1 Sterile Resources

In this chapter, we introduce the simple conception of a scarce resource, locally owned and globally traded. It is the classic Quadrant 1 resource, valuable and scarce. Production amounts to making it available for sale into an economic market in which its scarcity affects its price. Accordingly, price is endogenous to the resource system. Production is the same as consumption, which is tantamount to destruction: irreversible conversion to other chemical forms with no recycling. The owner's basic decision is how fast to produce.

That the resource is finite is a first principle. The fact that some portion of the resource is undiscovered at any point in time does not change its finiteness. What does change, over time, is the improving state of knowledge about how much of the resource there is. Decisions about how fast to produce are always reached within an environment of imperfect knowledge and speculation about future discoveries. There is a need to make decisions in this uncertain environment and a need to adjust continually as new information becomes available. Exploration reduces, but does not eliminate, uncertainty.

This case is extreme in its simplicity. It is elaborated below; the example of petroleum is used throughout. Many critical concepts of resource economics are introduced and carried forward into subsequent chapters.

1.1 COSTLESS PRODUCTION OF A STERILE RESOURCE

1.1.1 Base Case

This is the simplest case of exhaustion of a finite resource. We will use the terminology

$S(t)$ = amount of the resource remaining to be produced and sold
$X(t)$ = production rate
$P(t)$ = market price per unit of production

It is assumed that the resource is owned unambiguously, that it costs nothing to produce it, and that S, X, and P are known with perfect certainty. Three relations govern:

Mass balance:

$$\frac{dS}{dt} = -X \tag{1.1}$$

Price-sensitive demand:

$$X = \frac{a}{P^{\beta}} \tag{1.2}$$

Optimal economic decision making:

$$\frac{dP}{dt} = rP \tag{1.3}$$

The decision equation is reached by considering a tradeoff between a unit of resource produced and sold today versus waiting and doing the same later. If P grows faster than r, the interest rate available for investment of money, then conserving the resource for later sale is profitable and producers will do so – the value of the resource grows faster than money. If, on the other hand, P grows slower than r, then conservation is a bad investment and selling now is preferable – money grows faster than the value of the resource, and a resource owner would prefer to produce now and invest the proceeds at rate r. The price equation expresses the point of indifference between these two options; it would be realized in a situation of competition among many producers. (This is Hotelling's Rule [42]. There will be more to say about this later.)

The solution for P is

$$P = P_0 e^{rt} \tag{1.4}$$

and thus we have the production rate X, from the demand function

$$X = \frac{a}{{P_0}^{\beta}} e^{-\beta rt} \tag{1.5}$$

and the initial production rate is

$$X_0 = \frac{a}{{P_0}^{\beta}} \tag{1.6}$$

Since $dS/dt = -X$, we have

$$S(t) = S_0 - \int_0^t X dt = S_0 - \frac{a}{{P_0}^{\beta} \beta r} \left[1 - e^{-\beta rt}\right] \tag{1.7}$$

We require two conditions to close the system: S_0, the present amount of the resource, and P_0, the initial price.

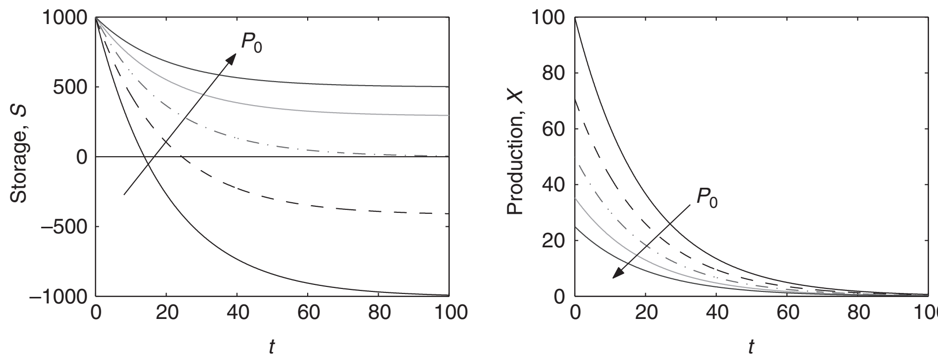

Figure 1.1. Five different depletion histories, identical except for initial price. P_0 increases by factors of 2 in the direction of the arrows.

S_0 is presumed known; P_0 is not. If P_0 is set too high, the demand will be stunted and the resource will go unutilized. If P_0 is set too low, the demand will be too large and the resource will be depleted prematurely, leaving our mathematics of decision making invalid (Figure 1.1). The system is closed by invoking the Terminal Condition (TC): complete resource exhaustion as time goes to infinity:

$$S \to 0 \quad \text{as } t \to \infty \tag{1.8}$$

Thus,

$$S_0 = \frac{a}{{P_0}^{\beta} \beta r} \tag{1.9}$$

The initial price is thus

$$P_0 = \left[\frac{a}{S_0 \beta r} \right]^{\frac{1}{\beta}} \tag{1.10}$$

and the initial production rate is

$$X_0 = \frac{a}{{P_0}^{\beta}} = \beta r S_0 \tag{1.11}$$

If P_0 is too high (X_0 too low), then S is never exhausted. If P_0 is too low (X_0 too high), then the resource is exhausted prematurely. In either case, production would be adjusted to satisfy the TC (Figure 1.2).

Rent is the integrated present worth of all net revenues:

$$R = \int_0^{\infty} e^{-rt} X(t) P(t) dt \tag{1.12}$$

Since $P = P_0 e^{rt}$, we have

$$R = P_0 \int_0^{\infty} X(t) dt = P_0 S_0 \tag{1.13}$$

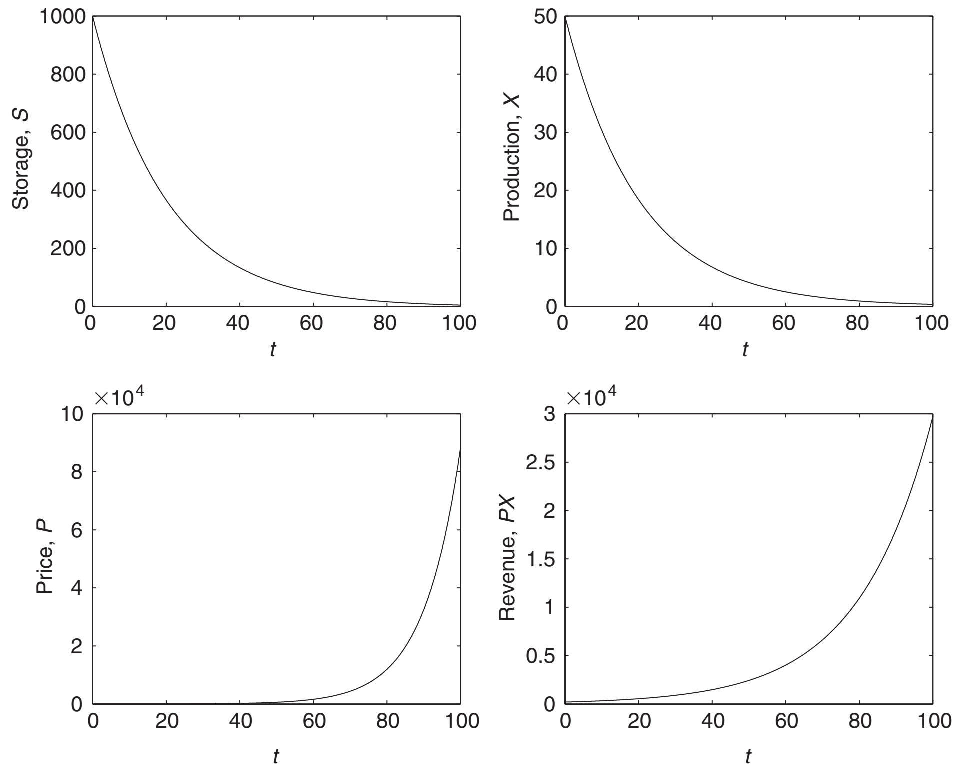

Figure 1.2. Exhaustion history that matches the TC. Demand is $X = a/P^{\beta}$, with $(a, \beta) = (100, 0.5)$.

For this simple case, the present worth of all future production is equal to today's price times today's total supply.

Program **Oil1** illustrates the exhaustion history under these conditions. The 2 ODE's are integrated forward in time with an explicit (Euler) forward-difference method. The initial price P_0 needs to be adjusted manually to satisfy the TC. Because numerical integration is not perfect, the relations developed above using the calculus correspond only approximately to the **Oil1** simulation; the discrepancies vanish as the numerical timestep Δt becomes infinitesimally small.

1.1.2 Finite Demand

Next, add a ceiling price $\overline{P}$, which limits demand (Figure 1.3). Above this price, customers purchase a substitute product. The previous solution, in which P rises without bound, is invalid. Equations 1.1–1.3 still govern, but the TC needs to be altered. The correct TC in this case is

$$S \to 0 \quad \text{as } P \to \overline{P} \tag{1.14}$$

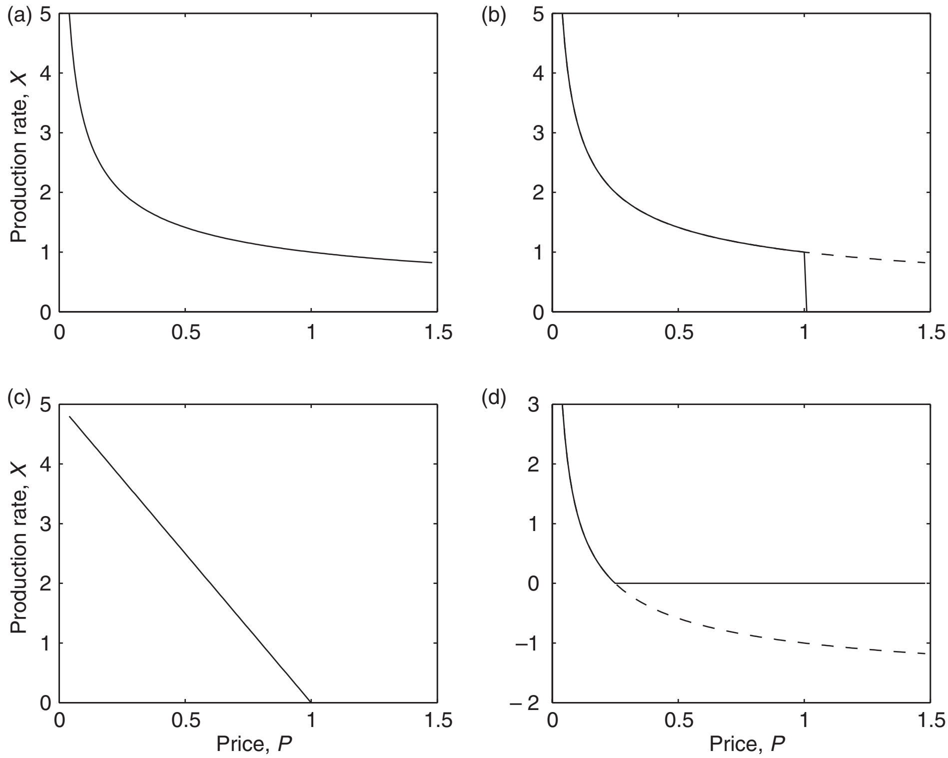

Figure 1.3. Four different demand functions $X(P)$. (a) base case $X_a = P^{-\beta}$; (b) base case with $P \leq \overline{P} = 1$; (c) linear demand $X_c = X_0(1 - P)$; (d) base case shifted, $X_d + 2 = P^{-\beta}$. The dash lines indicate the continuation of the base case curve beyond $X \leq 0$. Cases b, c, and d have finite demand.

which leads to exhaustion at finite time T. From Equations 1.1 and 1.3, we have

$$\overline{P} = P_0 e^{rT} \tag{1.15}$$

$$S_0 = \frac{a}{{P_0}^{\beta} \beta r} \left[1 - e^{-\beta r T} \right] \tag{1.16}$$

from which we obtain the final results

$$P_0 = \left[\frac{a}{\beta r S_0 + \frac{a}{\overline{P}^{\beta}}} \right]^{\frac{1}{\beta}} \tag{1.17}$$

$$X_0 = \beta r S_0 + \frac{a}{\overline{P}^{\beta}} \tag{1.18}$$

$$T = \frac{1}{\beta r} \ln \left[\frac{\beta r S_0 \overline{P}^{\beta}}{a} + 1 \right] \tag{1.19}$$

These relations reduce to the previous ones as $\overline{P} \to \infty$.

