Chapter 3

In[1]:= **Needs["DiscreteMath`RSolve`"]**

à Question 1

(i)	$y_{t+2} = y_{t+1} - 0.5\, y_t + 1$	second - order, linear, autonomous, non - homogeneous
(ii)	$y_{t+2} = 2\, y_t + 3$	second - order, linear, autonomous, non - homogeneous
(iii)	$\frac{y_{t+1} - y_t}{y_t} = 4$	first - order, linear, autonomous, homogeneous
(iv)	$y_{t+2} - 2\, y_{t+1} + 3\, y_t = t$	second - order, linear, non - autonomous, non - homogeneous

à Question 2

In[2]:= **RSolve[{P[t] == (1 + r) P[t - 1] - R, P[0] == P0}, P[t], t]**

Out[2]= $\left\{\left\{P[t] \to \frac{P0\, r\, (1+r)^t + R - (1+r)^t R}{r}\right\}\right\}$

à Question 3

For periods 0 to n,

0 P_0

1 $(1+r)\, P_0 - R_1$

2 $(1+r)[(1+r)\, P_0 - R_1] - R_2 = (1+r)^2\, P_0 - (1+r)\, R_1 - R_2$

3 $(1+r)[(1+r)^2\, P_0 - (1+r)\, R_1 - R_2] - R_3 = (1+r)^3\, P_0 - (1+r)^2\, R_1 - (1+r)\, R_2 - R_3$

4 $(1+r)[(1+r)^3\, P_0 - (1+r)^2\, R_1 - (1+r)\, R_2 - R_3] - R_4 =$

$(1+r)^4\, P_0 - (1+r)^3\, R_1 - (1+r)^2\, R_2 - (1+r)\, R_3 - R_4$

$\vdots$

n $(1+r)^n\, P_0 - (1+r)^{n-1}\, R_1 - (1+r)^{n-2}\, R_2 \cdots\ - (1+r)\, R_{n-1} - R_n$

Hence

$$P_n = (1+r)^n\, P_0 - (1+r)^{n-1}\, R_1 - (1+r)^{n-2}\, R_2 \cdots\ - (1+r)\, R_{n-1} - R_n$$

It is possible to check the consistency of this answer with the previous one. Let $R_k = R$ for all k. Then

$$P_n = (1+r)^n\, P_0 - (1+r)^{n-1}\, R - (1+r)^{n-2}\, \mathrm{R} \cdots\ - (1+r)\, R - R$$

Consider the terms involving R. Let the sum of these be denoted S. Then

In[3]:= $\mathbf{S = \sum_{k=0}^{n-1} (1 + r)\,\hat{}\,k\, R}$

Out[3]= $\frac{(-1 + (1+r)^n)\, R}{r}$

Substituting this into P_n we obtain

$$P_n = (1+r)^n P_0 - \frac{R}{r}[(1+r)^n - 1] = (1+r)^n (P_0 - \frac{R}{r}) + \frac{R}{r}$$

à Question 4

(i)

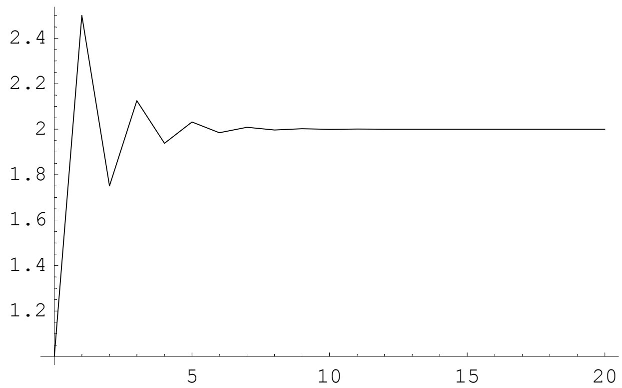
```
In[4]:= RSolve[ {y[t + 1] == - (1 / 2) y[t] + 3, y[0] == y0}, y[t], t]
```

Out[4]= $\{\{y[t] \to 2 (1 - (-\frac{1}{2})^t + (-1)^t 2^{-1-t} y0)\}\}$

In[5]:= $y1 = 2 + (-\frac{1}{2})^t (-2 + y0)$

Out[5]= $2 + (-\frac{1}{2})^t (-2 + y0)$

```
In[6]:= points1 = Table[ {t, y1}, {t, 0, 20}] /. y0 -> 1;

In[7]:= ListPlot[points1, PlotJoined -> True, PlotRange -> All];
```



(ii)

```
In[8]:= RSolve[ {2 * y[t + 1] == -3 * y[t] + 4, y[0] == y0}, y[t], t]
```

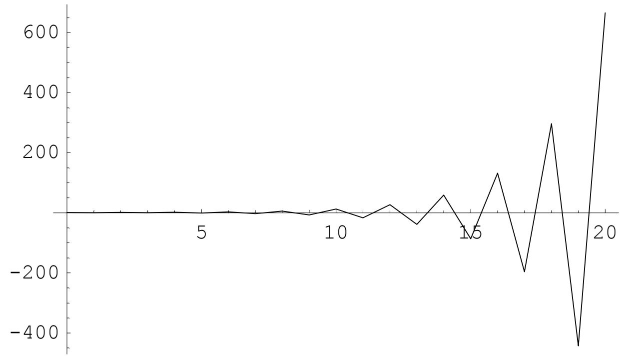
Out[8]= $\{\{y[t] \to \frac{1}{5} (-4 (-1 + (-\frac{3}{2})^t) + 5 (\frac{3}{2})^t e^{i \pi t} y0)\}\}$

In[9]:= $y2 = \frac{4}{5} + \frac{1}{5} (-\frac{3}{2})^t (-4 + 5 y0)$

Out[9]= $\frac{4}{5} + \frac{1}{5} (-\frac{3}{2})^t (-4 + 5 y0)$

```
In[10]:= points2 = Table[ {t, y2}, {t, 0, 20}] /. y0 -> 1;
```

```
In[11]:= ListPlot[points2, PlotJoined -> True, PlotRange -> All];
```



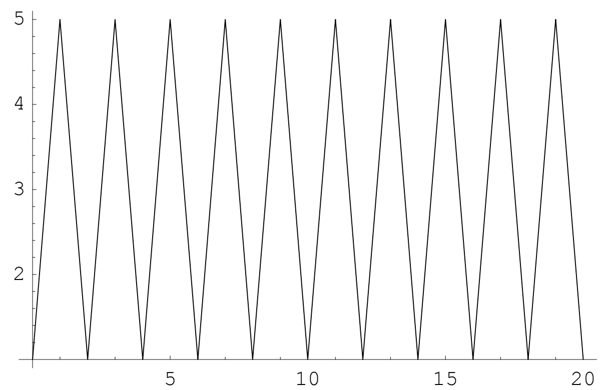
(iii)

```
In[12]:= RSolve[{y[t + 1] == -y[t] + 6, y[0] == y0}, y[t], t]

Out[12]= {{y[t] → If[Even[t], y0, 6 - y0]}}
```

