5 Chapter 5 Solution Set

Problems

5.1 Derive the ISP integral equation Eq.(5.1a) in terms of boundary value data as given by Eq.(5.1b) by applying standard Green function techniques to the two wave equations satisfied by the radiated field $u_+(\mathbf{r}', t')$ and the free field propagator $g_f(\mathbf{r}-\mathbf{r}', t-t')$.

We have that

$$[\nabla^2_{r'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}]u_+(\mathbf{r}', t') = q(\mathbf{r}', t'),$$
$$[\nabla^2_{r'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}]g_f(\mathbf{r}-\mathbf{r}', t-t') = 0.$$

Using our (by now) standard Green function techniques we obtain

$$g_f(\mathbf{r}-\mathbf{r}', t-t')[\nabla^2_{r'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}]u_+(\mathbf{r}', t')$$
$$-u_+(\mathbf{r}', t')[\nabla^2_{r'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}]g_f(\mathbf{r}-\mathbf{r}', t-t') = q(\mathbf{r}', t')g_f(\mathbf{r}-\mathbf{r}', t-t').$$

We now integrate both sides of the above equation over a finite volume $\tau \supset \tau_0$ and from $t' = -\infty$ to $+\infty$ to obtain

$$\int_{-\infty}^{\infty} dt' \int_{\partial\tau} dS' \, [g_f(\mathbf{r}-\mathbf{r}', t-t')\frac{\partial}{\partial n'}u_+(\mathbf{r}', t') - u_+(\mathbf{r}', t')\frac{\partial}{\partial n'}g_f(\mathbf{r}-\mathbf{r}', t-t')]$$
$$+\frac{1}{c^2}\int_{\tau} d^3r' \, [u_+(\mathbf{r}', t')\frac{\partial}{\partial t'}g_f(\mathbf{r}-\mathbf{r}', t-t') - g_f(\mathbf{r}-\mathbf{r}', t-t')\frac{\partial}{\partial t'}u_+(\mathbf{r}', t')]|_{-\infty}^{\infty}$$
$$= \int_0^{T_0} dt' \int_{\tau_0} d^3r' \, q(\mathbf{r}', t')g_f(\mathbf{r}-\mathbf{r}', t-t'),$$

where the normal derivatives are directed out of the interior τ into the exterior $\tau^{\perp}$. Since the volume τ is finite the last term in the second term in the above equation vanishes which then yields the ISP integral equation in terms of over specified boundary value data.

5.2 Derive the ISP integral equation Eq.(5.1a) in terms of Cauchy data as given by Eq.(5.2) by applying standard Green function techniques to the two wave equations satisfied by the radiated field $u_+(\mathbf{r}', t')$ and the free field propagator $g_f(\mathbf{r}-\mathbf{r}', t-t')$.

Following identical lines as used in the previous problem we find using standard Green function techniques that

$$g_f(\mathbf{r}-\mathbf{r}',t-t')[\nabla^2_{r'}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}]u_+(\mathbf{r}',t')$$
$$-u_+(\mathbf{r}',t')[\nabla^2_{r'}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}]g_f(\mathbf{r}-\mathbf{r}',t-t')=q(\mathbf{r}',t')g_f(\mathbf{r}-\mathbf{r}',t-t').$$

We now integrate both sides of the above equation over an infinite volume and from $t'=0-\epsilon$ to $t_0>T_0$ to obtain

$$\int_{-\epsilon}^{t_0}dt'\int_{\Sigma_\infty}dS'\,[g_f(\mathbf{r}-\mathbf{r}',t-t')\frac{\partial}{\partial n'}u_+(\mathbf{r}',t')-u_+(\mathbf{r}',t')\frac{\partial}{\partial n'}g_f(\mathbf{r}-\mathbf{r}',t-t')]$$
$$+\frac{1}{c^2}\int d^3r'\,[u_+(\mathbf{r}',t')\frac{\partial}{\partial t'}g_f(\mathbf{r}-\mathbf{r}',t-t')-g_f(\mathbf{r}-\mathbf{r}',t-t')\frac{\partial}{\partial t'}u_+(\mathbf{r}',t')]|_{t'=-\epsilon}^{t_0}$$
$$=\int_0^{T_0}dt'\int_{\tau_0}d^3r'\,q(\mathbf{r}',t')g_f(\mathbf{r}-\mathbf{r}',t-t'),$$

where Σ_∞ is the surface of a sphere having infinite radius. Since the radiated field is causal it will vanish over Σ_∞ throughout the time interval $[-\epsilon,t_0]$ so the first term in the above equation will vanish as will the second term at the end point $t'=-\epsilon$. We thus arrive at the result

$$\int_0^{T_0}dt'\int_{\tau_0}d^3r'\,q(\mathbf{r}',t')g_f(\mathbf{r}-\mathbf{r}',t-t')$$
$$=\frac{1}{c^2}\int d^3r'\,[u_+(\mathbf{r}',t_0)\frac{\partial}{\partial t_0}g_f(\mathbf{r}-\mathbf{r}',t-t_0)-g_f(\mathbf{r}-\mathbf{r}',t-t_0)\frac{\partial}{\partial t_0}u_+(\mathbf{r}',t_0)]$$

which is the required result.

5.3 Prove that the ISP integral equation holds under the replacement of $g_f(\mathbf{r}-\mathbf{r}',t-t')$ by any function $\hat{g}_f(\mathbf{r}-\mathbf{r}',t-t')$ that satisfies the homogeneous wave equation over all of space-time.

This result follows from the fact that the derivations of the ISP integral equation employed in the previous two problems remains valid under the replacement of g_f with $\hat{g}_f$.

5.4 Derive the frequency domain back propagated field given in terms of Cauchy data in Eq.(5.3d) by Fourier transformation of Eq.(5.2).

The time-dependent back propagated field in terms of Cauchy data at time $t=t_0>T_0$ is given by

$$\phi(\mathbf{r},t)=\frac{1}{c^2}\int d^3r'\,[u_+(\mathbf{r}',t_0)g_f'(\mathbf{r}-\mathbf{r}',t-t_0)-g_f(\mathbf{r}-\mathbf{r}',t-t_0)u_+'(\mathbf{r}',t_0)],$$

where the primes denote differentiation w.r.t. t_0. We can express the free field propagator in the Fourier integral

$$g_f(\mathbf{r}-\mathbf{r}',t-t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,G_f(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t_0)},$$

from which we conclude that

$$g'_f(\mathbf{r}-\mathbf{r}',t-t_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, i\omega G_f(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t_0)}.$$

Using the above two Fourier expansions in the back propagated field and performing some elementary algebra yields

$$\phi(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\{\frac{e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)Gf(\mathbf{r}-\mathbf{r}',\omega)-G_f(\mathbf{r}-\mathbf{r}',\omega)u'_+(\mathbf{r}',t_0)\}e^{-i\omega t},$$

which then yields the required result

$$\Phi(\mathbf{r},\omega)=\frac{e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)-u'_+(\mathbf{r}',t_0)]G_f(\mathbf{r}-\mathbf{r}',\omega).$$

5.5 Use the multipole expansion of the free field propagator given in Eq.(5.4) of Example 5.1 in Eq.(5.3d) to derive the expansion

$$\Phi(\mathbf{r},\omega)=\sum_{l,m}\Phi_l^m(\omega)j_l(kr)Y_l^m(\hat{\mathbf{r}})$$

where

$$\Phi_l^m(\omega)=-2ki\frac{e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)-\frac{\partial}{\partial t_0}u_+(\mathbf{r}',t_0)]j_l(kr')Y_l^{m*}(\hat{\mathbf{r}}').$$

