Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014)
CHAPTER 2

PROBLEM 2.1. Torque on Plasma.
Do the ideal MHD equations conserve angular momentum? What is the torque on the plasma?

SOLUTION. In the same way that momentum conservation implies that the momentum can be increased or decreased by forces acting on a plasma, so the angular momentum can be changed by torques.

Consider, for example, an MHD plasma operated on by just a plasma pressure gradient and a Lorentz force. Then the torque on the plasma may be found from the curl of the equation of motion, namely,

$$\nabla \times \rho \frac{d\mathbf{v}}{dt} = \nabla \times (\mathbf{j} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{j} - (\mathbf{j} \cdot \nabla)\mathbf{B}.$$

In general this will not vanish and so the angular momentum is changed by the torque.

PROBLEM 2.2. Unidirectional Field.
If the magnetic field is unidirectional, pointing everywhere in the same direction, why can the magnetic field have no gradient in that direction?

SOLUTION. Suppose the magnetic field is pointing in the x-direction, with $\mathbf{B}(x, y, z) = B(x, y, z)\hat{\mathbf{x}}$. Then the equation $\nabla \cdot \mathbf{B} = 0$ implies that $\partial B/\partial x = 0$. In other words, the magnetic field does not vary with x and there is no gradient in that direction.

PROBLEM 2.3. Consistency of MHD Equations.
The time-dependent MHD equations appear at first sight to represent a set of 10 equation for 9 variables. Why is this not overprescribed? How is the argument changed for the steady-state equations and for the equilibrium equations without flow?

SOLUTION.

(i) Unsteady case. The set of unsteady MHD equations is

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$
$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$
$$\rho \frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B} + \mathbf{F},$$
$$p = \frac{k_B}{m} \rho T \quad \left(\frac{\tilde{R}}{\tilde{\mu}} \rho T \right),$$
$$\frac{\rho^\gamma}{\gamma - 1} \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = -\boldsymbol{\nabla} \cdot \mathbf{q} - L_r + j^2/\sigma + H, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0.$$

This represents a set of 10 differential equations for the 9 variables $B_x, B_y, B_z, v_x, v_y, v_z, \rho, p, T$. However, the last equation ($\boldsymbol{\nabla} \cdot \mathbf{B} = 0$) does not have the same status as the other differential equations, and is really only an initial condition, since, if we assume $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ at $t = 0$ and take the divergence of the first equation, then we find

$$\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{B} = 0.$$

In other words, if $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ holds initially, then the other equations imply that it also holds for all time. Thus, we really have 9 differential equations for 9 variables that need to be solved at each moment of time.

(ii) Steady case. Now the above equations reduce to

$$\mathbf{0} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$
$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$
$$\rho (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B} + \mathbf{F},$$
$$p = \frac{k_B}{m} \rho T,$$
$$\frac{\rho^\gamma}{\gamma - 1} (\mathbf{v} \cdot \boldsymbol{\nabla}) \left(\frac{p}{\rho^\gamma} \right) = -\boldsymbol{\nabla} \cdot \mathbf{q} - L_r + j^2/\sigma + H, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0.$$

At first sight there is a paradox, since the number of equations is 10 and the number of variables is only 9 ($v_x, v_y, v_z, B_x, B_y, B_z, \rho, p, T$). The paradox may be resolved by recognising that the first equation represents only 2 independent equations rather than 3, and so the numbers of independent equations and variables is the same. This may be shown as follows in two ways, by writing the equation as $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$.

First of all, suppose the z- and x-components of $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$ hold everywhere and the y-component holds just on a plane $y = constant$. Then we can show that the y-component holds everywhere as follows.

The z- and x-components are

$$\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}, \qquad\qquad \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y}.$$

Then

$$\frac{\partial}{\partial y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial E_x}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial E_y}{\partial x}\right) = 0.$$

Thus, after integrating over y, we have

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = f(x, z),$$

where $f(x, z)$ is an arbitrary function. However, if $\partial E_z/\partial x - \partial E_x/\partial z = 0$ on $y = constant$, then $f(x, z) \equiv 0$ and so

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0,$$

as required.

The alternative approach is use the general solution of $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$, namely, $\mathbf{E} = \boldsymbol{\nabla}\Phi$ and to suppose that $E_x(x, y, z)$ and $E_y(x, y, z)$ are given functions of (x, y, z) everywhere and E_z is given as a boundary condition on, say, the z-axis as $E_z(0, 0, z)$. Then we can determine $E_z(x, y, z)$ everywhere in terms of them, as follows.