The above relations must govern at any time during the extraction history, else the trajectory would not be optimal and it would be altered, contrary to hypothesis.

Hence, we may drop the subscripts "0" and it is always true that

$$P = \left[\frac{a}{\beta r S + \frac{a}{\overline{P}^{\beta}}}\right]^{\frac{1}{\beta}} \tag{1.20}$$

$$X = \beta r S + \frac{a}{\overline{P}^{\beta}} \tag{1.21}$$

$$T = \frac{1}{\beta r} \ln\left[\frac{\beta r S \overline{P}^{\beta}}{a} + 1\right] \tag{1.22}$$

with S the remaining unexploited resource at any time t and P, X, T the contemporary price, production rate, and remaining time to exhaustion. Equations 1.20, 1.21, and 1.22 completely characterize the solution to Equations 1.1, 1.2, and 1.3, subject to the TC of exhaustion as price reaches the ceiling $\overline{P}$.

Program **Oil1a** illustrates these relationships. Figure 1.4 displays simulation results for finite $\overline{P}$ and T, achieved with the decision rule $X = X(S)$ (Equation 1.21).

Rent is, as above, the integrated present worth of all future production:

$$R = \int_0^{\infty} PXe^{-rt}dt = P_0 \int_0^{T} Xdt = P_0 S_0 \tag{1.23}$$

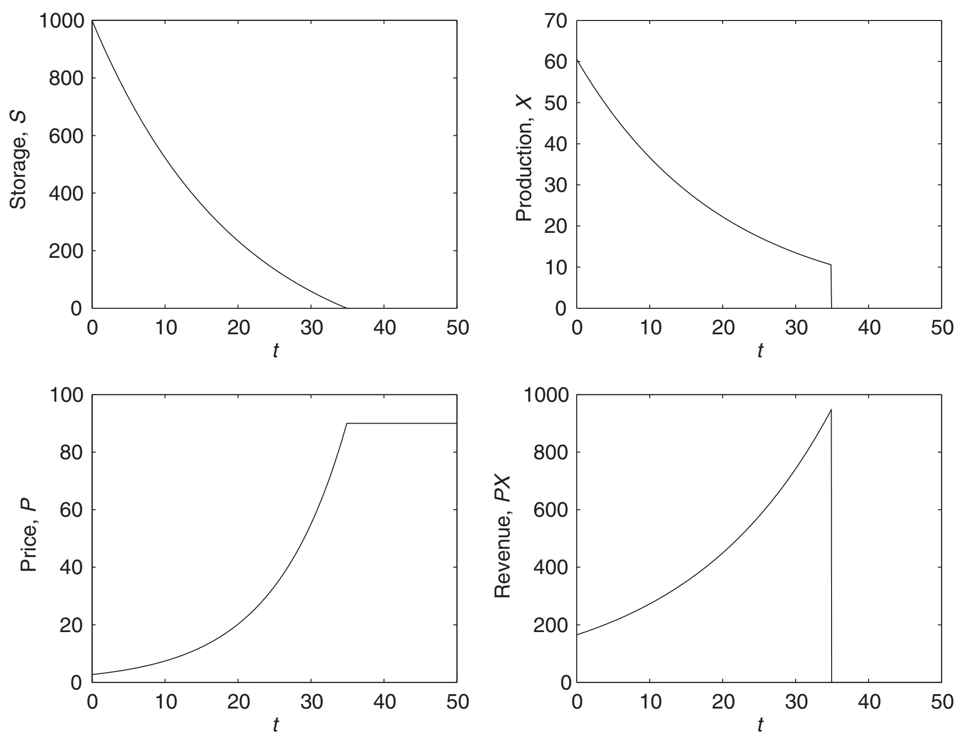

Figure 1.4. Extraction history with finite demand, $X_a = 100P^{-\beta}$ with $\overline{P} = 90$ (case (b) in Figure 1.3). This leads to exhaustion at finite time as shown.

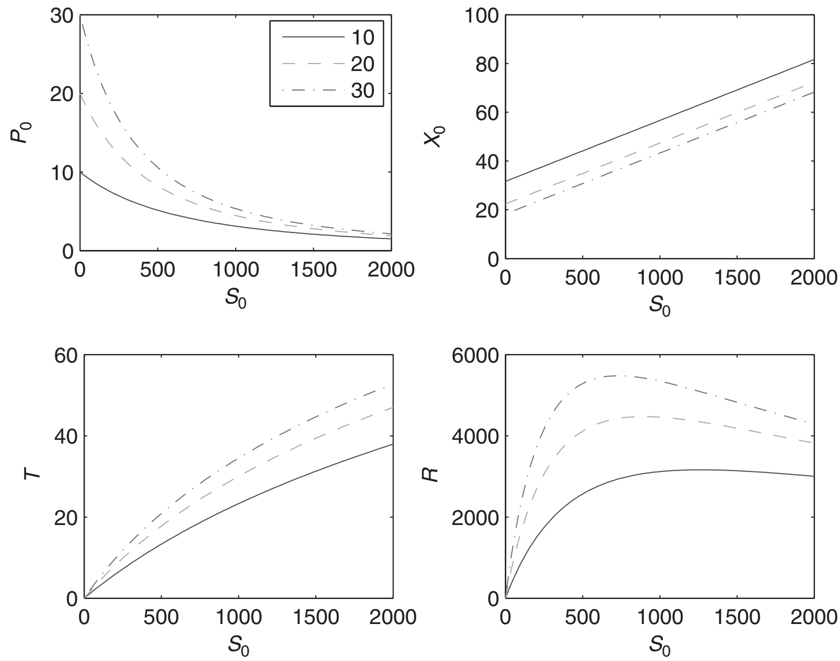

Figure 1.5. P, X, and T as funtions of the total reserve S at any time (Equations 1.20–1.22). Demand: $X = aP^{-\beta}$; $a = 100$; $\beta = 0.5$; $r = 0.05$; $\overline{P} = (10, 20, 30)$ as indicated by the linestyles.

This result is unchanged by the imposition of a ceiling price and the resultant finite T. Since P_0 decreases as $\overline{P}$ decreases, ceiling price has the effect of diminishing overall rent, in accord with intuition.

Equations 1.20–1.22 give P, X, and T as funtions of the total reserve S at any time, assuming complete exhaustion, $X = aP^{-\beta}$ and $P \leq \overline{P}$. Figure 1.5 plots these for three different values of $\overline{P}$. Rent peaks and begins to decline with S at high abundance in this scenario.

Consumers' Surplus

Consumption at price P indicates a willingness to pay at least P – that is, the value of the consumption $V \geq P$. The consumer obtains a surplus equal to the difference $V - P$. Figure 1.6 illustrates a demand curve made up of individual uses ΔX, each with its own value V. If price is set at P, those users with higher value will purchase, and those with lower value will not. The consumers' surplus (CS) is their accumulation:

$$\mathrm{CS} = \sum (V - P)\Delta X \tag{1.24}$$

for all increments where $(V - P) > 0$. In the limit,

$$\mathrm{CS}(P) = \int_0^{X(P)} (V - P)dx \tag{1.25}$$

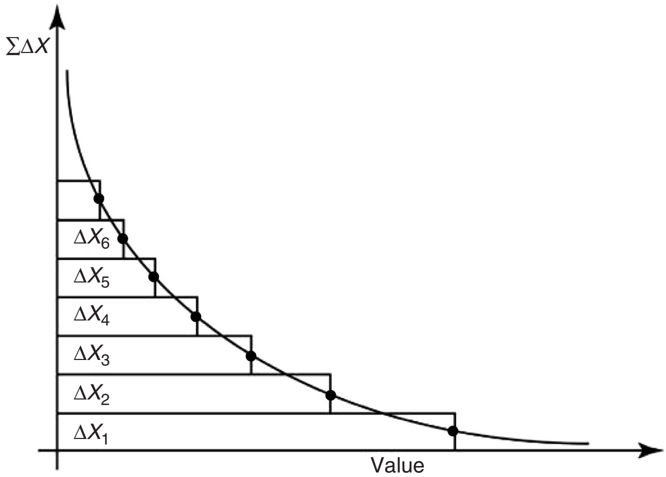

Figure 1.6. Demand curve built up of individual increments $\mathbf{\Delta}X$, ordered by decreasing individual value, plotted on the horizontal axis.

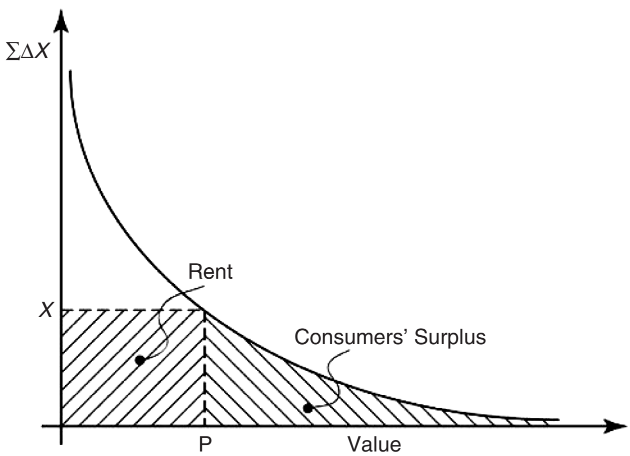

Figure 1.7. Demand curve as in Figure 1.6, adding the actual price P. The area to the right of the price line is the consumers' surplus; that to the left is the rent transferred to the seller.

Clearly, CS is a function of P. Graphically, this is illustrated in Figure 1.7 as the area "under the demand curve, to the right of P". The amount PX shown graphically is the total rent transferred to the seller. So transactions at P generate consumers' surplus as well as rent.

Graphically, it is easy to see that an equivalent integral is

$$\mathrm{CS}(P) = \int_{P}^{\overline{P}} X dV \tag{1.26}$$

An analogous concept of producers' surplus (PS) divides the rent into production cost plus surplus: net rent. When production is costly, the producers' surplus is the net rent.

Consumers' surplus is a static concept; time is fixed in its construction. Clearly in a depletion context, as P rises over time, CS will decrease: $\mathrm{CS}(P) = \mathrm{CS}(P(t))$. Suppose we have the base case demand $X = aP^{-\beta}$, with a ceiling price $\overline{P}$. Then it is easy to verify that CS may be integrated to obtain

$$\mathrm{CS}(P) = \frac{a}{1-\beta}\left[\overline{P}^{1-\beta} - P^{1-\beta}\right] \tag{1.27}$$

The present worth of the consumers' surplus is explored in Problem 34.

1.1.3 Linear Demand

As an extension of the preceding, consider the alternative demand function

$$P = \overline{P} - bX \tag{1.28}$$

We still have the requirement of exponential price growth

$$P = P_0 e^{rt} \tag{1.29}$$

and thus

$$X = \frac{\overline{P} - P_0 e^{rt}}{b} \tag{1.30}$$

Integrating $dS/dt = -X$ gives

$$S(t) = S_0 + \frac{1}{b}\left[\frac{P_0}{r}(e^{rt} - 1) - \overline{P}t\right] \tag{1.31}$$

The Terminal Condition is

$$S(T) \to 0 \quad \text{as } P(T) \to \overline{P} \tag{1.32}$$

and therefore

$$P_0 e^{rT} = \overline{P} \tag{1.33}$$

$$rT = \ln\left(\frac{\overline{P}}{P_0}\right) \tag{1.34}$$

and

$$S_0 = -\frac{1}{b}\left[\frac{P_0}{r}(e^{rT} - 1) - \overline{P}T\right] \tag{1.35}$$

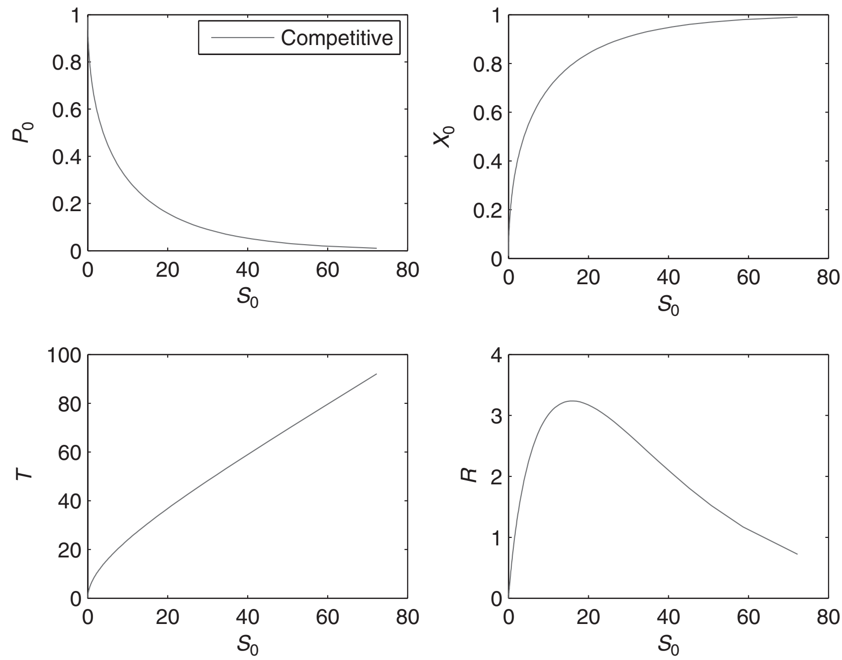

Figure 1.8. Optimal extraction relations for the linear demand case: $r = 0.05$; $P = 1-X$; competitive case.