We have from Eq.(5.4)

$$G_f(\mathbf{R},\omega)=-2ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(kr)j_l(kr')Y_l^m(\hat{\mathbf{r}})Y^{*m}_{\ l}(\hat{\mathbf{r}}')$$

which when employed in Eq.(5.3d) yields

$$\Phi(\mathbf{r},\omega)=\frac{e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)-u'_+(\mathbf{r}',t_0)]\overbrace{-2ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(kr)j_l(kr')Y_l^m(\hat{\mathbf{r}})Y^{*m}_{\ l}(\hat{\mathbf{r}}')}^{G_f(\mathbf{r}-\mathbf{r}',\omega)}$$

$$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\{-2ik\frac{e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)-u'_+(\mathbf{r}',t_0)]j_l(kr')Y^{*m}_{\ l}(\hat{\mathbf{r}}')\}j_l(kr)Y_l^m(\hat{\mathbf{r}})$$

5.6 Use the expansions of the back propagated field obtained in problem 5.5 and in Example 5.1 to establish a relationship between Dirichlet data on a sphere surrounding the source and Cauchy conditions acquired after a source ceases to radiate.

We have from Example 5.1 that

$$\Phi(\mathbf{r},\omega)=2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{u_l^m(\omega)}{h_l^+(kr_0)}j_l(kr)Y_l^m(\hat{\mathbf{r}})$$

where

$$u_l^m(\omega)=\int d\Omega\,U_+(\mathbf{r},\omega)|_{r=r_0}Y^{*m}_{\ l}(\hat{\mathbf{r}}).$$

On the other-hand, we found in problem 5.5 that

$$\Phi(\mathbf{r},\omega)=\sum_{l,m}\Phi_l^m(\omega)j_l(kr)Y_l^m(\hat{\mathbf{r}})$$

where

$$\Phi_l^m(\omega)=-2ki\frac{e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)-\frac{\partial}{\partial t_0}u_+(\mathbf{r}',t_0)]j_l(kr')Y_l^{m*}(\hat{\mathbf{r}'}).$$

On equating terms in the two representations we find that

$$2\frac{u_l^m(\omega)}{h_l^+(kr_0)}=\Phi_l^m(\omega)$$

which yields

$$\int d\Omega\,U_+(\mathbf{r},\omega)|_{r=r_0}Y^{*m}_{\ l}(\hat{\mathbf{r}})=-ki\frac{h_l^+(kr_0)e^{i\omega t_0}}{c^2}\int d^3r'\,[i\omega u_+(\mathbf{r}',t_0)$$
$$-\frac{\partial}{\partial t_0}u_+(\mathbf{r}',t_0)]j_l(kr')Y_l^{m*}(\hat{\mathbf{r}'})$$

which is the desired result.

5.7 Show that the PB integral equation for a source distributed over the surface of a sphere and where the data consists of boundary value data of any kind over the surfaces of two concentric spheres one interior and one exterior to the source sphere is incomplete and only involves the data on the exterior sphere. This is an example where the formulation of the ISP in terms of the PB integral equation fails and the more powerful SVD based approach is required (cf. Problem 5.18).

The general expression for a source distributed over the surface of a sphere of radius a_0 is found following the same general procedure as employed for the planar source in Section 5.2 and is given by

$$Q(\mathbf{r},\omega)=Q_s(\hat{\mathbf{r}},\omega)\delta(r-a_0)+\frac{1}{r^2}Q_d(\hat{\mathbf{r}},\omega)\frac{\partial}{\partial r}[r^2\delta(r-a_0)].$$

where the unit vector $\hat{\mathbf{r}}=\mathbf{r}/r$ denotes the two angular coordinates on the surface of the sphere. The Porter-Bojarski integral equation then becomes

$$\Phi(\mathbf{r},\omega)=\int_{\tau_0}d^3r'\,G_f(\mathbf{r}-\mathbf{r}',\omega)Q(\mathbf{r}',\omega)$$
$$=a_0^2\int d\Omega'\,[Q_s(\hat{\mathbf{r}}',\omega)G_f(\mathbf{r}-a_0\hat{\mathbf{r}}',\omega)-Q_d(\hat{\mathbf{r}}',\omega)\frac{\partial}{\partial a_0}G_f(\mathbf{r}-a_0\hat{\mathbf{r}}',\omega)]$$

We have from Example 5.1

$$G_f(\mathbf{R},\omega)=-2ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(kr)j_l(kr')Y_l^m(\hat{\mathbf{r}})Y^{*m}_{\ l}(\hat{\mathbf{r}}')$$

which when employed in the above PB integral equation yields

$$\Phi(\mathbf{r},\omega) = a_0^2 \int d\Omega' \{Q_s(\hat{\mathbf{r}}',\omega) \overbrace{-2ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr)j_l(ka_0)Y_l^m(\hat{\mathbf{r}})Y^{*m}_l(\hat{\mathbf{r}}')}^{G_f(\mathbf{r}-a_0\hat{\mathbf{r}}',\omega)}$$

$$-Q_d(\hat{\mathbf{r}}',\omega) \overbrace{-2ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} kj_l(kr)j_l'(ka_0)Y_l^m(\hat{\mathbf{r}})Y^{*m}_l(\hat{\mathbf{r}}')}^{\frac{\partial}{\partial a_0}G_f(\mathbf{r}-a_0\hat{\mathbf{r}}',\omega)}\}$$

$$= -2ika_0^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\{\int d\Omega' \, [j_l(ka_0)Q_s(\hat{\mathbf{r}}',\omega) - kj_l'(ka_0)Q_d(\hat{\mathbf{r}}',\omega)]Y^{*m}_l(\hat{\mathbf{r}}')\}j_l(kr)Y_l^m(\hat{\mathbf{r}}),$$

which we can write in the compact form

$$\Phi(\mathbf{r},\omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\{-2ika_0^2[j_l(ka_0) < Y_l^m, Q_s > -kj_l'(ka_0) < Y_l^m, Q_d >]\}j_l(kr)Y_l^m(\hat{\mathbf{r}}), \tag{5.1}$$

where $< x, y >$ denote the inner product of x and y over the unit sphere.

The surface source is bounded by the union of the interior sphere and exterior sphere so that the back propagated field appearing in the l.h.s. of the above PB integral equation requires the sum of the back propagated fields from both data spheres into the annulus lying between these two data spheres. The back propagated field from the exterior sphere having radius $r_>$ is readily computed and found using the results of Example 5.1 to be expressible in terms of Dirichlet data as

$$\Phi_>(\mathbf{r},\omega) = 2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{u^m_{l\,>}(\omega)}{h_l^+(kr_>)}j_l(kr)Y_l^m(\hat{\mathbf{r}}),$$

where

$$u^m_{l\,>}(\omega) = \int d\Omega \, U_+(\mathbf{r},\omega)|_{r=r_>}Y^{*m}_l(\hat{\mathbf{r}}) =< Y_l^m, U_+(r_>\hat{\mathbf{r}}) >,$$

where, as before, $< x, y >$ denotes an inner product over the unit sphere.

