Solutions to exercises in chapter 4

1. **Resonance and absorption**

   $m = 0.50(1)$ kg;k= 5.0(1) N/m; $f = 0.90(9) kg/s$; $F_{ext} = F_0 cos(\Omega t)$; $F_0 = 2.0(2)N$; $2\pi/\Omega = 1.00(1)$ s.

   a) The natural frequency of this spring pendulum is $\omega = \sqrt{k/m} = \sqrt{5N/m/0.5kg} = \sqrt{10kg/(s^2kg)} \simeq \pi s^{-1}$.

   b) The quantity whose error we want to determine is the square root of a ratio, hence we best work with relative errors and have $r_\omega = 1/2\sqrt{r_k^2 + r_m^2}$, mit $r_k = r_m = 0.02$, which yields $r_\omega = 1/2\sqrt{0.02^2 + 0.02^2} = 0.02/\sqrt{2} \simeq 1.4\%$

   c) The amplitude is given by $x_0 = \frac{F_0}{\sqrt{(m\Omega^2-k)^2+\Omega^2f^2}}$. With $\Omega = 2\pi s^{-1}$ given, we have $\Omega^2 \simeq 40s^{-2}$, and hence $m\Omega^2 - k = 40s^{-2}0.5kg - 5N/m = 20 - 5kg/s^2 = 15kg/s^2$. Furthermore we have $(f\Omega)^2 = 40s^{-2}(0.9)^2kg^2s^{-2} = 8.1 \cdot 10^{-1}4 \cdot 10kg^2s^{-4} = 32kg^2s^{-4}$, which yields $(m\Omega^2 - k)^2 + \Omega^2 f^2 = 257kg^2s^{-4}$ and $\sqrt{(m\Omega^2 - k)^2 + \Omega^2 f^2} = \sqrt{257}kgs^{-2} \simeq \sqrt{256}kgs^{-2} = 16kgs^{-2}$ so we finally get the amplitude: $x_0 = 2N/16kgs^{-2} = 1/8m$.

   d) Error propagation tells us that we need to know: $\frac{\partial x_0}{\partial F_0}, \frac{\partial x_0}{\partial \Omega}, \frac{\partial x_0}{\partial m}, \frac{\partial x_0}{\partial k}, \frac{\partial x_0}{\partial f}$. We will use relative errors.

   $\frac{\partial x_0}{\partial F_0} = \frac{1}{\sqrt{(m\Omega^2-k)^2+\Omega^2f^2}} = x_0/F_0$

   $\frac{\partial x_0}{\partial \Omega} = -\frac{F_0}{\sqrt{(m\Omega^2-k)^2+\Omega^2f^2}^3}(\Omega f^2 + (m\Omega^2 - k)2m\Omega) = -x_0/\Omega\frac{\Omega^2f^2+(m\Omega^2-k)2m\Omega^2}{(m\Omega^2-k)^2+\Omega^2f^2} = -x_0/\Omega\frac{32+15\cdot40}{256} = -x_0/\Omega \cdot 632/256$

   $\frac{\partial x_0}{\partial m} = -\frac{F_0}{\sqrt{(m\Omega^2-k)^2+\Omega^2f^2}^3}((m\Omega^2 - k)\Omega^2) = -x_0/m\frac{(m\Omega^2-k)m\Omega^2}{(m\Omega^2-k)^2+\Omega^2f^2} = -x_0/m\frac{15\cdot20}{256} = -x_0/m \cdot 300/256$

   $\frac{\partial x_0}{\partial k} = -\frac{F_0}{\sqrt{(m\Omega^2-k)^2+\Omega^2f^2}^3}((m\Omega^2 - k)(-1)) = x_0/m\frac{(m\Omega^2-k)k}{(m\Omega^2-k)^2+\Omega^2f^2} = x_0/m \cdot 75/256$

   $\frac{\partial x_0}{\partial f} = -\frac{F_0}{\sqrt{(m\Omega^2-k)^2+\Omega^2f^2}^3}(f\Omega^2) = -x_0/f\frac{(f^2\Omega^2}{(m\Omega^2-k)^2+\Omega^2f^2} = -x_0/f \cdot 32/256$