In[13]:= y3 = 3 + (-1)^{t} (-3 + y0)

Out[13]= 3 + (-1)^{t} (-3 + y0)

```
In[14]:= points3 = Table[ {t, y3}, {t, 0, 20}] /. y0 -> 1;

In[15]:= ListPlot[points3, PlotJoined -> True, PlotRange -> All];
```



(iv)

```
In[16]:= RSolve[ {y[t + 1] == (1 / 2) y[t] + 3, y[0] == y0}, y[t], t]
```

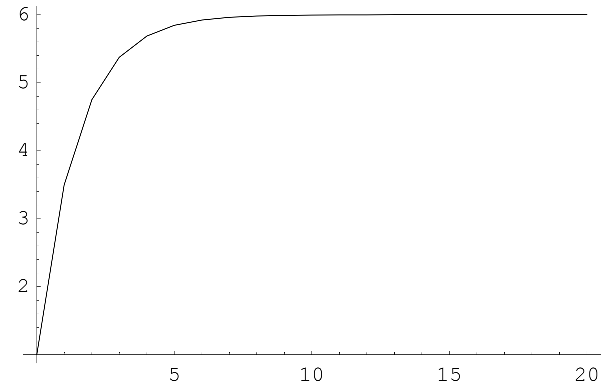
Out[16]= {{y[t] → 2^{-t} (6 (-1 + 2^{t}) + y0)}}

In[17]:= y4 = 6 + 2^{-t} (-6 + y0)

Out[17]= 6 + 2^{-t} (-6 + y0)

```
In[18]:= points4 = Table[ {t, y4}, {t, 0, 20}] /. y0 -> 1;
```

```
In[19]:= ListPlot[points4, PlotJoined -> True, PlotRange -> All];
```



(v)

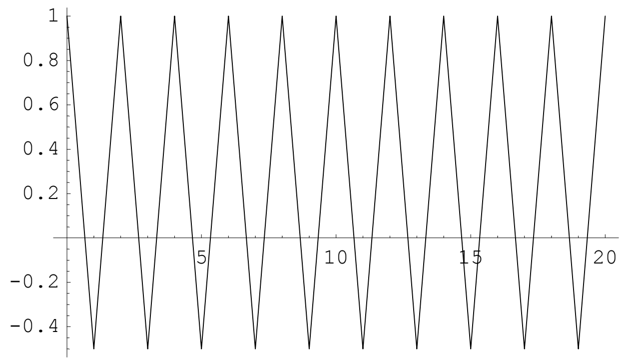
```
In[20]:= RSolve[ {4 * y[t + 2] + 4 * y[t + 1] - 2 == 0, y[0] == y0}, y[t], t]
```

Out[20]= $\{\{y[t] \to \text{If}[\text{Even}[t], y0, \frac{1}{2}(1 - 2\,y0)]\}\}$

In[21]:= $y5 = \frac{1}{4} + \frac{1}{4}(-1)^t(-1 + 4\,y0)$

Out[21]= $\frac{1}{4} + \frac{1}{4}(-1)^t(-1 + 4\,y0)$

```
In[22]:= points5 = Table[ {t, y5}, {t, 0, 20}] /. y0 -> 1;

In[23]:= ListPlot[ points5, PlotJoined -> True, PlotRange -> All];
```



à Question 5

(i)

```
In[24]:= Solve[a == a^3 - a^2 + 1, a]
```

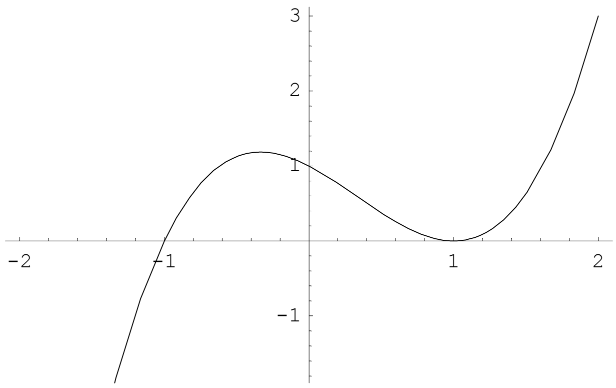
Out[24]= $\{\{a \to -1\}, \{a \to 1\}, \{a \to 1\}\}$

```
In[25]:= Factor[a^3 - a^2 - a + 1]
```

Out[25]= $(-1 + a)^2 (1 + a)$

(ii)

```
In[26]:= f[y_] := y^3 - y^2 - y + 1

In[27]:= f[1]

Out[27]= 0

In[28]:= Plot[f[y], {y, -2, 2}];
```

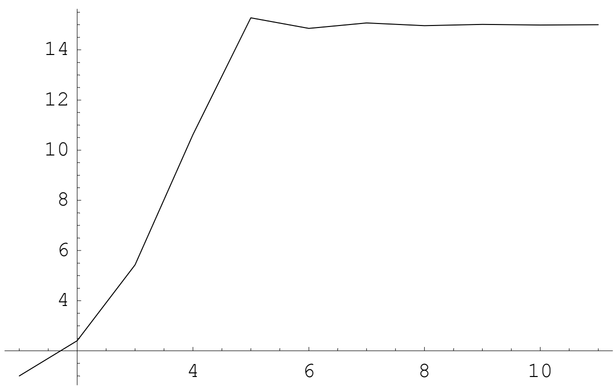


à Question 6

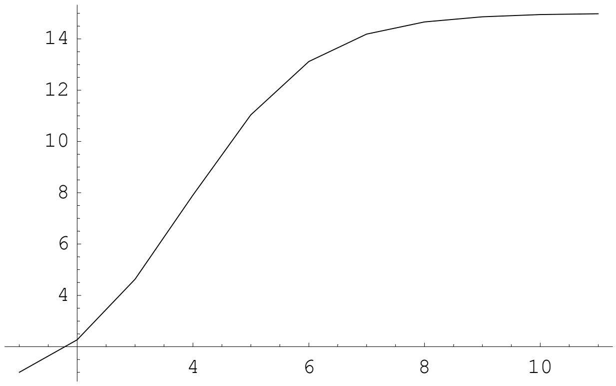
Although this requests using a spreadsheet, we can use *Mathematica* equally well.