The back propagated field from data acquired over the *inner* concentric sphere unfortunately vanishes so that the data over this sphere does not come into play in the solution of the problem so this approach fails. The reason for this is that the field within the interior of the surface source must satisfy the *homogeneous* Helmholtz equation so that at any point within this region we find using standard Green function techniques that

$$\int_{\partial\tau_<} dS'[U_+(\mathbf{r}',\omega)\frac{\partial}{\partial n'}G_\pm(\mathbf{r}-\mathbf{r}',\omega) - G_\pm(\mathbf{r}-\mathbf{r}',\omega)\frac{\partial}{\partial n'}U_+(\mathbf{r}',\omega)] = U_+(\mathbf{r},\omega),$$

if $\mathbf{r}$ is in the interior of the data sphere $\tau_<$ and zero if it lies in the exterior of this sphere and where $G_\pm$ is either the outgoing or incoming wave Green function. It then follows that the back propagated field will vanish throughout the annulus lying between these two data spheres (indeed will be identically zero over all of space) so that the PB integral equation will only involve the data on the exterior bounding sphere and, hence, is incomplete. Indeed, on substituting the back propagated field from the exterior sphere into Eq.(5.1) we obtain

$$-2ika_0^2[j_l(ka_0) < Y_l^m, Q_s > -kj_l'(ka_0) < Y_l^m, Q_d >] = \frac{< Y_l^m, U_+(r_>\hat{\mathbf{r}}) >}{h_l^+(kr_>)}.$$

The above equation has two unknowns $< Y_l^m, Q_s >$ and $< Y_l^m, Q_d >$ and only one known $< Y_l^m, U_+(r_>\hat{\mathbf{r}}) >$ and, hence, is incomplete.

5.8 Derive the most general form of a surface source distributed over an infinite plane and that is NR throughout one of the two half-spaces bounded by the plane.

By a straightforward generalization of Eq.(5.13) we find that the field radiated by the surface source is given by

$$U_+(\mathbf{r},\omega) = \int_{z=0} d^2\rho' \, [G_+(\mathbf{r}-\boldsymbol{\rho}',\omega)Q_s(\boldsymbol{\rho}',\omega) + Q_d(\boldsymbol{\rho}',\omega)\frac{\partial}{\partial z}G_+(\mathbf{r}-\boldsymbol{\rho}',\omega)].$$

If we substitute the Weyl expansion from Eq.(4.4a) into the above equation we obtain

$$U_+(\mathbf{r},\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty} \frac{d^2K_\rho}{\gamma}[\tilde{Q}_s(\mathbf{K}_\rho,\omega) \pm \gamma\tilde{Q}_d(\mathbf{K}_\rho,\omega)]e^{\pm i\gamma z}e^{i\mathbf{K}\cdot\boldsymbol{\rho}}$$

where the + sign is used in the r.h.s. $z > 0$ and the minus sign if $z < 0$. We conclude that the field will vanish in the r.h.s. half-space if

$$\tilde{Q}_s(\mathbf{K}_\rho,\omega) + \gamma\tilde{Q}_d(\mathbf{K}_\rho,\omega) = 0$$

and throughout the l.h.s. if

$$\tilde{Q}_s(\mathbf{K}_\rho,\omega) - \gamma\tilde{Q}_d(\mathbf{K}_\rho,\omega) = 0.$$

We can thus take the singlet component $Q_s(\boldsymbol{\rho},\omega)$ to be arbitrary and select the doublet component $Q_d(\boldsymbol{\rho},\omega)$ to satisfy one of the two above equations to obtain a surface source that is NR throughout one of the two half-spaces.

5.9 Derive the most general form a a surface source distributed over the surface of a sphere and that is NR throughout the interior (exterior) of the sphere.

This is solved in a parallel fashion to the previous problem. The field radiated by a surface source distributed over a sphere of radius a_0 is easily found to be (cf., Problem 5.7)

$$U_+(\mathbf{r},\omega) = a_0^2\int d\Omega' \, [Q_s(\hat{\mathbf{r}}',\omega)G_+(\mathbf{r}-a_0\hat{\mathbf{r}}',\omega) - Q_d(\hat{\mathbf{r}}',\omega)\frac{\partial}{\partial a_0}G_+(\mathbf{r}-a_0\hat{\mathbf{r}}',\omega)].$$

In place of the Weyl expansion used in the previous problem we now use the multipole expansion of the outgoing wave Green function given in Eq.(**??**) to find that

$$U_+(\mathbf{r},\omega) = -ika_0^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m j_l(kr) Y_l^m(\hat{\mathbf{r}}), \quad r < a_0,$$
$$U_+(\mathbf{r},\omega) = -ika_0^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_l^m h_l^+(kr) Y_l^m(\hat{\mathbf{r}}), \quad r > a_0,$$

where

$$a_l^m = h_l^+(ka_0) < Y_l^m, Q_s > -kh_l^{+\prime}(ka_0) < Y_l^m, Q_d >,$$
$$b_l^m = j_l^+(ka_0) < Y_l^m, Q_s > -kj_l'(ka_0) < Y_l^m, Q_d >,$$

and where $< x, y >$ denote the inner product over the unit sphere of x with y. If we then require a_l^m to be zero we have a surface source that will be NR throughout the entire of the sphere and if $b_l^m = 0$ it will be NR throughout the exterior of the sphere.

5.10 Prove the following theorem which is the frequency domain version of the "source decomposition theorem" Theorem 1.3 proven in Section 1.7 of Chapter 1: Let $Q(\mathbf{r},\omega)$ be a square integrable source compactly supported within τ_0. Then this source can be uniquely decomposed into an NR component $Q_{nr}(\mathbf{r},\omega)$ and a minimum norm component $\hat{Q}(\mathbf{r},\omega)$ such that

$$\int_{\tau_0} d^3r\, Q_{nr}(\mathbf{r},\omega)\hat{Q}(\mathbf{r},\omega) = 0,$$
$$[\nabla_r^2 + k^2]\hat{Q}(\mathbf{r},\omega) = 0,$$
$$Q_{nr}(\mathbf{r},\omega) = [\nabla_r^2 + k^2]\Pi(\mathbf{r},\omega),$$

where $\Pi(\mathbf{r},\omega)$ is a square integrable function supported in τ_0 that has continuous first partial derivatives.

5.11 Verify Eqs.(5.27).

We wish to show that with $-a_0 \le z \le +a_0$ we have that

$$\lim_{a_0\to 0} \frac{\cos\gamma z}{a_0(1 + j_0(2\gamma a_0))} \to \delta(z), \quad \lim_{a_0\to 0} \frac{\gamma \sin\gamma z}{a_0(1 - j_0(2\gamma a_0))} \to -\delta'(z).$$

The above results are easily verified by using the definitions of the delta function and doublet; i.e.,

$$\int_{-\infty}^{\infty} dz\, g(z)\delta(z) = g(0), \quad \int_{-\infty}^{\infty} dz\, g(z)\delta'(z) = -g'(0),$$

for any function $g(z)$ that is analytic in the neighborhood of the origin.