   Thus the relative error of x_0 is:

   $r_{x_0}^2 = r_{F_0}^2 + (\frac{\Omega^2f^2+(m\Omega^2-k)2m\Omega^2}{(m\Omega^2-k)^2+\Omega^2f^2})^2r_\Omega^2 + (\frac{(m\Omega^2-k)m\Omega^2}{(m\Omega^2-k)^2+\Omega^2f^2})^2r_m^2 + (\frac{(m\Omega^2-k)k}{(m\Omega^2-k)^2+\Omega^2f^2})^2r_k^2 + (\frac{(f^2\Omega^2}{(m\Omega^2-k)^2+\Omega^2f^2})^2r_f^2$

   Numerically:

   $r_{x_0}^2 = r_{F_0}^2 + (632/256)^2r_\Omega^2 + (300/256)^2r_m^2 + (75/256)^2r_k^2 + (32/256)^2r_f^2 \simeq r_{F_0}^2 + (2.5)^2r_\Omega^2 + (1.2)^2r_m^2 + (0.3)^2r_k^2 + (0.125)^2r_f^2$

   With the relative errors: $r_{F_0} = 0.1, r_\Omega = 0.01, r_m = 0.02, r_k = 0.02, r_f = 0.1$

   and hence $r_{x_0}^2 \simeq 0.01 + 0.0006 + 0.0006 + 0.00004 + 0.0001 \simeq 0.01$ also $r_{x_0} \simeq 10\%$

   e) No it doesn't. The oscillation is driven, such that the damping is compensated by the external force.

2. **Elastic waves**

   a) The string is fixed at both ends, such that the wave-length is $\lambda = 2L$ for the basic mode. Together with the frequency $\nu = 440$ Hz t: his gives the speed of the wave $v = \nu\lambda$. For a wave on a taught string, this is given by: $v = sqrt\sigma/\rho$, where $\sigma = Z/(\pi r^2)$ is the mechanical tension. We therefore obtain: $Z = \rho\lambda^2\nu^2\pi r^2 = 4 \cdot \rho L^2\nu^2\pi r^2$. Numerically: $Z = 4 \cdot 8 \cdot 10^3kg/m^3 0.3^2m^2 4.4^2 \cdot 10^4Hz^2\pi 0.2^2 \cdot 10^{-6}m^2 = 4 \cdot 8 \cdot 0.09 \cdot \pi \cdot 19.3 \cdot 0.04 \cdot 10N = 4 \cdot 8 \cdot \pi \cdot 0.9 \cdot 1.9 \cdot 0.4N \simeq 70N$.

   b) From a) we have: $Z = 4\rho L^2\nu^2\pi r^2$. We use relative errors to obtain: $r_Z^2 = 4(r_r^2 + r_\nu^2 + r_L^2)$. Numerically we have: $r_r = 0.1; r_L = 1/300; r_\nu = 0.05$. This means that we can safely neglect r_L, as it is more than 10 times smaller than r_ν and obtain $r_Z = 2\sqrt{5}0.05 = 4.5 \cdot 0.05 \simeq 0.22$ or $\sigma_Z \simeq 15N$.

c) From a) we have $\sqrt{\sigma/\rho} = 2L\nu$, so the maximum frequency is: $\nu_{max} = 1/(2L)\sqrt{sigma_m/\rho}$. Numerically: $\nu_{max} = 1/(0.6m)\sqrt{7 \cdot 10^8 N/m^2/(8 \cdot 10^3 kg/m^3)} = \sqrt{7/8 \cdot 10^5} m/s \cdot 1/(0.6m) = \sqrt{70/8} 10^2/0.6Hz \simeq \sqrt{36/4} 10^2/0.6Hz = 6/2 10^2/0.6Hz = 10^2/0.2Hz = 500Hz$.

3. **Elastic waves 2**

   The wave train will move with a speed of $v = sqrt\sigma/\rho$, where σ is determined by the restoring force due to gravity, i.e. $\sigma = F/A = mg/A = \rho Axg/A = \rho xg$. This means the speed of the wave is given by: $v = \sqrt{xg}$. To travel a distance L therefore takes a time of $T = \int_0^L dx/\sqrt{xg} = 2\sqrt{L/g}$. For comparison, the falling time of an object falling the same distance is $T_{fall} = \sqrt{2L/g}$.