(i)

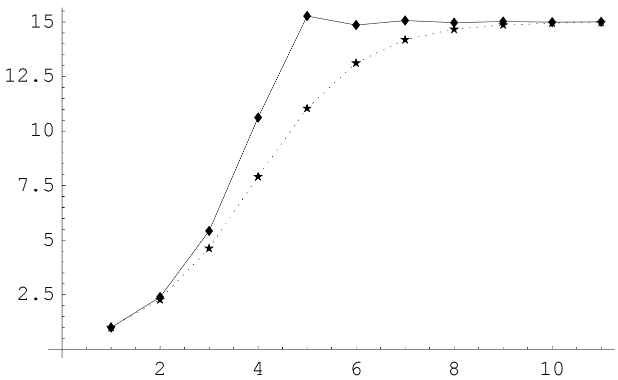
```
In[29]:= Clear[f, g]

In[30]:= f[y_] := (1 + a) * y - b * y^2

In[31]:= data1A = NestList[f, 1, 10] /. {a -> 1.5, b -> 0.1};

In[32]:= g[y_] := ((1 + a) * y) / (1 + b * y)

In[33]:= data1B = NestList[g, 1, 10] /. {a -> 1.5, b -> 0.1};

    General::spell1 :
     Possible spelling error: new symbol name "data1B" is similar to existing symbol "data1A".

In[34]:= graph1A = ListPlot[data1A, PlotJoined -> True];
```



```
In[35]:= graph1B = ListPlot[data1B, PlotJoined -> True];

General::spell1 :
 Possible spelling error: new symbol name "graph1B" is similar to existing symbol "graph1A".
```

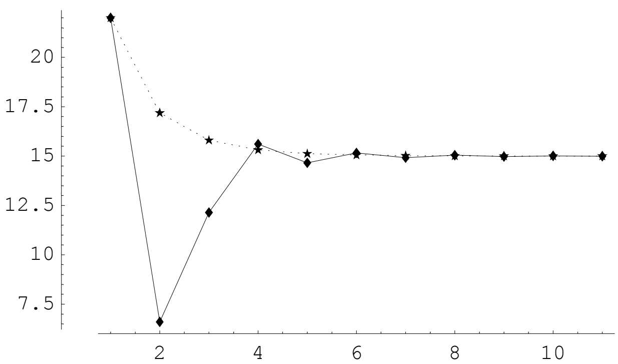



```
In[36]:= Needs["Graphics`MultipleListPlot`"]

In[37]:= MultipleListPlot[data1A, data1B, PlotJoined -> True];
```



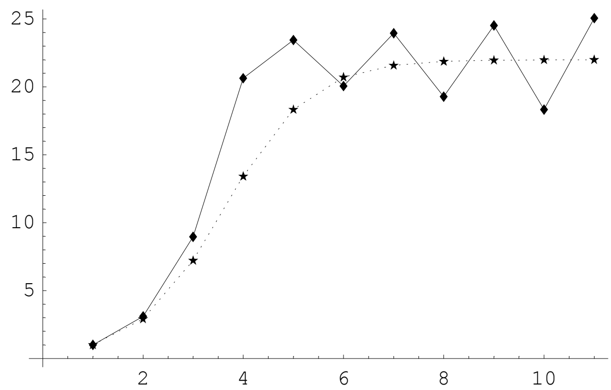
(ii)

```
In[38]:= data2A = NestList[f, 22, 10] /. {a -> 1.5, b -> 0.1};

In[39]:= data2B = NestList[g, 22, 10] /. {a -> 1.5, b -> 0.1};

General::spell1 :
 Possible spelling error: new symbol name "data2B" is similar to existing symbol "data2A".

In[40]:= MultipleListPlot[data2A, data2B,
           PlotJoined -> True, PlotRange -> All, AxesOrigin -> {0, 6}];
```



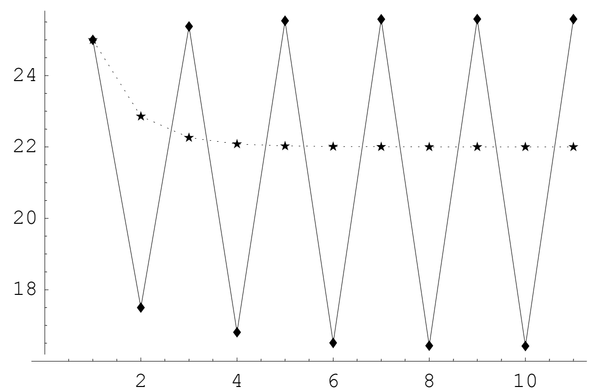
(iii)

```
In[41]:= data3A = NestList[f, 1, 10] /. {a -> 2.2, b -> 0.1};

In[42]:= data3B = NestList[g, 1, 10] /. {a -> 2.2, b -> 0.1};

General::spell1 :
 Possible spelling error: new symbol name "data3B" is similar to existing symbol "data3A".

In[43]:= MultipleListPlot[data3A, data3B, PlotJoined -> True];
```



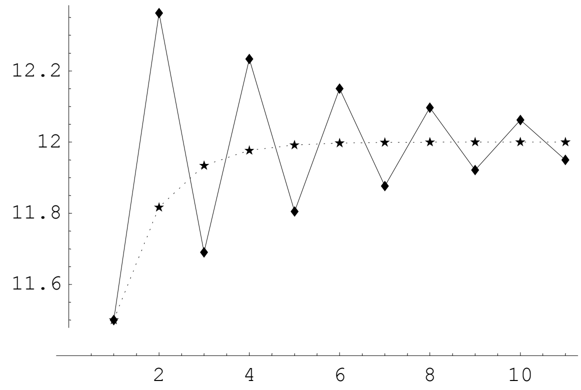
(iv)

```
In[44]:= data4A = NestList[f, 25, 10] /. {a -> 2.2, b -> 0.1};

In[45]:= data4B = NestList[g, 25, 10] /. {a -> 2.2, b -> 0.1};

General::spell1 :
 Possible spelling error: new symbol name "data4B" is similar to existing symbol "data4A".

In[46]:= MultipleListPlot[data4A, data4B, PlotJoined -> True, AxesOrigin -> {0, 16}];
```




```
In[47]:= data5A = NestList[f, 11.5, 10] /. {a -> 1.8, b -> 0.15};

In[48]:= data5B = NestList[g, 11.5, 10] /. {a -> 1.8, b -> 0.15};

General::spell1 :
 Possible spelling error: new symbol name "data5B" is similar to existing symbol "data5A".
```

In[49]:= `MultipleListPlot[data5A, data5B, PlotJoined -> True, AxesOrigin -> {0, 11.4}];`



à Question 7

In[50]:= `Clear[a, b, c, d, p]`

In[51]:= `RSolve[ {p[t] == ((a - c) / b) - (d / b) * p[t - 1], p[0] == p0}, p[t], t]`

Out[51]= $\left\{\left\{p[t] \to \frac{b^{-t}\,(-b^t\,c + a\,(b^t - (-1)^t\,d^t) + (-1)^t\,b\,d^t\,p0 + (-1)^t\,d^t\,(c + d\,p0))}{b + d}\right\}\right\}$