We have that

$$\frac{\cos\gamma z}{a_0(1 + j_0(2\gamma a_0))} = \frac{\cos\gamma z}{2a_0 + O(a_0^3)},$$
$$\frac{\gamma\sin\gamma z}{a_0(1 - j_0(2\gamma a_0))} = \frac{\gamma\sin\gamma z}{a_0\frac{(2\gamma a_0)^2}{3!} + O(a_0^5)}$$

Using these results we find that

$$\lim_{a_0\to 0}\int_{-a_0}^{a_0} dz\, g(z)\frac{\cos\gamma z}{a_0(1+j_0(2\gamma a_0))} = \lim_{a_0\to 0}\int_{-a_0}^{+a_0} dz g(z)\frac{\cos\gamma z}{2a_0+O(a_0^3)}$$
$$= \lim_{a_0\to 0}\frac{1}{2a_0+O(a_0^3)}\int_{-a_0}^{+a_0} dz\, g(z)\cos\gamma z = g(0), \tag{5.2}$$

a result that follows from expanding the analytic function $g(z)$ into a Taylor series about the origin. We also find that

$$\lim_{a_0\to 0}\int_{-a_0}^{+a_0} dz\, g(z)\frac{\gamma\sin\gamma z}{a_0(1-j_0(2\gamma a_0))}$$
$$= \lim_{a_0\to 0}\frac{\gamma}{a_0\frac{(2\gamma a_0)^2}{3!}+O(a_0^5)}\int_{-a_0}^{+a_0} dz\, g(z)\sin\gamma z.$$

If we again expand $g(z)$ in a Taylor series about the origin we find that

$$\int_{-a_0}^{+a_0} dz\, g(z)\sin\gamma z = \int_{-a_0}^{+a_0} dz\,[g(0)+g'(0)z+O(z^2)][\gamma z-\frac{1}{3!}(\gamma z)^3+O(z^5)]$$
$$= g'(0)[\gamma\frac{z^3}{3}-\frac{1}{3!}\frac{\gamma^3 z^4}{4}+O(z^6)]|_{-a_0}^{+a_0} = g'(0)[2\gamma\frac{a_0^3}{3}+O(a_0^4)].$$

Substituting this into Eq.(5.2) we obtain

$$\lim_{a_0\to 0}\int_{-a_0}^{a_0} dz\, g(z)\frac{\cos\gamma z}{a_0(1+j_0(2\gamma a_0))}$$
$$= \lim_{a_0\to 0}\frac{\gamma}{a_0\frac{(2\gamma a_0)^2}{3!}+O(a_0^5)}g'(0)[2\gamma\frac{a_0^3}{3}+O(a_0^4)]$$

which simplifies to $g'(0)$ thus establishes the required result.

5.12 Show that the adjoint of a compact operator is also compact. Hint: Show that it is Hilbert-Schmidt.

For $A^\dagger : H_f \to H_e$ to be Hilbert-Schmidt and, hence, compact we must show that there exists a complete O.N. sequence f_m, $m = 1, 2\cdots \in H_f$ such that

$$I_f = \sum_m ||A^\dagger f_m||^2 < \infty.$$

We now show that this holds so long as $A : H_e \to H_f$ is Hilbert-Schmidt.

To do this we take f_m, $m = 1, 2\cdots \in H_f$ and e_n, $n = 1, 2\cdots \in H_e$ to be *any* two complete O.N. sequences in their respective Hilbert spaces. Since $\{e_n\}$ is complete and O.N. we have that

$$A^\dagger f_m = \sum_n < e_n, A^\dagger f_m >_{H_e} e_n$$

so that

$$I_f = \sum_m ||A^\dagger f_m||^2 = \sum_m\sum_n | < e_n, A^\dagger f_m >_{H_e} |^2 = \sum_m\sum_n | < Ae_n, f_m >_{H_f} |^2.$$

But by Bessel's inequality we have that

$$I_f = \sum_m \sum_n | < Ae_n, f_m >_{H_f} |^2 \leq \sum_n | < Ae_n >_{H_e} |^2 < \infty$$

since A is compact so that there exists at least one complete O.N. sequence satisfying the above inequality.

5.13 Prove that $\hat{T}^\dagger \hat{T}$ and $\hat{T}\hat{T}^\dagger$ are compact if $\hat{T}$ is compact.

Since $\hat{T} : H_e \to H_f$ is compact it follows from the previous problem that so to is $\hat{T}^\dagger : H_f \to H_e$. Now consider

$$I = \sum_m ||\hat{T}\hat{T}^\dagger f_m||^2$$

where f_m, $m = 1, 2, \cdots \in H_f$ is a complete O.N. sequence. Define

$$\phi_m = \hat{T}^\dagger f_m.$$

Then we can express I in the form

$$I = \sum_m ||\hat{T}\phi_m||^2.$$

But $\hat{T}$ is compact by hypothesis and, hence, bounded so that

$$||\hat{T}\phi_m||^2 \leq M||\phi_m||^2, \quad \forall \phi_m \in H_f.$$

Using this result we then find that

$$I \leq M \sum_m ||\phi_m||^2 < \infty$$

with the last inequality following from the fact that $\hat{T}^\dagger$ is compact so that

$$\sum_m ||\phi_m||^2 = \sum_m ||\hat{T}^\dagger f_m||^2 < \infty.$$

The proof that $\hat{T}^\dagger \hat{T}$ is also compact is proven in exactly the same manner.

5.14 Let H be an $N \times M$ matrix with complex elements $h_{n,m}$ and possessing the singular set v_p, u_p, σ_p. Then show that it admits the SVD

$$H = U \Sigma V^\dagger$$

where U is the $N \times N$ matrix whose column vectors $u_1, u_2, \cdots, u_N$, V is the $M \times M$ matrix with column vectors $v_1, v_2, \cdots, v_M$ and Σ is the $N \times M$ diagonal matrix with elements $\sigma_1, \sigma_2, \cdots, \sigma_P$ where $P = \text{Min}\,(N, M)$.

Let $\mathcal{H}_v$ and $\mathcal{H}_u$ denote the Hilbert spaces generated by the $\{v_p\}$ and $\{u_p\}$ respectively. Then since H is an $N \times M$ matrix $\mathcal{H}_v$ and $\mathcal{H}_u$ are C^M and C^N respectively whose elements can be taken to be M and N dimensional complex column vectors.

Now let $v \in \mathcal{H}_v$ be an arbitrary element in $\mathcal{H}_v$. Then since $\mathcal{H}_v$ is generated by $\{v_p\}$ we have that

$$v = \sum_p < v_p, v >_{\mathcal{H}_v} v_p$$

and

$$Hv = \sum_p < v_p, v >_{\mathcal{H}_v} Hv_p = \sum_p \sigma_p < v_p, v >_{\mathcal{H}_v} u_p$$

which must hold for any $v \in \mathcal{H}_v$. We can rewrite the above equation in the form

$$Hv = \sum_p \sigma_p u_p v_p^\dagger v$$

where u_p is an N column vector and $v_p^\dagger$ denotes the Hermitian adjoint (complex row vector) of the M column vector v_p. Since the above equation must hold for any $v \in \mathcal{H}_v$ we have that the $N \times M$ matrix H admits the decomposition

$$H = \sum_p \sigma_p u_p v_p^\dagger. \tag{5.3}$$

The representation given in Eq.(5.3) is in the form of a weighted sum of outer products of vectors. It can be converted into a triple matrix product using standard methods of matrix algebra. Here we will demonstrate this for the simple case where $N = M = 2$. In this case we can express Eq.(5.3) in the form