The x-component of $\mathbf{E} = \boldsymbol{\nabla}\Phi$, namely, $\partial\Phi/\partial x = E_x(x, y, z)$ can be integrated to give

$$\Phi(x, y, z) = \int_{x=0}^{x} E_x dx + g(y, z),$$

where $g(y, z)$ is an arbitrary function of integration to be determined.

Next, we can substitute this expression for $\Phi(x, y, z)$ into the y-component of $\mathbf{E} = \boldsymbol{\nabla}\Phi$, namely, $\partial\Phi/\partial y = E_y(x, y, z)$ to give

$$\int_{x=0}^{x} \frac{\partial E_x}{\partial y} dx + \frac{\partial g(y, z)}{\partial y} = E_y(x, y, z).$$

In other words,

$$\frac{\partial g(y, z)}{\partial y} = h(y, z) \equiv E_y(x, y, z) - \int_{x=0}^{x} \frac{\partial E_x}{\partial y} dx.$$

This can be integrated to yield an expression for $g(y, z)$ as

$$g(y, z) = \int_{y=0}^{y} h(y, z) dy + c(z),$$

in terms of an unknown function of integration $c(z)$.

Then the z-component of $\mathbf{E} = \boldsymbol{\nabla}\Phi$ determines E_z everywhere as

$$E_z(x, y, z) = \frac{\partial\Phi}{\partial z} = \int_{x=0}^{x} \frac{\partial E_x}{\partial z} dx + \frac{\partial g(y, z)}{\partial z},$$

or, after substituting for $g(y, z)$,

$$E_z(x, y, z) = \frac{\partial\Phi}{\partial z} = \int_{x=0}^{x} \frac{\partial E_x}{\partial z} dx + \int_{y=0}^{y} \frac{\partial h}{\partial z} dy + \frac{dc}{dz}$$

However, dc/dz is determined by the boundary condition on E_z as

$$\frac{dc}{dz} = E_z(0, 0, z).$$

In other words, we have determined E_z everywhere, as required.

(iii) Equilibrium case. For the equilibrium case with no flow, the MHD equations reduce further to

$$\begin{aligned}
&\mathbf{0} = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B} + \mathbf{F}, \\
&p = \frac{k_B}{m} \rho T, \\
&0 = -\boldsymbol{\nabla} \cdot \mathbf{q} - L_r + j^2/\sigma + H, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0.
\end{aligned}$$

Thus, we have now have 6 equations for 6 unknowns $(B_x, B_y, B_z, \rho, p, T)$, and there is no longer any apparent paradox.

PROBLEM 2.4. **Incompressibility.**
(i) Show that for an adiabatic variation, the incompressible limit may be obtained formally by letting γ tend to infinity. (ii) Since γ is in reality finite, establish the condition for incompressibility in MHD.

SOLUTION. **(i)** Consider a steady situation, for which the continuity and adiabatic equations are

$$(\mathbf{v} \cdot \mathbf{\nabla})\rho + \rho \mathbf{\nabla} \cdot \mathbf{v} = 0 \tag{1}$$

and

$$(\mathbf{v} \cdot \mathbf{\nabla})p = \frac{\gamma p}{\rho}(\mathbf{v} \cdot \mathbf{\nabla})\rho \tag{2}$$

One proof would be to assume that $(\mathbf{v} \cdot \mathbf{\nabla})p$, p and ρ all remain finite while $\gamma \to \infty$. Then Eq.(1) implies that

$$(\mathbf{v} \cdot \mathbf{\nabla})\rho = 0,$$

so that there are no changes in density following the flow – i.e., it is incompressible.

An alternative proof would be to realise that Eq.(2) implies that the variations δp and $\delta \rho$ in pressure and density are related by

$$\frac{\delta p}{p} = \gamma \frac{\delta \rho}{\rho}.$$

Thus, if $\delta p / p$ remains finite but $\gamma \to \infty$, then $\delta \rho / \rho \to 0$. In other words, there is no change in density and the plasma is incompressible.