In[52]:= `sol1 = RSolve[ {p[t] == ((a - c) / b) - (d / b) * p[t - 1], p[0] == p0}, p[t], t] /. {a -> 10, b -> 3, c -> 2, d -> 1, p0 -> 1}`

Out[52]= $\left\{\left\{p[t] \to \frac{1}{4}\,3^{-t}\,(6\,(-1)^t - 2\,3^t + 10\,(-(-1)^t + 3^t))\right\}\right\}$

In[53]:= $\text{Simplify}\left[\frac{1}{4}\,3^{-t}\,(6\,(-1)^t - 2\,3^t + 10\,(-(-1)^t + 3^t))\right]$

Out[53]= $2 - \left(-\frac{1}{3}\right)^t$

In[54]:= `sol2 = RSolve[ {p[t] == ((a - c) / b) - (d / b) * p[t - 1], p[0] == p0}, p[t], t] /. {a -> 25, b -> 4, c -> 3, d -> 4, p0 -> 1}`

Out[54]= $\{\{p[t] \to 2^{-3-2t}\,(7\,(-4)^t - 3\,4^t + (-1)^t\,4^{1+t} + 25\,(-(-4)^t + 4^t))\}\}$

In[55]:= $\text{Simplify}[2^{-3-2t}\,(7\,(-4)^t - 3\,4^t + (-1)^t\,4^{1+t} + 25\,(-(-4)^t + 4^t))]$

Out[55]= $\frac{1}{4}\,(11 - 7\,(-1)^t)$

In[56]:= `sol3 = RSolve[ {p[t] == ((a - c) / b) - (d / b) * p[t - 1], p[0] == p0}, p[t], t] /. {a -> 45, b -> 5 / 2, c -> 5, d -> 15 / 2, p0 -> 1}`

Out[56]= $\left\{\left\{p[t] \to 2^{-1+t}\,5^{-1-t}\left((-3)^t\,2^{-1-t}\,5^{1+t} - 2^{-t}\,5^{1+t} + (-3)^t\,2^{-1-t}\,5^{2+t} + 45\left(-\left(-\frac{15}{2}\right)^t + \left(\frac{5}{2}\right)^t\right)\right)\right\}\right\}$

In[57]:= $\text{Simplify}\left[2^{-1+t}\,5^{-1-t}\left((-3)^t\,2^{-1-t}\,5^{1+t} - 2^{-t}\,5^{1+t} + (-3)^t\,2^{-1-t}\,5^{2+t} + 45\left(-\left(-\frac{15}{2}\right)^t + \left(\frac{5}{2}\right)^t\right)\right)\right]$

Out[57]= $4 - (-1)^t\,3^{1+t}$

In[58]:= **sol1[[1, 1, 2]]**

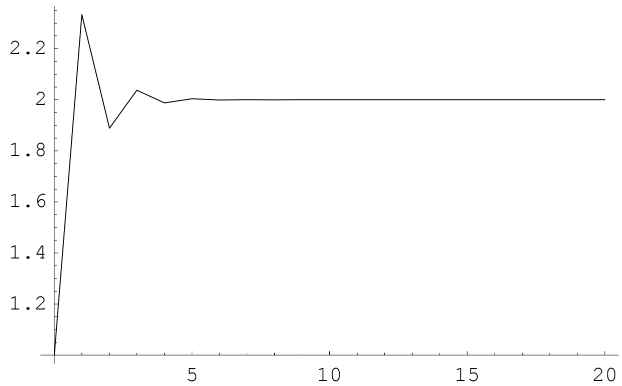
Out[58]= $\frac{1}{4}\, 3^{-t}\, (6\, (-1)^{t} - 2\, 3^{t} + 10\, (-(-1)^{t} + 3^{t}))$

In[59]:= **Simplify**$\left[\frac{1}{4}\, 3^{-t}\, (6\, (-1)^{t} - 2\, 3^{t} + 10\, (-(-1)^{t} + 3^{t}))\right]$

Out[59]= $2 - \left(-\frac{1}{3}\right)^{t}$

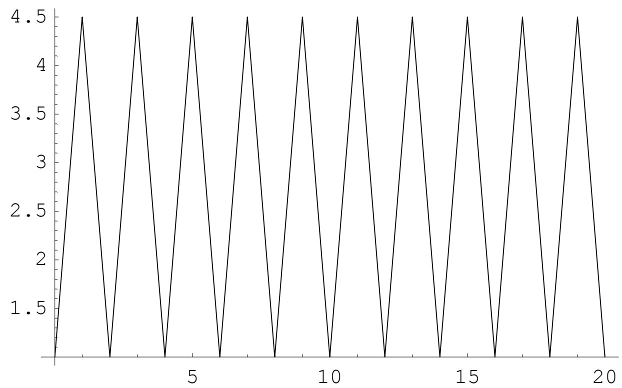
In[60]:= **p1 = Table[{t, sol1[[1, 1, 2]]} , {t, 0, 20}];**

In[61]:= **ListPlot[p1, PlotJoined -> True, PlotRange -> All];**



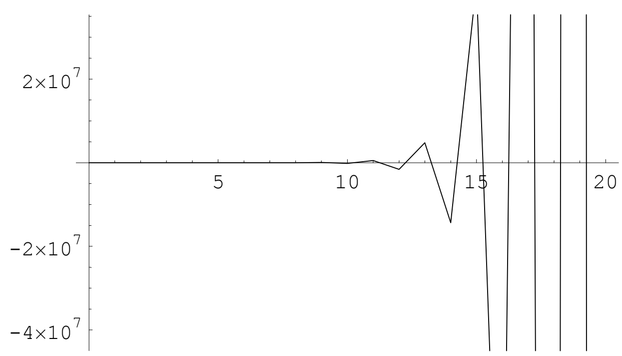
In[62]:= **p2 = Table[{t, sol2[[1, 1, 2]]} , {t, 0, 20}];**

In[63]:= **ListPlot[p2, PlotJoined -> True, PlotRange -> All];**



In[64]:= **p3 = Table[{t, sol3[[1, 1, 2]]} , {t, 0, 20}];**

In[65]:= **ListPlot[p3, PlotJoined -> True];**



à Question 8

The system of equations is

$C_t = a + b\,Y_{t-1}$

$E_t = C_t + I_t + G_t$

$Y_t = E_t$

Substituting, where I and G are exogenous, we obtain

$Y_t = (a + I + G) + b\,Y_{t-1}$

```
In[66]:= Solve[Y == (a + I + G) + b Y, Y]
```

$$\text{Out[66]}= \left\{\left\{Y \to -\frac{i + a + G}{-1 + b}\right\}\right\}$$

```
In[67]:= RSolve[{Y[t] == (a + I + G) + b Y[t - 1], Y[0] == Y0}, Y[t], t]
```

$$\text{Out[67]}= \left\{\left\{Y[t] \to \frac{(-1 + b^t)\,(i + a + G - Y0) + (-1 + b^{1+t})\,Y0}{-1 + b}\right\}\right\}$$

$$\text{In[68]}:= \text{Apart}\left[\frac{(-1 + b^t)\,(I + a + G - Y0) + (-1 + b^{1+t})\,Y0}{-1 + b}\right]$$

$$\text{Out[68]}= -\frac{i + a + G}{-1 + b} + \frac{b^t\,(i + a + G - Y0 + b\,Y0)}{-1 + b}$$

Hence, the general solution is

$$Y_n = \left(\frac{a+I+G}{1-b}\right) + b^n\left(Y_0 - \frac{a+I+G}{1-b}\right)$$

which is stable so long as $0 < b < 1$.

