Chapter 6: Mean Value Theorems

Part A: Mean Value Theorems, Indeterminate Forms, Taylor Polynomials

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Racetrack and Mean Value Inequalities

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Theorem 1

Let f, g be differentiable functions from an interval I to $\mathbb{R}$.

1. *(Racetrack Inequality) If $a, b \in I$, $f(a) \le g(a)$ and $f'(x) \le g'(x)$ on $[a, b]$ then $f(x) \le g(x)$ on $[a, b]$.*

Racetrack and Mean Value Inequalities

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2. *(Mean Value Inequality) If $a, b \in I$ and $m \le f'(x) \le M$ on $[a, b]$ then $m(x-a) \le f(x) - f(a) \le M(x-a)$ on $[a, b]$.*

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3. *If $a \in I$ and $|f'(x)| \le M$ on I, then $|f(x) - f(a)| \le M|x-a|$ on I.*

If any of the inequalities involving f' is strict, so is the corresponding inequality for f.

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3. *If $a \in I$ and $|f'(x)| \leq M$ on I, then $|f(x) - f(a)| \leq M|x - a|$ on I.*

If any of the inequalities involving f' is strict, so is the corresponding inequality for f.

Proof. Part (a) is a consequence of the Monotonicity Theorem. Part (b) follows by applying (a) to the functions $m(x - a)$, $f(x) = f(a)$, and $M(x - a)$. The $x \geq a$ case of (c) is covered by (b). The $x < a$ case is obtained by symmetry. □

(Lagrange's) Mean Value Theorem

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Theorem 2

Let $f : [a, b] \to \mathbb{R}$ *satisfy the following:*

1. *It is continuous on* $[a, b]$.
2. *It is differentiable on* (a, b).

Then there is a $c \in (a, b)$ *such that* $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

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Proof. Define $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$. Observe that $g(a) = g(b) = 0$, g is continuous on $[a, b]$ and differentiable on (a, b).

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Proof. Define $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$. Observe that $g(a) = g(b) = 0$, g is continuous on $[a, b]$ and differentiable on (a, b). By the Extreme Value Theorem, g achieves a maximum value M and a minimum value m on $[a, b]$. If $M = m$, then g is constant and hence zero. In this case, every $c \in (a, b)$ has the desired property.

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If $M \neq m$ then at least one of the maximum and minimum values of g is achieved at an interior point $c \in (a, b)$.

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If $M \neq m$ then at least one of the maximum and minimum values of g is achieved at an interior point $c \in (a, b)$.

By Fermat's Theorem, $g'(c) = 0$. Hence $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Rolle's Theorem

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Theorem 3

Let $f : [a, b] \to \mathbb{R}$ satisfy the following:

1. *It is continuous on $[a, b]$.*
2. *It is differentiable on (a, b).*
3. *$f(a) = f(b)$.*

Then there is a $c \in (a, b)$ such that $f'(c) = 0$.

Task 1

Suppose $a < b < c$, $f : [a, c] \to \mathbb{R}$ is twice continuously differentiable (i.e., the function $f'' = (f')'$ is continuous), and $f(a) = f(b) = f(c) = 0$. Show that there is $\alpha \in (a, c)$ such that $f''(\alpha) = 0$.

(Cauchy's) Mean Value Theorem

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Theorem 4

Let $f, g\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $g'(x) \neq 0$ for every $x \in (a, b)$. Then there is a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(Cauchy's) Mean Value Theorem

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Proof. First note that $g(a) \neq g(b)$. If they were equal, Rolle's Theorem would give a $c \in (a, b)$ such that $g'(c) = 0$. Thus both ratios in the theorem's conclusion are defined.

(Cauchy's) Mean Value Theorem

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Let $f, g\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $g'(x) \neq 0$ for every $x \in (a,b)$. Then there is a $c \in (a,b)$ such that

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Proof. First note that $g(a) \neq g(b)$. If they were equal, Rolle's Theorem would give a $c \in (a,b)$ such that $g'(c) = 0$. Thus both ratios in the theorem's conclusion are defined.