$$H = \sigma_1 \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} \begin{bmatrix} v_{11}^* & v_{12}^* \end{bmatrix} + \sigma_2 \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} \begin{bmatrix} v_{21}^* & v_{22}^* \end{bmatrix}$$

where $v_p = [v_{p1}, v_{p2}]$, $p = 1, 2$ and similarly for u_p. If we expand each of the outer products in the above equation we obtain

$$H = \sigma_1 \begin{bmatrix} u_{11}v_{11}^* & u_{11}v_{12}^* \\ u_{12}v_{11}^* & u_{12}v_{12}^* \end{bmatrix} + \sigma_2 \begin{bmatrix} u_{21}v_{21}^* & u_{21}v_{22}^* \\ u_{22}v_{21}^* & u_{22}v_{22}^* \end{bmatrix}$$

which can be further reduced to

$$H = \begin{bmatrix} (\sigma_1 u_{11}v_{11}^* + \sigma_2 u_{21}v_{21}^*) & (\sigma_1 u_{11}v_{12}^* + \sigma_2 u_{21}v_{22}^*) \\ (\sigma_1 u_{12}v_{11}^* + \sigma_2 u_{22}v_{21}^*) & (\sigma_1 u_{12}v_{12}^* + \sigma_2 u_{22}v_{22}^*) \end{bmatrix}.$$

The final reduction takes two steps:

$$H = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} \sigma_1 v_{11}^* & \sigma_1 v_{12}^* \\ \sigma_2 v_{21}^* & \sigma_2 v_{22}^* \end{bmatrix} = \overbrace{\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{bmatrix}}^{U} \overbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}^{\Sigma} \overbrace{\begin{bmatrix} v_{11}^* & v_{12}^* \\ v_{21}^* & v_{22}^* \end{bmatrix}}^{V^\dagger}.$$

5.15 Compute the singular system for the antenna synthesis problem addressed in Section 5.5 by first solving for the singular functions u_p and then computing the rest of the system from these functions.

The singular functions u_p satisfy the normal equation Eq.(5.55b)

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mu_l^2 < Y_l^m, u_p >_{\mathcal{H}_f} Y_l^m(\mathbf{s}) = \sigma_p^2 u_p(\mathbf{s}, \omega)$$

and are thus linear combinations of the spherical harmonics. On setting

$$u_p(\mathbf{s},\omega)=\sum_{l,m}C_l^m(p)Y_l^m(\mathbf{s})$$

we then obtain the equation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\mu_l^2C_l^m(p)Y_l^m(\mathbf{s})=\sigma_p^2\sum_{l,m}C_l^m(p)Y_l^m(\mathbf{s})$$

which requires that $C_l^m(p)=0$ unless $l=p$ and $\mu_p=\sigma_p$. The index $p=l$ is degenerate so that the singular functions for fixed p can be taken to be the spherical harmonics

$$u_p(\mathbf{s})=Y_l^m(\mathbf{s}),\quad -l\le m\le +l,$$

where p now represents the double index l,m.

The singular functions v_p are obtained using $T^\dagger u_p=\sigma_p^2v_p$. The adjoint operator is given in Eq.(5.52d) which then yields

$$\begin{aligned}\hat{T}^\dagger u_p&=-\mathcal{M}_{\tau_0}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}i^{l'}j_{l'}^*(kr)Y_{l'}^{m'}(\hat{\mathbf{r}})\int_{4\pi}d\Omega_s\,Y_{l'}^{m'^*}(\mathbf{s})Y_l^m(\mathbf{s})\\&=-\mathcal{M}_{\tau_0}i^lj_l^*(kr)Y_l^m(\hat{\mathbf{r}})=\sigma_pv_p,\end{aligned}$$

thus yielding the singular functions defined in Eqs.(5.56a) with $p=(l,m)$.

5.16 1. Show that the singular values σ_p and singular vectors u_p for the one-dimensional far field version of the ISP defined in Example 5.5 satisfy the normal equation

$$\overbrace{\frac{1}{4|k|^2}\mathcal{M}\sum_{s=\pm1}e^{ik^*sz}\int_{L_0}dz'\,e^{-iksz'}v_p(z',\omega)}^{\hat{T}^\dagger\hat{T}v_p}=\sigma_p^2v_p(z,\omega).$$

We showed in Example 5.5 that

$$\hat{T}=-\frac{i}{2k}\int_{L_0}dz'\,e^{-iksz'},\quad \hat{T}^\dagger=\frac{i}{2k^*}\mathcal{M}\sum_{s=\pm1}e^{ik^*sz}.$$

The normal equation satisfied by v_p is then found to be

$$T^\dagger Tv_p=\{\frac{i}{2k^*}\mathcal{M}\sum_{s=\pm1}e^{ik^*sz}\}\{-\frac{i}{2k}\int_{L_0}dz'\,e^{-iksz'}\}v_p(z',\omega)=\sigma_p^2v_p(z,\omega),$$

which simplifies to

$$\frac{1}{4|k|^2}\mathcal{M}\sum_{s=\pm1}e^{ik^*sz}\int_{L_0}dz'\,e^{-iksz'}v_p(z',\omega)=\sigma_p^2v_p(z,\omega).\tag{5.4}$$

2. Give an argument why the singular functions v_p for $\sigma_p > 0$ satisfy the homogeneous Helmholtz equation with wavenumber k^* everywhere *inside* the interval L_0 and are, in fact, a linear combination of the two functions $\exp(\pm ik^*z)$.

   The plane waves $\exp(ik^*z)$ satisfy the 1D Helmholtz equation with wavenumber k^*. It then follows that if $z \in L_0$ we can commute the Helmholtz operator with the masking operator $\mathcal{M}$ proving that the v_p satisfy the homogeneous Helmholtz equation with wavenumber k^* so long as $z \in L_0$. If $z \notin L_0$ we cannot commute the two operators so that these singular functions are generated by singularities on the two ends of L_0.

3. Using the above result show that the singular functions can be expressed in the form

$$v_p(z,\omega) = \mathcal{M} \sum_{s'=\pm 1} A_{s'}(p) e^{ik^*s'z}, \quad \sigma_p > 0,$$

   where the Fourier coefficients A_s, $s = \pm 1$ satisfy the matrix equation

$$\begin{bmatrix} \mathrm{sinc}[a_0(k-k^*)] & \mathrm{sinc}[a_0(k+k^*)] \\ \mathrm{sinc}[a_0(k+k^*)] & \mathrm{sinc}[a_0(k-k^*)] \end{bmatrix} \begin{bmatrix} A_{-1}(p) \\ A_{+1}(p) \end{bmatrix} = \sigma_p^2 \begin{bmatrix} A_{-1}(p) \\ A_{+1}(p) \end{bmatrix}.$$

   Since the singular functions v_p with $\sigma_p > 0$ satisfy the homogeneous Helmholtz equation with wavenumber k^* we can express them in the form

$$v_p(z) = \sum_{s=\pm 1} A_s e^{ik^*sz}$$

   which, when used in Eq.(5.4) yield

$$\frac{1}{4|k|^2}\mathcal{M} \sum_{s=\pm 1} e^{ik^*sz} \int_{L_0} dz'\, e^{-iksz'} [\sum_{s'=\pm 1} A_{s'} e^{ik^*s'z'}] = \sigma_p^2 \sum_{s=\pm 1} A_s e^{ik^*sz}.$$

   The plane waves $\exp(\pm ik^*z)$ are linearly independent so that the above equation requires that

$$\frac{1}{4|k|^2} \int_{L_0} dz'\, e^{-iksz'} [\sum_{s'=\pm 1} A_{s'} e^{ik^*s'z'}] = \sigma_p^2 A_s$$

   which reduces to

$$\frac{1}{2|k|^2} \sum_{s'=\pm 1} \frac{\sin[(ks - k^*s')a_0]}{ks - k^*s'} A_{s'} = \sigma_p^2 A_s.$$

   The above equations then simplify to the matrix equations in the problem statement on setting $s = \pm 1$.