à Question 9

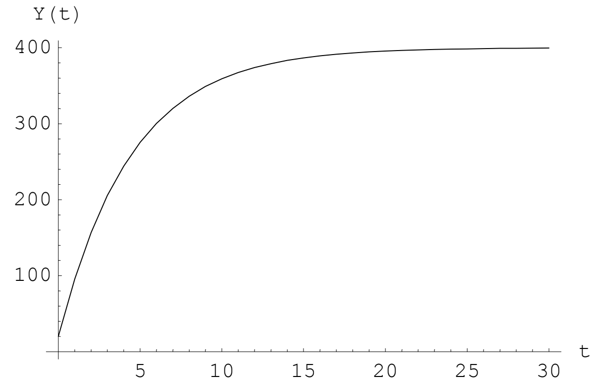
Readily obtain the difference equation $Y(t) = (a + I + G) + b\,Y(t-1)$

```
In[69]:= Clear[Y]

In[70]:= solY = RSolve[ {Y[t] == (50 + 10 + 20) + 0.8 * Y[t - 1], Y[0] == 20}, Y[t], t]
```

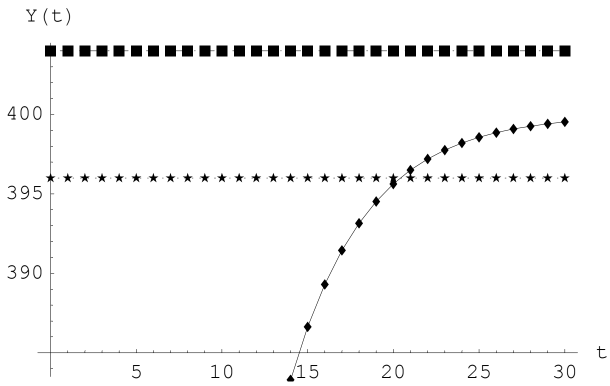
$$\text{Out[70]}= \{\{Y[t] \to 1.25\,(-304.\ 0.8^t + 320.\ 1.^t)\}\}$$

```
In[71]:= Ystar = (50 + 10 + 20) / (1 - 0.8)

Out[71]= 400.

In[72]:= Yu = (1 + 0.01) * Ystar

Out[72]= 404.

In[73]:= Yl = (1 - 0.01) * Ystar

Out[73]= 396.

In[74]:= dataY = Table[ {t, solY[[1, 1, 2]]}, {t, 0, 30}];
```

```
In[75]:= ListPlot[dataY, PlotJoined -> True, AxesLabel -> {"t", "Y(t)"}];
```



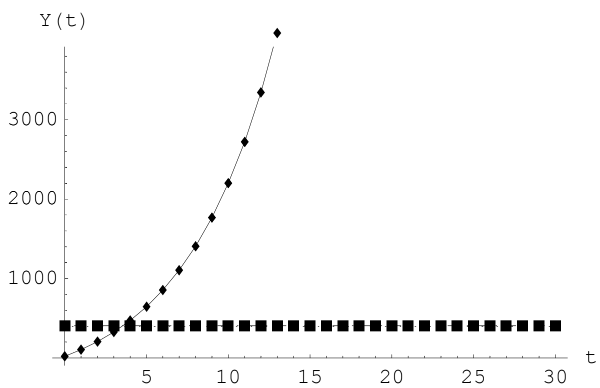

```
In[76]:= lowerline = Table[{t, Yl}, {t, 0, 30}];

In[77]:= upperline = Table[{t, Yu}, {t, 0, 30}];

In[78]:= MultipleListPlot[dataY, lowerline, upperline,
           PlotJoined -> True, AxesLabel -> {"t", "Y(t)"} ];
```



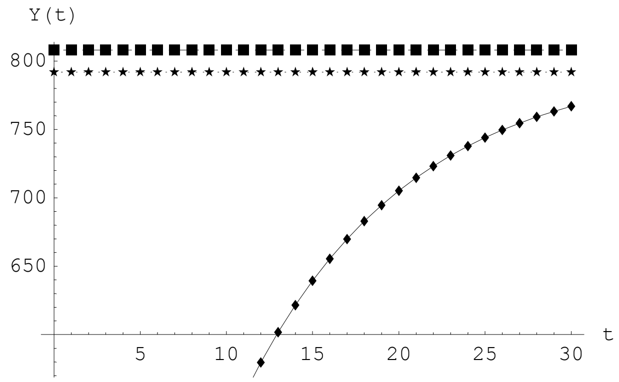

```
In[79]:= solY2 = RSolve[ {Y[t] == (50 + 10 + 20) + 1.2 * Y[t - 1], Y[0] == 20}, Y[t], t]

         General::spell1 :
          Possible spelling error: new symbol name "solY2" is similar to existing symbol "sol2".
```

Out[79]= $\{\{Y[t] \to 0.833333\,(-480.\,1.^{t} + 504.\,1.2^{t})\}\}$

```
In[80]:= dataY2 = Table[ {t, solY2[[1, 1, 2]]}, {t, 0, 30}];

In[81]:= MultipleListPlot[dataY2, lowerline,
           upperline, PlotJoined -> True, AxesLabel -> {"t", "Y(t)"}];
```



```
In[82]:= solY3 = RSolve[ {Y[t] == (50 + 10 + 20) + 0.9 * Y[t - 1] , Y[0] == 20}, Y[t] , t]

         General::spell1 :
          Possible spelling error: new symbol name "solY3" is similar to existing symbol "sol3".

Out[82]= {{Y[t] → 1.11111 (-702. 0.9^t + 720. 1.^t)}}

In[83]:= dataY3 = Table[{t, solY3[[1, 1, 2]]}, {t, 0, 30}];

In[84]:= Ystar3 = (50 + 10 + 20) / (1 - 0.9)

Out[84]= 800.

In[85]:= Yl3 = (1 - 0.01) * Ystar3

Out[85]= 792.

In[86]:= Yu3 = (1 + 0.01) * Ystar3

Out[86]= 808.

In[87]:= lowerline3 = Table[{t, Yl3}, {t, 0, 30}];

In[88]:= upperline3 = Table[{t, Yu3}, {t, 0, 30}];

In[89]:= MultipleListPlot[dataY3, lowerline3,
           upperline3, PlotJoined -> True, AxesLabel -> {"t", "Y(t)"}];
```



à Question 10

(i)

Substituting we obtain the difference equation

$$p(t) = \frac{a-c}{b} - \left(\frac{d}{b}\right)(1-e)\,p(t-1) - \left(\frac{d\,e}{b}\right)p(t-2)$$

which is a second-order nonhomogeneous difference equation.