(ii) We have

$$\frac{\delta \rho}{\rho} = \frac{\delta p}{\gamma p}$$

and in a nonmagnetic fluid (i.e., $\beta \gg 1$) $\rho(\mathbf{v} \cdot \mathbf{\nabla})\mathbf{v} = -\mathbf{\nabla} p$ implies that

$$\delta p = \rho v \ \delta v,$$

so that

$$\frac{\delta\rho}{\rho} = \frac{v\delta v}{\gamma p/\rho} = \frac{v\delta v}{c_s^2}.$$

Thus, we have an incompressible variation, i.e.,

$$\frac{\delta\rho}{\rho} \ll \frac{\delta v}{v},$$

when

$$v^2 \ll c_s^2. \tag{3}$$

In other words, in a highly subsonic fluid flow, the density variations are much smaller than the velocity variations and the flow is roughly incompressible – i.e., compressibility is important only for fast flows.

In MHD there are several possibilities. First of all, if $c_s \sim v_A$, then $v^2 \ll c_s^2$ implies $v^2 \ll v_A^2$.

On the other hand, if $v \sim v_A$, this condition becomes

$$v_A^2 \ll c_s^2$$

or

$$\beta \gg \frac{2}{\gamma} \sim 1. \tag{4}$$

In other words, with the above assumptions, the condition for incompressibility is (3) or (4). Note that the argument would be different when $\beta \ll 1$.

PROBLEM 2.5. **Frozen Flux.**
Confirm that Eq.2.53 (namely, $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}$) implies Eq.2.54 (namely, $\mathbf{v}_\perp = \mathbf{E} \times \mathbf{B}/B^2$).

SOLUTION.
We have

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}.$$

The vector product of this equation with $\mathbf{B}$ gives

$$\mathbf{E} \times \mathbf{B} + (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} = \mathbf{0}$$

or

$$\mathbf{E} \times \mathbf{B} + \mathbf{B}(\mathbf{v} \cdot \mathbf{B}) - \mathbf{v}B^2 = \mathbf{0}.$$

Thus, if the flow is perpendicular to the field ($\mathbf{v} \cdot \mathbf{B} = 0$), we have

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2},$$

as required.

PROBLEM 2.6. **Diffusion.**
Show that, when there is no $E_\parallel$ and $\mathbf{w}$ exists, in resistive MHD the slippage velocity is simply

$$\mathbf{w} - \mathbf{v} = \frac{\mathbf{j} \times \mathbf{B}}{\sigma\ B^2}.$$

SOLUTION. In resistive MHD, Ohm's law is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma},$$

and the flux velocity $\mathbf{w}$ satisfies

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \boldsymbol{\nabla}\Phi,$$

where the condition that $E_\parallel = 0$ implies that $\mathbf{B} \cdot \boldsymbol{\nabla}\Phi = 0$. Thus $\boldsymbol{\nabla}\Phi$ is normal to $\mathbf{B}$ and so we may include it in $\mathbf{w}$, so that the equation for $\mathbf{w}$ becomes

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \mathbf{0}.$$

Now, the vector products of these equations with $\mathbf{B}$ give

$$\mathbf{B} \times \mathbf{E} + \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \times \frac{\mathbf{j}}{\sigma}$$

and

$$\mathbf{B} \times \mathbf{E} + \mathbf{B} \times (\mathbf{w} \times \mathbf{B}) = \mathbf{0}$$

or

$$\mathbf{B} \times \mathbf{E} + \mathbf{v}B^2 - \mathbf{B}(\mathbf{v} \cdot \mathbf{B}) = \mathbf{B} \times \frac{\mathbf{j}}{\sigma}$$

and

$$\mathbf{B} \times \mathbf{E} + \mathbf{w}B^2 - \mathbf{B}(\mathbf{w} \cdot \mathbf{B}) = \mathbf{0}$$

Then, if we assume that the component of $\mathbf{w}$ along the magnetic field is the same as that of $\mathbf{v}$ (i.e., $\mathbf{v} \cdot \mathbf{B} = \mathbf{w} \cdot \mathbf{B}$), and subtract one equation from the other, we find

$$\mathbf{w} - \mathbf{v} = \frac{\mathbf{j} \times \mathbf{B}}{\sigma\ B^2},$$

as required.

PROBLEM 2.7. Field Lines.
For the magnetic field $\mathbf{B} = -y\ \hat{\mathbf{x}} + \hat{\mathbf{y}}$ sketch the magnetic field lines. From the sketch, what do you expect qualitatively the magnetic pressure force and magnetic tension force to be (a) at a point on the x-axis and (b) at a location where $y > 0$. Verify your intuition by calculating the magnetic pressure force, magnetic tension force and Lorentz force explicitly.