5.17 Set up and solve the ISP for a source compactly supported between two parallel planes and Dirichlet data over two bounding parallel planes using the SVD. Compare and contrast your solution with that found in Section 5.3.

We begin by formulating the problem using the angular spectrum expansion of the radiated field. We showed in Section 4.2 that the angular spectrum is related to the source via the equation

$$A(\mathbf{k}^{\pm},\omega) = -\frac{1}{4\pi}\tilde{Q}(\mathbf{k}_{\pm},\omega)$$

and in Section 4.3 related the angular spectrum to Dirichlet conditions over any plane $z = z_0$ lying outside the source support volume τ_0 via the equation

$$A(\mathbf{k}^{\pm},\omega) = \frac{\gamma}{2\pi i}\tilde{U}_+(\mathbf{K}_\rho, z_0,\omega)e^{\pm i\gamma z_0}.$$

In these equations $\mathbf{k}^{\pm} = \mathbf{K}_\rho \pm \gamma\hat{\mathbf{z}}$ and $\mathbf{K}_\rho$ is the spatial frequency vector relative to the x, y plane. We will take the two measurement planes to lie on either side of the source region at $z = \pm a_0$ and we then obtain the imaging model

$$-\frac{1}{4\pi}\tilde{Q}(\mathbf{k}_{\pm},\omega) = \frac{\gamma}{2\pi i}\tilde{U}_+(\mathbf{K}_\rho, \pm a_0,\omega)e^{\pm i\gamma a_0}. \tag{5.5a}$$

At this point it is preferable to set $\mathbf{k}_{\pm} = \mathbf{K}_\rho \pm \gamma\hat{\mathbf{z}}$ and write the above image model in the form

$$-\frac{1}{4\pi}\int_{-a_0}^{a_0} dz\,\overline{Q}(\mathbf{K}_\rho, z,\omega)e^{\mp i\gamma z} = \frac{\gamma}{2\pi i}\tilde{U}_+(\mathbf{K}_\rho, \pm a_0,\omega)e^{\pm i\gamma a_0} \tag{5.5b}$$

where

$$\overline{Q}(\mathbf{K}_\rho, z,\omega) = \int d^2\rho\, Q(\boldsymbol{\rho}, z,\omega)e^{-i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}.$$

We can then write the imaging equation in the standard form $\hat{T}\overline{Q} = f$ with

$$\hat{T} = -\int_{-a_0}^{a_0} dz\, e^{-is\gamma z}$$

with

$$f(s) = \frac{\gamma}{2\pi i}\tilde{U}_+(\mathbf{K}_\rho, sa_0,\omega)e^{is\gamma a_0}, \quad s = \pm 1.$$

The actual source $Q(\mathbf{r},\omega)$ is recovered from $\overline{Q}(\mathbf{K}_\rho, z,\omega)$ via an inverse 2D Fourier transform.

We see that by writing the imaging model in the form Eq.(5.5b) rather than (5.5a) we have effectively reduced the dimensionality of the ISP to one dimension which, we will find, yields an inverse problem that is completely analogous to the 1D ISP considered in Example 5.5 and Problem 5.16. The transverse wavevector $\mathbf{K}_\rho$ now plays the role of a parameter and the ISP is then solved for $\overline{Q}(\mathbf{K}_\rho, z,\omega)$ for each value of this wavevector. The Hilbert space $\mathcal{H}_{\overline{Q}}$ is $L^2(-a_0, +a_0)$ and the data space $\mathcal{H}_f$ is simply C^2 corresponding to the two complex numbers $f(s)$ with $s = \pm 1$. The inner products in the two spaces are

$$< v_2, v_1 >_{\mathcal{H}_{\overline{Q}}} = \int_{-a_0}^{a_0} dz\, v_2^*(z)v_1(z), \quad < u_2, u_1 >_{\mathcal{H}_f} = \sum_{s=\pm 1} u_2^*(s)u_1(s).$$

Using the above inner products we find that

$$\hat{T}^\dagger = -\frac{1}{4\pi}\mathcal{M}\sum_{s=\pm 1} e^{is\gamma^* z},$$

where $\mathcal{M} = 1$, $z \in [-a_0, +a_0]$ and zero otherwise is the masking operator. The composite operator $\hat{T}^\dagger\hat{T}$ is easily found to be

$$\hat{T}^\dagger\hat{T} = (\frac{1}{4\pi})^2\mathcal{M}\sum_{s=\pm 1} e^{is\gamma^* z}\int_{-a_0}^{a_0} dz'\, e^{-isz'}.$$

We introduce the SVD in the usual way with $v_p(z) \in \mathcal{H}_{\overline{Q}}$ and $u_p(s) \in \mathcal{H}_f$. The normal equations for the v_p are found to be

$$\overbrace{(\frac{1}{4\pi})^2\mathcal{M}\sum_{s=\pm 1} e^{is\gamma^* z}\int_{-a_0}^{a_0} dz'\, e^{-isz'} v_p(z')}^{\hat{T}^\dagger\hat{T}v_p} = \sigma_p^2 v_p(z). \tag{5.6}$$

The computation of the singular functions $v_p(z)$ follows from the observation that the $v_p(z)$ with $\sigma_p > 0$ are linear combinations of $\exp(\pm i\gamma^* z)$ and can thus be represented in the form

$$v_p(z) = \mathcal{M}\sum_{s=\pm 1} A_p(s)e^{is\gamma^* z}. \tag{5.7}$$

On substituting the above representation into the left hand side of the normal equations and performing some elementary calculus we find that

$$\frac{a_0}{2}\mathcal{M}\sum_{s=\pm 1} e^{is\gamma^* z}\sum_{s'=\pm 1} A_p(s')\text{Sinc }[(s\gamma - s'\gamma^*)a_0] = \sigma_p^2 v_p(z),$$

where $\text{Sinc }(x) = \sin x/x$ is the Sinc function. It then follows from the above two equations that

$$\frac{a_0}{2}\sum_{s'=\pm 1} A_p(s')\text{Sinc }[(s\gamma - s'\gamma^*)a_0] = \sigma_p^2 A_p(s),$$

which can be expressed in matrix form as

$$\begin{bmatrix} \text{sinc}[a_0(\gamma-\gamma^*)] & \text{sinc}[a_0(\gamma+\gamma^*)] \\ \text{sinc}[a_0(\gamma+\gamma^*)] & \text{sinc}[a_0(\gamma-\gamma^*)] \end{bmatrix}\begin{bmatrix} A_p(s=-1) \\ A_p(s=+1) \end{bmatrix} \tag{5.8a}$$

$$= \frac{2\sigma_p^2}{a_0}\begin{bmatrix} A_p(s=-1) \\ A_p(s=+1) \end{bmatrix}. \tag{5.8b}$$

The singular values σ_p are obtained as solutions to

$$\det\begin{bmatrix} \frac{2\sigma_p^2}{a_0} - \text{sinc}[a_0(\gamma-\gamma^*)] & \text{sinc}[a_0(\gamma+\gamma^*)] \\ \text{sinc}[a_0(\gamma+\gamma^*)] & \frac{2\sigma_p^2}{a_0} - \text{sinc}[a_0(\gamma-\gamma^*)] \end{bmatrix} = 0$$

Evaluating the determinant and performing some elementary algebra then leads to the result

$$\sigma_p^2 = \frac{a_0}{2}\left[\mathrm{sinc}[a_0(\gamma-\gamma^*)] + p\ \mathrm{sinc}[a_0(\gamma+\gamma^*)]\right], \tag{5.8c}$$

with $p = \pm 1$.

