Appendix G
Fourier transform: an overview

In this appendix, we provide an overview of the key concepts and properties relating to the Fourier analysis of functions of time and space variables that are essential to a deeper understanding of the digital filters (Chapter 21). In the following, we use the term functions and signals interchangeably.

G.1 Fourier transforms of continuous signals (infinite duration)

A signal $g : \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to a class D of functions if (a) g is *piecewise smooth* and (b) is *absolutely integrable*, that is

$$\int_{-\infty}^{\infty} |g(t)|\mathrm{d}t < \infty. \tag{G.1.1}$$

It can be verified that

$$\begin{aligned} g^{+}(t) &= \tfrac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}} \quad \text{for } t \geq 0 \\ &= 0 \qquad \text{for } t < 0 \end{aligned} \tag{G.1.2}$$

belongs to the class D since $g^{+}(t)$ has one discontinuity at the origin $t = 0$ and

$$\int_{-\infty}^{\infty} g^{+}(t)\mathrm{d}t = 1.$$

But the periodic and continuous function $g(t) = \sin 2\pi f t$ does *not* belong to class D.

The (direct or forward) *continuous time Fourier transform* (CTFT) $G(f)$ of $g(t)$ is a function $G : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} G(f) &= F[g(t)] \\ &= \textstyle\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t \end{aligned} \tag{G.1.3}$$

where $\mathrm{i} = \sqrt{-1}$, the standard unit imaginary number and the variable f is called the (rotational) *frequency* and $w = 2\pi f$ is called the *angular frequency*.

The importance and the utility of the Fourier transform lies in the fact that we can recover the function $g(t)$ from $G(f)$ using the *inverse transform* defined by

$$\begin{aligned} g(t) &= F^{-1}[G(f)] \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} G(f)\mathrm{e}^{\mathrm{i}2\pi f t}\mathrm{d}f. \end{aligned} \tag{G.1.4}$$

Since

$$g(t) = F^{-1}[G(f)] = F^{-1}[F(g(t))]$$

and

$$G(f) = F[g(t)] = F[F^{-1}(G(f))]$$

$g(t)$ and $G(f)$ are two equivalent but different representations of the same function. $G(f)$ is often called the *spectral* representation of $g(t)$.

Example G.1.1 The Fourier transform of $g^+(t)$ in (G.1.2) is given by

$$\begin{aligned} G^+(f) &= F[g^+(t)] \\ &= \tfrac{1}{\lambda}\int_0^{\infty} \mathrm{e}^{-\frac{t}{\lambda}}\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t \\ &= \tfrac{1}{\lambda}\int_0^{\infty} \mathrm{e}^{-(\frac{1+\mathrm{i}2\pi f\lambda}{\lambda})t}\mathrm{d}t \\ &= \tfrac{1}{1+\mathrm{i}2\pi f}. \end{aligned} \tag{G.1.5}$$

Clearly, G(f) is a (rational) complex function in the frequency f whose amplitude and phase is given by

$$|G(f)| = \frac{1}{1+4\pi^2 f^2\lambda^2} \tag{G.1.6}$$

and

$$\mathrm{Arg}[G(f)] = \tan^{-1}[-2\pi f\lambda]. \tag{G.1.7}$$

For completeness, we now demonstrate the computation of the inverse transform of $G(f)$ in (G.1.5)

$$F^{-1}[G(f)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}2\pi f t}}{1+\mathrm{i}2\pi f t}\mathrm{d}f.$$

Example G.1.2 Consider the so-called *Gaussian pulse*

$$g(t) = \mathrm{e}^{-\pi t^2}. \tag{G.1.8}$$

Its Fourier transform $G(f)$ is given by

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} \mathrm{e}^{-\pi t^2}\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t \\ &= \mathrm{e}^{\pi f^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\pi(t+\mathrm{i}f)^2}\mathrm{d}t. \end{aligned}$$

Setting

$$\sqrt{\pi}(t + \mathrm{i}f) = \frac{x}{\sqrt{2}} \quad \text{with} \quad \mathrm{d}t = \frac{1}{\sqrt{2\pi}}\mathrm{d}x$$

we obtain

$$\begin{aligned} G(f) &= \mathrm{e}^{-\pi f^2}\left[\tfrac{1}{\sqrt{2\pi}}\textstyle\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x\right] \\ &= \mathrm{e}^{-\pi f^2} \end{aligned} \tag{G.1.9}$$

since the integral within the square bracket is the integral of the *standard Gaussian* or *normal distribution* whose value is unity.

We encourage the reader to verify that the inverse transform of $G(f)$ in (G.1.9) gives $g(t)$ in (G.1.8).

G.2 Properties of Fourier transform

We now state several basic properties of the Fourier transform pairs.