Define $h(x) = g(x)\dfrac{f(b)-f(a)}{g(b)-g(a)}$. We have $h(b) - h(a) = f(b) - f(a)$.

(Cauchy's) Mean Value Theorem

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Applying Rolle's theorem to $f(x) - h(x)$, we get a $c \in (a,b)$ such that $f'(c) - h'(c) = 0$. Then,

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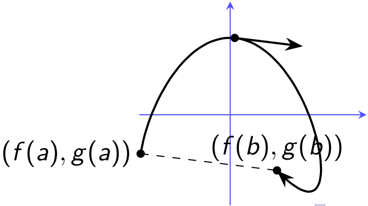
Applying Rolle's theorem to $f(x) - h(x)$, we get a $c \in (a,b)$ such that $f'(c) - h'(c) = 0$. Then,

$$0 = f'(c) - h'(c) = f'(c) - g'(c)\frac{f(b) - f(a)}{g(b) - g(a)} \implies \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Remarks

Remark 1: Taking $g(x) = x$ in Cauchy's Mean Value Theorem gives Lagrange's Mean Value Theorem. So it is also called the **Extended** or the **Generalized Mean Value Theorem**.

Remark 2: Its motivation comes from the motion of a particle in a plane. Let the location at any time t, $a \leq t \leq b$, be given by $(g(t), f(t))$. Then its net displacement is $(g(b) - g(a), f(b) - f(a))$, while its velocity vector at any time t is $(g'(t), f'(t))$. Thus Cauchy's Mean Value Theorem says that there is a time instant c when the velocity vector is parallel to the total displacement.



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0/0 Indeterminate Form

A limit of the type $\lim_{x \to a} \frac{f(x)}{g(x)}$ with $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ is said to be an **indeterminate form** of type $\frac{0}{0}$. (The limits could also be one-sided)

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Theorem 5

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, *$f$ and g are differentiable at a, and* $g'(a) \neq 0$. *Then* $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

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Proof. Since f, g are differentiable at a, they are continuous there, and so $f(a) = g(a) = 0$.

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A limit of the type $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ with $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ is said to be an **indeterminate form** of type $\dfrac{0}{0}$. (The limits could also be one-sided)

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Proof. Since f, g are differentiable at a, they are continuous there, and so $f(a) = g(a) = 0$. Further, $g'(a) \neq 0$ implies that there is an interval centered at a in which $g(x)$ is never zero.

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Proof. Since f, g are differentiable at a, they are continuous there, and so $f(a) = g(a) = 0$. Further, $g'(a) \neq 0$ implies that there is an interval centered at a in which $g(x)$ is never zero.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{(f(x) - f(a))/(x-a)}{(g(x)-g(a))/(x-a)} = \frac{f'(a)}{g'(a)}.$$

□

Example

We have to evaluate $\lim_{x \to 0} \frac{x + \sin x}{\log(1 - x)}$.

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We calculate:

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{1 + \cos x}{-1/(1 - x)} = \lim_{x \to 0} (x - 1)(1 + \cos x) = -2.$$

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Therefore, $\lim_{x\to 0} \dfrac{x+\sin x}{\log(1-x)} = -2$.

L'Hôpital's Rule

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Theorem 6

Let $f, g\colon (a,b) \to \mathbb{R}$ *be differentiable functions which satisfy the following.*

1. $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0$,
2. $g'(x) \neq 0$ *for every* $x \in (a,b)$,
3. $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$.

Then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Proof on next slide.

L'Hôpital's Rule – Proof

Proof. Extend the domain of f, g to $[a, b)$ by defining $f(a) = g(a) = 0$. Then f, g become continuous on $[a, b)$. Further, by Rolle's Theorem, $g(x) \neq 0$ for every $x \in (a, b)$.