As a final step the spectral amplitudes $A_p(s)$ are obtained from the solutions to the matrix equation Eq.(5.8b) using the singular values computed in Eq.(5.8c). The singular functions $v_p(z)$ are then found from Eqs.(5.7) and the singular functions $u_p(s)$ are then obtained from $\hat{T}^\dagger v_p = \sigma_p u_p$ and the minimum norm least squares solution to the ISP is obtained in the usual way via Eqs.(5.48b).

We note that this problem solution differs from that obtained using the BP integral equation in Section 5.3 in that it includes homogeneous and evanescent data. The back propagation operator employed in the BP integral equation destroys evanescent waves so that the solution includes on the homogeneous wave data.

5.18 Setup and solve the ISP for a source distributed over the surface of a sphere and where the data consists of Dirichlet data over the surfaces of two concentric spheres one interior and one exterior to the source sphere.

We express the radiated field from the surface source using Eq.(2.57)

$$U_+(\mathbf{r},\omega) = \int_{\partial\tau_0} dS_0\, [Q_s(\mathbf{r}_0,\omega)G_+(\mathbf{r}-\mathbf{r}_0,\omega) - Q_d(\mathbf{r}_0,\omega)\frac{\partial}{\partial n'}G_+(\mathbf{r}-\mathbf{r}_0,\omega)].$$

For this problem $\partial\tau_0$ is the surface of a sphere of radius a_0 and the radiated field is measured over two concentric spheres which having radii $a_+ > a_0$ and $a_- < a_0$. Using the multipole expansion of the Green function obtained in Section 3.4 we obtain the equations

$$\begin{aligned} U_+(a_+\hat{\mathbf{r}},\omega) &= -ika_0^2\sum_{l,m}[< Y_l^m, Q_s > j_l(ka_0)h_l^+(ka_+) \\ &\quad - < Y_l^m, Q_d > a_0 j_l'(ka_0)h_l^+(ka_+)]Y_l^m(\hat{\mathbf{r}}), \\ U_+(a_-\hat{\mathbf{r}},\omega) &= -ika_0^2\sum_{l,m}[< Y_l^m, Q_s > h_l^+(ka_0)j_l(ka_-) \\ &\quad - < Y_l^m, Q_d > a_0 h_l^{+\prime}(ka_0)j_l(ka_-)]Y_l^m(\hat{\mathbf{r}}), \end{aligned}$$

from which we obtain the data model

$$\begin{aligned} < Y_l^m, U_+(a_+\hat{\mathbf{r}},\omega) >&= -ika_0^2[< Y_l^m, Q_s > j_l(ka_0)h_l^+(ka_+) \\ &\quad - < Y_l^m, Q_d > a_0 j_l'(ka_0)h_l^+(ka_+)] \\ < Y_l^m, U_+(a_-\hat{\mathbf{r}},\omega) >&= -ika_0^2[< Y_l^m, Q_s > h_l^+(ka_0)j_l(ka_-) \\ &\quad - < Y_l^m, Q_d > a_0 h_l^{+\prime}(ka_0)j_l(ka_-)]. \end{aligned}$$

The above equations can be cast into matrix form as

$$\left[\begin{array}{cc} j_l(ka_0)h_l^+(ka_+) & -a_0 j_l'(ka_0)h_l^+(ka_+) \\ h_l^+(ka_0)j_l(ka_-) & -a_0 h_l^{+'}(ka_0)j_l(ka_-) \end{array}\right]\left[\begin{array}{c} < Y_l^m, Q_s > \\ < Y_l^m, Q_d > \end{array}\right]$$

$$= \frac{i}{ka_0^2}\left[\begin{array}{c} < Y_l^m, U_+(a_+\hat{\mathbf{r}},\omega) > \\ < Y_l^m, U_+(a_-\hat{\mathbf{r}},\omega) > \end{array}\right].$$

The solution to the problem is then obtained in by solving the above matrix equation and expressing the two source components as generalized Fourier series in terms of the spherical harmonics.

5.19 Compute the singular system given in Example 5.7.

The normal equations for the singular functions $\{v_p(x)\}$ are given in the example as

$$\hat{T}^\dagger\hat{T}v_p = \mathcal{M}_{a_0}\frac{1}{2\pi}\int_{-\infty}^{\infty} dK e^{iKx}\int_{-a_0}^{a_0} dx'\, e^{-iKx'}v_p(x') = \sigma_p^2 v_p(x).$$

Since the domain of the singular functions $v_p(x)$ is the finite interval $[-a_0, +a_0]$ they can be expanded into a Fourier series in the form

$$v_p(x) = \frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx}, \quad x \in [-a_0, +a_0]$$

which, when substituted into the normal equations yields

$$\mathcal{M}_{a_0}\frac{1}{2\pi}\int_{-\infty}^{\infty} dK e^{iKx}\int_{-a_0}^{a_0} dx'\, e^{-iKx'}\{\frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx'}\}$$

$$= \sigma_p^2\mathcal{M}_{a_0}\frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx}$$

The integral over K on the l.h.s. of the equation yields $2\pi\delta(x-x')$ so the equation simplifies to become

$$\mathcal{M}_{a_0}\int_{-a_0}^{a_0} dx'\,\delta(x'-x)\{\frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx'}\} = \mathcal{M}_{a_0}\frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx}$$

so the normal equations become

$$\mathcal{M}_{a_0}\frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx} = \sigma_p^2\mathcal{M}_{a_0}\frac{1}{2a_0}\sum_{n=-\infty}^{\infty} C_n(p)e^{i\frac{\pi}{a_0}nx}.$$

It follows from the normal equations that $\sigma_p = 1$ and that the complex exponentials $1/\sqrt{2a_0}\mathcal{M}_{a_0}\exp(i\pi/a_0 nx)$ form an O.N. basis for the singular functions $v_p(x)$. We can thus arbitrarily select this basis to be the singular functions under the assignment of $n = p$. The singular functions $u_p(K)$ are then found to be

$$u_p(K) = \hat{T}v_p = \frac{1}{\sqrt{2\pi}}\int_{-a_0}^{a_0} dx\, \overbrace{\frac{1}{\sqrt{2a_0}}e^{i\frac{\pi}{a_0}px}}^{v_p(x)}\, e^{-iKx} = \frac{\text{sinc } \frac{a_0}{\pi}(K - \frac{\pi}{a_0}p)}{\sqrt{\frac{\pi}{a_0}}},$$

where

$$\text{sinc } x = \frac{\sin \pi x}{\pi x}$$

is the "Sinc function."