(a) **Linearity** If $g(t)$ and $h(t)$ are two functions in class D, and a and b are two real constants, then

$$F[ag(t) + bh(t)] = aF[g(t)] + bF[h(t)].$$

(b) **Shifting property** For any real τ,

$$F[g(t - \tau)] = \mathrm{e}^{-\mathrm{i}2\pi f\tau} F[g(t)].$$

(c) **Stretching property** For any non-zero real constant a,

$$F[g(at)] = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

where as usual $G(f) = F[g(t)]$.

(d) **Derivative/Integral of functions** If $G(f) = F[g(t)]$, then

$$F\left[\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right] = \mathrm{i}2\pi f G(f)$$

and

$$F\left[\int_{-\infty}^{t} g(\tau)\mathrm{d}\tau\right] = \frac{G(f)}{\mathrm{i}2\pi f}.$$

(e) **Moment property** The nth derivative of $G(f)$

$$\frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n} = (-\mathrm{i}2\pi)^n F[t^n g(t)].$$

Hence, the nth moment of $g(t)$ is given by

$$\int_{-\infty}^{\infty} t^n g(t)\mathrm{d}t = \left(\frac{\mathrm{i}}{2\pi}\right)^n \frac{\mathrm{d}^n G(f)}{\mathrm{d}f^n}|_{f=0}.$$

(f) **Parseval's theorem** The *total energy* in the signal is the same in both the representations:

$$\int_{-\infty}^{\infty} |g(t)|^2\mathrm{d}t = \int_{-\infty}^{\infty} |G(f)|^2\mathrm{d}f.$$

(g) **Time reversal property**

$$\begin{aligned} F[g(-t)] &= \textstyle\int_{-\infty}^{\infty} g(-t)\mathrm{e}^{-\mathrm{i}2\pi f t}\mathrm{d}t \\ &= G(-f) \end{aligned}$$

by changing the variable $-t = x$ and simplifying.

Example G.2.1 Using $g^+(t)$ in (G.1.2), define

$$\begin{aligned} g^-(t) = g^+(-t) &= \tfrac{1}{\lambda}\mathrm{e}^{\frac{t}{\lambda}} \text{ for } t \le 0 \\ &= 0 \text{ for } t > 0. \end{aligned} \tag{G.2.1}$$

Then

$$G^-(f) = F[g^-(t)] = \frac{1}{1 - \mathrm{i}2\pi f\lambda}. \tag{G.2.2}$$

(h) **Convolution theorem** If $g(t)$ and $h(t)$ are functions in class D, then the convolution of $g(t)$ and $h(t)$ denoted by $(g * h)(t)$ is defined by

$$(g * h)(t) = \int_{-\infty}^{\infty} g(r)h(t - r)\mathrm{d}r. \tag{G.2.3}$$

The following properties of the convolution are easily verified:

(1) $g * h$	$= h * g$	*Commutative*
(2) $g * (h * f)$	$= (g * h) * f$	*Associative*
(3) $g * (f + h)$	$= (g * f) + (g * h)$	*Distributive*

We now state one of the most fundamental properties that is widely used in signal analysis. The Fourier transform of the convolution of $g(t)$ and $h(t)$ is the product of their individual Fourier transforms, that is,

$$F[g * h] = F(g)F(h). \tag{G.2.4}$$

Example G.2.2 Let

$$\begin{aligned} g^+(t) &= \tfrac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}} && \text{if } t \ge 0 \\ &= 0 && \text{if } t < 0 \end{aligned}$$

and

$$g^-(t) = \tfrac{1}{\lambda}\mathrm{e}^{\frac{t}{\lambda}} \qquad \text{if } t \le 0$$
$$= 0 \qquad \text{if } t > 0.$$

Let $u(t)$ denote the *unit step* function

$$u(t) = 1 \qquad \text{if } t \ge 0$$
$$= 0 \qquad \text{if } t < 0.$$

Then

$$\left.\begin{aligned} &g^+(t) = \tfrac{1}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}u(t) \qquad \text{for } 0 < t < \infty \\ &\text{and} \\ &g^-(t) = \tfrac{1}{\lambda}\mathrm{e}^{\frac{t}{\lambda}}u(-t) \qquad \text{for } 0 < t < \infty \end{aligned}\right\} \tag{G.2.5}$$

The convolution of $g^+(t)$ and $g^-(t)$ in (G.1.1) is given by [since $u(-t) = (1 - u(t))$]

$$\begin{aligned} s(t) &= (g^- * g^+)(t) \\ &= \int_{-\infty}^{\infty} g^-(\tau)\ g^+(t-\tau)\mathrm{d}\tau \\ &= \frac{1}{\lambda^2}\int_{-\infty}^{\infty}[1-u(\tau)]u(t-\tau)\mathrm{e}^{\frac{t}{\tau}}\mathrm{e}^{-(\frac{t-\tau}{\lambda})}\mathrm{d}\tau. \end{aligned}$$

