Reliability and Availability Engineering: Modeling, Analysis, Applications

Chapter 9 - Continuous Time Markov Chain: Availability Models

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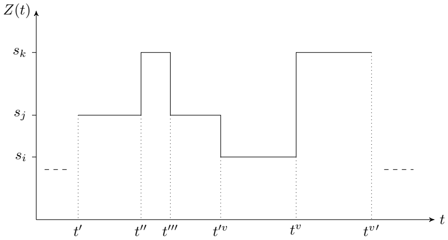
Jan 2017

Outline

Continuous Time Markov Process

A stochastic process $Z(t)$ defined over a discrete state space Ω of cardinality $n = |\Omega|$ is a continuous-time discrete-state Markov process (or Markov chain - CTMC) if:



for any ordered sequence of time instants

$$(0 < t_1 < t_2 < \ldots < t_{m-1} < t_m)$$

The following property holds:

$$\begin{aligned} P\{Z(t_m) &= s_{j_m} | Z(t_{m-1}) = s_{j_{m-1}}, \ldots, Z(t_1) = s_{j_1}\} \\ &= P\{Z(t_m) = s_{j_m} | Z(t_{m-1}) = s_{j_{m-1}}\} \end{aligned}$$

Transition Probability Matrix

Let us introduce the following notation:

$$p_{ij}(u, t) \; = \; P\{ Z(t) = j \,|\, Z(u) = i \} \qquad (u \leq t)$$

With:

$$p_{ii}(t, t) = 1 \qquad ; \qquad p_{ij}(t, t) = 0$$

where $p_{ij}(u, t)$ is the conditional probability that the Markov chain $Z(t)$ is in state j at time t, given it was in state i at time u (*transition probability*).

$$\pi_i(t) \; = \; P\{ Z(t) = s_i \}$$

$\pi_i(t)$ is the (unconditional) probability that $Z(t)$ is in state i at time t and is called the state occupancy probability, or simply the *state probability*. From the above definitions:

$$\sum_{j=1}^{n} p_{ij}(u, t) = 1 \qquad ; \qquad \sum_{i=1}^{n} \pi_i(t) = 1$$

Matrix notation

$\boldsymbol{P}(u,t) = [p_{ij}(u,t)]$ is a square matrix of dimension $(n \times n)$ and is called the *transition probability matrix* of the CTMC.

$\boldsymbol{\pi}(t) = [\pi_i(t)]$ is a row vector of dimension $(1 \times n)$ and is called the *(transient) state probability vector* of the CTMC.

In matrix notation, the initial condition assumes the form:

$$\boldsymbol{P}(t,t) = \boldsymbol{I} \, ,$$

where $\mathbf{I}$ is the identity matrix of appropriate dimensions, and the normalization condition, the form

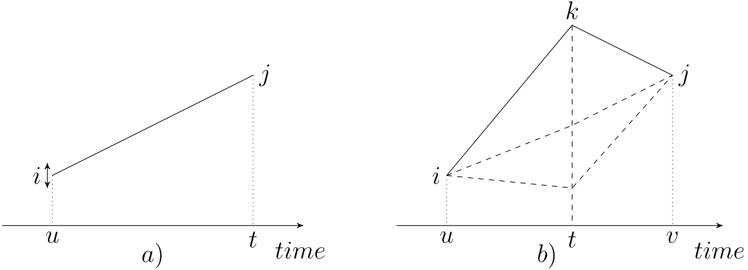
$$\boldsymbol{P}(t,t)\,\boldsymbol{e}^T = \boldsymbol{e}^T \qquad ; \qquad \boldsymbol{\pi}(0)\,\boldsymbol{e}^T = 1$$

where $\boldsymbol{e}$ is a row vector of appropriate dimension with all entries equal to one, and ^{T} means transposition.