L'Hôpital's Rule – Proof

Proof. Extend the domain of f, g to $[a, b)$ by defining $f(a) = g(a) = 0$. Then f, g become continuous on $[a, b)$. Further, by Rolle's Theorem, $g(x) \neq 0$ for every $x \in (a, b)$.
Let $0 < h < b - a$. Then, for each such h, the functions $f(x)$ and $g(x)$ satisfy the hypotheses of Cauchy's Mean Value Theorem on the interval $[a, a + h]$. Hence there is a $c_h \in (a, a + h)$ such that

$$\frac{f'(c_h)}{g'(c_h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f(a+h)}{g(a+h)}.$$

L'Hôpital's Rule – Proof

Proof. Extend the domain of f, g to $[a, b)$ by defining $f(a) = g(a) = 0$. Then f, g become continuous on $[a, b)$. Further, by Rolle's Theorem, $g(x) \neq 0$ for every $x \in (a, b)$.

Let $0 < h < b - a$. Then, for each such h, the functions $f(x)$ and $g(x)$ satisfy the hypotheses of Cauchy's Mean Value Theorem on the interval $[a, a + h]$. Hence there is a $c_h \in (a, a + h)$ such that

$$\frac{f'(c_h)}{g'(c_h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f(a+h)}{g(a+h)}.$$

We have $a < c_h < a + h$. So the Sandwich Theorem implies that $c_h \to a+$ as $h \to 0+$. Hence

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{h \to 0+} \frac{f(a+h)}{g(a+h)} = \lim_{h \to 0+} \frac{f'(c_h)}{g'(c_h)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)} = L.$$

□

Other Versions

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Theorem 7

1. *(Left-hand limit) Suppose* f, g *are differentiable on* (a, b), $g'(x) \neq 0$ *for every* $x \in (a, b)$, $\lim_{x \to b-} f(x) = \lim_{x \to b-} g(x) = 0$, *and* $\lim_{x \to b-} \frac{f'(x)}{g'(x)} = L$. *Then* $\lim_{x \to b-} \frac{f(x)}{g(x)} = L$.
2. *(Two-sided limit) Suppose* $a < b < c$, f, g *are differentiable on* $I = (a, b) \cup (b, c)$, $g'(x) \neq 0$ *for every* $x \in I$, $\lim_{x \to b} f(x) = \lim_{x \to b} g(x) = 0$, *and* $\lim_{x \to b} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. *Then* $\lim_{x \to b} \frac{f(x)}{g(x)} = L$.

Example

Example 8

Consider $\lim_{x \to 1} \dfrac{\log x}{x-1}$.

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This is an indeterminate form of the type $0/0$ and $f(x) = \log x$, $g(x) = x - 1$ satisfy the first three hypotheses of L'Hôpital's Rule for two-sided limits with $a = 0$, $b = 1$, $c = 2$.

Example

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Further,

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{1/x}{1} = 1$$

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Further,

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Hence $\lim_{x \to 1} \frac{\log x}{x-1} = 1$. (We could also have used Theorem 5.)

Example

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Example 9

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$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

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So the gap $\sin x - x$ is much smaller than x^2. Let's compare with x^3:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3\,x^2} = \lim_{x\to 0} \frac{-\sin x}{3!\,x} = -\frac{1}{3!}.$$

Example

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We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. This means that for small x, $\sin x \approx x$. To improve this approximation we use L'Hôpital's Rule to compare $\sin x - x$ with higher powers of x. First with x^2:

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Thus $\frac{\sin x - x}{x^3} \approx -\frac{1}{3!}$, or $\sin x \approx x - \frac{x^3}{3!}$ for small x. This process can be continued to get better and better polynomial approximations to $\sin x$.