Since $[1 - u(t)] = 0$ for $\tau > 0$, we obtain

$$s(t) = \frac{\mathrm{e}^{-\frac{t}{\lambda}}}{\lambda^2}\int_{-\infty}^{\infty}[1-u(\tau)]u(t-\tau)\mathrm{e}^{\frac{2\tau}{\lambda}}\mathrm{d}\tau.$$

At $t = 0$

$$s(0) = \frac{1}{\lambda^2}\int_{-\infty}^{0}\mathrm{e}^{\frac{2\tau}{\lambda}}\mathrm{d}\tau = \frac{1}{2\lambda}.$$

For any $t > 0$,

$$s(t) = \frac{\mathrm{e}^{-\frac{t}{\lambda}}}{\lambda^2}\int_{-\infty}^{0}\mathrm{e}^{\frac{2\tau}{\lambda}}\mathrm{d}\tau$$

$$= \frac{1}{2\lambda}\mathrm{e}^{-\frac{t}{\lambda}}.$$

For any $t < 0$

$$s(t) = \frac{\mathrm{e}^{-\frac{t}{\lambda}}}{\lambda^2}\int_{-\infty}^{t}\mathrm{e}^{\frac{2\tau}{\lambda}}\mathrm{d}\tau$$

$$= \frac{1}{2\lambda}\mathrm{e}^{\frac{t}{\lambda}}.$$

Hence,

$$s(t) = \frac{1}{2\lambda} e^{-\frac{|t|}{\lambda}} \tag{G.2.6}$$

which is an *even symmetric* function since $s(t) = s(-t)$.

By the convolution theorem,

$$\begin{aligned} F[s(t)] &= F[g^-(t) * g^+(t)] \\ &= F[g^-(t)]F[g^+(t)] \\ &= \frac{1}{1 + 4\pi^2 f^2 t^2}. \end{aligned} \tag{G.2.7}$$

Example G.2.3 We leave it to the reader to verify that the convolution of $s(t)$ with itself is given by

$$\begin{aligned} s^{*2}(t) &= (s * s)(t) \\ &= \int_{-\infty}^{\infty} s(\tau)s(t - \tau)\mathrm{d}\tau \\ &= \frac{(\lambda + |t|)}{4\lambda^2} e^{-\frac{|t|}{\lambda}}. \end{aligned} \tag{G.2.8}$$

It is interesting to explore what happens if we take repeated convolution of a function with itself. To this end, let

$$g^{*n}(t) = \underbrace{(g * g * \cdots * g)}_{n \text{ times}}(t)$$

denote the n-fold convolution of $g(t)$ with itself. It turns out that under some broad conditions on $g(t)$, the limit of $g^{*n}(t)$, as n becomes large, is independent of $g(t)$.

(i) **Repeated convolution: a form of a central limit theorem** Let $g(t) \in D$ be such that for some positive constants α and β, its Fourier transform can be approximated by

$$G(f) = \alpha - \beta f^2 \tag{G.2.9}$$

for small f. Then

$$\lim_{n \longrightarrow \infty} \left[\frac{\sqrt{n} g(\sqrt{n} t)}{\alpha} \right]^{*n} = \sqrt{\frac{\pi \alpha}{\beta}} \exp\left[-\frac{\pi^2 \alpha}{\beta} t^2 \right]. \tag{G.2.10}$$

Since this result is central to deriving properties of *(spatial) recursive filters* in Chapter 21, we now provide a proof of this result.

By the stretching property, we obtain $n > 0$

$$F\left[[\sqrt{n} g(\sqrt{n} t)] \right] = G\left(\frac{f}{\sqrt{n}} \right). \tag{G.2.11}$$

Now using the well-known result,

$$\lim_{n\longrightarrow\infty}\left(1-\frac{a}{n}\right)^n = \mathrm{e}^{-a} \tag{G.2.12}$$

and the convolution theorem, we get

$$\begin{aligned} F\left[\tfrac{1}{\alpha^n}[\sqrt{n}g(\sqrt{n}t)]^{*n}\right] &= \tfrac{1}{\alpha^n}G^n\left(\tfrac{f}{\sqrt{n}}\right) \quad \text{[by convolution theorem]} \\ &= \tfrac{1}{\alpha^n}\left[\alpha-\beta\left(\tfrac{f^2}{n}\right)\right]^n \quad \text{[by (G.2.9)]} \\ &= \left[1-\tfrac{\beta}{\alpha}\tfrac{f^2}{n}\right]^n \\ &\approx \mathrm{e}^{-\frac{\beta}{\alpha}f^2} \quad \text{[by (G.2.12)].} \end{aligned} \tag{G.2.13}$$