Chapman-Kolmogorov (CK) Equations

The Markov property, combined with the theorem of the total probability, implies the following Chapman-Kolmogorov (CK) equations:

$$\begin{aligned} \pi_j(t) &= \textstyle\sum_i \pi_i(u) \cdot p_{ij}(u,t) \\ p_{ij}(u,v) &= \textstyle\sum_k p_{ik}(u,t) \cdot p_{kj}(t,v) \qquad \text{for } u \le t \le v \end{aligned}$$



In matrix notation

$$\begin{aligned} \boldsymbol{\pi}(t) &= \boldsymbol{\pi}(u) \cdot \boldsymbol{P}(u,t) \\ \boldsymbol{P}(u,v) &= \boldsymbol{P}(u,t) \cdot \boldsymbol{P}(t,v) \end{aligned}$$

Homogeneous CTMC

A Markov chain is said to be homogeneous when the transition probabilities in matrix $\mathbf{P}(u, x)$ depend only on the length of the time interval $(x - u)$ and not on the values of the time instants x and u.

Formally given time instants t_1 and t_2, the time homogeneous property is written as:

$$\mathbf{P}(t_1, t_1 + x) = \mathbf{P}(t_2, t_2 + x) = \mathbf{P}(0, x)$$

Most of the modeling techniques in availability and reliability analysis are based on homogeneous Markov chains.

Homogeneous CTMC

If the Markov chain is homogeneous, by substituting $u = 0$ and $\theta = v - t$ in the CK equations, we get:

$$
\begin{array}{lcll}
\boldsymbol{\pi}(t) & = & \boldsymbol{\pi}(0) \cdot \boldsymbol{P}(t) \quad \text{given} & \boldsymbol{\pi}(0) = \pi_0 \\
\boldsymbol{P}(t+\theta) & = & \boldsymbol{P}(t) \cdot \boldsymbol{P}(\theta) & \boldsymbol{P}(0) = \boldsymbol{I}
\end{array}
$$

where $\boldsymbol{\pi}(0)$ is the initial state probability vector of the CTMC.

The Infinitesimal Generator Matrix

Define, for $i \neq j$ and for $\Delta\, t \geq 0$:

$$q_{ij} = \left. \frac{d\, p_{ij}(t)}{d\, t} \right|_{t=0} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t) - p_{ij}(0)}{\Delta\, t} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta\, t} \qquad (1)$$

From equation (1) it is easy to see that $q_{ij} \geq 0$. Similarly, define for $i = j$ and for $\Delta\, t \geq 0$:

$$q_{ii} = \left. \frac{d\, p_{ii}(t)}{d\, t} \right|_{t=0} = \lim_{\Delta t \to 0} \frac{p_{ii}(\Delta t) - p_{ii}(0)}{\Delta\, t} = - \lim_{\Delta t \to 0} \frac{1 - p_{ii}(\Delta t)}{\Delta\, t} \qquad (2)$$

From equation (2) it can be seen that $q_{ii} \leq 0$.

Physical interpretation of the quantities q_{ij}

We rewrite the above equations in the following form:

$$p_{ij}(\Delta\, t) \;=\; P\,\{\,Z(t\,+\,\Delta\, t)\}\,=\,j\,|\,Z(t)\,=\,i\}\,=\,q_{ij}\,\Delta t\,+\,o(\Delta\, t)$$

$$p_{ii}(\Delta\, t) \;=\; P\,\{\,Z(t\,+\,\Delta\, t)\}\,=\,i\,|\,Z(t)\,=\,i\}\,=\,1\,+\,q_{ii}\,\Delta t\,+\,o(\Delta t)$$

$q_{ij}\,\Delta\, t$ is the conditional probability of jumping to state j in interval $\Delta\, t$, given that the CTMC was in state i at the beginning of the interval.

The quantities q_{ij} are called the transition rates of the CTMC.