(ii)

```
In[90]:= Clear[p, p0, p1]
```

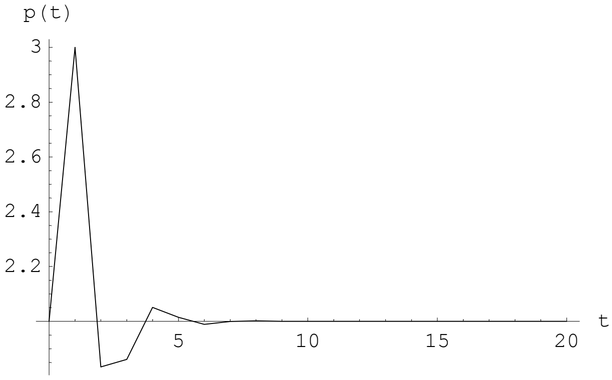
Substituting the values $a = 10$, $b = 3$, $c = 2$, $d = 1$ and $e = 0.5$ we can solve the following difference equation

```
In[91]:= solp = RSolve[ {p[t] == (8 / 3) - (1 / 6) p[t - 1] - (1 / 6) p[t - 2],
            p[0] == 2, p[1] == 3}, p[t], t]
```

Out[91]=
$$\left\{\left\{p[t] \to \frac{1}{23} 2^{1-2t} 3^{-t} \left(23\, 12^t + 3\, i \sqrt{23} \left(-1 - i\sqrt{23}\right)^t - 3\, i \sqrt{23} \left(-1 + i \sqrt{23}\right)^t\right)\right\}\right\}$$

```
In[92]:= seriesp = Table[{t, N[solp[[1, 1, 2]]]}, {t, 0, 20}];
```

```
In[93]:= ListPlot[seriesp, PlotJoined -> True,
           PlotRange -> All, AxesLabel -> {"t", "p(t)"}];
```



Furthermore, $q(t) = c + d[p(t-1) - e(p(t-1) - p(t-2)]$. Substituting the same values we find $q(t) = 2 + 0.5\, p(t-1) + 0.5\, p(t-2)$

```
In[94]:= pricelist = Table[N[solp[[1, 1, 2]]], {t, 0, 20}];
```

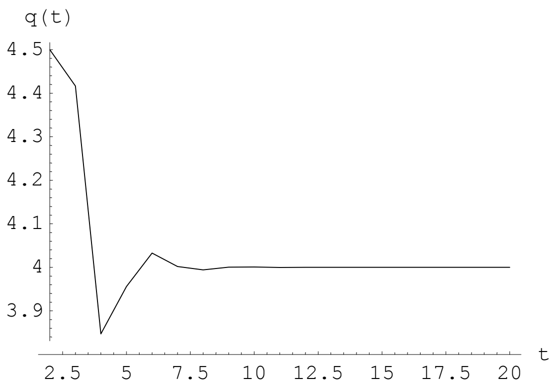
```
In[95]:= pricelist2 = Table[p[t] = N[solp[[1, 1, 2]]], {t, 0, 20}]
```

```
Out[95]= {2., 3. + 0. i, 1.83333 + 0. i, 1.86111 + 0. i, 2.05093 + 0. i, 2.01466 + 0. i,
          1.98907 + 0. i, 1.99938 + 0. i, 2.00193 + 0. i, 1.99978 + 0. i,
          1.99972 + 0. i, 2.00008 + 0. i, 2.00003 + 0. i, 1.99998 + 0. i,
          2. + 0. i, 2. + 0. i, 2. + 0. i, 2. + 0. i, 2. + 0. i, 2. + 0. i, 2. + 0. i}
```

```
In[96]:= qlist = Table[{t, 2 + 0.5 * p[t - 1] + 0.5 * p[t - 2]}, {t, 2, 20}]
```

```
Out[96]= {{2, 4.5 + 0. i}, {3, 4.41667 + 0. i}, {4, 3.84722 + 0. i}, {5, 3.95602 + 0. i},
          {6, 4.03279 + 0. i}, {7, 4.00186 + 0. i}, {8, 3.99422 + 0. i}, {9, 4.00065 + 0. i},
          {10, 4.00085 + 0. i}, {11, 3.99975 + 0. i}, {12, 3.9999 + 0. i},
          {13, 4.00006 + 0. i}, {14, 4.00001 + 0. i}, {15, 3.99999 + 0. i},
          {16, 4. + 0. i}, {17, 4. + 0. i}, {18, 4. + 0. i}, {19, 4. + 0. i}, {20, 4. + 0. i}}
```

```
In[97]:= ListPlot[qlist, PlotJoined -> True, PlotRange -> All,
           AxesOrigin -> {2, 3.8}, AxesLabel -> {"t", "q(t)"}];
```



à Question 11

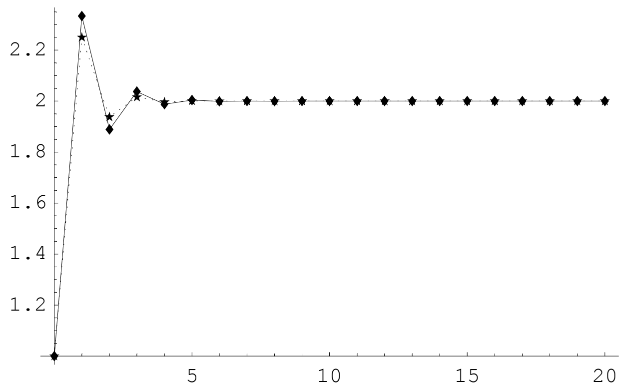
```
In[98]:= Clear[p]

In[99]:= ds1 = RSolve[{p[t] == (8 / 3) - (1 / 3) * p[t - 1], p[0] == 1}, p[t], t]
```

Out[99]= $\{\{p[t] \to 2 - \left(-\frac{1}{3}\right)^t\}\}$

```
In[100]:= ds2 = RSolve[ {p[t] == (10 / 4) - (1 / 4) * p[t - 1], p[0] == 1}, p[t], t]
```

Out[100]= $\{\{p[t] \to 2 - \left(-\frac{1}{4}\right)^t\}\}$

```
In[101]:= ds1list = Table[{t, N[ds1[[1, 1, 2]]]}, {t, 0, 20}];

In[102]:= ds2list = Table[{t, N[ds2[[1, 1, 2]]]}, {t, 0, 20}];

In[103]:= MultipleListPlot[ds1list, ds2list, PlotJoined -> True, PlotRange -> All];
```




```
In[104]:= dsdata =
  TableForm[Table[{t, N[ds1[[1, 1, 2]]], N[ds2[[1, 1, 2]]]}, {t, 0, 20}] ]
```

Out[104]//TableForm=

0	1.	1.
1	2.33333	2.25
2	1.88889	1.9375
3	2.03704	2.01563
4	1.98765	1.99609
5	2.00412	2.00098
6	1.99863	1.99976
7	2.00046	2.00006
8	1.99985	1.99998
9	2.00005	2.
10	1.99998	2.
11	2.00001	2.
12	2.	2.
13	2.	2.
14	2.	2.
15	2.	2.
16	2.	2.
17	2.	2.
18	2.	2.
19	2.	2.
20	2.	2.