4. **Sound intensity**

   Decibel(dB) is a logarithmic unit of sound intensity. 70 dB corresponds to an intensity of $10^{-5} W/m^2$. A decrease of 10dB corresponds to a decrease of a factor of 10 in intensity. We are thus looking for intensities of $10^{-6}, 10^{-7}, 10{-}8$ W/m^2. A straight highway of length L emits sound at a distance r through a surface of area $A = \pi r L$. Since the emitted power is constent, the intensity thus decreases as $1/A \propto 1/r$. Therefore we need to increase our distance by factors of ten, i.e. 500 m, 5 km and 50 km.

5. **Hearing threshold**

   For a length of 2.5 cm to have the largest amplitude of excitation while being attached at one end, the mode of excitation needs to be the basic standing wave. Therefore the corresponding wavelength needs to be $\lambda = 10$ cm. In order to obtain the frequency of sound that this belongs to, we use the fact that the speed of sound is given by the product of wavelength and frequency, i.e. $\nu = v_S/\lambda$. With a speed of sound of 300 m/s, we obtain a frequency of $\nu \simeq 3$ kHz.

6. **Hearing threshold 2**

   Thermal energy at room temperature is $k_B T = 4 \cdot 10^{-21}$ J. This energy needs to act onto the area $A = 0.25 \cdot 10^{-4}\text{m}^2$ during a time $\tau = 0.3 \cdot 10^{-3}$ s. Then the intensity would be $I = k_B T/(\tau A) = 3 \cdot 4 \cdot 10^{-21}\text{J}/(10^{-3}\text{s} 0.25 \cdot 10^{-4}\text{ m}^2) = 48 \cdot 10^{-21}/10^{-7}\text{ W/m}^2 \simeq 5 \cdot 10^{-13}\text{ W/m}^2$. This is only about half of the hearing threshold of the human ear! If the ear would be more sensitive than it is, we would actually constantly hear thermal noise!

7. **Sound waves**

   The speed of sound is given by the ratio of the density to the kompressional modulus. While the densities of gases are smaller (by roughly a factor of 1000), the modulus of compression increases by many orders of magnitude in liquids (by at least a factor of 10^4 up to 10^6)). This more than compensates the increase in density and thus leads to a larger speed of sound in solids and liquids compared to gases by a factor of 3 to 30.

8. **Sound intensity 2**

   a) A sperical wave emits into an area $A = 4\pi r^2$. THe maximum intensity at a distance r therefore is $I(r) = \frac{P_0}{4\pi r^2}$. In other words, the distance we are looking for is given by: $r^2 = \frac{P_0}{4\pi I}$. An intensity of 120 dB corresponds to 1 W/m^2. With $P_0 = 125$ W and $4\pi \simeq 12.5$, we obtain: $r^2 = 10$ m^2 or $r \simeq \pi$ m.

   b) An intensity of 65 dB corresponds to a decrease of 60 dB or a factor of 10^6. We are therefore looking for a time where $\exp(-t/\tau) = 10^{-6}$. Taking the (natural) log on both sides gives: $t/\tau = 6 \cdot \ln(10) = 6 \cdot 2.3 \simeq 14$. The time we are looking for thus is: $t = 14\tau = 140s$.

9. **Fourier decomposition**

   a) The reverberation time τ is the time on which the energy of the oscillation decreases.The energy is proportional to the amplitude squared, hence the reverberation time is $\tau = m/f$.

   b) The width (full width at half maximum) of the resonance curve is $\Delta\omega = f/m$.

c) According to a) and b) we have $\Delta\omega = 1/\tau$. A good frequency resolution corresponds to a long reverberation time! For the numerical example, we have 5% resolution at a frequency of 100 Hz or a $\Delta\omega = 100Hz \cdot 0.05 \cdot 2\pi = 10\pi s^{-1} \simeq 30s^{-1}$. The reverberation time is thus about 1/30 s. A faster sequence of tones at 100Hz cannot be resolved by the human ear.