Example

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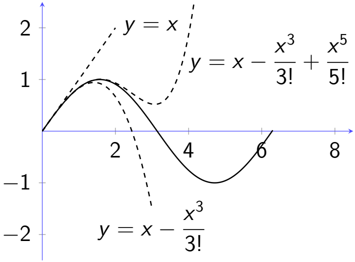
Task 2

Use L'Hôpital's Rule to compare $\sin x - x + x^3/3!$ *with* x^5 *near zero and obtain the approximation* $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ *for small* x.

The graph below shows the progressive improvements in these approximations to $\sin x$:



Limits at Infinity

Theorem 10

Suppose f and g are differentiable on (a,∞), $g'(x)\neq 0$ for every $x\in(a,\infty)$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Limits at Infinity

Theorem 10

Suppose f and g are differentiable on (a,∞), $g'(x) \neq 0$ for every $x \in (a,\infty)$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Proof. We begin by recalling that $\lim_{x\to\infty} f(x) = \lim_{t\to 0+} f(1/t)$. Hence,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0+} \frac{f(1/t)}{g(1/t)} = \lim_{t\to 0+} \frac{(f(1/t))'}{(g(1/t))'} = \lim_{t\to 0+} \frac{-f'(1/t)/t^2}{-g'(1/t)/t^2}$$
$$= \lim_{t\to 0+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

□

∞/∞ Form and Infinite Limits

Theorem 11 (L'Hôpital's Rule)

1. *Each version of L'Hôspital's Rule also holds if* $f, g \to \pm\infty$.
2. *Each version of L'Hôspital's Rule also holds if* $L = \pm\infty$.

∞/∞ Form and Infinite Limits

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∞/∞ Form and Infinite Limits

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1. Consider the left-hand limit at $c \in \mathbb{R}$. Let $\lim\limits_{x\to c-} \dfrac{f'(x)}{g'(x)} = L$. For any $\epsilon > 0$ there is an x_0 such that $x_0 \leq x < c$ implies

$$L - \epsilon < \frac{f'(x)}{g'(x)} < L + \epsilon.$$

∞/∞ Form and Infinite Limits

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$$L - \epsilon < \frac{f'(x)}{g'(x)} < L + \epsilon.$$

Now take an $x \in (x_0, c)$. By Cauchy's Mean Value Theorem there is a $\xi \in (x_0, x)$ such that

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

(continued ...)

∞/∞ Form and Infinite Limits

(...continued)

Hence $L-\epsilon < \dfrac{f(x)-f(x_0)}{g(x)-g(x_0)} < L+\epsilon$, which we rearrange to

$$L-\epsilon < \frac{f(x)/g(x)-f(x_0)/g(x)}{1-g(x_0)/g(x)} < L+\epsilon,$$

∞/∞ Form and Infinite Limits

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Hence $L-\epsilon < \dfrac{f(x)-f(x_0)}{g(x)-g(x_0)} < L+\epsilon$, which we rearrange to

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and then,

$$(L-\epsilon)\left(1-\frac{g(x_0)}{g(x)}\right)+\frac{f(x_0)}{g(x)} < \frac{f(x)}{g(x)} < (L+\epsilon)\left(1-\frac{g(x_0)}{g(x)}\right)+\frac{f(x_0)}{g(x)}.$$

∞/∞ Form and Infinite Limits

(...continued)

Hence $L - \epsilon < \dfrac{f(x) - f(x_0)}{g(x) - g(x_0)} < L + \epsilon$, which we rearrange to

$$L - \epsilon < \frac{f(x)/g(x) - f(x_0)/g(x)}{1 - g(x_0)/g(x)} < L + \epsilon,$$

and then,

$$(L - \epsilon)\Big(1 - \frac{g(x_0)}{g(x)}\Big) + \frac{f(x_0)}{g(x)} < \frac{f(x)}{g(x)} < (L + \epsilon)\Big(1 - \frac{g(x_0)}{g(x)}\Big) + \frac{f(x_0)}{g(x)}.$$

As $x \to c-$, $g(x_0)/g(x) \to 0$ and $f(x_0)/g(x) \to 0$. Hence by taking x close to c we get

$$L - 2\epsilon < \frac{f(x)}{g(x)} < L + 2\epsilon.$$

∞/∞ Form and Infinite Limits

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(...continued)