A little rearrangement leads to

$$T = \frac{1}{r}\ln\left(\frac{\overline{P}}{P_0}\right) \tag{1.36}$$

$$S_0 = \frac{\overline{P}}{br}\left[\frac{P_0}{\overline{P}} - 1 + \ln\left(\frac{\overline{P}}{P_0}\right)\right] \tag{1.37}$$

$$X_0 = \frac{\overline{P} - P_0}{b} \tag{1.38}$$

These last three equations relate S, X, and T to P_0; they comprise implicit functions $X(S)$, $T(S)$, and $P_0(S)$. There are no simple closed-form solutions, but $X(S)$, $T(S)$, and $P_0(S)$ can be evaluated numerically as in **Oil6M+C**; the plots shown therein are reproduced in Figure 1.8. They characterize this system under linear demand, as did the closed-form Equations 1.20–1.22 for the earlier demand function.

As before, rent $R = P_0S_0$. It is interesting to note here that as S increases, P_0 ultimately decreases toward the limiting case $P_0 \to 0$ as $S \to \infty$. As a result, the R initially grows with S but ultimately peaks and then decreases with increasing S. The point of maximum rent is found by setting $dR/dt = 0$; the result is

$$\ln\left(\frac{\overline{P}}{P_0}\right) = 2\left(1 - \frac{P_0}{\overline{P}}\right) \tag{1.39}$$

There is one root of this equation in the range $\frac{P_0}{\overline{P}} < 1$: $\frac{P_0}{\overline{P}} \sim 0.2$. The peak value of S is

$$S^* = \frac{\overline{P}}{br}\left[1 - \frac{P_0}{\overline{P}}\right] \sim \frac{\overline{P}}{br}\,[.8] \tag{1.40}$$

More resource beyond this limit results in less overall rent. Consumers' surplus for this case is explored in Problem 35.

The program **Oil6M+C** provides a simulation of X, P, S, and R versus time.

1.1.4 Expanding Demand

Going back to the base case of an unbounded demand curve, Equation 1.2, consider the case $a = a(t)$, an exogenous trend toward higher demand. This represents the same general price sensitivity, but growth in a represents an outward shift in demand:

$$X = \frac{a(t)}{P^{\beta}} \tag{1.41}$$

This is illustrated in Figure 1.9. For example, a simple doubling of the number of resource-consuming machines would be expected to double the demand for the resource, all other things (including P) being constant. Increasing the intrinsic use-per-machine would have the same effect. Growth in a would reflect the product of these two effects, extrinsic and intrinsic resource use or, equivalently, the resource intensity of individual use and the number of uses.

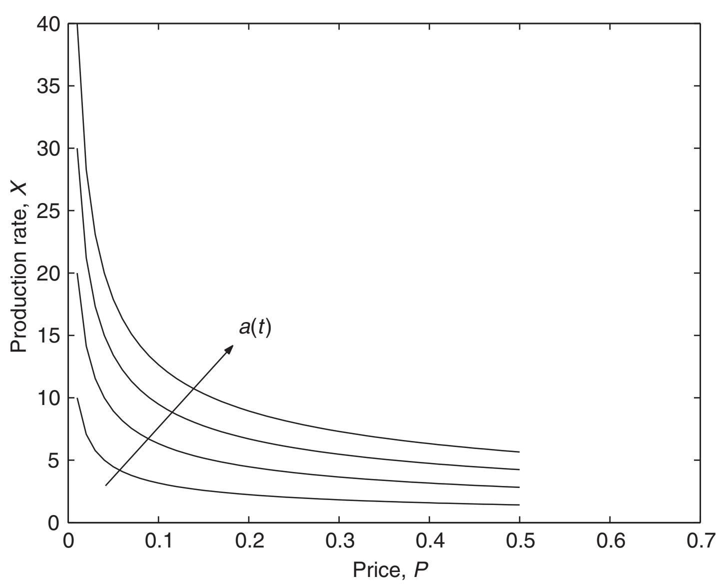

Figure 1.9. Expanding demand curve, Equation 1.41, with $\beta = 0.5$.

As before we have

$$\frac{dS}{dt} = -X \tag{1.42}$$

$$\frac{dP}{dt} = rP \tag{1.43}$$

Integrating the P equation gives the price and production history:

$$P(t) = P_0 e^{rt} \tag{1.44}$$

$$X(t) = \frac{a(t)}{P_0^{\beta}} e^{-\beta rt} \tag{1.45}$$

Exponential Demand Growth

For this case,

$$a(t) = a_0 e^{gt} \tag{1.46}$$

$$X(t) = \frac{a_0}{P_0^{\beta}} e^{(g-\beta r)t} = X_0 e^{(g-\beta r)t} \tag{1.47}$$

With $g < \beta r$, we have an unbounded future of finite production and the TC of resource exhaustion at $T \to \infty$. Integrating gives us

$$S_0 = \frac{a_0}{P_0^{\beta}} \frac{1}{\beta r - g} = \frac{X_0}{\beta r - g} \tag{1.48}$$

and finally,

$$X_0 = (\beta r - g)S_0 \tag{1.49}$$

$$P_0 = \left[\frac{a_0}{(\beta r - g)S_0}\right]^{\frac{1}{\beta}} \tag{1.50}$$

By comparison with the no-growth case, X is initially smaller and decays at the slower rate $e^{(g-\beta r)t}$. Price is initially higher, and its growth rate is unchanged, e^{rt}. Rent $= P_0 S_0$ is higher, proportional to P_0. As the demand growth g approaches βr, pumping becomes infinitesimally small and nearly constant over time; price responds inversely and grows without bound. For $g > \beta r$, the resource is never produced; the owner conserves in the face of rapidly escalating demand.

The case of declining demand (contraction) is the reverse: $g < 0$. In this case, X is initially larger than the constant-demand ($g = 0$) reference case, and its decay rate is faster.

Linear Demand Growth

Next we examine the linear growth case

$$a(t) = a_o + a_1 t \tag{1.51}$$

with exponential price growth

$$P = P_0 e^{rt} \tag{1.52}$$

Production is again according to the demand curve

$$X = \left[\frac{a_o + a_1 t}{P_0^\beta} \right] e^{-\beta rt} \tag{1.53}$$

At long time, $X \to 0$; but early growth in X is possible. Differentiating,

$$\frac{dX}{dt} = \frac{1}{P_0^\beta} \left[a_1 - \beta r (a_0 + a_1 t) \right] e^{-\beta rt} \tag{1.54}$$

This is positive at $t = 0$ when

$$a_1 > a_0 \beta r \tag{1.55}$$

and if that is the case, there is a maximum in X at t^*:

$$t^* = \frac{(a_1 - a_0 \beta r)}{a_1 \beta r} \tag{1.56}$$

For $t > t^*$, production declines toward zero at $T \to \infty$. For exhaustion at that point, we require

$$S_0 = \frac{a_0}{P_0^\beta \beta r} \left[1 + \frac{a_1}{a_0 \beta r} \right] = \frac{X_0}{\beta r} \left[1 + \frac{a_1}{a_0 \beta r} \right] \tag{1.57}$$

Equivalently,

$$X_0 = \frac{\beta r S_0}{\left[1 + \frac{a_1}{a_0 \beta r} \right]} \tag{1.58}$$

Compared with the no-growth case ($a_1 = 0$), growth introduces a smaller initial production that peaks early, then decays to zero, asymptotically as $te^{-\beta rt}$. (Small demand growth may not produce the peak.) Initial price is higher, growing at the rate e^{rt}.

$$P_0 = \left[\frac{a_0 \left[1 + \frac{a_1}{a_0 \beta r} \right]}{\beta r S_0} \right]^{\frac{1}{\beta}} \tag{1.59}$$

Rent is accordingly higher, $P_0 S_0$.

Figure 1.10 illustrates the production history in this case, for peaking parameters $a_1 > a_0 \beta r$.

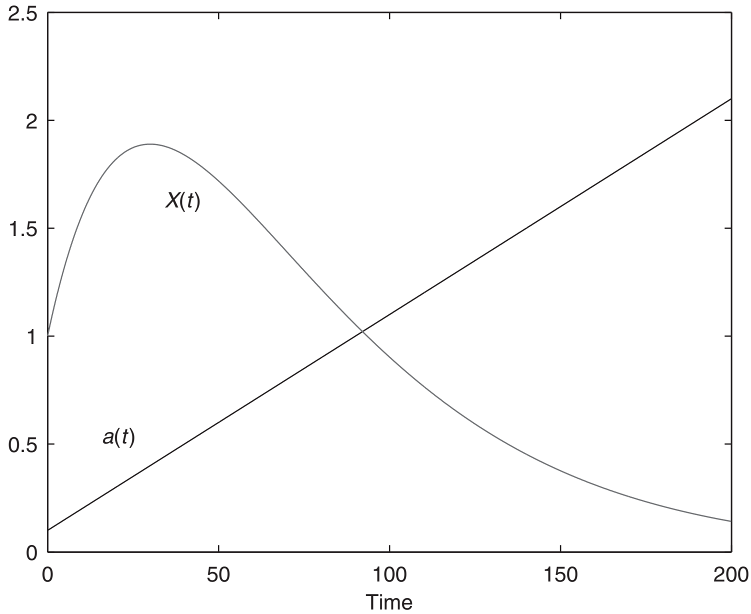

Figure 1.10. Time history of production with linear growth in demand. Parameters: $a_0 = 0.1$; $a_1 = 0.01$; $X_0 = 1.$; $\beta = 0.5$; $r = .05$. Production peaks at $t^* = 30$.

Saturating Demand Growth

Consider the demand function in Equation 1.41 with

$$a(t) = a_0\left(1 - e^{-st}\right) \tag{1.60}$$

Demand is initially zero and rises toward the saturation level a_0 at the exponential rate s. Again with $P = P_0 e^{rt}$, we have the production rate

$$X = \frac{a_0}{P_0^{\beta}}\left(1 - e^{-st}\right)e^{-\beta rt} \tag{1.61}$$

and the reserve history

$$S(t) = S_0 - \frac{a_0}{P_0^{\beta}}\left[\frac{1 - e^{-\beta rt}}{\beta r} - \frac{1 - e^{-(s+\beta r)t}}{s + \beta r}\right] \tag{1.62}$$

Applying the TC of complete exhausition as $t \to \infty$ gives

$$S_0 = \frac{a_0}{\beta r P_0^{\beta}}\left[\frac{s}{s + \beta r}\right] \tag{1.63}$$

and Equations 1.61 and 1.62 for production and remaining stock become

$$X(t) = S_0\left[\frac{s + \beta r}{s}\right]\left[1 - e^{-st}\right]e^{-\beta rt} \tag{1.64}$$

$$S(t) = \beta r S_0 \left[\frac{s+\beta r}{s}\right]\left[1 - \left(\frac{\beta r}{s+\beta r}\right)e^{-st}\right]e^{-\beta r t} \tag{1.65}$$

Starting from zero, demand growth produces early growth in production X, a peak at t^*, and a decline at the rate βr once demand has reached saturation at a_0. The peak production occurs at

$$t^* = \frac{1}{s}\ln\left[\frac{s+\beta r}{\beta r}\right] \tag{1.66}$$

In the limit of extremely fast saturation (very large $s \to \infty$), we recover the base case: instantaneous onset of production, followed by monotonic decay ($t^* = 0$) at the rate $-\beta r$. For intermediate saturation rates ($s > \beta r$), production at long time approaches the limiting relation $X(t) = \beta r S(t)$, both decaying at the asymptotic rate βr. Essentially, at saturated demand we recover asymptotically the initial base case of constant demand.