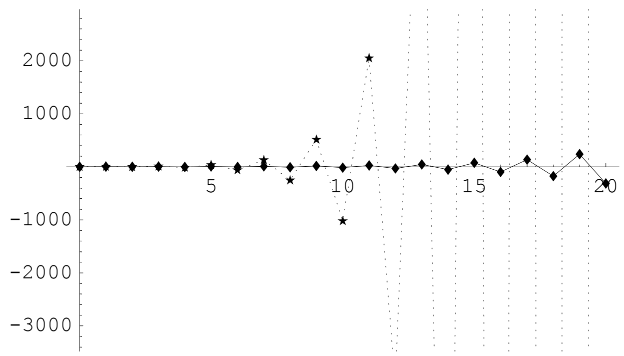
```
In[105]:= pstar1 = Solve[p == (8 / 3) - (1 / 3) p, p]
Out[105]= {{p → 2}}
In[106]:= Clear[p]
In[107]:= pstar2 = Solve[p == (10 / 4) - (1 / 4) p, p]
Out[107]= {{p → 2}}
In[108]:= ds3 = RSolve[{p[t] == (14 / 3) - (4 / 3) p[t - 1], p[0] == 1}, p[t], t]
```

Out[108]= $\left\{\left\{p[t] \to \frac{1}{7}\left(14 - 11\left(-\frac{4}{3}\right)^{t} + \left(-\frac{1}{3}\right)^{t} 4^{1+t}\right)\right\}\right\}$

```
In[109]:= ds4 = RSolve[ {p[t] == 6 - 2 p[t - 1], p[0] == 1}, p[t], t]
```

Out[109]= $\{\{p[t] \to 2 - 2^{t}\, e^{i \pi t}\}\}$

```
In[110]:= ds3list = Table[{t, N[ds3[[1, 1, 2]]]}, {t, 0, 20}];
In[111]:= ds4list = Table[{t, N[ds4[[1, 1, 2]]]}, {t, 0, 20}];
In[112]:= MultipleListPlot[ds3list, ds4list, PlotJoined -> True];
```



à Question 12

```
In[113]:= Clear[p]
In[114]:= pbar = Solve[25 - 4 * p == 3 + 4 p, p]
```

Out[114]= $\left\{\left\{p \to \frac{11}{4}\right\}\right\}$

```
In[115]:= qbar = 25 - 4 * pbar[[1, 1, 2]]
    General::spell1 :
     Possible spelling error: new symbol name "qbar" is similar to existing symbol "pbar".
Out[115]= 14
In[116]:= newp = RSolve[{25 - 4 p[t] == 3 + 4 p[t - 1], p[0] == p0}, p[t], t]
```

Out[116]= $\left\{\left\{p[t] \to \text{If}\left[\text{Even}[t],\ p0,\ \frac{1}{2}\ (11 - 2\, p0)\right]\right\}\right\}$

```
In[117]:= newlistp = Table[p[t] = N[newp[[1, 1, 2]]], {t, 0, 20}] /. p0 -> 1

Out[117]= {1, 4.5, 1, 4.5, 1, 4.5, 1, 4.5, 1,
           4.5, 1, 4.5, 1, 4.5, 1, 4.5, 1, 4.5, 1, 4.5, 1}

In[118]:= newlistp2 = Table[p[t] = N[newp[[1, 1, 2]]], {t, 0, 20}] /. p0 -> 3

Out[118]= {3, 2.5, 3, 2.5, 3, 2.5, 3, 2.5, 3,
           2.5, 3, 2.5, 3, 2.5, 3, 2.5, 3, 2.5, 3, 2.5, 3}
```

à Question 13

Here we use *Mathematica* to solve the 3-cycle problem.

```
In[119]:= Clear[f, x]

In[120]:= f[x_] = 3.84 x (1 - x)

Out[120]= 3.84 (1 - x) x

In[121]:= eq1 = f[f[f[x]]]

Out[121]= 56.6231 (1 - x) x (1 - 3.84 (1 - x) x) (1 - 14.7456 (1 - x) x (1 - 3.84 (1 - x) x))

In[122]:= sol1 = Solve[eq1 == x, x]

Out[122]= {{x → 0.}, {x → 0.149407}, {x → 0.169434}, {x → 0.488004},
           {x → 0.540388}, {x → 0.739583}, {x → 0.953736}, {x → 0.959447}}
```

à Question 14

Assume $y_1 = Y^*$ and $y_2 = \text{t}Y^*$ are linearly dependent, then

$$b_1\, Y^* + b_2\, t\, Y^* = 0$$

which implies that $b_1 + b_2\, t = 0$ for all t only if $b_1 = b_2 = 0$. Hence Y^* and $t\, Y^*$ are linearly independent.

à Question 15

(i) Assume F if linearly homogeneous, then

$$\frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) = f(\hat{k}_t)$$

$$i.e. \quad \hat{y}_t = f(\hat{k}_t)$$

Since $K_{t+1} - (1-\delta) K_t = s\,Y_t$ then

$$\frac{K_{t+1}}{A_t L_t} - (1-\delta)\frac{K_t}{A_t L_t} = \frac{s\,Y_t}{A_t L_t}$$

$$\left(\frac{K_{t+1}}{A_{t+1} L_{t+1}}\right) A_{t+1} \frac{L_{t+1}}{A_t L_t} - (1-\delta)\frac{K_t}{A_t L_t} = \frac{s\,Y_t}{A_t L_t}$$

$$\hat{k}_{t+1}\left(\frac{A_{t+1} L_{t+1}}{A_t L_t}\right) - (1-\delta)\hat{k}_t = s\,f(\hat{k}_t)$$

But

$$\frac{A_{t+1} L_{t+1}}{A_t L_t} = \left(\frac{\gamma^{t+1} A_0}{\gamma^t A_0}\right)\frac{L_{t+1}}{L_t} = \gamma(n+1)$$