Since the transition from state i to some state $j \in \Omega$ in the interval $(t, t + \Delta t]$ is a certain event:

$$1 \,=\, \sum_{j} p_{ij}(\Delta t) \,=\, 1\,+\,q_{ii}\,\Delta t\,+\,\sum_{j:\,j\neq i} q_{ij}\,\Delta t$$

Hence,

$$q_{ii} \,=\, -\sum_{j:\,j\neq i} q_{ij}$$

The Infinitesimal Generator Matrix

The values q_{ij} can be grouped into matrix $\boldsymbol{Q} = [q_{ij}]$ called the infinitesimal generator matrix of the CTMC. Matrix $\boldsymbol{Q}$ is defined as:

$$\boldsymbol{Q} = [\, q_{ij} \,] \qquad \text{where} \qquad \begin{array}{ll} q_{ij} \geq 0 & i \neq j \\ q_{ii} \leq 0 & q_{ii} = -\displaystyle\sum_{j:\, j \neq i} q_{ij} \end{array} .$$

the off-diagonal entries are non negative, the diagonal entries are non positive and the row sum is equal to 0.

The off diagonal entries of row i represent the transitions out of state i, while off diagonal entries of column i the transitions into state i.

Kolmogorov Differential Equation

The CK equations, in a small interval $(t, t + \Delta t]$ can be written as:

$$\begin{aligned} p_{ij}(t+\Delta\, t) &= \sum_k p_{ik}(t)\, p_{kj}(\Delta\, t) \\ &= p_{ij}(t)(1 + q_{jj}\,\Delta t) + \sum_{k:k\neq j} p_{ik}(t)\, q_{kj}\,\Delta\, t + o(\Delta t)\,. \end{aligned}$$

Rearranging, we get:

$$\begin{aligned} \frac{p_{ij}(t+\Delta\, t) - p_{ij}(t)}{\Delta\, t} &= p_{ij}(t)\, q_{jj} + \sum_{k:k\neq j} p_{ik}(t)\, q_{kj} + \frac{o(\Delta t)}{\Delta t} \\ &= \sum_k p_{ik}(t)\, q_{kj} + \frac{o(\Delta t)}{\Delta t}\,. \end{aligned}$$

Taking the limit as $\Delta t \to 0$,

$$\frac{d\, p_{ij}(t)}{d\, t} = \sum_k p_{ik}(t)\, q_{kj} \quad \text{with initial condition} \quad p_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Kolmogorov Differential Equation

In matrix notation:

$$\frac{d\,\boldsymbol{P}(t)}{d\,t} = \boldsymbol{P}(t) \cdot \boldsymbol{Q} \qquad ; \qquad \boldsymbol{P}(0) = \boldsymbol{I}\,.$$

Let $\boldsymbol{\pi}(t)$ be the (transient) state probability vector at time t.
Differentiating both sides, we have:

$$\frac{d\,\boldsymbol{\pi}(t)}{d\,t} = \boldsymbol{\pi}(0) \cdot \frac{d\,\boldsymbol{P}(t)}{d\,t} = \boldsymbol{\pi}(0) \cdot \boldsymbol{P}(t) \cdot \boldsymbol{Q}\,.$$

From which we derive the state probability equation:

$$\frac{d\,\boldsymbol{\pi}(t)}{d\,t} = \boldsymbol{\pi}(t) \cdot \boldsymbol{Q} \quad \text{with initial condition} \quad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0\,. \tag{3}$$

These are the fundamental equations for CTMC, known as Kolmogorov differential equations.

The State Probability Vector - 2

Equation 3 is in the form of a *Ordinary Differential Equation Initial Value Problem (ODEIVP)*.

The ODEIVP Equation (3) has formal solution:

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0) \cdot e^{\boldsymbol{Q}\,t},$$

where $e^{\boldsymbol{Q}\,t} = \boldsymbol{P}(t)$ is defined by the following series expansion:

$$e^{\boldsymbol{Q}\,t} = \boldsymbol{I} + \boldsymbol{Q}\,t + \frac{1}{2}(\boldsymbol{Q}\,t)^2 + \frac{1}{3\,!}(\boldsymbol{Q}\,t)^3 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i\,!}(\boldsymbol{Q}\,t)^i \,.$$

Physical interpretation of the transition rates

Suppose a state i is directly connected to a state j and the two states differ by the value of a single binary variable X_k.

