr + \frac{\sqrt{L^2-r^2}}{L}\sinh(ct'/L)cdt', \\ \Rightarrow \qquad -dz_0^2 + dz_1^2 &= \frac{r^2dr^2}{L^2-r^2} - \frac{(L^2-r^2)}{L^2}c^2(dt')^2, \end{aligned}$$

while

$$dz_2^2 + dz_3^2 + dz_4^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

so

$$\begin{aligned} ds^2 &= \frac{r^2dr^2}{L^2-r^2} - \frac{(L^2-r^2)}{L^2}c^2(dt')^2 + dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2) \\ &= -\left(1-\frac{r^2}{L^2}\right)c^2(dt')^2 + \frac{L^2dr^2}{L^2-r^2} + dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned}$$

which is what was to be proven. This puts the de Sitter metric into the Schwarzschild form

$$ds^2 = -f^2(r)c^2dt'^2 + \frac{dr^2}{f^2(r)} + r^2(d\theta^2+\sin^2\theta d\phi^2)$$

with $f^2(r) = 1 - \frac{r^2}{L^2}$, for $r \le L$.

This is the same line element as question 6 of chapter 6 with $M = 0$.

6) With $M = 0$, the line element is

$$ds^2 = -\left(1-\frac{r^2}{L^2}\right)c^2dt^2 + \frac{dr^2}{\left(1-\frac{r^2}{L^2}\right)} + r^2(d\theta^2+\sin^2\theta d\phi^2).$$

Using the Lagrangian

$$\mathcal{L} = -\left(1-\frac{r^2}{L^2}\right)c^2\dot{t}^2 + \frac{\dot{r}^2}{\left(1-\frac{r^2}{L^2}\right)} + r^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

the equations of motion are

$$\left(1-\frac{r^2}{L^2}\right)\dot{t}=k=const.,$$
$$\frac{d}{d\tau}\left(\frac{\dot{r}}{1-\frac{r^2}{L^2}}\right)=r(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)+\frac{rc^2\dot{t}^2}{L^2},$$
$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right)=\sin\theta\cos\theta\dot{\phi}^2,$$
$$\frac{d}{d\tau}\left(r^2\sin^2\theta\dot{\phi}\right)=l=const.$$

We can set $\theta=\pi/2$ to be constant in the equatorial plane, so $r^2\dot{\phi}=l$. Rather than tackle the r equation directly it is simpler to choose τ to be the proper time (for a massive particle) and set $\mathcal{L}=-c^2$,

$$c^2=\frac{c^2k^2}{1-\frac{r^2}{L^2}}-\frac{\dot{r}}{1-\frac{r^2}{L^2}}-\frac{l^2}{r^2}$$
$$\dot{r}^2=c^2k^2-c^2\left(1-\frac{r^2}{L^2}\right)-\frac{l^2\left(1-\frac{r^2}{L^2}\right)}{r^2}$$
$$=c^2(k^2-1)+\frac{c^2r^2}{L^2}-\frac{l^2}{r^2}+\frac{l^2}{L^2}$$
$$\Rightarrow \qquad \dot{r}=\pm\sqrt{c^2(k^2-1)+\frac{l^2}{L^2}+\frac{c^2r^2}{L^2}-\frac{l^2}{r^2}}$$
$$\Rightarrow \qquad \frac{dr}{dt}=\pm\frac{\left(1-\frac{r^2}{L^2}\right)}{k}\sqrt{c^2(k^2-1)+\frac{l^2}{L^2}+\frac{c^2r^2}{L^2}-\frac{l^2}{r^2}}.$$

Since ϕ is a constant, $l=0$ and

$$\frac{v}{c}=\frac{1}{c}\frac{dr}{dt}=\pm\frac{1}{k}\left(1-\frac{r^2}{L^2}\right)\sqrt{k^2-1+\frac{r^2}{L^2}}.$$

When $r>L$, $g_{00}=-\left(1-\frac{r^2}{L^2}\right)>0$ and t is no longer a time-like coordinate, it makes no sense to think of $\frac{dr}{dt}$ as a velocity.

Taking the plus sign, $v(r)$ has a maximum at $r=\frac{\sqrt{9-6k^2}}{3}$, where

$$v=\frac{2k^3}{3\sqrt{3}},$$

which is the largest value $v(r)$ can take.

7) With

$$\Omega_\Lambda=\frac{\Lambda c^2}{3H_0^2}=0.7$$

and

$$\frac{3H_0^2}{8\pi G} = 10^{-26}\text{kg m}^{-3}, \qquad \text{(equation [7.16] of the text)}$$

the cosmological constant generates a negative pressure

$$P = -\frac{\Lambda c^4}{8\pi G} = -\left(\frac{3H_0^2\Omega_\Lambda}{c^2}\right)\left(\frac{c^4}{8\pi G}\right) = -\Omega_\Lambda c^2\left(\frac{3H_0^2}{8\pi G}\right) \approx 10^{-9}\text{kg m}^{-1}\text{s}^{-2},$$

that is 10^{-9} Pa, or 10^{-14} atmospheres.

8) Creating a $\mu^+ - \mu^-$ pair requires an energy of 212 MeV, corresponding to a temperature of 2×10^{10} K. From [7.35] this corresponds to a time

$$t \approx \frac{1}{200^2} = 2.5 \times^{-5} \text{s},$$

or 25 μs after the Big Bang.