Therefore

$$\gamma(n+1)\,\hat{k}_t - (1-\delta)\,\hat{k}_t = s\,f(\hat{k}_t)$$

or

$$\hat{k}_{t+1} = \frac{(1-\delta)\,\hat{k}_t + s\,f(\hat{k}_t)}{\gamma(1+n)}$$

(ii) Let $\hat{k}^*$ denote the positive equilibrium, then

$$\hat{k}_{t+1} = f(\hat{k}^*) + f'(\hat{k}^*)\left(\hat{k}_t - \hat{k}^*\right)$$

But $f(\hat{k}^*) = \hat{k}^*$ and

$$f'(\hat{k}^*) = \frac{1-\delta}{\gamma(1+n)} + \frac{s\,f'(\hat{k}^*)}{\gamma(1+n)}$$

Hence

$$\hat{k}_{t+1} = \hat{k}^* + \left(\frac{1-\delta}{\gamma(1+n)} + \frac{s\,f'(\hat{k}^*)}{\gamma(1+n)}\right)\left(\hat{k}_t - \hat{k}^*\right)$$

à Question 16

Solving for $p^e(t-1)$ gives $p^e(t-1) = \frac{a-c}{d} - \left(\frac{b}{d}\right) p(t-1)$, using this we obtain the difference equation

$$p_t = \left(1 - \frac{\lambda\,(b+d)}{b}\right) p_{t-1} + \frac{\lambda\,(a-c)}{b}$$

```
In[123]:= Clear[pbar, qbar, p, q, a, b, c, d]

In[124]:= pbar = Solve[p == (1 - λ (b + d)/b) p + λ (a - c)/b, p]

Out[124]= {{p → -(-a + c)/(b + d)}}

In[125]:= qbar = a - b * pbar[[1, 1, 2]]

Out[125]= a + b (-a + c)/(b + d)
```

In[126]:= $\frac{b\,c + a\,d}{b + d}$

Out[126]= $\frac{b\,c + a\,d}{b + d}$

In[127]:= RSolve$\left[\left\{u[t] == \left(1 - \frac{\lambda\,(b + d)}{b}\right) u[t - 1], u[0] == u0\right\}, u[t], t\right]$

Out[127]= $\{\{u[t] \to (-1)^t\, b^{-t}\, u0\, (b\,(-1 + \lambda) + d\,\lambda)^t\}\}$

In[128]:= $\left\{\left\{u[t] \to u0 \left(1 - \frac{(b + d)\,\lambda}{b}\right)^t\right\}\right\}$

Out[128]= $\left\{\left\{u[t] \to u0 \left(1 - \frac{(b + d)\,\lambda}{b}\right)^t\right\}\right\}$

In[129]:= Solve$\left[1 - \frac{(b + d)\,\lambda}{b} == -1, \lambda\right]$

Out[129]= $\left\{\left\{\lambda \to \frac{2\,b}{b + d}\right\}\right\}$

In[130]:= Solve$\left[1 - \frac{(b + d)\,\lambda}{b} == 1, \lambda\right]$

Out[130]= $\{\{\lambda \to 0\}\}$

à Question 17

Our equation is $P_n = (1 + r)^n\,(P_0 - \frac{R}{r}) + \frac{R}{r}$. In the present problem, taking note that we are dealing in months and the interest rate is per annum, we require zero payment after n = 3x12 = 36 monthly payments. Hence:

$P_n = 0$, $P_0 =$ 8000, $r = \frac{0.075}{12}$ = 0.00625 and $R = m$ = fixed monthly payment.

In[131]:= NSolve$\left[(1 + 0.00625)^{36} \left(8000 - \frac{m}{(0.075 / 12)}\right) + \frac{m}{(0.075 / 12)} == 0, m\right]$

Out[131]= $\{\{m \to 248.849745285249\}\}$

Hence the monthly payment is £248.85

à Question 18

Since

$n_t = 2\,n_{t-1}$ and $n_0 = 1$

```
In[132]:= RSolve[{n[t] == 2 n[t - 1], n[0] == 1}, n[t], t]
```

Out[132]= $\{\{n[t] \to 2^t\}\}$

In[133]:= NSolve$[5\;10^6 == 2^t, t]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

Out[133]= $\{\{t \to 22.2535\}\}$

Hence, the bacteria becomes contagious in just over twenty-two minutes.

à Question 19

Given the recursive equation
$x_{t+1} = \frac{x_t}{1+x_t}$ with $x(0) = x_0$

```
In[134]:= RSolve[{x[t + 1] == x[t] / (1 + x[t]), x[0] == x0}, x[t], t]

Out[134]= RSolve[{x[1 + t] == x[t]/(1 + x[t]), x[0] == x0}, x[t], t]
```

The fact that *Mathematica* returns the expression as entered means that it cannot solve it.
But now define

```
In[135]:= f[x_] := x/(1 + x)

In[136]:= Simplify[NestList[f, x0, 10]]

Out[136]= {x0, x0/(1 + x0), x0/(1 + 2 x0), x0/(1 + 3 x0), x0/(1 + 4 x0),
  x0/(1 + 5 x0), x0/(1 + 6 x0), x0/(1 + 7 x0), x0/(1 + 8 x0), x0/(1 + 9 x0), x0/(1 + 10 x0)}
```

The solution is clearly, then,
$x_n = \frac{x_0}{1+n\,x_0}$

à Question 20

■ (a)

Here we shall use *Mathematica* to do this.

```
In[137]:= x[0] := 1; x[1] := 1;
          x[n_] := x[n] = x[n - 1] + x[n - 2]
          data = Table[x[n], {n, 0, 10}]

        General::spell1 :
         Possible spelling error: new symbol name "data" is similar to existing symbol "dataY".

Out[139]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89}
```

■ (b)

Solving

```
In[140]:= Clear[x]
```

```
In[141]:= sol = RSolve[{x[n] == x[n - 1] + x[n - 2], x[0] == 1, x[1] == 1}, x[n], n]
```

Out[141]= $\left\{\left\{x[n] \to \frac{2^{-1-n}\left(-\left(1-\sqrt{5}\right)^{1+n}+\left(1+\sqrt{5}\right)^{1+n}\right)}{\sqrt{5}}\right\}\right\}$

```
In[142]:= Table[N[sol], {n, 0, 10}]

Out[142]= {{{x[0.] → 1.}}, {{x[1.] → 1.}}, {{x[2.] → 2.}},
  {{x[3.] → 3.}}, {{x[4.] → 5.}}, {{x[5.] → 8.}}, {{x[6.] → 13.}},
  {{x[7.] → 21.}}, {{x[8.] → 34.}}, {{x[9.] → 55.}}, {{x[10.] → 89.}}}
```

Which gives the same sequence of numbers.