Assume that in the source state i, $X_k = 1$ (k-th component up) and in the destination state j, $X_k = 0$ (k-th component down).

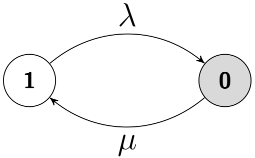
Then the transition $i \rightarrow j$ represents the failure of component k.

$q_{ij}\,\Delta\, t$ is the probability of a transition to state j in the interval $(t\,,t\,+\,\Delta\,t]$ given that the CTMC was in state i at time t, but because of the physical meaning of transition $i \rightarrow j$, in our case, q_{ij} coincides with the definition of the failure rate of component k in state i.

One repairable component: Availability

The CTMC state diagram for this system, whose state space consists of two states, is shown in Figure

$$\boldsymbol{Q} = \begin{array}{c|cc|} & (1) & (0) \\ \hline (1) & -\lambda & \lambda \\ (0) & \mu & -\mu \end{array}$$



$$\left[\begin{array}{cc} \dfrac{d\,\pi_1(t)}{d\,t} & \dfrac{d\,\pi_0(t)}{d\,t} \end{array} \right] = \left[\begin{array}{cc} \pi_1(t) & \pi_0(t) \end{array} \right] \cdot \left[\begin{array}{cc} -\lambda & \lambda \\ \mu & -\mu \end{array} \right]$$

From the above matrix equation, we obtain:

$$\begin{cases} \dfrac{d\,\pi_1(t)}{d\,t} = -\lambda\,\pi_1(t) + \mu\,\pi_0(t) \\ \dfrac{d\,\pi_0(t)}{d\,t} = \lambda\,\pi_1(t) - \mu\,\pi_0(t) \end{cases}$$

We assume as initial probability vector $\boldsymbol{\pi}(0) = \left[\begin{array}{cc} 1 & 0 \end{array} \right]$

One repairable component: Availability

Taking the Laplace transform on both sides, we can write:

$$\begin{cases} s\,\pi_1^*(s) - 1 & = & -\lambda\,\pi_1^*(t) + \mu\,\pi_0^*(t) \\ s\,\pi_0^*(s) & = & \lambda\,\pi_1^*(t) - \mu\,\pi_0^*(t) \end{cases}$$

Solving the algebraic set of equations in the Laplace domain, we obtain:

$$\begin{cases} \pi_1^*(s) & = & \dfrac{s+\mu}{s\,(s+\lambda+\mu)} \\ \pi_0^*(s) & = & \dfrac{\lambda}{s\,(s+\lambda+\mu)} \end{cases}$$

Taking the inverse Laplace transform we obtain in the time domain:

$$\begin{cases} \pi_1(t) & = & \dfrac{\mu}{\lambda+\mu} + \dfrac{\lambda}{\lambda+\mu}\,e^{-(\lambda+\mu)\,t} \\ \pi_0(t) & = & \dfrac{\lambda}{\lambda+\mu} - \dfrac{\lambda}{\lambda+\mu}\,e^{-(\lambda+\mu)\,t} \end{cases}$$

One repairable component: Availability

The pointwise or instantaneous system availability $A(t)$ is the probability that the system is up at time t.

In the two-state example, we have $A(t) = \pi_1(t)$.

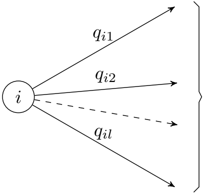
Correspondingly, the unavailability $U(t)$ is obtained in this case as $U(t) = 1 - A(t) = \pi_0(t)$.

The asymptotic solution for the steady-state availability exists, and is given by:

$$\begin{cases} \lim_{t\to\infty} \pi_1(t) = \pi_1 = \dfrac{\mu}{\lambda + \mu} \\ \lim_{t\to\infty} \pi_0(t) = \pi_0 = \dfrac{\lambda}{\lambda + \mu} \end{cases}$$

Sojourn time in state i

We isolate state i by deleting transitions entering state i. We have:



$$\frac{d\,\pi_i(t)}{d\,t} = -\,\pi_i(t)\,q_i$$

$$\pi_i(t) = e^{-\,q_i\,t} \qquad \pi_i(0) = 1$$

Where: $q_i = -\,q_{ii} = \sum_{j:\,j\neq i} q_{ij}$ is a negative constant equal to the sum of the rates out of state i.