Now taking the inverse transform of both sides of (G.2.13), we get

$$\begin{aligned} \tfrac{1}{\alpha^n}[\sqrt{n}g(\sqrt{n}t)]^{*n} &= F^{-1}\left[\mathrm{e}^{-\frac{\beta}{\alpha}f^2}\right] \\ &= F^{-1}\left[\mathrm{e}^{-\pi\left(\sqrt{\frac{\beta}{\alpha\pi}}f\right)^2}\right] \\ &= F^{-1}\left[\mathrm{e}^{-\pi\left[\frac{f}{\sqrt{\frac{\alpha\pi}{\beta}}}\right]^2}\right] \\ &= \sqrt{\frac{\alpha\pi}{\beta}}\mathrm{e}^{-\pi\left[t\sqrt{\frac{\alpha\pi}{\beta}}\right]^2} \\ &= \sqrt{\frac{\alpha\pi}{\beta}}\mathrm{e}^{-\frac{\alpha}{\beta}\pi^2t^2} \end{aligned} \tag{G.2.14}$$

where we have used the result relating to the Gaussian pulse from Example G.1.2 and the stretching property. Clearly, the r.h.s. of (G.2.14) is independent of $g(t)$.

We now need to extract the n-fold convolution of $g(t)$ from that of $\sqrt{n}g(\sqrt{n}t)$. To this end, observe from the derivation leading to (G.2.13) that

$$G^n\left(\frac{f}{\sqrt{n}}\right) \approx \alpha\mathrm{e}^{-\frac{\alpha}{\beta}f^2}. \tag{G.2.15}$$

By changing the variables, we immediately get

$$G^n(f) \approx \alpha^n\mathrm{e}^{-\frac{n\beta}{\alpha}f^2}. \tag{G.2.16}$$

Taking the inverse transform of both sides,

$$\begin{aligned} g^{*n}(t) &= F^{-1}[G^n(f)] \\ &= F^{-1}\left[\alpha^n\mathrm{e}^{-\frac{n\beta}{\alpha}f^2}\right] \\ &= \alpha^nF^{-1}\left[\mathrm{e}^{-\pi}\left(\tfrac{f}{r}\right)^2\right] \end{aligned} \tag{G.2.17}$$

where

$$r = \sqrt{\frac{\pi\alpha}{\beta n}}. \tag{G.2.18}$$

Again, from Example G.1.2 and the stretching property, we obtain

$$\begin{aligned} g^{*n}(t) &= \alpha^n r \mathrm{e}^{-\pi(rt)^2} \\ &= \alpha^n \sqrt{\tfrac{\pi\alpha}{\beta n}} \mathrm{e}^{-\frac{\pi^2\alpha}{\beta n}t^2} \end{aligned} \tag{G.2.19}$$

which is the desired result.

Example G.2.4 We compute the n-fold convolution of $s(t)$ in (G.2.2). The Fourier transform pairs are such that

$$s(t) = \frac{1}{2\pi}\mathrm{e}^{-\frac{|t|}{\lambda}} \Longleftrightarrow \frac{1}{1+4\pi^2 f^2\lambda^2} = s(f). \tag{G.2.20}$$

This $s(f)$ satisfies the condition (G.2.9) since

$$s(f) = [1 + 4\pi^2 f^2 \lambda^2]^{-1} \approx 1 - (4\pi^2\lambda^2) f^2.$$

Comparing this with (G.2.9), we get $\alpha = 1$ and $\beta = 4\pi^2\lambda^2$. Substituting this into (G.2.19), we readily obtain

$$\begin{aligned} [s(t)]^{*n} &= \tfrac{1}{2\lambda\sqrt{\pi n}} \exp\left[-\tfrac{t^2}{4\lambda^2 n}\right] \\ &= \tfrac{1}{\sqrt{2\pi}\sigma} \exp\left[-\tfrac{t^2}{2\sigma^2}\right] \end{aligned} \tag{G.2.21}$$

where

$$\sigma^2 = 2\lambda^2 n \tag{G.2.22}$$

that is, $[s(t)]^{*n}$ has the shape of the Gaussian with mean zero and variance $\sigma^2 = 2\lambda^2 n$.

G.3 Fourier transform of discrete signals (infinite duration)

Let $z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ be the set of all integers, $N^+ = \{0, 1, 2, 3, \ldots\}$ be the set of all nonnegative integers. The function $g : z \longrightarrow \mathbb{R}$ is called a two-way infinite sequence and is denoted by $\{\ldots, g_{-2}, g_{-1}, g_0, g_1, g_2, \ldots\}$. Likewise $g^+ : N^+ \longrightarrow \mathbb{R}$ is denoted by $\{g_0, g_1, g_2, \ldots\}$ and $g^- : N^- \longrightarrow \mathbb{R}$, is denoted by $\{g_0, g_{-1}, g_{-2}, \ldots\}$ are one-way finite sequences.