When demand development is very slow, $s < \beta r$, then its development dominates production; but the peak production time t^* remains early and insensitive to s. Expanding t^* in a Taylor series gives

$$t^* \to \frac{1}{\beta r} \tag{1.67}$$

This limit recovers the comparable result for linear demand growth, Equation 1.56, as it should.

Figure 1.11 illustrates the production history for demand saturating exponentially at a finite rate. Figure 1.12 illustrates the effect of s with all other parameters constant, including S_0.

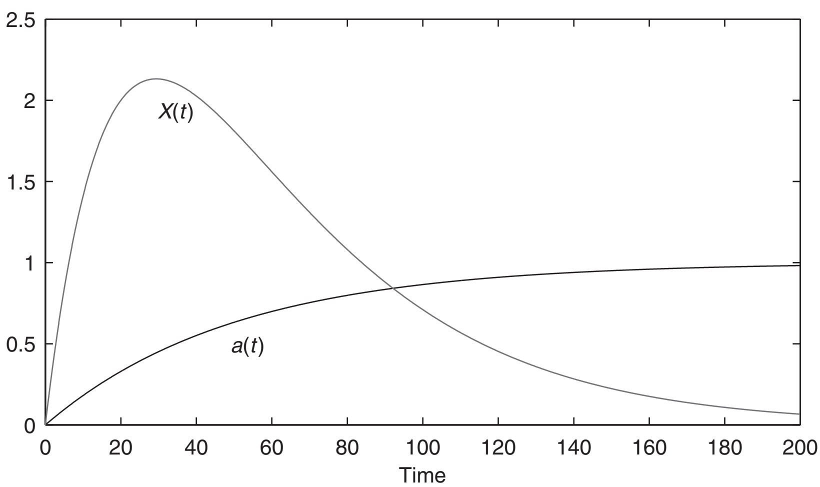

Figure 1.11. Time history of production with exponentially saturating growth in demand. Parameters: $a_0 = 1$; $s = 0.02$; $\beta = 0.5$; $r = 0.05$. Peak production at $t^* = 29.4$.

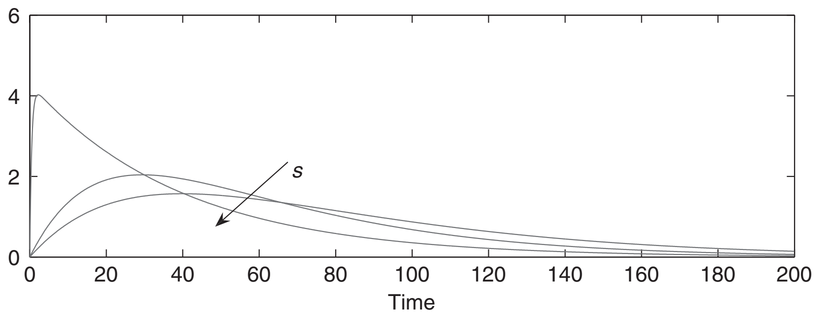

Figure 1.12. Time history of production X for slow, intermediate, and fast growth saturation. Parameters: $a_0 = 1$; $s = 0.0002$, 0.02, and 2.0; $\beta = 0.5$; $r = 0.05$; initial supply fixed at $S_0 = 170$.

Note that this analysis requires knowledge from the beginning that demand will ultimately reach a_0. That knowledge is embedded in the decision making.

Endogenous Demand Growth

It is interesting to speculate on what drives demand growth. We imagine a resource, like oil, which requires capitalization in combustion (fuels) and/or chemical manufacturing (plastics) in order to be used; and these secondary products have their own consumer markets. In these cases, capital formation has an intrinsic lifetime of order 5–20 years (decay rate $\rho = 0.05 - 0.20$). So we speculate that the growth in a would be related to capital formation in the secondary industry:

$$\frac{da}{dt} + \rho a = f(S, X, S/X, aS/X, P, \pi) \tag{1.68}$$

Increases in a could be driven by perception of large reserves S, large production X, large increases in reserves dS/dt, the exhaustion timescale S/X, changes in P as a metric of scarcity, and/or the rate of rent formation π. Clearly, the dynamics of the secondary industry, and their linkage to the primary resource industry, become critical here. This is beyond our scope.

1.2 DECISION RULES

The decision rule used so far has been

$$\frac{dP}{dt} = rP \tag{1.69}$$

with r the interest rate applicable to investment of the proceeds of resource sale. This has been applicable to costless production in a competitive market. Three complications need to be explored: taxation, the finite cost of production, and the possibility of monopoly production.

1.2.1 Taxation

If we introduce a tax θ, we have a wedge between the price seen by the consumer P and the net rent $P - \theta$ accruing to the producer. Adapting the principle of intertemporal maximization of net rent (as used above), we have the decision rule

$$\frac{d(P-\theta)}{dt} = r(P-\theta) \tag{1.70}$$

and

$$(P-\theta) = (P_0 - \theta)e^{rt} \tag{1.71}$$

Two cases are interesting. First, consider a *constant tax* θ, which might be conceived as a *consumption tax*. In this case, we have

$$P(t) = \theta + (P_0 - \theta)e^{rt} \tag{1.72}$$

With θ fixed, it loses relevance as P grows. Consumption will be according to the demand function – for example,

$$X(t) = \frac{a}{P^\beta} = a\left[\theta + (P_0 - \theta)e^{rt}\right]^{-\beta} \tag{1.73}$$

All other things remaining the same, initial price P_0 will rise with θ, but it will not completely offset it; $P_0 - \theta$ will be less, so that initial net rent is less because of θ. This effect will be less pronounced as production proceeds; ultimately, growth in P overwhelms constant θ.

For example, for $[S_0, r, \beta, a, \Delta t] = [1000, 0.1, 0.5, 100, 0.5]$ and exhaustion at $t \to \infty$: for $\theta = 0$, we find $[P_0, X_0] = [4.4, 47.67]$. The tax case $\theta = 2$ gives $[P_0, X_0] = [5.75, 41.70]$. The early production is delayed by the imposition of the tax; later production is affected less. Program **Oil1-tax** simulates this.

A second case is a *proportional tax*, $\theta = \tau P$. This amounts to a tax on *rent*, not on consumption per se. Following (1.70), we have

$$\frac{dP(1-\tau)}{dt} = rP(1-\tau) \tag{1.74}$$

With τ constant, the factor $(1-\tau)$ cancels, leaving $P = P_0e^{rt}$ and the whole production history unaffected by τ. The only effect is to redirect a portion of the rent, τR, to the public treasury, leaving the balance $(1-\tau)R$ for the original resource owner.

1.2.2 Costly Production

Again we generalize the concept of rent here as the present worth of resource sales minus expenses:

$$PX - CX \tag{1.75}$$

where P and X are the price and extraction rate as before and C is the unit cost of production. We will consider the case where C depends on the remaining stock of resource, $C = C(S)$. This would reflect the case where production becomes more costly as extraction proceeds, with C increasing as S decreases. The contributions to rent R from two adjacent production periods separated in time by Δt are

$$R = \left\{ [P_1 - C(S_1)]\, X_1 + \frac{1}{1 + r\Delta t}\, [P_2 - C(S_2)]\, X_2 \right\} \tag{1.76}$$

We wish to discover the effect of adjusting a hypothetical production pattern by moving a quantity Δ of production forward in time, at the expense of later production, leaving all other things the same. We introduce the perturbations

$$\begin{aligned} \Delta X_1 &= \Delta \\ \Delta X_2 &= -\Delta \\ \Delta S_1 &= 0 \\ \Delta S_2 &= -\Delta \cdot \Delta t \end{aligned} \tag{1.77}$$

Perturbing R with the changes ΔX and ΔS, we obtain

$$\Delta R = \left\{ [P_1 - C(S_1)]\, \Delta X_1 - X_1 \frac{dC}{dS} \Delta S_1 + \frac{1}{1 + r\Delta t} \left[[P_2 - C(S_2)]\, \Delta X_2 - X_2 \frac{dC}{dS} \Delta S_2 \right] \right\} \tag{1.78}$$

With the perturbations above, we obtain

$$\Delta R = \left\{ [P_1 - C(S_1)]\, \Delta - \frac{1}{1 + r\Delta t} \left[[P_2 - C(S_2)]\, \Delta - X_2 \frac{dC}{dS} \Delta \cdot \Delta t \right] \right\} \tag{1.79}$$

The point of indifference to such an adjustment is $\Delta R = 0$:

$$[P_1 - C(S_1)] = \frac{1}{1 + r\Delta t} \left[[P_2 - C(S_2)] - X_2 \frac{dC}{dS} \Delta t \right] \tag{1.80}$$

Rearranging this, we obtain

$$[P_2 - C(S_2)] = [1 + r\Delta t]\, [P_1 - C(S_1)] + X_2 \frac{dC}{dS} \Delta t \tag{1.81}$$

and taking the limit as $\Delta t \to 0$, we have

$$\frac{d(P - C)}{dt} = r(P - C) + X \frac{dC}{dS} \tag{1.82}$$

Here we find two effects:

1. The rent rate $(P - C)$, not price alone, must generally increase at the rate r with time.

2. There is an additional effect of variable costs, XdC/dS, called the "stock effect." With $X > 0$ and $dC/dS < 0$ (C increasing as resource exhaustion proceeds), this term will be negative. Thus, the rate of rise in $(P - C)$ will be slowed by the stock effect.

Finally, we can rearrange this equation to a more convenient form by noting that $-X = dS/dt$ and therefore by the chain rule,

$$X\frac{dC}{dS} = -\frac{dC}{dS}\frac{dS}{dt} = -\frac{dC}{dt} \tag{1.83}$$

Thus, we have the simpler form of the **decision rule under costly production:**

$$\frac{dP}{dt} = r(P - C) \tag{1.84}$$

Again it is clear that as C becomes significant, it slows the exponential growth that would characterize P in the costless production case. As the resource dwindles, we expect growth in C to bring rent toward zero, with growth in P slowing and P finally stabilizing at $\overline{P}$. The TC becomes $R \to 0$, $P \to \overline{P}$, $C \to \overline{P}$, and $S \to \underline{S}$, where $\underline{S}$ is the small residual left in storage that is not ecomonical to extract:

$$C(\underline{S}) = \overline{P} \tag{1.85}$$

Naturally, these relations reduce to the costless case studied earlier when $C = 0$.

Program **Oil2** simulates the simple case of constant cost – for example, a tax on production. In this case, we have the simple rule

$$\frac{d(P - C)}{dt} = r(P - C) \tag{1.86}$$

that is, there is no stock effect. Demand remains sensitive only to P alone, so the solution is not simply to shift P from the costless case. **Oil4** simulates the stock effect with cost function

$$C(S) = \gamma/S^{\delta} \tag{1.87}$$

1.2.3 Monopoly versus Competitive Production

Next we turn to the monopoly case, wherein a single producer supplies the whole market. Under this condition, production decisions directly affect price; while under competetive conditions, the effect is only via the aggregate of several producers' independent decisions. The effect is on the revenue from sales PX. Perturbing this product gives two terms:

$$\Delta(PX) = P\Delta X + X\Delta P \tag{1.88}$$

And with the demand function $P = P(X)$, we have

$$\Delta(PX) = P\Delta X + X\frac{dP}{dX}\Delta X \tag{1.89}$$

Defining the demand **elasticity** $\epsilon \equiv \frac{X}{P}\frac{dP}{dX}$, we have

$$\Delta(PX) = P(1+\epsilon)\Delta X \tag{1.90}$$

Redoing the perturbation analysis in the previous section, we obtain the additional elasticity terms as follows:

$$\frac{dP(1+\epsilon)}{dt} = r\,[P(1+\epsilon) - C] \tag{1.91}$$

which is the **decision rule under costly monopoly production**. (The reader should check this.)