The sojourn time in each state is exponentially distributed with a rate equal to the sum of the exit rates.

The probability that the sojourn time in state i terminates by a transition toward state j, is:

$$p_{ij}(t) = \frac{q_{ij}}{q_i}\,(1 - e^{-\,q_i\,t})$$

Classification of states and stationary distribution

The states of a CTMC can be partitioned in two classes: *recurrent* states and *transient* states.

A state is *recurrent* if the CTMC will eventually return to that state with probability 1, otherwise the state is *transient*.

The state space of a CTMC can always be partitioned into a set of (zero or more) transient states and one or more closed sets of recurrent states.

A state j is *reachable* from i for some $t > 0$, if $p_{ij}(t) > 0$. A closed set of recurrent states is a set in which all pairs of states are mutually reachable. Once the CTMC has reached a state in a closed set of recurrent states, it never leaves the set again.

If a closed set of recurrent states contains a single state, this state is an *absorbing* state, and the corresponding row of the infinitesimal generator matrix has only 0 entries.

Irreducible Markov Chain

A CTMC is *irreducible* if its state space is formed by a single set of recurrent states, so that every state is reachable from every other state.

For an irreducible CTMC the state probabilities reach an asymptotic value as the time goes to infinity and this asymptotic value is independent of the initial condition.

We call the asymptotic solution the *steady state* solution.

If the steady state solution exists, then, for any i:

$$\lim_{t \to \infty} \pi_i(t) = \pi_i \qquad \lim_{t \to \infty} \frac{d\,\pi_i(t)}{d\,t} = 0$$

and in matrix form:

$$\boldsymbol{\pi} \cdot \boldsymbol{Q} = \boldsymbol{0} \qquad \text{with} \qquad \boldsymbol{\pi}\,\boldsymbol{e}^T = 1\,. \tag{4}$$

Where $\mathbf{0}$ is the zero vector of appropriate dimension.

Properties of the steady-state distribution

The steady-state distribution for an irreducible CTMC has the following properties:

- for all initial conditions, the occupancy state probability $\pi_i(t)$ tends to a constant value π_i as $t \to \infty$, and the π_i's form a probability distribution.
- if the initial probability is $\pi_i, \forall i$, then $\pi_i(t) = \pi_i$ for all t;
- the fraction of time spent in state i during the interval $(0, t]$ tends to π_i as $t \to \infty$. In steady state, the fraction of time spent by the CTMC in state i is equal to π_i.

Irreducible Markov Chain

The steady state equation can be written as:

$$\boldsymbol{\pi} \cdot \boldsymbol{Q} = \mathbf{0} \qquad \text{with} \qquad \boldsymbol{\pi}\, \mathbf{e}^T = 1$$

The above Equation is a linear homogeneous set of n equations with constant coefficients, subject to the normalization condition.

To force the set of equations to have a single positive solution, we can incorporate the normalization condition into the the set of equations by replacing any one of the n equations with the normalization condition.

$$[\pi_1\, \pi_2\, \ldots\, \pi_{n-1}\, \pi_n] \begin{bmatrix} q_{11} & q_{12} & \ldots & q_{1,n-1} & 1 \\ q_{21} & q_{22} & \ldots & q_{2,n-1} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ q_{n-1,1} & q_{n-1,2} & \ldots & q_{n-1,n-1} & 1 \\ q_{n,1} & q_{n2} & \ldots & q_{n,n-1} & 1 \end{bmatrix} = [0\,0\,\ldots\,0\,1] \qquad (5)$$

The balance equation

The steady state equation can be interpreted as a probability *balance equation*.

The balance equation means that for every state the probability flow-in equals the probability flow-out.