Hence $L-\epsilon < \dfrac{f(x)-f(x_0)}{g(x)-g(x_0)} < L+\epsilon$, which we rearrange to

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$$(L-\epsilon)\left(1-\frac{g(x_0)}{g(x)}\right)+\frac{f(x_0)}{g(x)} < \frac{f(x)}{g(x)} < (L+\epsilon)\left(1-\frac{g(x_0)}{g(x)}\right)+\frac{f(x_0)}{g(x)}.$$

As $x \to c-$, $g(x_0)/g(x) \to 0$ and $f(x_0)/g(x) \to 0$. Hence by taking x close to c we get

$$L-2\epsilon < \frac{f(x)}{g(x)} < L+2\epsilon.$$

This gives $\displaystyle\lim_{x\to c-}\frac{f(x)}{g(x)} = L$. (continued ...)

∞/∞ Form and Infinite Limits

(...continued)

2. We show how to modify the proof of the first part of this theorem for the $L = \infty$ case. Given any $M \in \mathbb{R}$ there is an x_0 such that $x_0 \leq x < c$ implies

$$M + 1 < \frac{f'(x)}{g'(x)}.$$

∞/∞ Form and Infinite Limits

(...continued)

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$$M + 1 < \frac{f'(x)}{g'(x)}.$$

Applying Cauchy's Mean Value Theorem and proceeding as before we reach

$$(M+1)\left(1 - \frac{g(x_0)}{g(x)}\right) + \frac{f(x_0)}{g(x)} < \frac{f(x)}{g(x)}, \quad \text{for } x \in (x_0, c).$$

∞/∞ Form and Infinite Limits

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By taking x close to c we get: $M < \dfrac{f(x)}{g(x)}$. $\square$

Example

Example 12

Consider $\lim_{x\to\infty} \frac{x}{e^x}$.

This is an $\frac{\infty}{\infty}$ form, the numerator and denominator are differentiable on $(0, \infty)$, and the derivative of the denominator is always non-zero.

So L'Hôpital's Rule can be applied:

$$\lim_{x\to\infty} \frac{x'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \implies \lim_{x\to\infty} \frac{x}{e^x} = 0.$$

Example

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Example 13

On occasion we may need to apply L'Hôpital's Rule repeatedly. Consider $\lim_{x\to\infty} \frac{x^2}{e^x}$. This is again an $\frac{\infty}{\infty}$ form, the numerator and denominator are continuously differentiable on $(0, \infty)$, and the derivative of the denominator is always non-zero. Further,

$$\lim_{x\to\infty} \frac{(x^2)'}{(e^x)'} = 2 \lim_{x\to\infty} \frac{x}{e^x}.$$

The second limit, by another application of L'Hôpital's Rule (previous example), is 0. Hence $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$.

We can repeat this argument to show that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any $n \in \mathbb{N}$. Thus the exponential function grows faster than any power of x.

Example

Example 14

Consider $\lim_{x\to\infty} \frac{x^p}{\log x}$, with $p > 0$.

This is an $\frac{\infty}{\infty}$ form, the numerator and denominator are continuously differentiable on $(0,\infty)$ and the derivative of the denominator is always non-zero. Now

$$\lim_{x\to\infty} \frac{(x^p)'}{(\log x)'} = \lim_{x\to\infty} \frac{p\,x^{p-1}}{1/x} = \lim_{x\to\infty} p\,x^p = \infty.$$

Hence $\lim_{x\to\infty} \frac{x^p}{\log x} = \infty$.

Other Indeterminate Forms

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Type $\infty - \infty$: These have the form $f(x) - g(x)$ where $f(x), g(x) \to \infty$. The result depends on which term dominates. For example, $\lim_{x\to\infty}(x - x) = 0$ and $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty$.