For the demand function

$$X = \frac{a}{P^{\beta}} \tag{1.92}$$

we have

$$\epsilon = -\frac{1}{\beta} \tag{1.93}$$

that is, a constant; for this demand function, monopoly and competitive producers would behave the same. For the linear demand function

$$P = \overline{P} - bX \tag{1.94}$$

we have

$$\epsilon = -\left(\frac{\overline{P}}{P} - 1\right) \tag{1.95}$$

Thus, in this case ϵ is negative and increases toward 0 as P increases toward $\overline{P}$.
In the monopoly case, it is convenient to introduce the variable $Q \equiv P(1+\epsilon)$. The equation

$$\frac{dQ}{dt} = r(Q - C) \tag{1.96}$$

then replaces its competitive version with P replacing Q. The example below illustrates this.

Monopoly Production under Linear Demand (Costless)

As an example, consider the linear demand case

$$P = \overline{P} - bX \tag{1.97}$$

for which we have

$$\epsilon = -\left(\frac{\overline{P}}{P} - 1\right) \tag{1.98}$$

This case can be integrated in closed form as follows. First, introduce the quantity $Q \equiv P(1+\epsilon)$. In the present case,

$$Q = 2P - \overline{P} \tag{1.99}$$

and equivalently

$$P = \frac{1}{2}(Q + \overline{P}) \tag{1.100}$$

Since $dQ/dt = rQ$, we have

$$Q = Q_0 e^{rt} \tag{1.101}$$

and from the demand function,

$$X = \frac{\overline{P} - Q_0 e^{rt}}{2b} \tag{1.102}$$

Integrating $\frac{dS}{dt} = -X$ gives

$$S(t) = S_0 - \frac{1}{2b}\left[\overline{P}t - \frac{Q_0}{r}\left(e^{rt} - 1\right)\right] \tag{1.103}$$

The various Initial Conditions are

$$\begin{aligned} Q_0 &= 2P_0 - \overline{P} \\ P_0 &= \frac{1}{2}(Q_0 + \overline{P}) \\ X_0 &= \frac{1}{b}(\overline{P} - P_0) \end{aligned} \tag{1.104}$$

The Terminal Condition requires $S(T) \to 0$ as $P(T) \to \overline{P}$, and therefore we have

$$\begin{aligned} S(T) &= 0 \\ P(T) &= \overline{P} \\ Q(T) &= \overline{P} \end{aligned} \tag{1.105}$$

and since $Q(T) = Q_0 e^{rT}$,

$$rT = \ln\left(\frac{Q(T)}{Q_0}\right) \tag{1.106}$$

After some manipulations, we arrive at the final set of relations among (S_0, X_0, T) and P_0:

$$T = \frac{1}{r}\ln\left(\frac{\overline{P}}{2P_0 - \overline{P}}\right) \tag{1.107}$$

$$S_0 = \frac{1}{rb}\left[\frac{\overline{P}}{2}\ln\left(\frac{\overline{P}}{2P_0 - \overline{P}}\right) + \left(P_0 - \overline{P}\right)\right] \tag{1.108}$$

$$X_0 = \frac{1}{b}\left(\overline{P} - P_0\right) \tag{1.109}$$

This solution requires $P_0 > \overline{P}/2$.

The reader is encouraged to compare these relations with the comparable results for competitive production in Equations 1.36–1.38 (Problems 6 and 7). Generally, monopoly production results in higher initial prices, longer exhaustion times, and higher overall rent than competitive production. There would be an incentive for competitive producers to form a monopoly and share the extra rent generated. It is interesting that this leads to slower overall resource exhaustion.

Total rent under this scenario is

$$\begin{aligned}
R &= \int_0^{\infty} P(t)X(t)e^{-rt}\,dt \\
&= \int_0^{T} \frac{(Q_0 e^{rt} + \overline{P})}{2}\frac{(\overline{P} - Q_0 e^{rt})}{2b} e^{-rt}\,dt \\
&= \frac{1}{4b}\int_0^{T} (\overline{P}^2 - Q_0^2 e^{2rt})e^{-rt}\,dt \\
&= \frac{1}{4b}\int_0^{T} (\overline{P}^2 e^{-rt} - Q_0^2 e^{rt})\,dt \\
&= \frac{1}{4rb}(\overline{P}^2[1 - e^{-rT}] + Q_0^2[1 - e^{rT}])
\end{aligned} \tag{1.110}$$

With some effort, one can simplify this as follows. Using

$$1 - e^{-rT} = 2\left(\frac{\overline{P} - P_0}{\overline{P}}\right) \quad \text{and} \quad 1 - e^{rT} = 2\left(\frac{P_0 - \overline{P}}{2P_0 - \overline{P}}\right) \tag{1.111}$$

the final result becomes

$$R = \frac{1}{br}\left(\overline{P} - P_0\right)^2 \tag{1.112}$$

Monopoly rent in this case is clearly always nonnegative. There is no peak at intermediate S, unlike the competitive case illustrated in Figure 1.8.

Figure 1.13 illustrates these relations among X_0, T, P_0, S_0, and R for monopoly production. The competitive case plotted in Figure 1.8 is replicated for comparison. **Oil6M+C** provides a simulation of this system's time evolution.

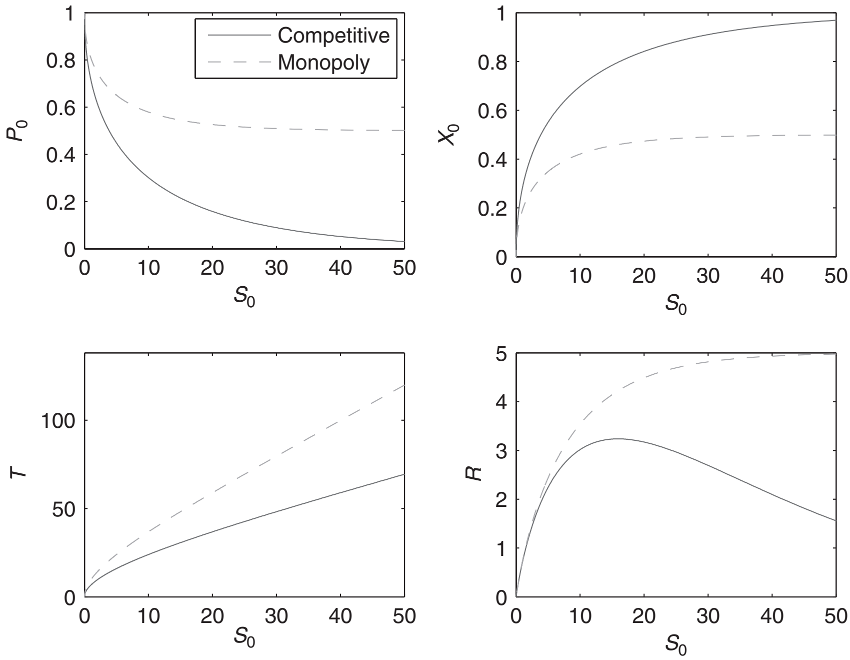

Figure 1.13. Optimal extraction relations for the linear demand case: $r = 0.05$; $P = 1 - X$; monopoly case (dash line). The competitive case (solid line) is replicated from Figure 1.8 for comparison.

1.3 DISCOVERY

Next we add an inventory of undiscovered resource U to the picture and the rate of discovery D:

$$\frac{dS}{dt} = -X + D \tag{1.113}$$

$$\frac{dU}{dt} = -D \tag{1.114}$$

$(S + U)$ is the total unproduced resource; initial values are S_0 and U_0. To close the equations, we need expressions for X and D.

1.3.1 Exogenous Discovery

Consider first the simple case where discovery is exogenous – that is, it is completely independent of P, X, and S. A example would be the first-order discovery rate

$$D = \rho U \tag{1.115}$$

This discovery rate is proportional to the remaining undiscovered resource; it is high when there is a lot to find, and it decreases as the discovery history proceeds and the remaining U decreases.

For production, we invoke the result of the previous analyses: With demand $X = a/P^\beta$, ceiling price $\overline{P}$, and zero production cost, we have

$$X = \beta r S + \frac{a}{\overline{P}^\beta} \tag{1.116}$$

Essentially, we assume the production decision is pessimistic, based on a forecast of zero future discovery. This production relation is rational (optimal) according to the previous analyses.

The complete system is now

$$\frac{dS}{dt} + \beta r S = \rho U - \frac{a}{\overline{P}^\beta} \tag{1.117}$$

$$\frac{dU}{dt} + \rho U = 0 \tag{1.118}$$

The solution for U is straightforward:

$$U = U_0 e^{-\rho t} \tag{1.119}$$

and for S, we now have

$$\frac{dS}{dt} + \beta r S = \rho U_0 e^{-\rho t} - \frac{a}{\overline{P}^\beta} \tag{1.120}$$

The solution for S will have two parts: a "homogeneous" part of the form $e^{-\beta r t}$, as in the cases without discovery; and a "particular" part of the form dictated by the right-hand side, $e^{-\rho t}$ plus a constant:

$$S(t) = Ae^{-\beta r t} + Be^{-\rho t} + C \tag{1.121}$$

Plugging 1.121 into 1.120, we obtain

$$A\left[-\beta r + \beta r\right] e^{-\beta r t} + B\left[-\rho + \beta r\right] e^{-\rho t} + C\left[\beta r\right] = \rho U_0 e^{-\rho t} - \frac{a}{\overline{P}^\beta} \tag{1.122}$$

The first term vanishes identically. The balance of the equation requires

$$B = U_0 \left[\frac{\rho}{\beta r - \rho}\right] \tag{1.123}$$

$$C = \left[-\frac{a}{\beta r \overline{P}^\beta}\right] \tag{1.124}$$

and the Initial Condition requires $S_0 = A + B + C$, so

$$A = S_0 - B - C \tag{1.125}$$

The complete solution is

$$S(t) = [S_0 - B - C]\, e^{-\beta r t} + B e^{-\rho t} + C \tag{1.126}$$

Two cases are interesting. If production is fast relative to discovery, then S decays monotonically, with the slow discovery persisting and extending production over a longer time than in the no-discovery case. The criterion for this case is $dS/dt < 0$ at $t = 0$. This requires

$$\frac{\beta r(S_0 - C)}{\rho U_0} > 1 \tag{1.127}$$

If, on the other hand, initial discovery is fast relative to production, then S rises initially. In this case, there will be a peak in S at intermediate time, after which discovery is largely over, production dominates, and S declines. It is easy to find the peak time t_p by setting dS/dt to zero. The result for the case $C = 0$ ($\overline{P} = \infty$) is

$$t_p = \left(\frac{1}{\rho - \beta r}\right) \ln \left[\frac{\rho}{\beta r \left(1 - \frac{S_0}{B}\right)}\right] \tag{1.128}$$

(The student should verify this and develop the case $C \neq 0$.) Beyond $t = t_p$, we are essentially in a depletion phase with discovery largely over.

For example: with $r = 10\%$ per year, $\rho = 10\%$ per year, $\beta = .5$, $\overline{P} = \infty$, $S_0 = 10$, and $U_0 = 10$, we have S peaking at 11.25, at $t_p = 5.75$ years. If, instead, $U_0 = 50$, then S peaks at 30.26 and t_p occurs at 11.96 years. These are illustrated in Figure 1.14. Figure 1.15 illustrates the slower discovery rate, $\rho = 0.04$, which does not exhibit a peak. The Excel and Matlab programs **Oil5a** simulate this system.

More-complex discovery relations are interesting, and their solution is possible via simulation. For example, the price-sensitive discovery rate

$$D = \rho U \left(\frac{P}{P_0}\right)^{\gamma} \tag{1.129}$$

simulates enhanced exploration effort as the resource scarcity causes price to increase. This may be examined in **Oil5a**. Additional features include the introduction of stochastic disturbances to the otherwise smooth discovery rate. That is available in **Oil5aRandom**.

1.3.2 Discovery Rate: Effort and Efficiency

It is useful to pick apart the various factors influencing the discovery rate: the discovery efficiency; the discovery effort; and the amount left to be discovered.