This equation can be written directly from the observation of the CTMC transition graph, without the need of deriving the infinitesimal generator.

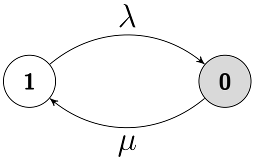
Solving the steady-state equation for state i, we can write

$$\pi_i q_{ii} \; = \; \pi_1 q_{1i} + \pi_2 q_{2i} + \ldots + \pi_n q_{ni}$$

where the left-hand-side is the probability flow out of state i and the right-hand-side is the probability flow in state i.

One repairable component: Availability

Application of (5) provides:



$$[\pi_1 \;\pi_0] \begin{bmatrix} -\lambda & 1 \\ \mu & 1 \end{bmatrix} = [0 \; 1] \;. \quad (6)$$

Expanding, we get:

$$\begin{cases} -\lambda\,\pi_1 + \mu\,\pi_0 & = & 0 \\ \pi_1 + \pi_0 & = & 1\,. \end{cases}$$

Note that the first equation is the probability balance equation of state 1 that could have been written directly.
From the above equation we obtain the steady state availability and unavailability, respectively as:

$$\begin{cases} A_{ss} & = & \pi_1 = \dfrac{\mu}{\lambda+\mu} = \dfrac{\text{MTTF}}{\text{MTTR}+\text{MTTF}} \\ U_{ss} & = & \pi_0 = \dfrac{\lambda}{\lambda+\mu} = \dfrac{\text{MTTR}}{\text{MTTR}+\text{MTTF}} \end{cases}$$

Expected State Occupancy in $(0, t]$

The total time spent by the CTMC $Z(t)$ in a generic state i during the interval $(0, t]$ is a random variable whose expected value is denoted by $b_i(t)$. Note that such a sojourn can be collected over multiple visits to the state.

We introduce an indicator variable $I_i(t)$ defined as follows:

$$\begin{cases} I_i(t) = 1 & \text{if} \quad Z(t) = i \\ I_i(t) = 0 & \text{if} \quad Z(t) \neq i\,. \end{cases}$$

Then, the total time spent by the CTMC in state i in the interval $(0, t]$ is:

$$\text{total time spent in state } i = \int_0^t I_i(u)\, du$$

with initial condition $b_i(0) = 0$.

Expected State Occupancy in $(0, t]$

Hence, the expected value $b_i(t)$ becomes:

$$\begin{aligned} b_i(t) &= E\left[\int_0^t I_i(u)\,du\right] = \int_0^t E\,[\,I_i(u)\,]\,du \\ &= \int_0^t [0 \cdot P\{I_i(u) = 0\} + 1 \cdot P\{I_i(u) = 1\}]du \\ &= \int_0^t \pi_i(u)\,du\,. \end{aligned}$$

Define the vector $\boldsymbol{b}(t) = [\,b_i(t)\,]$, then:

$$\boldsymbol{b}(t) = \int_0^t \boldsymbol{\pi}(u)\,du\,.$$

Direct integration of Equation (3), yields:

$$\frac{d\,\boldsymbol{b}(t)}{d\,t} = \boldsymbol{b}(t)\,\boldsymbol{Q} + \boldsymbol{\pi}(0)\,,$$

under the initial condition $\boldsymbol{b}(0) = \mathbf{0}$.

Expected State Occupancy in $(0, t]$

The time-averaged expected state occupancy, denoted by the vector $\boldsymbol{g}(t) = (1/t)\,\boldsymbol{b}(t)$, is defined as:

$$\frac{d\,\boldsymbol{g}(t)}{d\,t} = \boldsymbol{g}(t)\left(\boldsymbol{Q} - \frac{\boldsymbol{I}}{t}\right) + \frac{\boldsymbol{\pi}(0)}{t}, \qquad \text{with} \quad \mathbf{g}(0)=\mathbf{0}. \tag{7}$$

The following normalization condition for $\boldsymbol{b}(t)$ and $\boldsymbol{g}(t)$, hold:

$$\boldsymbol{b}(t)\,\boldsymbol{e}^T = t \qquad \text{and} \qquad \boldsymbol{g}(t)\,\boldsymbol{e}^T = 1\,.$$

The sum of the expected times spent over all states equals the length of the interval.