5.20 Derive the singular system of the Slepian Polack problem given in Example 5.8.

We begin with Eq.(5.42) in Example 5.8:

$$\kappa_n S_{0,n}(c,\omega) = \int_{-1}^{1} d\xi\, S_{0,n}(c,\xi)e^{ic\omega\xi}, \quad -1 < \omega < +1,$$

where c is a constant parameter and

$$K_n = \frac{2i^n}{\sqrt{2\pi}} R_{on}^{(1)}(c,1),$$

with $R_{on}^{(1)}(c,1)$ being the Radial Prolate Spheroidal Wave Functions that are orthogonal over the intervals $-1 < \xi < +1$ and $-1 < \omega < +1$ with norm

$$||S_{0,n}||^2 = \int_{-1}^{+1} d\xi\, |S_{0,n}(c,\xi)|^2.$$

We now set $c = K_0 a_0$ and make the transformation $\xi = x/a_0$ to obtain

$$\kappa_n S_{0,n}(c,\omega) = \frac{1}{a_0}\int_{-a_0}^{a_0} dx\, S_{0,n}(c, x/a_0)e^{iK_0\omega x}, \quad -1 < \omega < +1.$$

On setting $\omega = K/K_0$ the above converts to

$$\kappa_n S_{0,n}(c, K/K_0) = \frac{1}{a_0}\int_{-a_0}^{a_0} dx\, S_{0,n}(c, x/a_0)e^{iKx}, \quad -K_0 < K < +K_0.$$

Two final steps are required. First we wish to normalize the two functions appearing on either side of the equation so that we have orthonormal singular functions. This is easily accomplished noting that

$$||S_{0,n}(c, K/K_0)||^2 = \int_{-K_0}^{K_0} dK\, |S_{0,n}(c, K/K_0)|^2 = K_0 \int_{-1}^{1} d\omega\, |S_{0,n}(c,\omega)|^2 = K_0 ||S_{0,n}||^2$$

with a similar result for $||S_{0,n}(c, x/a_0)||^2$. Thus we find that

$$\sqrt{K_0}\kappa_n \frac{S_{0,n}(c, K/K_0)}{\sqrt{K_0}||S_{0,n}||} = \frac{1}{\sqrt{a_0}}\int_{-a_0}^{a_0} dx\, \frac{S_{0,n}(c, x/a_0)}{\sqrt{a_0}||S_{0,n}||}e^{iKx}, \quad -K_0 < K < +K_0.$$

so that now

$$\frac{S_{0,n}(c, K/K_0)}{\sqrt{K_0}||S_{0,n}||}, \quad \frac{S_{0,n}(c, x/a_0)}{\sqrt{a_0}||S_{0,n}||}$$

are each orthonormal over their respective domains. The final step is simply to note that $\kappa_n = |\kappa_n| i^n$ to find that

$$\overbrace{\sqrt{K_0 a_0}|\kappa_n|}^{\sigma_n} \frac{i^{n/2} S_{0,n}(c, K/K_0)}{\sqrt{K_0}||S_{0,n}||} = \int_{-a_0}^{a_0} dx \, \frac{i^{-n/2} S_{0,n}(c, x/a_0)}{\sqrt{a_0}||S_{0,n}||} e^{iKx}, \quad -K_0 < K < +K_0,$$

which is the desired result.

5.21 By following identical steps as used in solving the full view 2D ISP problem for a cylindrical source region show that in the limited view problem that $\hat{T}$ and $\hat{T}^\dagger$ can be expressed in the form

$$\hat{T} = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (-i)^n e^{in\alpha_j} \int_0^{a_0} r dr \int_{-\pi}^{\pi} d\phi \, J_n(kr) e^{-in\phi},$$

$$\hat{T}^\dagger = \mathcal{M}_{\tau_0} \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} i^n e^{in\phi} J_n^*(kr) \sum_{j=1}^{N} e^{-in\alpha_j}.$$

This results from a straightforward modification of the procedure employed in the full view 2D ISP.

5.22 Using the expressions for $\hat{T}$ and $\hat{T}^\dagger$ found in the previous problem show that the 2D composite operators $\hat{T}^\dagger\hat{T}$ and $\hat{T}\hat{T}^\dagger$ are given by

$$\hat{T}^\dagger\hat{T} = \mathcal{M}_{\tau_0} \frac{1}{2\pi} \sum_{n,n'} r(n,n') e^{in\phi} J_n^*(kr) \int_0^{a_0} r' dr' \int_{-\pi}^{\pi} d\phi' \, J_{n'}(kr') e^{-in'\phi'}$$

$$\hat{T}\hat{T}^\dagger = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \nu_n^2(ka_0) e^{in\alpha_j} \sum_{j=1}^{N} e^{-in\alpha_{j'}}$$

where

$$r(n,n') = i^{(n-n')} \sum_{j=1}^{N} e^{-i(n-n')\alpha_j}.$$

This is a straightforward calculation using the definitions of $\hat{T}$ and $\hat{T}^\dagger$ given in the previous problem and the definitions of the inner products for the limited view problem.

5.23 Derive Eq.(5.88).

We have that

$$\chi_j(\mathbf{r}) = \frac{1}{\sqrt{2\pi}} \mathcal{M}_{\tau_0} e^{ik^* \mathbf{s}_j \cdot \mathbf{r}}$$

where τ_0 is the interior of a circle of radius a_0 centered at the origin. We then find that

$$< \chi_j, \chi_{j'} >_{\mathcal{H}_Q} = \int_{\tau_0} d^2 r \, \chi_j^*(\mathbf{r}) \chi_{j'}(\mathbf{r}) = \frac{1}{2\pi} \int_0^{a_0} r dr \int_0^{2\pi} d\phi \, e^{-ik\mathbf{s}_j \cdot \mathbf{r}} e^{ik^* \mathbf{s}_{j'} \cdot \mathbf{r}}.$$

We now make use of the multipole expansion of the plane wave given in

Eq.(5.76)

$$e^{-ikr\cos(\phi-\alpha)} = e^{-ik\mathbf{s}_j\cdot\mathbf{r}} = \sum_{n=-\infty}^{\infty} (-i)^n e^{in\alpha} J_n(kr) e^{-in\phi},$$

where $\mathbf{s}_j = (\cos\alpha, \sin\alpha)$. We then obtain

$$< \chi_j, \chi_{j'} >_{\mathcal{H}_Q} = \frac{1}{2\pi}\int_0^{a_0} rdr \int_0^{2\pi} d\phi \overbrace{\sum_{n=-\infty}^{\infty} (-i)^n e^{in\alpha_j} J_n(kr) e^{-in\phi}}^{e^{-ik\mathbf{s}_j\cdot\mathbf{r}}}$$

$$\times \overbrace{\sum_{n=-\infty}^{\infty} (i)^{n'} e^{in'\alpha_{j'}} J^*_{n'}(kr) e^{in'\phi}}^{e^{ik^*\mathbf{s}_{j'}\cdot\mathbf{r}}} = \int_0^{a_0} rdr \sum_{n=-\infty}^{\infty} e^{in(\alpha_j-\alpha_{j'})} |J_n(kr)|^2$$

$$= \sum_{n=-\infty}^{\infty} \nu_n^2(ka_0) e^{in(\alpha_j-\alpha_{j'})}$$

where

$$\nu_n^2(ka_0) = \int_0^{a_0} rdr |J_n(kr)|^2.$$