The efficiency of discovery embodies accumulated geological knowledge, including both where not to look (already looked there) and what features to look for. Thus, we expect a monotonically rising curve as discovery proceeds – a learning curve,

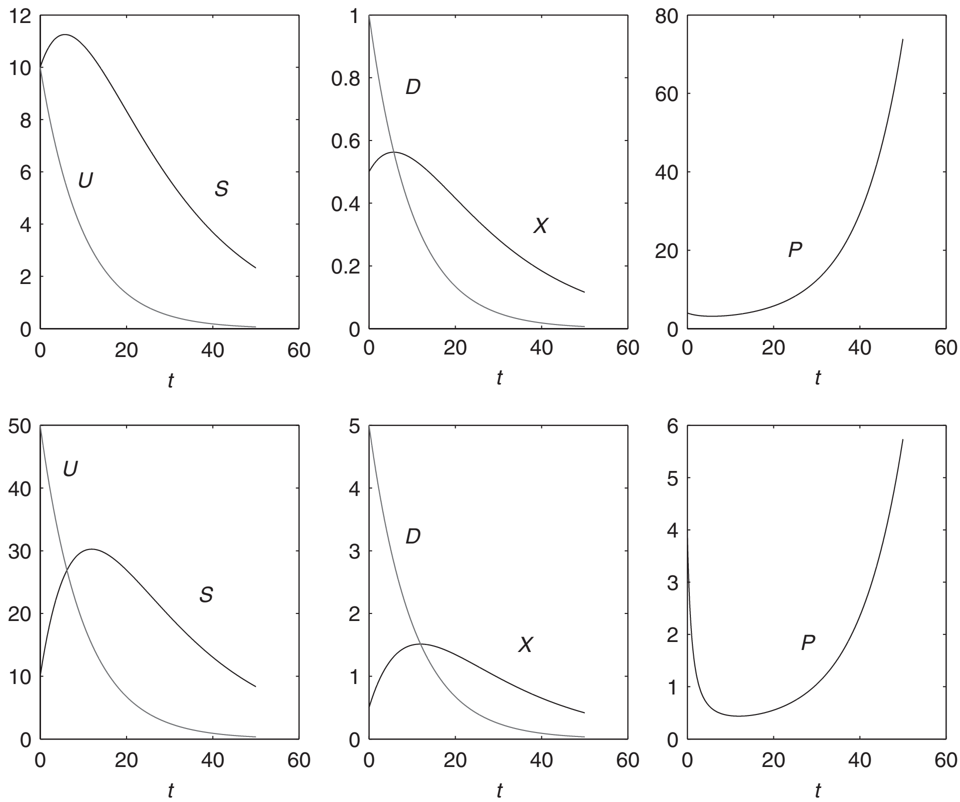

Figure 1.14. Discovery and extraction history, as in the example

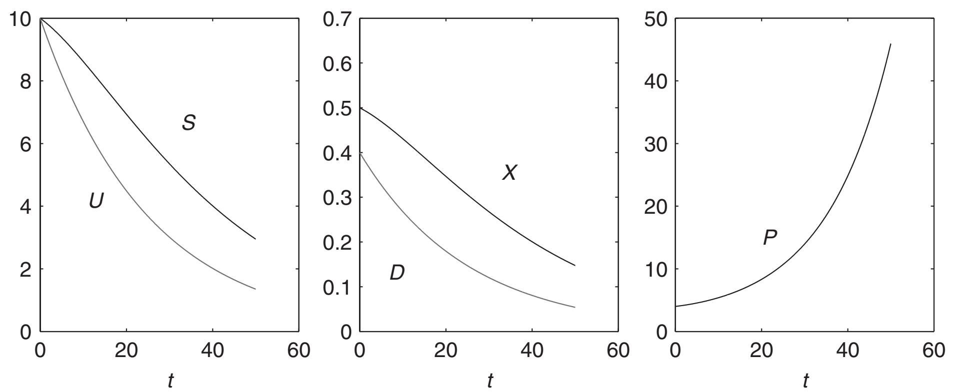

Figure 1.15. Discovery and extraction history, as in Figure 1.14; but ρ is slower, changed from 0.10 to 0.04.

for example, of the form $\left[1 - e^{-(U_0 - U)/U^*}\right]$, with $U_0 - U$ the cumulative discovery and U^* a scaling factor. In addition, we have the inescapable effect of technological invention of devices, sensors, information bases, etc., which we expect will only rise over time in their amplification of discovery rate by the exogenous multiplier $A(t)$.

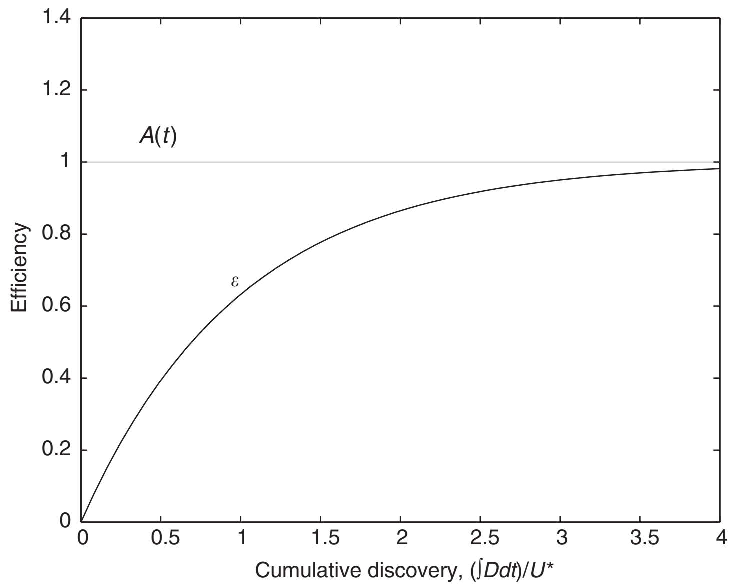

Figure 1.16. Discovery efficiency ϵ versus cumulative discovery, as in Equation 1.130.

The composite of these leads to the hypothetical efficiency function

$$\epsilon = A(t)\left[1 - e^{-(U_0 - U)/U^*}\right] \tag{1.130}$$

This relation is sketched in Figure 1.16. Saturation of the learning curve at its technological limit A occurs when cumulative discovery reaches $4U^*$.

The effect of discovery effort E is monotonically related to the discovery rate. At low levels of effort, the relationship should be linear. At higher levels of effort, we expect decreasing returns to scale due to crowding and duplication of effort. An example of such an effort function, $f(E)$, is

$$f(E) = \left[1 - e^{-E/E^*}\right] \tag{1.131}$$

where E^* is a scaling factor. Like the learning curve above, it saturates at $E \approx 4E^*$. This function is illustrated in Figure 1.17.

These relations together give us the discovery rate D:

$$D = \epsilon \cdot f(E) \cdot U \tag{1.132}$$

At low effort, $E \ll E^*$, we have the linear relationship

$$f(E) \simeq E/E^* \tag{1.133}$$

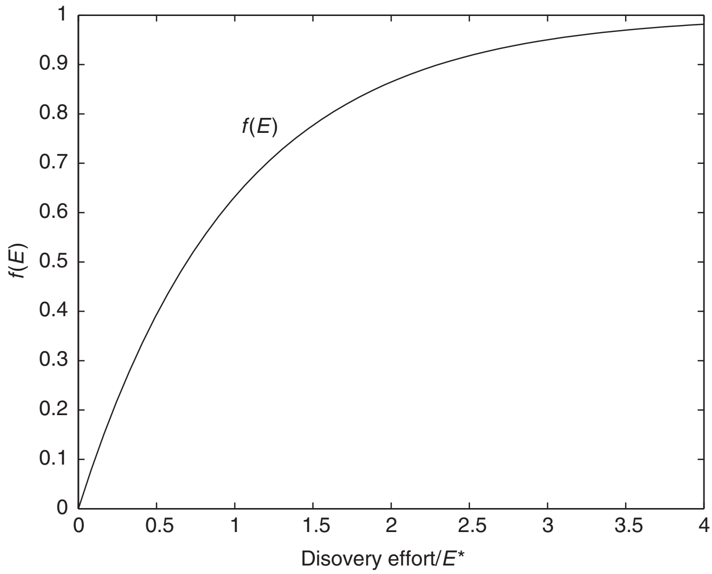

Figure 1.17. Discovery effort multiplier $f(E)$, an increasing function of effort, as in Equation 1.131. This is the effect of effort on discovery rate.

and thus

$$D \simeq \frac{\epsilon}{E^*} EU \tag{1.134}$$

1.3.3 Effort Level and Exploration Profit

So far, we have a dynamical system involving proven and undiscovered reserves S and U, the production rate X, which is closed in terms of S, and the discovery rate D, which is closed in terms of discovery effort E.

$$\frac{dS}{dt} = -X + D \tag{1.135}$$

$$\frac{dU}{dt} = -D \tag{1.136}$$

$$X = \beta r S + \frac{a}{P^\beta} \tag{1.137}$$

$$D = \epsilon \cdot f(E) \cdot U \tag{1.138}$$

What sets the level of effort?

To fix ideas, imagine an exploration company selling rights to new discoveries, at the price p_d, to production companies. What is at stake in selling such a new discovery is the present worth of all future rent that could be derived from producing that resource; the rate of rent transfer to the selling company would be $p_d \cdot D$ if a competitive market existed among buyers. (In a costless production economy, p_d would be equal to the market price P of produced resource; otherwise, it would be less.)

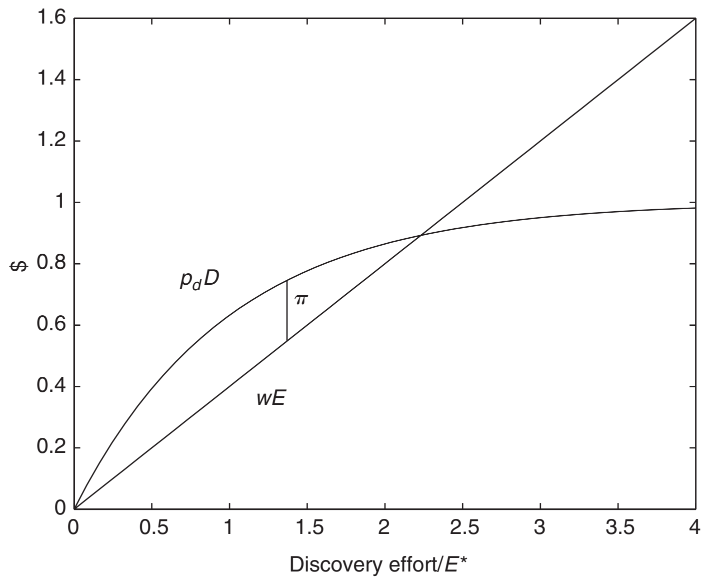

Figure 1.18. Profitability of exploration as a function of effort E. The curved line is the value of discovery; the straight line is its cost. The profitability π is the difference. Parameters: $p_d \epsilon U = 1.$; $wE^* = .4$.

Assume that the cost of exploration is proportional to effort, wE. Then we have the rate of profit earned by exploration as

$$\pi = p_d D - wE = p_d \epsilon f(E) U - wE \tag{1.139}$$

This is illustrated in Figure 1.18. Competition in the exploration business would drive the effort to its maximum profitable rate, where $\pi = 0$:

$$E = \frac{p_d}{w} D \Rightarrow \frac{f(E)}{E} = \frac{w}{p_d \epsilon U} \tag{1.140}$$

This is the "open access," or "free entry," situation. All effort is free to mobilize and enter the exploration competitively. It does so until profitability becomes zero.

If, on the other hand, discovery effort could be controlled, then effort would be limited to the point of maximum π, where $d\pi/dE = 0$:

$$\frac{df}{dE} = \frac{w}{p_d \epsilon U} \tag{1.141}$$

This is the "controlled access," or "restricted entry," case. A monopoly on exploraton would presumably keep effort at this lower level. For the parameters illustrated in Figure 1.18, visual inspection shows that $E \approx 2.2E^*$ for the open, competitive case, and $E \approx .75E^*$ for the controlled-access case. Competition in exploration gives higher effort, faster discovery, and therefore early production.