10. **Fourier transform**

a) With $A(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(t)\cos(\omega t)dt$ and $u(t) = u_0 \exp(-\gamma t)\cos(\Omega t)$ we have $A(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u_0 \exp(-\gamma t)\cos(\Omega t)$ c

Writing the cosine in terms of exponentials we get:

$A(\omega) = u_0\sqrt{\frac{1}{8\pi}} \int_0^\infty \exp(-\gamma t)(\exp(i\Omega t) + \exp(-i\Omega t))(\exp(i\omega t) + \exp(-i\omega t))dt.$

We thus have to evaluate four different integrals of the exponential function, namely:

$\exp((-\gamma + i(\Omega+\omega))t)$, $\exp((-\gamma + i(\Omega-\omega))t)$, $\exp((-\gamma + i(-\Omega+\omega))t)$ and $\exp((-\gamma - i(\Omega+\omega))t)$.

This is always of the form $\int_0^\infty \exp(-at)dt = -\frac{1}{a}\exp(-at)|_0^\infty = \frac{1}{a}$.

Therefore we obtain for the Fourier transform:

$A(\omega) = u_0\sqrt{\frac{1}{8\pi}}(\frac{1}{\gamma-i(\Omega+\omega)} + \frac{1}{\gamma-i(\Omega-\omega)} + \frac{1}{\gamma+i(\Omega-\omega)} + \frac{1}{\gamma+i(\Omega+\omega)}).$

Which yields:

$A(\omega) = u_0\sqrt{\frac{1}{2\pi}}(\frac{\gamma}{\gamma^2+(\Omega+\omega)^2} + \frac{\gamma}{\gamma^2+(\Omega-\omega)^2})$

or:

$A(\omega) = u_0\sqrt{\frac{1}{2\pi}}(\frac{\gamma(\gamma^2+(\Omega+\omega)^2+\gamma^2+(\Omega-\omega)^2)}{(\gamma^2+(\Omega+\omega)^2)(\gamma^2+(\Omega-\omega)^2)}$

$A(\omega) = u_0\sqrt{\frac{2}{\pi}}(\frac{\gamma(\gamma^2+\Omega^2+\omega^2)}{\gamma^4+\gamma^2(\Omega+\omega)^2+\gamma^2(\Omega-\omega)^2+(\Omega+\omega)^2(\Omega-\omega)^2}$

$A(\omega) = u_0\sqrt{\frac{2}{\pi}}(\frac{\gamma(\gamma^2+\Omega^2+\omega^2)}{\gamma^4+2\gamma^2(\Omega^2+\omega^2)+(\Omega^2-\omega^2)^2}$

With $\Omega^2 = \omega_0^2 - \gamma^2$ we obtain:

$A(\omega) = u_0\sqrt{\frac{2}{\pi}}(\frac{\gamma(\gamma^2+\omega_0^2-\gamma^2+\omega^2)}{\gamma^4+2\gamma^2(\omega_0^2-\gamma^2+\omega^2)+(\omega_0^2-\gamma^2-\omega^2)^2}$

$A(\omega) = u_0\sqrt{\frac{2}{\pi}}(\frac{\gamma(\omega_0^2+\omega^2)}{\gamma^4+2\gamma^2\omega_0^2-2\gamma^4+2\gamma^2\omega^2)+\gamma^4-2\gamma^2(\omega_0^2-\omega^2)+(\omega_0^2-\omega^2)^2}$

$A(\omega) = u_0\sqrt{\frac{2}{\pi}}(\frac{\gamma(\omega_0^2+\omega^2)}{4\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}$

11. **Fourier transform 2**

$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty z(t)\cos(\omega t)dt$

and

$B(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty z(t)\sin(\omega t)dt.$

As $z(t)$ is non-zero only in the interval between zero and τ, we can change to borders of the integral accordingly:

$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\tau \cos(\omega t)dt$

and

$B(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\tau \sin(\omega t)dt.$

This gives:

$A(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sin(\omega\tau)}{\omega}$

and

$B(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1-\cos(\omega\tau)}{\omega}.$

The absolute value then gives $\sqrt{A^2+B^2} = \frac{1}{\sqrt{2\pi}} \frac{2\sin(\omega\tau/2)}{\omega}.$

12. **Standing waves**

    a) Closed on top means that the pipe is closed on both sides, such that the wave-length of the basic mode is twice the length of the pipe. THe frequency is then $\nu = v/\lambda = v/(2h)$. Numerically: $\nu = 340 m/s/(2 \cdot 1.7)$ m = 100 Hz.

    b) The relative error is given by: $r_\nu = \sqrt{r_v^2 + r_\lambda^2}$. Numerically: $r_v = r_\lambda/2 = 1/340$: $r_\nu = 1/340\sqrt{5} \simeq$ 0.6%.