The sequence $g : z \longrightarrow \mathbb{R}$ is said to be *absolutely* summable if

$$\sum_{n=-\infty}^{\infty} |g_n| < \infty. \tag{G.3.1}$$

The *discrete time Fourier transform* (DTFT) of g is given by

$$G(f) = \sum_{n=-\infty}^{\infty} g_n \mathrm{e}^{-\mathrm{i}2\pi f n}. \tag{G.3.2}$$

Since

$$\mathrm{e}^{\pm \mathrm{i}2\pi f} = \cos 2\pi f \pm \sin 2\pi f$$

it follows that $G(f)$ is *periodic* with period = 1, that is

$$G(f+1) = G(f). \tag{G.3.3}$$

Henceforth, we define conveniently in the range $[-1/2, 1/2)$. The inverse transform is defined by

$$g_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} G(f)\mathrm{e}^{\mathrm{i}2\pi f n}\mathrm{d}f \quad n = 0, \pm 1, \pm 2, \ldots \tag{G.3.4}$$

Example G.3.1 For some $0 < \alpha < 1$, let $g^+ : z \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} g_n^+ &= (1-\alpha)\alpha^n \quad \text{for } n \geq 0 \\ &= 0 \quad \text{for } n < 0 \end{aligned} \tag{G.3.5}$$

and its *dual* $g^- : z \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} g_n^- &= (1-\alpha)\alpha^{-n} \quad \text{for } n \leq 0. \\ &= 0 \quad \text{for } n > 0. \end{aligned} \tag{G.3.6}$$

It can be verified that

$$\sum_{n=0}^{\infty} g_n^+ = \sum_{n=-\infty}^{-1} g_n^- = 1. \tag{G.3.7}$$

Given α, there exists a real constant $a > 0$ such that $\alpha = \mathrm{e}^{-a}$. Substituting in (G.3.5) and (G.3.6), we obtain,

$$\begin{aligned} g_n^+ &= (1-\mathrm{e}^{-a})\mathrm{e}^{-an} \quad \text{for } n \geq 0 \\ &= 0 \quad \text{for } n > 0 \end{aligned} \tag{G.3.8}$$

and

$$\begin{aligned} g_n^- &= (1-\mathrm{e}^{-a})\mathrm{e}^{an} \quad \text{for } n \leq 0 \\ &= 0 \quad \text{for } n > 0. \end{aligned} \tag{G.3.9}$$

Setting $a = 1/\lambda$, from

$$(1-\mathrm{e}^{-a}) = 1 - \mathrm{e}^{-\frac{1}{\lambda}} \approx \frac{1}{\lambda}.$$

the reader can ascertain that (G.3.5) and (G.3.6) are the discrete analogs of the exponential functions $g^+(t)$ and $g^-(t)$ in Example G.2.2.

The DTFT of g^+ in (G.3.5) is given by

$$\begin{aligned}
G^+(f) &= (1-\alpha)\sum_{n=0}^{\infty}\alpha^n \mathrm{e}^{-2\pi f n} \\
&= (1-\alpha)\sum_{n=0}^{\infty}\mathrm{e}^{-[a+\mathrm{i}2\pi f]n} \quad [\text{using } \alpha = \mathrm{e}^{-a}] \\
&= \frac{1-\alpha}{1-\mathrm{e}^{-[a+\mathrm{i}2\pi f]}} \\
&= \frac{1-\alpha}{1-\alpha \mathrm{e}^{-\mathrm{i}2\pi f}}. \qquad \text{(G.3.10)}
\end{aligned}$$

Similarly, the DTFT of g^- in (G.3.6) is given by

$$\begin{aligned}
G^-(f) &= (1-\alpha)\sum_{n=-\infty}^{0}\mathrm{e}^{[a-\mathrm{i}2\pi f]n} \\
&= (1-\alpha)\sum_{n=0}^{\infty}\mathrm{e}^{-[a-\mathrm{i}2\pi f]n} \\
&= \frac{1-\alpha}{1-\mathrm{e}^{-[a-\mathrm{i}2\pi f]}} \\
&= \frac{1-\alpha}{1-\alpha \mathrm{e}^{\mathrm{i}2\pi f}} = G^+(-f). \qquad \text{(G.3.11)}
\end{aligned}$$

It is readily verified that $G^+(f)$ and $G^-(f)$ in (G.3.10) and (G.3.11) are complex conjugates of each other, that is, they share a common amplitude but are of opposite phase.

Discrete convolution Let $g = \{g_n\}$ and $h = \{h_n\}$ be two sequences. The (discrete) convolution of g and h is a sequence $s = \{s_n\}$ denoted by $s = g * h = h * g$ where

$$s_n = \sum_{k=-\infty}^{\infty} g_k h_{n-k}. \qquad \text{(G.3.12)}$$

Example G.3.2 Let $s = \{s_n\}$ denote the convolution of the two sequences g^+ and g^- defined in Example G.3.1, where

$$s_n = \sum_{k=-\infty}^{\infty} g_k^- g_{n-k}^+. \qquad \text{(G.3.13)}$$

When $n = 0$, using the definition in (G.3.5) and (G.3.6),

$$\begin{aligned}
s_0 &= \textstyle\sum_{k=-\infty}^{\infty} g_k^- g_{-k}^+ \\
&= \textstyle\sum_{k=-\infty}^{0} g_k^- g_{-k}^+ \quad (\text{since } g_k^- = 0 \quad \text{for } k > 0) \\
&= g_0^- g_0^+ + g_{-1}^{-1} g_1^{+1} + g_{-2}^- g_2^+ + \cdots \\
&= (1-\alpha)^2 \textstyle\sum_{k=0}^{\infty}\alpha^{2k} \\
&= \frac{1-\alpha}{1+\alpha}.
\end{aligned}$$