1.3.4 Effort Dynamics

How does E change with time? In the competitive case, it is reasonable to postulate that rent attracts effort. Denoting by ν the first-order rate constant for this process, we have the 3-state system:

$$\frac{dE}{dt} = \nu\pi \tag{1.142}$$

$$\frac{dS}{dt} = -X + D \tag{1.143}$$

$$\frac{dU}{dt} = -D \tag{1.144}$$

$$X = \beta r S + \frac{a}{\overline{P}^{\beta}} \tag{1.145}$$

$$D = \epsilon \cdot f(E) \cdot U \tag{1.146}$$

$$\pi = (p_d D - wE) \tag{1.147}$$

$$P = \left(\frac{a}{X}\right)^{\frac{1}{\beta}}; \quad P(t) \le \overline{P} \tag{1.148}$$

The Excel program **Oil7** simulates these dynamics, under the assumption of costless production ($p_d = P$, the market value of production), efficiency saturation

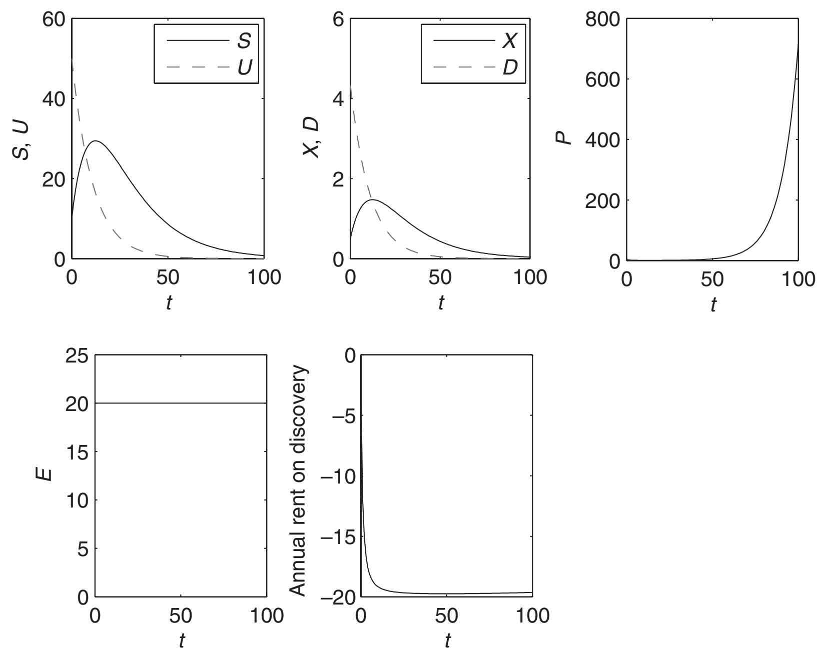

Figure 1.19. Time history with discovery adjusted as in Equations 1.142–1.148, with constant effort. Parameters: $r = .10$; $a = 1$; $\beta = .5$; $\overline{P} = \infty$; $\epsilon = .1$; $E^* = 10$; $p_d = P(t)$; $w = 1$; $\nu = 0$.

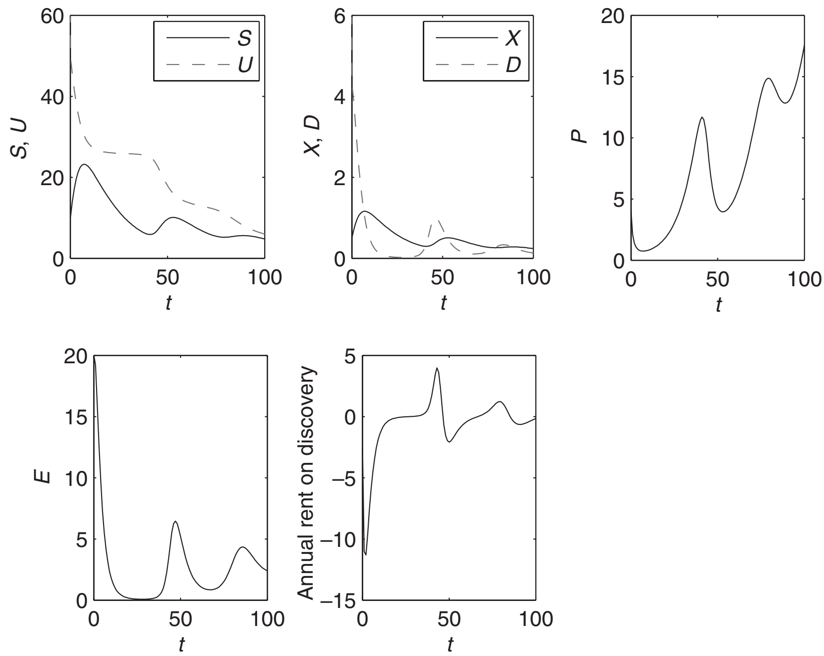

Figure 1.20. Same as in Figure 1.19, except rent-sensitive discovery, $\nu = 0.3$.

(extensive geological knowledge and exploration experience), and constant technology. For small ν, there is an orderly approach to quasi-equilibrium wherein exploration proceeds gradually as S is exhausted. Faster effort response (higher ν) generates a boom-or-bust cycle in exploration, with periodic episodes of discovery followed by layoffs in the exploration business and simple depletion of known reserves as production proceeds. When price rises sufficiently to reinvigorate exploration effort, the cycle is repeated.

An example is presented in Figures 1.19 and 1.20. In Figure 1.19, we plot the case $\nu = 0$, wherein the initial effort $E = 20$ is held constant. Discovery proceeds at a constant proportional rate. The production floods the market, suppressing the price. The exploration is not profitable, but there is no adjustment. Figure 1.20 is the same case, but with $\nu = 0.3$. The initial unprofitable level of effort is reduced quickly, essentially shutting down exploration until roughly $t = 40$, at which point it is profitable to re-initiate exploration effort. There is a complex cycling of effort.

1.4 SOME UNCLOSED ISSUES

The development above is a framework for examining the exhaustible resources. Several questions remain:

- Demand changes reflecting more uses or changes in efficiency; substitution dynamics; technical progress; changes in market participation in the resource consumption. What is da/dt?

- Discovery and the effect of technical progress. What is dA/dt?
- What is the effect of production capacity, the maximum X and dX/dt which is possible? Clearly, that puts a constraint on short-term changes in production.
- The cost of extraction generally: what is $C(S)$?
- How is rent distributed?
- The role of uncertainty, error, bogus data, and uncontrolled stochastic disturbances, generally, on resource management.

Each of these items deserves careful scrutiny in more-elaborate developments.

1.5 RECAP

As mentioned at the outset, this is an extreme case; the example used (petroleum) is not meant to be definitive, only illustrative and helpful to the exposition. Several key ideas emerge, including the role of the owner in setting the decision context and "collecting the rent"; the role of the demand funtion in representing value and "consumers' surplus"; the role of finite demand and substitution; the dynamic of public policy in conditioning decision making as well as rent dispersal; the role of discovery; and the view of exploration as a subsidiary industry. From a sustainability perspective, this resource has only one fate: exhaustion over a finite interval. The sustainable substitute needs to be found during the process: That can be the accumulated rent, treated as a renewable asset; the knowledge gained during the process of exhaustion ("learning how do do without"); the capitalization of a substitute; and/or the social progress achieved during resource exhaustion. Ultimately, substitution alone is inadequate if it necessarily progresses through a series of finite resource exhaustions. Without a sustainable finale, one simply exhausts all resources, irreversibly, leaving no opportunity.

There are several introductory texts on natural resource economics, including Conrad and Clark [13], Neher [68], and Conrad [12]. Griffin [35] covers water resources, which is a necessary supplement for natural resources generally. Conrad [12] includes a valuable annotated bibliography.

All of the above lead to the more general coverage of public goods – for example, Musgrave and Musgrave [66] and Cornes and Sandler [14]. Contemporary scholarship is reemphasizing the foundational notion of merit goods introduced by Musgrave (Ver Eecke [22]; Meier [63]). There are several works that have incorporated globalization as a starting point, notably the work of Kaul et al. [49, 47, 48]. Further study in all of these works is encouraged.

1.6 PROGRAMS

The following programs illustrate the ideas in these lectures. In most cases, there are .xls and .m versions:

- **Oil1:** Costless production, constant-elasticity demand. P_0 is adjustable to meet TC.
- **Oil1a:** Costless production, constant-elasticity demand. Simulation with $X = X(S)$.
- **Oil1-tax:** Costless production, constant-elasticity demand, constant tax.
- **Oil2:** Constant-cost production.
- **Oil4:** Variable-cost production: $C(S) = \gamma/S^{\delta}$.
- **Oil5a:** Simulates price-sensitive discovery and production.
- **Oil5aRandom:** Adds stochastic perturbation to discovery rate in **Oil5a**.
- **Oil6Monop:** Simulates costless production with linear demand; monopoly production.
- **Oil6Cmptn:** Simulates costless production with linear demand; competitive production.
- **Oil6M+C_Rent:** Combines the two **Oil6** programs; adds the calculation of P_0, X_0, T and the total present worth of rent, as functions of S_0 for the two cases monopoly and competitive production.
- **Oil7:** Like **OiL5a** but adds exploration effort as an endogenous state variable. Discovery rate is the product of fixed efficiency, an increasing function of exploration effort, and remaining undiscovered resource. Effort growth is proportional to the current profitability of exploration.

1.7 PROBLEMS

1 Costless production of a sterile resource:

$$\frac{dS}{dt} = -X \tag{1.149}$$

$$\frac{dP}{dt} = rP \tag{1.150}$$

$$X = \frac{a}{P^{\beta}} \tag{1.151}$$

where S is the amount in storage, X is the annual production, and P is the price. Simulate the evolution of this system using the following parameters:

$S_0 = 1,000$

$\beta = 0.5$

$r = .10/\text{year}$

$a = 100.$

Plot S, P, and X as functions of time. Confirm that P grows at the rate r and that S decays at the rate βr. By trial and error, confirm that the initial price that exhausts the resource at infinite time is

$$P_0 = \left[\frac{a}{S_0 \beta r}\right]^{\frac{1}{\beta}} \tag{1.152}$$

2 Derive Equations 1.17, 1.18, and 1.19, which summarize the solution with finite demand capped at the substitution price $\overline{P}$.

3 Repeat Problem 1 with $\overline{P} = 12$. What is the Terminal Condition, and what is the theoretical value of P_0 that achieves it? Does that work in the simulation? Compare results with the simulation of Problem 1, wherein $\overline{P}$ is unbounded.

4 The same as Problem 3, but add a sudden, unanticipated reduction of $\overline{P}$ from 12 to 8 at time = 10 years, due to unanticipated technical progress at $t = 10$. Clearly describe what changes need to occur at $t = 10$ and say why.

5 Redo the simulation in Problem 3, adding the unit cost of production C (dollars per unit produced):

$$C = \frac{\gamma}{S^{\frac{1}{3}}} \tag{1.153}$$

such that annual revenues are $= (P - C)X$. Study an interesting range of the parameter γ, ranging from 10 to 100. (Begin with $\gamma = 50$.) Describe how you obtained the initial price and how you altered the equations. Compare with the Problem 3 simulation in terms of initial price, time to exhaustion, production rate, and any other relevant quantities.

6 Costless production of a sterile resource:

$$\frac{dS}{dt} = -X \tag{1.154}$$

$$\frac{dP}{dt} = rP \tag{1.155}$$

$$X = \frac{5}{P} \tag{1.156}$$

where S is the amount in storage, X is the annual production, P is the price, and r is the interest rate.

Today, $S = 400$. There is no substitute for this resource. Interest rate $r = 0.10$ per year.

- **(a)** What is today's price?
- **(b)** At $t = 10$, what will the price be? What amount of resource will remain in storage?
- **(c)** At $t = 10$, there is a new discovery of 300 units of resource. What will the price change to?
- **(d)** Sketch the time history of P, S, and X, from $t = 0$, until the resource is exhausted.

7 Costless production of a sterile resource; with the linear demand function

$$P = \overline{P} - bX \tag{1.157}$$

In the text, we solve analytically for S, X, and P as functions of time under *competitive* extraction. Confirm that analysis by simulation using $\overline{P} = 1$ and $b = 1$ and P_0, which meets the appropriate Terminal Condition.

We also have expressions for S_0, X_0, and T as functions of the initial price P_0. Evaluate these functions over a useful range of P_0, using $\overline{P} = 1$ and $b = 1$. Use these evaluations to plot the implicit functions X_0, T, and P_0 as functions of the initial storage S_0.

8 Costless competitive production of a sterile resource with finite demand:

$$X + K = \frac{a}{P^\beta} \tag{1.158}$$

This is illustrated in Figure 1.3.

(a) What is the ceiling price $\overline{P}$?