13. **Standing waves 2**

    a) For a tube open at one end, we have: $\nu_n = \frac{(2n-1)}{4L} v$. Rearranging for the length: $L = \frac{(2n-1)}{4\nu_n} v$. The lower density of methane relative to air gives an increase in the speed of sound as in $v = c_{air}\sqrt{\frac{\rho_{air}}{\rho_M}}$. Hence the length of the tube becomes: $L = \frac{(2n-1)}{4\nu_n} c_L \sqrt{\frac{\rho_{air}}{\rho_M}}$, where n is the number of nodes. Numerically: $L = \frac{\sqrt{29}}{4 \cdot 440 Hz} 343 m/s = \frac{4.5 \cdot 343}{\sqrt{2440}} m = \frac{4.5 \cdot 3.43}{\sqrt{24.4}} m \simeq 2.5 m$

    b) Considering relative errors using the result from a): $r_L^2 = r_\nu^2 + \frac{1}{4} r_\rho^2$, where $r_\nu = 0.5\%$ and $r_\rho = 2\%$. This yields $r_L = \sqrt{1.25}\% \simeq 1\%$.

14. **Diffraction**

    a) The first diffraction minimum is at an angle $\sin(\phi) = \lambda/d$, where d is the width of the slit. Therefore, the wave-length of the microwaves is $\lambda = d\sin(\phi) = 5cm$. Microwave travel at the speed of light $c = 3 \cdot 10^8$ m/s, such that the frequency is $\nu = c/\lambda = \frac{3}{5} \cdot 10^{10}$ Hz = 6 GHz

    b) $\sigma_\lambda^2 = (\frac{\partial \lambda}{\partial d})^2 \sigma_d^2 + (\frac{\partial \lambda}{\partial \phi})^2 \sigma_\phi^2 = (\frac{\lambda}{d})^2 \sigma_d^2 + (\frac{\lambda \cos(\phi)}{\sin(\phi)})^2 \sigma_\phi^2$, where ϕ is measured in radians. THis implies for the relative errors: $r_\lambda^2 = r_d^2 + \cot^2(\phi)\sigma_\phi^2$. Finally, we have for an angle of 30°: $\cot(30°) = \sqrt{3}$, hence: $r_\lambda^2 = r_d^2 + 3\sigma_\phi^2 = 0.1^2 + \frac{3\pi^2}{60^2} = 0.01 + \frac{1}{120} \simeq 0.018$ and finally $r_\lambda = 13\%$.

15. **Interference**

    You want to moor in a minimum of the interference pattern. Therefore the angle towards the opening has to fulfil $\sin(\phi) = (m + 1/2)\lambda/d$. For the first minimum, this implies numerically $\phi = arcsin(\frac{1}{2} \cdot \frac{10m}{50m}) = arcsin(1/10) \simeq 6$ degrees.

16. **Sonar and Doppler**

    a) Die resolution limit for an object corresponds roughly to the wave-length. From the speed of sound and the frequency, we obtain: $x \simeq \lambda = v/\nu$. Numerically: λ = 340 m/s / 60 kHz = 3.4/6 $10^2/10^4$ m = 3.4/6 $\cdot 10^{-2}$ m $\simeq$0.55 cm.