Type $\mathbf{0} \cdot \infty$: These have the form $f(x)g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$.

Type $\mathbf{1}^{\infty}$: These have the form $f(x)^{g(x)}$ where $f(x) \to 1$ and $g(x) \to \infty$. A familiar example is $\lim_{x\to 0}(1+x)^{1/x} = e$.

Type $\infty^{\mathbf{0}}$: These have the form $f(x)^{g(x)}$ where $f(x) \to \infty$ and $g(x) \to 0$. Applying log converts this to a $0 \cdot \infty$ form.

Type $\mathbf{0}^{\mathbf{0}}$: These have the form $f(x)^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$. Applying log converts this to a $0 \cdot \infty$ form.

Example

Example 15

The right-hand side limit of $x \log x$ at 0 is a $0 \cdot \infty$ form. We can convert it to a ratio and apply L'Hôspital's Rule.

$$\lim_{x \to 0+} x \log x = \lim_{t \to \infty} \frac{\log(1/t)}{t} = -\lim_{t \to \infty} \frac{\log t}{t} = -\lim_{t \to \infty} \frac{1/t}{1} = 0.$$

Example

Example 16

Consider $\lim_{x\to\pi/2-}(\sin x)^{\tan x}$. This is a 1^∞ form. Let $y=(\sin x)^{\tan x}$.
Then $\log y=(\tan x)\log(\sin x)=\dfrac{\log(\sin x)}{\cot x}$ and $\lim_{x\to\pi/2-}\dfrac{\log(\sin x)}{\cot x}$ is an $\dfrac{\infty}{\infty}$ form. Apply L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}-}\log y &= \lim_{x\to\frac{\pi}{2}-}\frac{\log(\sin x)}{\cot x}=\lim_{x\to\frac{\pi}{2}-}\frac{\cot x}{-\csc^2 x}\\ &= -\lim_{x\to\frac{\pi}{2}-}(\cos x)(\sin x)=0.\end{aligned}$$

Finally, $\log y\to 0$ implies $y\to 1$.

Example

Example 17

Consider $\lim_{x \to 0+} x^x$. This is a 0^0 form.

Let $y = x^x$. Then $\log y = x \log x = \dfrac{\log x}{1/x}$, and $\lim_{x \to 0+} \dfrac{\log x}{1/x}$ is an $\dfrac{\infty}{\infty}$ form.

Apply L'Hôpital's Rule:

$$\lim_{x \to 0+} \frac{(\log x)'}{(1/x)'} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = -\lim_{x \to 0+} x = 0 \implies \lim_{x \to 0+} \log y = 0.$$

Hence $y \to 1$.

Alert

Example 18

Suppose $f(x) = e^{-x}$, $g(x) = 1/x$, and we have to calculate $\lim_{x\to\infty} \frac{f(x)}{g(x)}$.
This is an $\frac{\infty}{\infty}$ form and we are allowed to apply L'Hôpital's Rule. If we do, we get

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{-e^{-x}}{-1/x^2},$$

which is more complicated than the original limit! Of course, we can easily resolve this by first rearranging the expression to x/e^x.

Alert

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Example 19

Consider $f(x) = x + \sin x$ and $g(x) = 2x$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is an $\frac{\infty}{\infty}$ form and we may be tempted to calculate as follows:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1+\cos x}{2}, \text{ hence does not exist.}$$

However, this conclusion is not justified. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then L'Hôpital's Rule *fails to imply anything* about the original limit. In fact, we can apply the Sandwich Theorem to conclude that the original limit equals $1/2$.