(b) Confirm the following relations:

$$S_0(T) = \frac{K}{\beta r}\left[e^{\beta r T} - 1\right] + KT \tag{1.159}$$

$$X_0(T) = K\left[e^{\beta r T} - 1\right] \tag{1.160}$$

$$P_0(T) = \frac{a}{Ke^{\beta r T}} \tag{1.161}$$

At any time, these must relate S, X, and P to the remaining life T of the resource.

(c) Plot S, X, and P versus T.

(d) Using the same relations, plot X, P, and T versus S, the amount of resource remaining.

9 Repeat Problem 7 but use *monopoly* production. Compare the results with Problem 7. Do you find that monopoly production is more or less conservative than competitive production?

10 Costless production of a sterile resource is governed by

$$\frac{dS}{dt} = -X \tag{1.162}$$

$$\frac{dP}{dt} = rP \tag{1.163}$$

$$X = \frac{a}{P^\beta} \tag{1.164}$$

where S is the amount in storage, X is the annual production, and P is the price. Parameters are: $a = 20$; $\beta = 0.5$; and $r = 0.1$. Today ($t = 0$), $S = 100$.

(a) Calculate P, S, and X at $t = 0$ and at $t = 5$.

(b) At $t = 5$, new laws are enacted that will regulate and tax production. The new laws will become effective in 10 years, that is, at $t = 15$. Details are unclear; but it is highly likely that production will be either unprofitable, illegal, or both, beginning at $t = 15$. What adjustments to P and X will occur and why? Calculate the new values of P and X following this adjustment.

(c) Sketch the time history of P, X, and S from $t = 0$ to $t = 15$ and beyond.

11 Costless production of a sterile resource, $S_0 = 1{,}000$. Demand is

$$X = \frac{1}{\sqrt{P}} \tag{1.165}$$

subject to the ceiling price $\overline{P} = 15$. Interest rate is $r = .05$.

(a) What are today's price P? production rate X? time to exhaustion T?

(b) A new substitute is about to be announced; the net effect will be that demand will begin decaying over time:

$$X = \frac{e^{-0.1t}}{\sqrt{P}} \tag{1.166}$$

What changes will occur in P, X, and T as soon as this substitute is announced?

(c) What is the value of the research that led to the substitute? Who is happy, who is sad?

12 Costless, competitive production of a sterile resource. Demand is the standard used frequently in class:

$$X = \frac{100}{\sqrt{P}} \tag{1.167}$$

Owner is conservative in the face of discovery. Interest rate is 0.10. Price is observed to be constant over many years, at $P = 100$.

(a) What is the discovery rate?

(b) What is the amount of reserves (S) today?

13 For the case summarized in Equations 1.20–1.23: At what initial abundance S^* does the present worth of rent peak? Obtain this as a function of the parameters $r, a, \beta, \overline{P}$. Confirm your analysis graphically that rent decreases with S for $S > S^*$.

14 Costless, competitive production of a sterile resource. The demand function is given by

$$X = \frac{1}{b} \ln\left(\frac{\overline{P}}{P}\right) \tag{1.168}$$

or equivalently,

$$P = \overline{P}e^{-bX} \tag{1.169}$$

Find the initial price P_0 and the time to exhaustion T, both as functions of the initial storage S_0.

15 Costless production of a sterile resource, with demand given by

$$P = \overline{P} - bX \tag{1.170}$$

Price today is $.6\overline{P}$. The interest rate is 10%. There is no discovery. Predict the price in one year under

(a) monopoly production
(b) competitive production

16 Costless, competitive production of a sterile resource. Demand is

$$X = \frac{100}{\sqrt{P}} \tag{1.171}$$

with no price ceiling, interest rate $r = .10$, and total reserves today $S_0 = 1{,}000$.
It is widely accepted that another 200 units of resource are under a national park; but it is believed that production there will never be allowed.

(a) What are price P and production X today (i.e., year 2001)?
(b) Public policy is suddenly changed to allow production in the national park. What changes? Forecast P, X, and S five years from now (i.e., year 2006).
(c) After five years of drilling, it is realized that the 200 units under the park was overestimated by a factor of 2. What changes? What do P and X become now (year 2006, immediately following the realization)?
(d) Sketch the evolution of P, X, and S over time from year 2001 to 2010.

17 Costless competitive production of a sterile resource. Demand is

$$X = \frac{100}{P^{\frac{1}{3}}} \tag{1.172}$$

$S_0 = 1{,}000$, interest rate $r = .05$, and the ceiling price is infinite. You are considering selling production rights to all of this resource.

(a) What price should you expect?
(b) The buyer expects that there is a ceiling price $\overline{P} = 8$. What is the most this buyer will pay?
(c) You have secret knowledge about a substitute product that your own R&D unit has developed. This substitute makes $\overline{P} = 6$. With the offer from the buyer above in hand, you should be happy. Quantify your happiness.

18 Derive Equation 1.128 for the time to peak when discovery rate is fast relative to production.

19 Simulate the discovery system described in the text, Section 1.3.1, under three scenarios: (a) fast discovery ($\rho = .1$, $\beta r = .05$); (b) moderate discovery ($\rho = .02$, $\beta r = .05$); and (c) slow discovery ($\rho = .005$, $\beta r = .05$). Use $S_0 = 100$ and $U_0 = 600$; assume the ceiling price $\overline{P}$ is infinite. Do the simulation results agree with the text discussion?

20 Derive the discovery solutions for S and U as in the text, but with "optimistic" production that anticipates future discovery:

$$X = \beta r(S + \alpha U) + \frac{a}{\overline{P}^{\beta}} \tag{1.173}$$

with $0 < \alpha < 1$. (a) Compare the results with the pessimistic solutions given in the text. (b) What happens when $\alpha > 1$? Is there a critical (large) value of α? Explain why or why not.

21 Redo the simulations in Problem 19, but with "optimistic" production:

$$X = \beta r\left(S + \frac{U}{2}\right) \tag{1.174}$$

as in Problem 20. Assume the ceiling price $\overline{P}$ is infinite. Compare these simulations to a) the analytic solutions from Problem 20; and (b) the pessimistic production simulations in Problem 19.

22 Equations 1.39 and 1.40 give the point beyond which rent decreases as S increases, for costless competitive production with linear demand.

(a) Derive those equations.
(b) Confirm the root given in the text.
(c) Plot R versus S^* for several combinations of the fixed parameters $\overline{P}$, r, and b, and confirm that 1.39 and 1.40 are true.

23 Ten units of a nonrenewable resource are available under costless production. The market for it is described by the demand function

$$P = \sqrt{1 - X^2} \tag{1.175}$$

where P is the price and X is the production rate. There is no exploration; interest rate $r = 0.10$.

(a) Confirm: $\epsilon = -X^2/P^2$.
(b) Confirm: $Q \equiv P(1 + \epsilon) = 2P - 1/P$.
(c) Confirm: Given Q, then $P = \frac{1}{4}\left(Q + \sqrt{Q^2 + 8}\right)$.
(d) What are the limits on P? X? Q?
(e) What is the Terminal Condition?

Find the initial price, the initial production rate, and the time to exhaustion under (f) competitive production; and (g) monopoly production. Do (f) and (g) by simulation.

24 Costless production of a sterile nonrenewable resource; $S_0 = 100$; $r = 6\%$; $\beta = .5$. It is observed that demand is growing linearly with time; in the coming year, it will grow by 4% and will continue linearly forever.

(a) What is the production from now until exhaustion? Compute and plot X and P.
(b) Compare with the no-growth-in-demand scenario, all other things being unchanged.

(c) Simulate (a) and (b).
(d) Compute the rent under (a) and (b).

25 Costless production of a sterile resource. Demand is

$$X = \frac{1}{\sqrt{P}} \tag{1.176}$$

Interest rate $r = .10$. Today, it is known that $P = 25$. There is no ceiling price.
(a) What are S and X today?
(b) What is the financial value of exclusive production rights to this resource?
(c) How long will it take before S is depleted to 25% of today's amount?

26 Continuation of Problem 25. A new conservation technology is about to be introduced. As a result, demand will shrink gradually over time:

$$X = \frac{e^{-\alpha t}}{\sqrt{P}} \tag{1.177}$$

with $\alpha = .06$ and $r = .10$. Note: Here you may still assume that $\frac{dP}{dt} = rP$. P_0 may change, however.
(a) What are P, X, and S as functions of time?
(b) What is today's price? Today's production rate? How have they changed from your Problem 25 answers? (In problem 25, $\alpha = 0$.)
(c) How long will it take before S is depleted to 25% of today's amount?
(d) What is the total value today of this new technology? Who is happy, customers, technologists, and/or resource owners?

27 Redo Problems 25 and 26; but add a price ceiling $\overline{P} = 15$. In addition to the questions there, evaluate the resource lifetime in each case.

28 Costless, competitive production of a sterile resource. Demand is

$$X = \frac{200}{\sqrt{P}} \tag{1.178}$$

There is no ceiling price; interest rate $r = .10$. There is exploration. But producers always act conservatively, forecasting zero discovery until it actually happens.
Production one year ago was $X = 100$. Today, $X = 120$.
(a) What is the discovery rate?
(b) What are the total reserves S today?
(c) A company wants to buy exclusive rights to your resource. What will you sell for, today?
(d) The Supreme Court just ruled that 500 units of your S actually lie under a national park; your right to that amount is voided. Quantify your (un)happiness as an owner.

29 Costless production, no ceiling price, with constant-elasticity demand, shrinking exponentially:

$$X = \frac{e^{-\alpha t}}{P^{\beta}} \tag{1.179}$$

What is the relation between the decay rate α and P_0? between α and total rent?

30 Costless production, constant elasticity $\beta = .5$. Parameters: $S_0 = 1{,}000$, $r = .05$, $a = 1$.

- **(a)** Compute and plot X, S, and P for the first 10 years of production.
- **(b)** At $t = 10$, an attractive substitute for the resource becomes instantly available. As a result, demand begins to shrink exponentially, at the rate $g = -0.20$ per year. What changes in X and P will occur instantly?
- **(c)** Plot the time history of X, S, and P from $t = 0$ (as in (a) above) and continuing through the change in demand, out to $t = 30$.

31 You own 6,000 units of a sterile resource; production is costless, and there is no discovery. Demand is

$$X = \frac{7}{\sqrt{P}} \tag{1.180}$$

Interest rate $r = .05$ and there is no price ceiling.

- **(a)** What is today's price?
- **(b)** How much resource is left after 10 years?
- **(c)** At $t = 10$ years, you decide to sell all rights to future production. How much money will you sell out for?
- **(d)** Engineers in another firm have worked secretly on a new use for your resource. You are ignorant of this. This new technology is "ready to go" at $t = 10$. Once announced, it is expected that demand will instantly expand to

  $$X = \frac{9}{\sqrt{P}} \tag{1.181}$$

  Assuming they first buy the production rights from you as in (c) above, quantify their happiness.
- **(e)** The technical announcement about the new use in (d) is made, at $t = 10$. What immediate adjustments will occur in price, supply, and production rate? With this in hand, sketch the time history of these quantities from today ($t = 0$) through $t = 15$.

32 Costless production of a sterile resource, with demand

$$X = \frac{7}{\sqrt{P}} \tag{1.182}$$

Interest rate $r = .05$, and there is no price ceiling. You own 6,000 units of the resource today.

(a) What is today's price?

(b) What is today's production rate?

You read the following in the evening news: "*The government announced today that, effective in eight years, production of this resource will be illegal due to unacceptable health risks.*"

(c) What changes should you make?

(d) What is the new price, effective tomorrow morning?

(e) What is the new production rate, effective tomorrow morning?

(f) You will sue the government for damages. What is the dollar value of your suit?

33 Costless production of a sterile resource, with demand

$$X = \sqrt{1 - P} \tag{1.183}$$

Interest rate $r = .05$. It is known that in exactly four years, this resource will be exhausted.

(a) What is the production rate and price, both now and in four years, under *competitive* production? (Your answer should consist of four numbers.)

(b) Same, under *monopoly* production.

34 Compute the present worth of the consumers' surplus, $\int \text{CS} \cdot e^{-rt}\,dt$, as given in Equation 1.27, assuming $P(t) = P_0 e^{rt}$.

35 For linear demand given by

$$P = \overline{P} - bX \tag{1.184}$$

(a) Find an expression for consumers' surplus as a function of P.

(b) Find an expression for the present worth of consumers' surplus, $\int \text{CS} \cdot e^{-rt}\,dt$ assuming $P(t) = P_0 e^{rt}$.