SUPPLEMENTARY PROBLEMS

We split the problems into two sections, Section A and Section B. Section A problems are fairly standard, whilst Section B problems are a little more demanding. As usual, more difficult problems are denoted with an asterisk. In Section A, the first number denotes the chapter of the text the problem refers to. (So Problem 2.3 refers to a problem supplementary to Chapter 2). Double starred questions are problems that the author has not yet fully considered, and may be difficult.

Section A

1.1 You pay into an annuity a sum of $\$P$ dollars. This annuity pays you $\$\alpha$ per month. The interest is $r\%$ (calculated as simple interest on the remaining balance at the end of each month). If $A(n)$ is the amount remaining at the end of the nth month, $A(0) = P$, write down $A(n+1)$ in terms of $A(n)$, and deduce a closed formula for $A(n)$.

If $P = \$100,000$, $\alpha = \$500$ and the interest rate is 4% per month, find out how long the annuity will last.

1.2 Let $g(x) = x^2 + 1/4$. Use induction to show that:

(a) $g^n(x) \to 1/2$ as $n \to \infty$, if $-1/2 < x < 1/2$,

(b) $g^n(x) \to \infty$ as $n \to \infty$, if $|x| > 1/2$.

Deduce that $x = 1/2$ is a fixed point of g that is stable on the left and unstable on the right.

1.3 If $p(x)$ is a polynomial of degree n, what is the degree of (i) $p^2(x)$, (ii) $p^k(x)$?

2.1 Let $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$ (a C^1 function), with $|f'(x)| < 1$ for all $x \in (0, 1)$.

(a) Prove that $f(x)$ has a unique fixed point p in $[0, 1]$ (See 1.4 # 2).

(b) Prove that $f(x)$ cannot have a point of period 2 in $[0, 1]$ (Hint: Use the Mean Value Theorem).

(c) Prove that $f^n(x) \to p$ as $n \to \infty$, for all $x \in (0, 1)$.

2.2 Suppose that f is an odd function, $f(\alpha) \neq 0$ and $N_f(\alpha) = -\alpha$.

(a) Show that $f^n(x) - x$ and $N_f(x)$ are odd functions.

(b) Show that $\{-\alpha, \alpha\}$ is a 2-cycle for N_f. Use this to find the 2-cycles for N_f when (i) $f(x) = x^3 - 4x$, (ii) $f(x) = \sin(x)$. What happens with $f(x) = x^3 + 2x$.

(c) Show that if $\{\alpha, \beta\}$ is a 2-cycle for the odd function f, then so is $\{-\alpha, -\beta\}$. Give an example of an odd function f with a 2-cycle not of the form $\{-\alpha, \alpha\}$ (Hint: Consider $f(x) = x^3 - \lambda x$ for some $\lambda > 2$).

2.3 (a) Let $f(x)$ be a polynomial that is also an odd function. Note that $g(x) = ([f(x)]^2 - x^2)/x$ is a polynomial. Show that $g(x)$ divides $f^2(x) - x$. (Hint: Note that some 2-cycles arise from solving $f(x) = -x$).

(b) Check the above for (i) $f(x) = x^3 - x$, (ii) $f(x) = x^3 - \lambda x$ for $0 \le \lambda < 3$. Find all the two cycles for $f(x) = x^3 - 3x/2$, and deduce that there are some that don't arise from solving $f(x) = -x$.

2.4 Show that the fixed points of T (the standard tent map) are in the first and sixth positions of the fixed points of T^3 (we are using the ordering on the line real line).

2.5 (a) Show that if a function $f : \mathbb{R} \to \mathbb{R}$ has a 2-cycle $\{\alpha, \beta\}$, then the line $y = -x + a$ intersects the graph in the two points (α, β) and (β, α), for some $a \in \mathbb{R}$. Conversely, show that if the line $y = -x + a$ intersects the graph of f in two points equidistant from the line $y = x$, then f has a 2-cycle. Illustrate the situation graphically.

(b) Graph the function $f : [1, 10] \to [1, 10]$ defined by $f(x) = \begin{cases} x^2; & 1 \le x \le \sqrt{10} \\ x^2/10; & \sqrt{10} < x \le 10 \end{cases}$, and use (a) to show that f has a 2-cycle. Find the 2-cycle.

