Digital Logic Design: a rigorous approach ©

Chapter 14: Shifters

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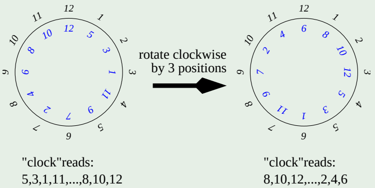
Example

- assume that we place the bits of $a[1:12]$ on a wheel.
- $a[1]$ is at one o'clock, $a[2]$ is at two o'clock, etc.
- rotate the wheel, and read the bits in clockwise order starting from one o'clock and ending at twelve o'clock.
- the resulting string is a cyclic shift of $a[1:12]$.



Definition of a Cyclic Shifter

We denote (a mod b) by $\mathrm{mod}(a, b)$.

Definition

The string $b[n-1:0]$ is a cyclic left shift by i positions of the string $a[n-1:0]$ if

$$\forall j: \quad b[j] = a[\mathrm{mod}(j-i, n)].$$

Example

Let $a[3:0] = 0010$. A cyclic left shift by one position of $\vec{a}$ is the string 0100. A cyclic left shift by 3 positions of $\vec{a}$ is the string 0001.

Definition of BARREL-SHIFTER(n)

Definition

A BARREL-SHIFTER(n) is a combinational circuit defined as follows:

Input: $x[n-1:0] \in \{0,1\}^n$ and $sa[k-1:0] \in \{0,1\}^k$ where $k = \lceil \log_2 n \rceil$.

Output: $y[n-1:0] \in \{0,1\}^n$.

Functionality: $\vec{y}$ is a cyclic left shift of $\vec{x}$ by $\langle \vec{sa} \rangle$ positions. Formally,

$$\forall j \in [n-1:0] : \quad y[j] = x[\mathrm{mod}(j - \langle \vec{sa} \rangle, n)].$$

We often refer to the input $\vec{x}$ as the data input and to the input $\vec{sa}$ as the shift amount input. **To simplify the discussion, we assume that n is a power of 2, namely, $n = 2^k$.**

$\text{BARREL-SHIFTER}(n)$ Implementation

We break the task of designing a barrel shifter into smaller sub-tasks of shifting by powers of two. We define this sub-task formally as follows.
A $\text{CLS}(n, 2^i)$ is a combinational circuit that implements a cyclic left shift by zero or 2^i positions depending on the value of its select input.

Definition

A $\text{CLS}(n, i)$ is a combinational circuit defined as follows:

Input: $x[n-1:0]$ and $s \in \{0, 1\}$.

Output: $y[n-1:0]$.

Functionality:

$$\forall j \in [n-1:0]: \quad y[j] = x[\mathrm{mod}(j - s \cdot i, n)].$$

Subtask: $\text{CLS}(n, i)$ Implementation

A $\text{CLS}(n, i)$ is quite simple to implement since:

- $y[j]$ is either $x[j]$ or $x[\text{mod}(j - i, n)]$.
- So all one needs is a MUX-gate to select between $x[j]$ or $x[\text{mod}(j - i, n)]$.
- The selection is based on the value of s.
- It follows that the delay of $\text{CLS}(n, i)$ is the delay of a MUX, and the cost is n times the cost of a MUX.

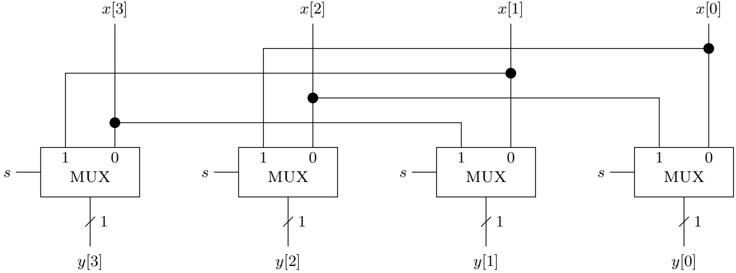


Figure: A row of multiplexers implement a $\text{CLS}(4, 2)$.

Back to BARREL-SHIFTER(n)

- The design of a BARREL-SHIFTER(n) is based on CLS($n, 2^i$) shifters.
- The implementation is based on k levels of CLS($n, 2^i$), for $i \in [k-1:0]$.
- The ith level is controlled by $sa[i]$.