Table of Contents

Approximation by Polynomials

To approximate $f(x)$ by a polynomial near $x = a$, we shall use polynomials of the form

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n.$$

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At the next iteration, we have

$$f^{(n-2)}(x) - f^{(n-2)}(a) - f^{(n-1)}(a)(x-a) - \frac{f^{(n)}(a)}{2}(x-a)^2 \le \frac{M}{3 \cdot 2}(x-a)^3.$$

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Continuing in this fashion, we finally obtain

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \le \frac{M}{(n+1)!}(x-a)^{n+1} \qquad \text{for } x > a. \qquad (1)$$

Taylor Polynomials

Similarly, if we have $m \leq f^{(n+1)}(x)$ on I, we get

$$\frac{m}{(n+1)!}(x-a)^{n+1} \leq f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad \text{for } x > a. \tag{2}$$

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The polynomial defined by

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

is called the n^{th} **Taylor polynomial** of $f(x)$ centred at $x = a$. When $a = 0$ the Taylor polynomials are also called the **Maclaurin polynomials**.

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Task 3

If T_n is the n^{th} Taylor polynomial of f centered at a, show that

$$T_n^{(k)}(a) = f^{(k)}(a) \text{ for } k = 0, 1, \ldots, n.$$

Example

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Example 20

Let us calculate the Taylor polynomials of $\sin x$ centered at $a = 0$:

$$
\begin{aligned}
f(x) &= \sin x & \Longrightarrow \quad & a_0 = f(0) = 0, \\
f'(x) &= \cos x & \Longrightarrow \quad & a_1 = f'(0) = 1, \\
f''(x) &= -\sin x & \Longrightarrow \quad & a_2 = \frac{f''(0)}{2!} = 0, \\
f'''(x) &= -\cos x & \Longrightarrow \quad & a_3 = \frac{f'''(0)}{3!} = -\frac{1}{3!}.
\end{aligned}
$$

We see that $a_k = 0$ when k is even. And for odd $k = 2\ell + 1$ we have $a_k = \dfrac{(-1)^\ell}{(2\ell+1)!}$. Thus the $(2n+1)^{\text{th}}$ Taylor polynomial has the form

$$T_{2n+1}(x) = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(2\ell+1)!} x^{2\ell+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + (-1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!}.$$

Example

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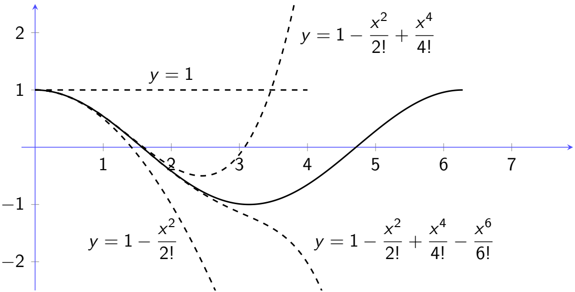
Example 21

The Taylor polynomials of $\cos x$ centred at $a = 0$ can be found similarly.

$$T_{2n}(x) = \sum_{\ell=0}^{n} \frac{(-1)^\ell}{(2\ell)!} x^{2\ell} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots + (-1)^{2n} \frac{x^{2n}}{(2n)!}.$$



Example

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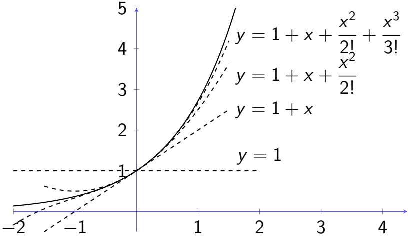
Example 22

The Taylor polynomials of e^x centred at $a = 0$ are

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$



Putting $x = 1$ gives $e \approx \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$.

Example

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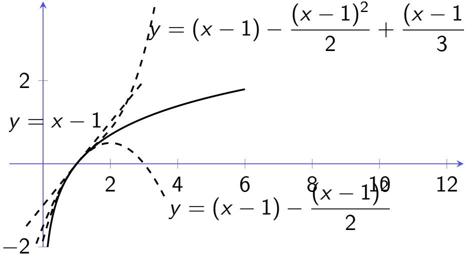
Example 23

The Taylor polynomials of $f(x) = \log x$ centred at $a = 1$ are:

$$T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k$$
$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n}.$$



Taylor's Theorem

Theorem 24

Let I be an interval, $f : I \to \mathbb{R}$, and $a \in I$.