(c)** Does the function f from (b) have any other periodic orbits? What about the function $g : [1, 10] \to [1, 10]$ defined by $g(x) = \begin{cases} 2x; & 1 \le x \le 5 \\ x/5; & 5 < x \le 10 \end{cases}$. Does g have any periodic points? Is g transitive.

2.6 $f_a : [0, 1] \to [0, 1]$ is defined by $f_a(x) = \begin{cases} ax^2; & 0 \le x \le 1/2 \\ a(1-x)^2; & 1/2 < x \le 1 \end{cases}$.

(a) Find the fixed points for each $a \in [0, 4]$ and determine their stability.

(b) For the fixed point $x = 0$, find the immediate basin of attraction. Deduce that f_a is not chaotic for $a \in [0, 4]$.

(c)** What is the basin of attraction of $x = 0$. Investigate the dynamics of f_a as a increases (use Mathematica).

3.1 Let $f : \mathbb{R} \to \mathbb{R}$ have a period-16 point. What other periodic points must f have? What if instead f has a period-18 point?

3.2 Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which there exists $y \in \mathbb{R}$ with $f^2(y) < y < f(y)$, then f has a fixed point. Give an example to show that f need not have a 2-cycle (See Exercises 3.2 # 12, where the requirement that $f : [a, b] \to [a, b]$ should be included).

4.1 (a) Let $f : X \to X$ be a continuous function on a metric space X. f is a *periodic function* if there exists $p \in \mathbb{Z}^+$ satisfying $f^p(x) = x$ for all $x \in X$ (the minimum such p is the *period* of f). Show that such a function is a homeomorphism and give two examples of (non-trivial) periodic functions of period $p > 1$.

(b) Let $f : [0, 1] \to [0, 1]$ be a continuous periodic function. What can you say about p?

5.1 Write down the definition of sensitive dependence on initial conditions as a quantified statement, and use it to write down its negation.

6.1 Show that if $f : X \to X$ is a continuous function on a metric space X for which f^2 is chaotic, then f is chaotic. Is the converse true?

7.1 $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 2x + 2$. Show that f is linearly conjugate to $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$.

7.2 A map $f : [0, 1] \to [0, 1]$ is defined by $f(x) = \begin{cases} -2x + 1; & 0 \leq x \leq 1/2 \\ 2x - 1; & 1/2 < x \leq 1 \end{cases}$. Show that f is conjugate to the tent map by constructing a conjugacy.

8.1 If $x = c$ is an attracting fixed point of a polynomial $p(x)$ of degree $n > 1$, we know that the immediate basin of attraction is an open interval. Why must the interval be bounded?

8.2 (a) Let $p(x) = (x-a)(x-b)(x-c)$ be a polynomial of degree 3 having three distinct real roots. Find the Schwarzian derivative and show directly that it is negative.

(b) If p is a polynomial of degree $n > 1$ having all roots real and distinct, show that p has a negative Schwarzian derivative.

8.3 Let $p(x)$ be a polynomial of degree $n > 1$.

(a) If p' has $n - 1$ real roots (some of which may be repeated), show that p has a negative Schwarzian derivative.

(b) If $p(x)$ has n real roots (not necessarily distinct), show that p has a negative Schwarzian derivative.

(c) Give an example of a degree 3 polynomial not having negative Schwarzian derivative everywhere.

8.4 Let $p(x) = x^3 + ax^2 + bx + c$, $a \neq 0$, where $p'(x) > 0$ for all $x \in \mathbb{R}$. Show that the Schwarzian derivative $Sp(x)$ is positive on an interval.

9.1 $f_a : [0, 1] \to [0, 1]$, $a > 1$, is defined by $f_a(x) = \begin{cases} ax; & 0 \leq x \leq 1/a \\ -a(x-1)/(a-1); & 1/a < x \leq 1 \end{cases}$.
f_a is a skewed tent map. Sketch the graph of f_a.

(a) Show that f_a is conjugate to the tent map. (Hint: Use the methods of Section 9.1).

(b) Show that if h is the conjugacy, then $h(0) = 0$, and that h cannot be linear near 0. (Hint: Consider $h^{-1} \circ f \circ h$ near 0).