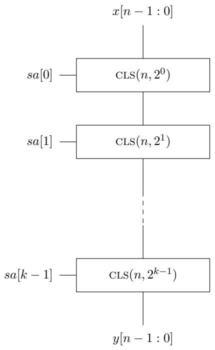


Figure: A BARREL-SHIFTER(n) built of k levels of CLS($n, 2^i$) ($n = 2^k$).

Correctness

Observation

For every $x, q \in \mathbb{Z}$,

$$\text{mod}(x, n) = \text{mod}(x + qn, n).$$

Observation

If $\alpha = \text{mod}(a, n)$ *and* $\beta = \text{mod}(b, n)$, *then*

$$\text{mod}(a - b, n) = \text{mod}(\alpha - \beta, n)\ .$$

Claim

The barrel shifter design depicted in the previous slide is correct.

Proof.

Prove by induction on i, that output of $\text{CLS}(n, 2^i)$ equals the cyclic left shift of x by $\langle sa[i:0] \rangle$. □

Cost & Delay

Claim

The cost and delay of BARREL-SHIFTER(n) *satisfy:*

$$c(\text{BARREL-SHIFTER}(n)) = n \log_2 n \cdot c(\text{MUX})$$
$$d(\text{BARREL-SHIFTER}(n)) = \log_2 n \cdot d(\text{MUX}).$$

Proof.

Follows from the fact that the design consists of $\log_2 n$ levels of CLS$(n, 2^i)$ shifters. □

The cone of the Barrel Shifter

Consider the output $y[0]$ of BARREL-SHIFTER(n) .

Claim

The cone of the Boolean function implemented by the output $y[0]$ contains at least n elements.

Corollary

The delay of BARREL-SHIFTER(n) is asymptotically optimal.

Logical Shift

Definition

The binary string $y[n-1:0]$ is a logical left shift by ℓ positions of the binary string $x[n-1:0]$ if

$$y[i] \triangleq \begin{cases} 0 & \text{if } i < \ell \\ x[i-\ell] & \text{if } \ell \leq i < n. \end{cases}$$

Example

$y[3:0] = 0100$ is a logical left shift of $x[3:0] = 1001$ by $\ell = 2$ positions. When we apply a logical left shift to $x[n-1:0]$ by ℓ positions, we obtain the string $x[n-1-\ell:0] \circ 0^{\ell}$.

Fast multiplication

In binary representation, logical shifting to the left by s positions corresponds to multiplying by 2^s followed by modulo 2^n.

Logical Shifters (cont.)

Definition

The binary string $y[n-1:0]$ is a logical right shift by ℓ positions of the binary string $x[n-1:0]$ if

$$y[i] \triangleq \begin{cases} 0 & \text{if } i \geq n-\ell \\ x[i+\ell] & \text{if } 0 \leq i < n-\ell. \end{cases}$$

Example

$y[3:0] = 0010$ is a logical right shift of $x[3:0] = 1001$ by $\ell = 2$ positions. When we apply a logical right shift to $x[n-1:0]$ by ℓ positions, we obtain the string $0^{\ell} \circ x[n-1:\ell]$.

Fast division

In binary representation, logical shifting to the right by s positions corresponds to the integer part of the quotient after division by 2^s.

Notation.

- Let $\text{LLS}(\vec{x}, i)$ denote the logical left shift of $\vec{x}$ by i positions.
- Let $\text{LRS}(\vec{x}, i)$ denote the logical right shift of $\vec{x}$ by i positions.

A bi-directional logical shifter

A bi-directional logical shifter is defined as follows.

Definition

A $\text{L-SHIFT}(n)$ is a combinational circuit defined as follows:

Input:

- $x[n-1:0] \in \{0,1\}^n$,
- $sa[k-1:0] \in \{0,1\}^k$, where $k = \lceil \log_2 n \rceil$, and
- $\ell \in \{0,1\}$.

Output: $y[n-1:0] \in \{0,1\}^n$.