SOLUTION. The magnetic field has components

$$B_x = -y, \qquad B_y = 1,$$

and so the fieldlines are given by

$$\frac{dx}{dy} = -\frac{1}{y},$$

or

$$x = -\tfrac{1}{2}y^2 + c.$$

A sketch of them is shown in Fig.2.1.

From the sketch, the curvature of the field lines produces a tension force to the right and the decrease in their spacing as one moves away from the x-axis produces a magnetic pressure force that acts towards the x-axis. Thus, one expects just a tension force at Q_1, as indicated by the force L, and a combination of pressure and tension forces at Q_2 acting inwards normal to the fieldline towards the centre of curvature of the fieldline, again as indicated by the direction L.

The magnetic pressure force is

$$\mathbf{P} \equiv -\frac{1}{2\mu}\boldsymbol{\nabla}(B^2) = -\frac{1}{2\mu}\frac{\partial}{\partial y}(1+y^2)\hat{\mathbf{y}} = -\frac{y}{\mu}\hat{\mathbf{y}},$$

which vanishes at Q_1 on the x-axis and is negative at Q_2 in $y > 0$, as indicated on Fig.**??**.

On the other hand, the tension force is

$$\mathbf{T} \equiv (\mathbf{B}\cdot\boldsymbol{\nabla})\frac{\mathbf{B}}{\mu} = \frac{1}{\mu}\frac{\partial}{\partial y}(B_x)\hat{\mathbf{x}} = -\frac{1}{\mu}\hat{\mathbf{x}},$$

which is uniform and negative, as indicated by the direction T at Q_2 and being the same as L at Q_2.

Finally, the electric current is

$$\mathbf{j} = \frac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{B} = -\frac{1}{\mu}\frac{\partial B_x}{\partial y}\hat{\mathbf{z}} = \frac{1}{\mu}\hat{\mathbf{z}},$$

and so the Lorentz force is

$$\mathbf{L} \equiv \mathbf{j}\times\mathbf{B} = -j_z B_y \hat{\mathbf{x}} + j_z B_x \hat{\mathbf{y}} = \frac{1}{\mu}(-1. - y),$$

which agrees with $\mathbf{P}+\mathbf{T}$ as expected and has the same directions at Q_1 and Q_2 as expected intuitively.

PROBLEM 2.8. **Flux Surfaces in Axisymmetric Cylindrical Polars.**

In axisymmetric cylindrical polars $(R, \phi.z)$ show that the magnetic field can be written in terms of the flux function $[F(R,z)]$ as

$$(B_R, B_z) = \left(\frac{1}{R}\frac{\partial F}{\partial z}, -\frac{1}{R}\frac{\partial F}{\partial R}\right).$$

SOLUTION. We follow the same lines as the proof for axisymmetric spherical polars in Sec.2.9.3 of the book. Thus, $\mathbf{B}$ and F need to satisfy two equations, namely,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \quad \text{(a)} \qquad \text{and} \qquad \mathbf{B}\cdot\boldsymbol{\nabla}F = 0 \quad \text{(b)}. \tag{5}$$

Eq.(5b) expresses the fact that, if magnetic field lines lie in surfaces $F = constant$, then the vector $\mathbf{B}$ will lie in those surfaces and $\boldsymbol{\nabla}F$ will be perpendicular to them.

In axisymmetric cylindrical polars (R, ϕ, z), we can satisfy Eq.(5a) by putting

$$\mathbf{B} = \boldsymbol{\nabla}\times[F(R,z)G(R,z)\hat{\boldsymbol{\phi}}] = -\frac{\partial}{\partial z}(FG)\,\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial}{\partial R}(RFG)\,\hat{\boldsymbol{z}}.$$

Also, Eq.(5b) becomes $B_R \partial F/\partial R + B_z \partial F/\partial z = 0$ or

$$\frac{\partial}{\partial z}(FG)\frac{\partial F}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}(RFG)\frac{\partial F}{\partial z}$$

which is satisfied by putting $G = -1/R$, so that

$$(B_R, B_z) = \left(\frac{1}{R}\frac{\partial F}{\partial z}, -\frac{1}{R}\frac{\partial F}{\partial R}\right),$$

as required.