9.2 The polynomial $p_k(\mu) = \mu^{k-1} - \mu^{k-2} - \mu^{k-3} + \mu^{k-4} - \mu^{k-5} + \cdots + \mu - 1$, $k \geq 5$ (odd), arises in the solution to Exercise 9.2 # 3(iii). Show that $p_k(\sqrt{2}) = 1 - \sqrt{2} < 0$ for $k = 5, 7, 9 \ldots$. Deduce that $T_{\sqrt{2}}$ (Tent map) cannot have points having odd periods (see also Section 2.8).

10.1 (a) Give a description of points in the Sierpinski Carpet in an analogous way to the ternary representation of points in the Cantor Set (as a first step in the construction, take the interval $[0, 1] \times [0, 1]$). Deduce that the Sierpinski Carpet is invariant under the Iterated Function System defined by the maps

$$f_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/3 + a_i \\ y/3 + b_i \end{pmatrix},$$

$i = 1, \ldots, 8$, where $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ are the points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$, $\begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$, $\begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ and $\begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$.

(b) Give a description of Menger sponge in terms of ternary expansions.

10.2 (a) Show that the Sierpinski triangle has topological dimension zero.

(b) Show that the Sierpinski triangle is invariant under the Iterated Function System given in Example 10.5.9.

10.3 For each $n \geq 1$, set $A_n = \{(k/n, \ell/n) : 0 \leq k \leq n,\ \ 0 \leq \ell \leq n\}$, a discrete subset of $B = [0,1] \times [0,1]$. If D denotes the Hausdorff metric on the compact subsets of $\mathbb{R}^2$, show that $\lim_{n\to\infty} A_n = B$ with respect to the Hausdorff metric.

10.4 Let $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ be an iterated function system given by $F_1(x) = x/4$ and $F_2(x) = x/2 + 1/2$. Describe the attractor and find its fractal dimension.

14.1 $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = \bar{z}$. Show that f is not differentiable at any point in $\mathbb{C}$.

14.2 A function $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = z^8$. Find the fixed points of f. Use your calculations to find the real linear and quadratic factors of the polynomial $p(z) = z^7 - 1$.

14.3. Prove that if $f_c(z) = z^2 + c$ has an attracting periodic point, then $c \in \mathcal{M}$, the Mandelbrot set. (Hint: Use any appropriate results from Section 14.5).

14.4. Show that $p(z) = az + b$, $a \neq 1$ on $\mathbb{C}$, is conjugate to $q(z) = az$.

14.5. Let $f_c(z) = z^2 + c$. Find the values of c so that $z = i$ is a period-2 point. Find the fixed points in each case and determine their stability. Does $c \in \mathcal{M}$? (Hint: It is useful to use Mathematica to find the square roots of complex numbers).

14.6 Show that the function $H(z) = \dfrac{z-i}{z+i}$ gives a conjugacy between the Newton map N_{f_1}, where $f_1(z) = z^2 + 1$, and the function $f_0(z) = z^2$. Deduce the Julia set of N_{f_1}, and that it is chaotic on its Julia set. (Hint: Find $H^{-1}(z)$ and set $z = e^{i\theta}$).

14.7 Is it possible for a polynomial $p(z)$ to satisfy $\dfrac{p^2(z) - z}{p(z) - z} = p(z) + z$? If so, find all such polynomials and their period-2 points. What does this say about the periodic points of the polynomial?

(b) What about $\dfrac{p^2(z) - z}{p(z) - z} = p(z) - z$?

Section B

1. Consider the family of functions $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = x^3 - \lambda x$, for a parameter $\lambda \in \mathbb{R}$.

(a) Find all the fixed points and determine their nature and where they are created as λ varies.

(b) Find where a two cycle is created (you may use a computer algebra facility), and give the graph of where this happens. Determine the stability of the hyperbolic 2-cycles.

(c) Use a computer algebra facility to find an approximate value of λ where the 3-cycle is created. Give the graph of this situation.

2. Let $f(x) = ax^3 + bx + c$ where a and b satisfy $a/b > 0$. Denote by N_f the corresponding Newton function.

(a) Show that N_f has a unique fixed point.

(b) Show that N_f cannot have any period-2 points.

(c) Why does it follow that N_f has no points of period n, $n > 2$?