When $n = 1$,

$$\begin{aligned} s_1 &= \textstyle\sum_{k=-\infty}^{0} g_k^- g_{1-k}^+ \\ &= g_0^- g_1^+ + g_{-1}^- g_2^+ + g_{-2}^- g_3^+ + \cdots \\ &= (1-\alpha)^2\{\alpha + \alpha^3 + \alpha^5 + \cdots\} \\ &= \alpha(1-\alpha)^2 \textstyle\sum_{k=0}^{\infty} \alpha^{2k} \\ &= \left(\frac{1-\alpha}{1+\alpha}\right)\alpha. \end{aligned}$$

Similarly, it can be verified that

$$s_{-1} = \left(\frac{1-\alpha}{1+\alpha}\right)\alpha.$$

Indeed, for any n, it can be verified that

$$s_{\pm n} = \left(\frac{1-\alpha}{1+\alpha}\right)\alpha^{|n|} \tag{G.3.14}$$

which is a *symmetric function*.

Discrete Convolution Theorem Let $g = \{g_n\}$ and $h = \{h_n\}$ be two sequences. The (discrete) convolution of g and h is a sequence $s = \{s_n\}$ denoted by $s = g * h$ where

$$s_n = \sum_{k=-\infty}^{\infty} g_k h_{n-k}. \tag{G.3.15}$$

The DTFT $S(f)$ of s_n is given by

$$\begin{aligned} S(f) &= \textstyle\sum_{n=-\infty}^{\infty} s_n \mathrm{e}^{-\mathrm{i}2\pi f n} \\ &= \textstyle\sum_{n=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} g_k h_{n-k}\right) \mathrm{e}^{-\mathrm{i}2\pi f n} \\ &= \textstyle\sum_{r=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_k h_r \mathrm{e}^{-i2\pi f(k+r)} \\ &= \textstyle\sum_{r=-\infty}^{\infty} h_r \mathrm{e}^{-\mathrm{i}2\pi f r} \sum_{k=-\infty}^{\infty} g_k \mathrm{e}^{-\mathrm{i}2\pi f k} \\ &= H(f)G(f) = G(f)H(f). \end{aligned}$$

That is, the DTFT of the convolution of two sequences g and h is the product of the DTFT of g and h.

Example G.3.3 We now compute the DTFT of the convolution of the two sequences g^+ and g^- described in Example G.3.1. Indeed, the DTFT $S(f)$ of the sequence s in (G.3.14) which is the convolution of g^+ and g^- in (G.3.5) and (G.3.6) is obtained as the product of the DTFT $G^+(f)$ and $G^-(f)$ in (G.3.10) and

(G.3.11). Thus,

$$
\begin{aligned}
S(f) &= G^+(f)G^-(f) \\
&= \frac{(1-\alpha)^2}{(1-\alpha e^{-i2\pi f})(1-\alpha e^{i2\pi f})} \\
&= \frac{(1-\alpha)^2}{(1-\alpha)^2 + 2\alpha[1-\cos 2\pi f]} \\
&= \frac{1}{1+\dfrac{\alpha}{(1-\alpha)^2}[2\sin \pi f]^2}.
\end{aligned} \tag{G.3.16}
$$

G.4 Fourier series

We begin by defining *orthogonal* functions over the interval $[a, b]$ where $a < b$ are two real numbers.

(A) **Functions in one dimension** Let f and g be two real-valued functions over $[a, b]$. The *inner product* $< f, g >$ of f and g is defined by

$$< f, g >= \int_a^b f(x)g(x)\mathrm{d}x. \tag{G.4.1}$$

The *norm* $\|f\|$ of $f(x)$ is given by

$$\|f\| =< f, f >= \int_a^b f^2(x)\mathrm{d}x > 0. \tag{G.4.2}$$

If $\|f\| = 1$, then f is said to be *normalized*. We say that the functions f and g are *orthogonal* when

$$< f, g >= 0. \tag{G.4.3}$$

A sequence of functions $\{g_k(x)\}_{k=0}^{\infty}$ is said to be a (pair-wise) *orthogonal* system if

$$< g_m, g_n >= 0 \quad \text{for } m \neq n \tag{G.4.4}$$

and

$$\|g_n\|^2 = \lambda_n^2 > 0 \ \text{ for all } m. \tag{G.4.5}$$

If $\lambda_n = 1$, then $\{g_k(x)\}_{k=0}^{\infty}$ is called an *orthonormal system.*