Equation (7) has a singularity at $t = 0$; however, as t increases, solving Equation (7) is numerically more convenient because of the following asymptotic property:

$$\lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} \frac{1}{t}\, b_i(t) = \lim_{t\to\infty} \pi_i(t) = \pi_i\,, \tag{8}$$

where $\boldsymbol{\pi}$ is the steady state probability vector of the CTMC.

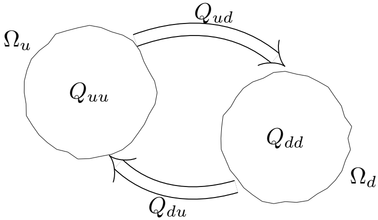
Outline

Dependability Models Defined on a CTMC

The state space Ω of a dependability model can be partitioned into a subset Ω_u of up states and a subset Ω_d of down states.

The states in Ω_u are the up states in which the structure function of the system is equal to 1, and the states in Ω_d are the down states in which the structure function of the system is equal to 0.

From the above, the infinitesimal generator matrix of the CTMC can be partitioned in the following way



$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{uu} & \boldsymbol{Q}_{ud} \\ \boldsymbol{Q}_{du} & \boldsymbol{Q}_{dd} \end{bmatrix}$$

Availability model

In an availability model both $\boldsymbol{Q}_{ud}$ and $\boldsymbol{Q}_{du}$ must have non-zero entries.

In a reliability model, the states in Ω_d are absorbing so that $\boldsymbol{Q}_{du}$ and $\boldsymbol{Q}_{dd}$ are zero matrices (matrices with all entries equal to 0).

The instantaneous system availability at time t is defined as the sum of the state probabilities at time t over the up states, and the unavailability at time t as the sum over the down state at time t.

$$A(t) = \sum_{i \in \Omega_u} \pi_i(t) \quad ; \quad U(t) = \sum_{j \in \Omega_d} \pi_j(t)$$

The steady-state availability and unavailability are defined in similar way utilizing the steady-state probability vector.

$$A = \sum_{i \in \Omega_u} \pi_i \quad ; \quad U = \sum_{j \in \Omega_d} \pi_j$$

Expected Uptime and Expected Downtime

The expected uptime $U_I(t)$ is defined as the expected total amount of time that the CTMC spends in the up states Ω_u in $(0, t]$:

$$U_I(t) = \sum_{i \in \Omega_u} b_i(t) \,.$$

In steady state, the expected uptime over an interval T_I turns out to be

$$U_I = T_I \sum_{i \in \Omega_u} g_i = T_I \, A \,.$$

Similarly, the expected downtime $D_I(t)$ is defined as the total amount of time that the CTMC spends in the down states Ω_d in $(0, t]$:

$$D_I(t) = \sum_{i \in \Omega_d} b_i(t)$$

and, in steady state, the expected downtime over any interval T_I turns out to be:

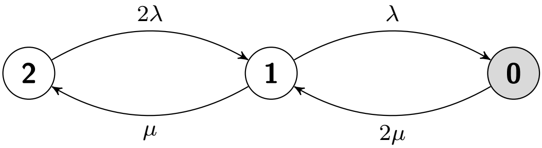
$$D_I = T_I \sum_{i \in \Omega_d} g_i = T_I \, U \,.$$

Two Repairable Components: Non-shared - 1

Non-shared (independent) repair

A parallel system consists of two identical repairable components with failure rate λ and repair rate μ.

There are as many repair persons as failed components.