    b) We divide the problem into two processes: (i) The frequency change experienced by the insect and (ii) that observed by the bat. From the point of view of the insect, the bat is the source and the insect the detector. Therefore, the sonar of the bat arrives at the insect with a frequency of $\nu_{insect} = \nu_0 \frac{1 + v_{insect}/c}{1 - v_{bat}/c}$. This frequency is reflected by the insect and observed by the bat, now with the insect as the source and the bat as the detector. Therefore: $\nu_{bat} = \nu_{insect} \frac{1 + v_{bat}/c}{1 - v_{insect}/c}$. Inserting the result from (i) we obtain: $\nu_{bat} = \nu_0 \frac{1 + v_{bat}/c}{1 - v_{insect}/c} \cdot \frac{1 + v_{insect}/c}{1 - v_{bat}/c} \simeq \nu_0(1 + 2(v_{bat} + v_{insect})/c$, where we have used that the speeds are small compared to the speed of sound in the final step. The relative frequency-shift therefore becomes $\Delta\nu/\nu_0 = 2(v_{bat} + v_{insect})/c$. Numerisch erhalten wir: $\Delta\nu/\nu_0 = 26/340 = 7.5\%$.

    c) The errors in the velocities add in squares, therefore the error of $v_{bat} + v_{insect}$ is $0.14 m/s$. The relative error of this velocity is also the relative error of the frequency change, i.e. 0.14/13 or 1%.

17. **Sonar and Doppler 2**

    a) Again, the resolution limit is roughly $x = \lambda$. With $v = \nu\lambda$, we obtain the wave-length from the speed of sound and the frequency. Using the compressional modulus and the density to determine the speed of sound, we obtain $v = \sqrt{K/\rho}$, and hence $x = \lambda = v/\nu = \sqrt{K/\rho}/\nu$.

Numerically: $v = \sqrt{K/\rho} = \sqrt{2 \cdot 10^9 Pa/10^3 kg/m^3} = \sqrt{2 \cdot 10^6 m^2/s^2} = \sqrt{2}10^3 m/s$ and hence $x \simeq \lambda = \sqrt{2}10^3 m/s/(1.5 \cdot 10^6 Hz) \simeq 10^{-3} m = 1mm$.

b) We are using a stationary sender and receiver to measure the blood flow. Hence we have to go from stationary source to moving observer (blood) and then back from moving source (blood) to stationary receiver. The frequency that we observe in the receiver therefore is $\nu'' = \nu \frac{1-v_B/v}{1+v_B/v}$, where v_B is the speed of the blood flow and v is the speed of sound, which is $v = \sqrt{2}10^3$ m/s according to a), thus $v_B/v = 10^{-4}/\sqrt{2}$, which is very small. We may thus Taylor approximate the relation of the frequency as: $\nu'' = \nu \frac{1-v_B/v}{1+v_B/v} \simeq \nu(1 - v_B/v)^2 \simeq \nu(1 - 2v_B/v)$. In other words, the relativ frequency change is: $\Delta\nu/\nu = -2v_B/v = -\sqrt{2} \cdot 10^{-4}$. Therefore the frequency change is about 200 Hz.

c) With the approximation used above, $\Delta\nu/\nu = -2v_B/v$, we can directly write down the relative error $r^2_{\Delta\nu/\nu} = r^2_{v_B} + r^2_v$ and with the relation for the speed of sound from a), we have the error of v: $r^2_v = 1/4(r^2_E + r^2_\rho)$ Thus: $r^2_{\Delta\nu/\nu} = r^2_{v_B} + 1/4(r^2_K + r^2_\rho)$ nach den Angaben ist $r_{v_B} = 2r_K = r_\rho = r = 0.1$ also erhalten wir: $r^2_{\Delta\nu/\nu} = r^2 + 1/4(4r^2 + r^2) = 2.25r^2$. The relative error thus is $r_{\Delta\nu/\nu} = 1.5 \cdot 0.1$. The error in the original frequency is negligibly small, hence $r_{\Delta\nu} = 0.15$

18. **Atomic Physics**

a) The Bohr radius is inversely proportional to the mass of the particle, hence for a muon the Bohr radius is roughly 200 times smaller, i.e. $0.5/200\ \mathring{A} = 0.25$ pm.

b) Since the masses are again a factor of 10 larger, the radius decreases by another factor of 10, i.e. $r = 25 fm$. To be exact, in this case, both particles have the same mass, such that they would both move around their centre of mass. This would in effect reduce the radius by another factor of two.