Example G.4.1 The trigonometric sequence

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \ldots, \frac{1}{\sqrt{\pi}}\sin kx, \frac{1}{\sqrt{\pi}}\cos kx, \ldots \right\}$$

is an *orthonormal* system over $[-\pi, \pi]$. That is,

$$g_0(x) = \frac{1}{\sqrt{2\pi}}$$

$$\left.\begin{aligned} g_{2k-1}(x) &= \tfrac{1}{\sqrt{\pi}} \sin kx \\ g_{2k}(x) &= \tfrac{1}{\sqrt{\pi}} \cos kx \end{aligned}\right\} \text{ for } k = 1, 2, 3, \ldots$$

Example G.4.2 The trigonometric sequence

$$\{1, \sin\frac{\pi x}{a}, \cos\frac{\pi x}{a}, \ldots \sin\frac{k\pi x}{a}, \cos\frac{k\pi x}{a} \ldots\}$$

is an *orthogonal system* over the interval $[-a, a]$. Thus,

$$g_0(x) = 1$$

$$\left.\begin{aligned} g_{2k-1}(x) &= \sin\tfrac{k\pi x}{a} \\ g_{2k}(x) &= \cos\tfrac{k\pi x}{a} \end{aligned}\right\} \text{ for } k = 1, 2, 3, \ldots$$

Similarly, it can be verified that

$$g_0(x) = 0 \text{ and } g_k(x) = \sqrt{\frac{2}{\pi}} \sin kx, \quad \text{for } k = 1, 2, 3, \ldots$$

is an orthonormal system over $[0, \pi]$.

Fourier series expansion Given an orthogonal system $\{g_k(x)\}_{k=0}^{\infty}$ over the interval $[a, b]$, any piecewise, continuous function $f(x)$ over $[a, b]$ can be represented by a (formal) series

$$f(x) = \sum_{k=0}^{\infty} c_k g_k(x) \tag{G.4.6}$$

called the *Fourier series* of $f(x)$ where the Fourier coefficients c_k are obtained using the orthogonality of $\{g_k\}_{k=0}^{\infty}$ as

$$c_k = \frac{1}{\|g_k\|} \int_a^b f(x) g_k(x) \mathrm{d}x. \tag{G.4.7}$$

Clearly, if $\{g_k\}$ is orthonormal, then $\|g_k\| = 1$ for all k.

Example G.4.3 Using the orthonormal system over $[-\pi, \pi]$ in Example G.4.1, it can be verified that the Fourier coefficients c_k are given by

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x$$

$$\left.\begin{aligned} c_{2k-1} &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(x)\sin kx\ \mathrm{d}x \\ c_{2k} &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(x)\cos kx\ \mathrm{d}x \end{aligned}\right\}\quad \text{for } k = 1, 2, 3, \ldots$$

Combining this with (G.4.6) and (G.4.7), we obtain the standard Fourier expansion

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}(c_{2k-1}\cos kx + c_{2k}\sin kx). \tag{G.4.8}$$

Example G.4.4 Let $f(x)$ be an odd periodic function with period 2π where

$$\begin{aligned} f(x) &= 1 \qquad \text{for } 0 < x < \pi \\ &= -1 \qquad \text{for } -\pi < x < 0. \end{aligned}$$

Then, it can be verified that the Fourier expansion of $f(x)$ is given by

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{(2k-1)}. \tag{G.4.9}$$

Example G.4.5 Let $f(x) = x^2$ for $0 < x < 2\pi$. Then,

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{\cos\ kx}{k^2} - 4\pi\sum_{k=1}^{\infty}\frac{\sin\ kx}{k}. \tag{G.4.10}$$

One of the most fundamental property of the Fourier series expansion is the inherent optimality property as expressed in the following.

Least squares property Let $\{g_k\}_{k=0}^{\infty}$ be an orthogonal system over $[a, b]$. Let the (finite) linear combination $\sum_{k=0}^{N} a_k g_k(x)$ with arbitrary coefficients $a_0, a_1, \ldots, a_N$ denote an approximation $f(x)$ over $[a, b]$. Then $[f(x) - \sum_{k=0}^{N} a_k g_k(x)]$ denotes the *error* in this approximation. The norm of this error is a minimum exactly when $a_k = c_k$, the Fourier coefficients defined in (G.4.7). That is,

$$\|f(x) - \sum_{k=0}^{N} c_k g_k(x)\| \leq \|f(x) - \sum_{k=0}^{N} a_k g_k(x)\| \tag{G.4.11}$$

for any arbitrary set of coefficients $a_0, a_1, \ldots, a_N$.

Remark G.4.1 This is the infinite-dimensional analog of the projection theorem described in Appendix A.