Functionality: The output $\vec{y}$ satisfies

$$\vec{y} \triangleq \begin{cases} \text{LLS}(\vec{x}, \langle \vec{sa} \rangle) & \text{if } \ell = 1, \\ \text{LRS}(\vec{x}, \langle \vec{sa} \rangle) & \text{if } \ell = 0. \end{cases}$$

A bi-directional logical shifter (cont.)

Example

- let $x[3:0] = 0010$.
- If $sa[1:0] = 10$ and $\ell = 1$, then L-SHIFT(4) outputs $y[3:0] = 1000$.
- If $\ell = 0$, then the output equals $y[3:0] = 0000$.

Implementation

As in the case of cyclic shifters, we break the task of designing a logical shifter into sub-tasks of logical shifts by powers of two.

Definition

An $\text{LBS}(n, i)$ is a combinational circuit defined as follows:

Input: $x[n-1:0]$ and $s, \ell \in \{0,1\}$.

Output: $y[n-1:0]$.

Functionality: The output $\vec{y}$ satisfies

$$\vec{y} \triangleq \begin{cases} \vec{x} & \text{if } s = 0, \\ \text{LLS}(\vec{x}, i) & \text{if } s = 1 \text{ and } \ell = 1, \\ \text{LRS}(\vec{x}, i) & \text{if } s = 1 \text{ and } \ell = 0. \end{cases}$$

The role of the input s in is to determine if a shift (in either direction) takes place at all. If $s = 0$, then $y[j] = x[j]$, and no shift takes place. If $s = 1$, then the direction of the shift is determined by ℓ.

LBS(n, i)

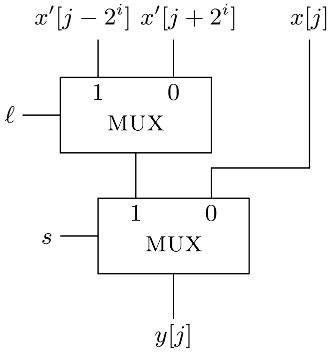


Figure: A bit-slice of an implementation of LBS($n, 2^i$).

question

Design a bi-directional logical shifter L-SHIFT(n).

Arithmetic Shifters

Arithmetic shifters are used for shifting binary strings that represent signed integers in two's complement representation. **Since left shifting is the same in logical shifting and in arithmetic shifting, we discuss only right shifting** (i.e., division by a power of 2).

Definition

The binary string $y[n-1:0]$ is an arithmetic right shift by ℓ positions of the binary string $x[n-1:0]$ if the following holds:

$$y[i] \triangleq \begin{cases} x[n-1] & \text{if } i \geq n-\ell \\ x[i+\ell] & \text{if } 0 \leq i < n-\ell. \end{cases}$$

Arithmetic Shifters (cont.)

Example

- $y[3:0] = 0010$ is an arithmetic shift of $x[3:0] = 0101$ by $\ell = -1$ positions.
- On the other hand, $y[3:0] = 1110$ is an arithmetic shift of $x[3:0] = 1001$ by $\ell = -2$ positions.
- When we apply an arithmetic shift by $\ell < 0$ positions to $x[n-1:0]$, we obtain the string $x[n-1]^{\ell} \circ x[n-1:\ell]$.

Notation.

Let $\text{ARS}(\vec{x}, i)$ denote the arithmetic right shift of $\vec{x}$ by i positions.

An arithmetic right shifter

Definition

An ARITH-SHIFT(n) is a combinational circuit defined as follows:

Input: $x[n-1:0] \in \{0,1\}^n$ and $sa[k-1:0] \in \{0,1\}^k$, where $k = \lceil \log_2 n \rceil$.

Output: $y[n-1:0] \in \{0,1\}^n$.

Functionality: The output $\vec{y}$ is a (sign-extended) arithmetic right shift of $\vec{x}$ by $\langle \vec{sa} \rangle$ positions. Formally,

$$y[n-1:0] \triangleq \text{ARS}(x[n-1:0], \langle \vec{sa} \rangle).$$

Example

Let $x[3:0] = 1001$. If $sa[1:0] = 10$, then ARITH-SHIFT(4) outputs $y[3:0] = 1110$.

question

Design an arithmetic right shifter ARITH-SHIFT(n).