3. This question is related to Exercise 4.2 # 8, parts (a), (b) and (c).

(a) Show that the function $f(x) = -1/(x+1)$ has the property that $f^3(x) = x$ for all $x \neq -1$, $x \neq 0$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on a set I, with $f^3(x) = x$ for all $x \in I$? Set $g(x) = f^2(x)$. Show that $g^3(x) = x$ for all $x \in I$. Deduce a different function from the one in (a) with this property.

(c) In general, show that such a function cannot have a 2-cycle.

(d) Deduce that a function $f : \mathbb{R} \to \mathbb{R}$ with the property $f^3(x) = x$ cannot be continuous (i.e., there must be some points of discontinuity for f, where we may not have $f^3(x) = x$).

(e) Show that f is one-to-one. Need f be onto? Show that f is strictly decreasing and investigate any symmetry. What if $f : [-\infty, \infty] \to [-\infty, \infty]$ - consider the example in (a) above?

(f) If $f'(x)$ exists for all $x \in I$, show that the 3-cycles are non-hyperbolic, (assume that $f(x)$ is not the function $y = x$) .

(g) Suppose that $f(x) = \dfrac{ax+b}{cx+d}$ satisfies $f^3(x) = x$, show that if $f(x)$ is not the function $y = x$, and $a \neq d$ then

$$a^2 + bc + ad + d^2 = 0.$$

(i) Show that if $ad - bc > 0$, then such a function cannot have any fixed points.

(ii) Deduce two other functions with the property that $f^3(x) = x$ wherever defined, one having fixed points and one with no fixed points.

4.** (check). If $f : \mathbb{D} \to \mathbb{D}$ has the property $|f'(z)| < 1$ for all $z \in \mathbb{D} \setminus \mathbb{S}^1$, show that $|f(z) - f(w)| < |z - w|$ for all $z, w \in \mathbb{D}$. Does it follow that f is a contraction mapping (consider some examples)? Show that f has a unique fixed point α. Why can we deduce that $B_f(\alpha) = \mathbb{D}$? ($\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$).

5. There is a correspondence between functions of the form $f(z) = \dfrac{az+b}{cz+d}$ and matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, when $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$. To be more precise, show that there is a group homomorphism $\Phi : \mathrm{GL}_2(\mathbb{R}) \to \mathcal{F}$, where $\mathrm{GL}_2(\mathbb{R})$ is the multiplicative group of 2×2 matrices having real entries and non-zero determinant, and $\mathcal{F}$ is the group (under composition) of linear fractional transformations $f(z) = \dfrac{az+b}{cz+d}$, ($ad - bc \neq 0$), where Φ is defined by $\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(z)$. Is the map Φ one-to-one or onto?

6. (a) Show that $f_c : \mathbb{R} \to \mathbb{R}$, $f_c(x) = x^2 + c$ has a 3-cycle for $c \leq -7/4$.

(b) Deduce that $L_\mu : \mathbb{R} \to \mathbb{R}$, $L_\mu(x) = \mu x(1-x)$ has a 3-cycle for $\mu \leq 1 - \sqrt{8}$.

7.* Use the following steps to show that the Koch curve is continuous, i.e., there is a continuous function $f : [0,1] \to \mathbb{R}^2$, whose range is the top part of the Koch curve (see Section 10.1):

(a) Denote by $f_n : [0,1] \to \mathbb{R}^2$ the piecewise defined function which is linear on each of the 4^n segments defined at the nth stage of the construction of the Koch curve. Show that $|f_n(t) - f_{n+1}(t)| \leq 3^{-n}$ for all $t \in [0,1]$.

(b) Deduce that $(f_n(t))$ is a Cauchy sequence for each $t \in [0,1]$, so that $f(t) = \lim_{n\to\infty} f_n(t)$ exists for all $t \in [0,1]$.

(c) Show that convergence of the sequence of functions (f_n) to f is *uniform* on $[0,1]$ (i.e., $\lim_{n\to\infty} \left[\sup_{0\leq t\leq 1} |f_n(t) - f(t)|\right] = 0$).

(d) Now complete the proof using the fact that the uniform limit of a sequence of continuous functions on an interval $[a,b]$ is a continuous function. That f is not differentiable anywhere can be deduced from the fact that the length of the graph of f is infinite on any arbitrarily small interval.