(B) **Functions in two dimensions** Consider a rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$ for some $a < b$ and $c < d$. The system $\{g_k(x, y)\}_{k=0}^{\infty}$ is said to be *orthogonal* over $[a, b] \times [c, d]$ if there inner product

$$< g_m, g_n >= \int_c^d \int_a^b g_m(x, y) g_n(x, y) \mathrm{d}x \mathrm{d}y = 0 \qquad \text{for } m \neq n \tag{G.4.12}$$

and

$$\|g_m\|^2 =< g_m, g_m >= \int_c^d \int_a^b g_m^2(x, y) \mathrm{d}x \mathrm{d}y > 0 \qquad \text{for all } m. \tag{G.4.13}$$

Again, if $\|g_k\| = 1$ for all k, then $\{g_k(x, y)\}$ is an *orthonormal* system.

Fourier series expansion Let $f(x, y)$ be given over $[a, b] \times [c, d]$. Then,

$$f(x, y) = \sum_{k=0}^{\infty} c_k g_k(x, y) \tag{G.4.14}$$

where

$$c_k = \frac{1}{\|g_k\|^2} \int_c^d \int_a^b f(x, y) g_k(x, y) \mathrm{d}x \mathrm{d}y \tag{G.4.15}$$

are the Fourier coefficients.

Let $\{g_k(x)\}_{i=0}^{\infty}$ and $\{h_j(y)\}_{j=0}^{\infty}$ be two orthogonal systems over the one-dimensional intervals $[a, b]$ and $[c, d]$ respectively. Then, it can be verified that the product system

$$\{g_i(x) h_j(y) \mid 0 \leq i < \infty \text{ and } 0 \leq j < \infty\}$$

is an orthogonal system over $[a, b] \times [c, d]$. Consequently,

$$\{\cos mx \cos ny, \cos mx \sin ny, \sin mx \cos ny, \sin mx \sin ny | 0 \leq m < \infty, 0 \leq n < \infty\}$$

is an orthogonal system over $[-\pi, \pi] \times [-\pi, \pi]$. Using (G.4.13) that $\|1\|^2 = 4\pi^2$,

$$\| \cos ny\|^2 = \| \cos mx\|^2 = \| \sin mx\|^2 = \| \sin ny\|^2 = 2\pi^2$$

$$\| \cos mx \cos ny\|^2 = \pi^2$$

$$\| \cos mx \sin ny\|^2 = \pi^2$$

$$\| \sin mx \sin ny\|^2 = \pi^2$$

$$\| \sin mx \cos ny\|^2 = \pi^2.$$

Hence, any piecewise continuous function $f(x)$ over $[a, b] \times [c, d]$ can be expressed in a *double Fourier Series* as

$$f(x, y) = \sum_{m,n} \lambda_{mn} [A_{mn} \cos mx \cos ny + B_{mn} \sin mx \cos ny + C_{mn} \cos mx \sin ny + D_{mn} \sin mx \sin ny] \tag{G.4.16}$$

where

$$\begin{aligned}\lambda_{mn} &= \tfrac{1}{4} \ \text{ if } m = 0, n = 0\\ &= \tfrac{1}{2} \ \text{ if } m = 0, n \neq 0, \ \text{ or } m \neq 0, n = 0\\ &= 1 \ \text{ if } m \neq 0, n \neq 0\end{aligned}$$

$$\begin{aligned}A_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\cos mx \cos ny \mathrm{d}x\mathrm{d}y\\ B_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\sin mx \cos ny \mathrm{d}x\mathrm{d}y\\ C_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\cos mx \sin ny \mathrm{d}x\mathrm{d}y\end{aligned}$$

and

$$D_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x, y)\sin mx \sin ny \mathrm{d}x\mathrm{d}y.$$

Example G.4.6 Let $f(x, y) = xy^2$ on $[-\pi, \pi] \times [-\pi, \pi]$. Then, the double Fourier series for $f(x, y)$ is

$$\begin{aligned}f(x, y) = \frac{2\pi^2}{3}&\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\sin mx\\ &- 8\sum_{m,n=1}^{\infty}\frac{(-1)^{m+1}}{mn^2}\sin mx \cos ny.\end{aligned} \tag{G.4.17}$$

Example G.4.7 Let $f(x, y) = xy$ on $[0, 2\pi] \times [0, 2\pi]$. Then,

$$\begin{aligned}f(x, y) = \pi^2 - 2\pi\sum_{m=1}^{\infty}&\frac{\sin mx}{m} - 2\pi \textstyle\sum_{n=1}^{\infty} \displaystyle\frac{\sin ny}{n}\\ &+ 4\sum_{m,n=1}^{\infty}\frac{\sin mx \sin ny}{mn}.\end{aligned} \tag{G.4.18}$$

The least squares property (G.4.11) also carries over to double Fourier Series.

Notes and references

Fourier analysis in one dimension is covered extensively in Gary and Goodman (1995), Papoulis (1962) and Bracewell (1965). For an introduction to Fourier analysis in multiple dimensions, refer to Walker (1988) and Rees et al. (1981). Our treatment in this Appendix follows from Gary and Goodman (1995) and Walker (1988).