1. *Let $f(x)$ be differentiable $n+1$ times on I, and suppose $|f^{(n+1)}(x)| \leq M$ on I.*
2. *Let $T_n(x)$ be the n^{th}-degree Taylor polynomial of $f(x)$ centred at a.*

Then, for each $x \in I$, $|f(x) - T_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$.

Taylor's Theorem

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Then, for each $x \in I$, $|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$.

Proof. We have already established this for $x \geq a$ in equations 1 and 2. The $x < a$ case can be converted to the $x > a$ case by reflection about $x = a$. □

Example

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Example 25

Suppose we need to approximate $\sin 1.2$ to 4 decimal places. Applying Taylor's theorem to $\sin x$ with $a = 0$ and $x = 1.2$, we find that $M = 1$ and

$$|\sin 1.2 - T_n(1.2)| \leq \frac{1.2^{n+1}}{(n+1)!}.$$

To ensure $T_n(1.2)$ is sufficiently accurate, we need to choose n such that $\frac{1.2^{n+1}}{(n+1)!} \leq 5 \times 10^{-5}$. If we take $n = 8$ we get $\frac{1.2^9}{9!} = 1.4 \times 10^{-5} < 5 \times 10^{-5}$. So the 8th degree Taylor polynomial suffices. However the degree 8 term is zero in the Taylor expansion of $\sin x$ and so we only need the terms up to degree 7.

$$\sin 1.2 \approx 1.2 - \frac{1.2^3}{3!} + \frac{1.2^5}{5!} - \frac{1.2^7}{7!} \approx 0.932025.$$

Example

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Example 26

Now let us approximate Euler's number e to 4 decimal places. Recall that we already know $e < 4$ and so the function e^x is bounded by 4 on $[0, 1]$. Therefore, applying Taylor's theorem to e^x with $a = 0$ and $x = 1$, we find that

$$|e - T_n(1)| \leq \frac{4}{(n+1)!}.$$

To ensure $T_n(1)$ is sufficiently accurate, we need to choose n such that $\frac{4}{(n+1)!} \leq 5 \times 10^{-5}$. Again, $n = 8$ does the job. Therefore,

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{8!} = 2.718278\ldots.$$

(The exact value is $2.718281\ldots$)

Remainder Theorem

Theorem 27

Let I be an interval, $f : I \to \mathbb{R}$ be $n+1$ times continuously differentiable, $a \in I$. Let $T_n(x)$ be the n^{th}-degree Taylor polynomial of $f(x)$ centered at a. Then, for each $x \in I$, there is a ξ between a and x such that

$$f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

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Hence, $m \le \left(f(x) - T_n(x)\right)\dfrac{(n+1)!}{(x-a)^{n+1}} \le M$.

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Now the Intermediate Value Theorem gives a $\xi \in (a, x)$ such that

$$f^{(n+1)}(\xi) = \left(f(x) - T_n(x)\right)\frac{(n+1)!}{(x-a)^{n+1}}.$$

Classifying Critical Points

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Theorem 28

Let f have a critical point at a and be n times continuously differentiable at a. Suppose $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$.

1. *If n is even and $f^{(n)}(a) > 0$ then f has a local minimum at a.*
2. *If n is even and $f^{(n)}(a) < 0$ then f has a local maximum at a.*
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Proof. By continuity, there is an open interval I containing a such that $f^{(n)}$ does not change sign in I. For each $x \in I$ there is a $\xi \in I$ such that

$$f(x) = f(a) + \frac{f^{(n)}(\xi)}{n!}(x-a)^n.$$

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If n is even and $f^{(n)}(a) > 0$ then we have $f^{(n)}(\xi) > 0$ for every $\xi \in I \setminus \{a\}$. It follows that $f(x) > f(a)$ for every $x \in I$ and hence there is a local minimum at a. The other cases are similar. □