Application of the balance equation gives:

$$\begin{cases} 2\,\lambda\,\pi_2 & = & \mu\,\pi_1 \\ (\lambda+\mu)\,\pi_1 & = & 2\,\lambda\,\pi_2 + 2\,\mu\,\pi_0 \\ \lambda\,\pi_1 & = & 2\,\mu\,\pi_0 \\ \pi_2+\pi_1+\pi_0 & = & 1 \end{cases}$$

Two Repairable Components: Non-shared - 2

$$\begin{cases} \pi_2 & = & (\mu/2\,\lambda)\,\pi_1 \\ \pi_1 & = & (2\,\mu/\lambda)\,\pi_0 \\ \dfrac{\mu}{2\,\lambda}\,\dfrac{2\,\mu}{\lambda}\,\pi_0 + \dfrac{2\,\mu}{\lambda}\,\pi_0 + \pi_0 & = & 1 \end{cases}$$

from which, the steady-state system unavailability U^{ns} is given as:

$$\pi_0 = U^{ns} = \frac{1}{1 + \dfrac{2\,\mu}{\lambda} + \dfrac{2\,\mu^2}{2\,\lambda^2}} = \frac{1}{\left(1 + \dfrac{\mu}{\lambda}\right)^2} = \left(\frac{\lambda}{\lambda + \mu}\right)^2$$

and the steady-state availability A^{ns} is given as,

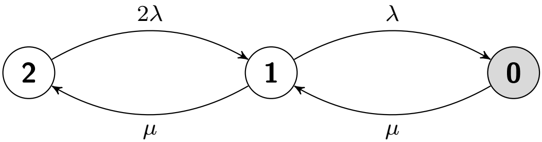
$$A^{ns} = 1 - U^{ns} = 1 - \left(\frac{\lambda}{\lambda + \mu}\right)^2 = \frac{\mu(2\lambda + \mu)}{(\lambda + \mu)^2}$$

Two Repairable Components: Shared - 3

Shared (dependent) repair

A parallel system consists of two identical repairable components with failure rate λ and repair rate μ.

Only one repair person is available.



Application of the balance equation gives:

$$\begin{cases} 2\,\lambda\,\pi_2 & = & \mu\,\pi_1 \\ (\lambda+\mu)\,\pi_1 & = & 2\,\lambda\,\pi_2 + \mu\,\pi_0 \\ \lambda\,\pi_1 & = & \mu\,\pi_0 \\ \pi_2+\pi_1+\pi_0 & = & 1 \end{cases}$$

Two Repairable Components: Shared - 4

$$\begin{cases} \pi_2 & = \quad (\mu/2\,\lambda)\,\pi_1 \\ \pi_1 & = \quad (\mu/\lambda)\,\pi_0 \\ \dfrac{\mu}{2\,\lambda}\,\dfrac{\mu}{\lambda}\,\pi_0 + \dfrac{\mu}{\lambda}\,\pi_0 + \pi_0 & = \quad 1 \end{cases}$$

from which, the steady-state system unavailability U^{sh} is given as:

$$U^{sh} = \pi_0 = \frac{1}{1 + \dfrac{\mu}{\lambda} + \dfrac{\mu^2}{2\,\lambda^2}} = \frac{2\,\lambda^2}{2\,\lambda^2 + 2\,\lambda\mu + \mu^2}$$

and the steady-state availability A^{sh} is given as,

$$A^{sh} = 1 - U^{sh} = \frac{\mu(2\lambda + \mu)}{2\,\lambda^2 + 2\,\lambda\mu + \mu^2}$$

Two Repairable Components: Comparison - 5

Comparing the availability in the two cases, we get:

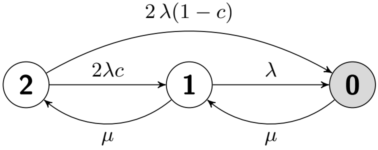
$$\frac{A^{ns}}{A^{sh}} = \frac{2\,\lambda^2 + 2\,\lambda\mu + \mu^2}{\lambda^2 + 2\,\lambda\mu + \mu^2} > 1$$

$$A^{ns} > A^{sh}$$

The non-shared, independent, case provides better availability.

Two Repairable Components: Imperfect coverage

Upon failure, the detection and recovery process may complete successfully with a coverage probability c, and with probability $(1-c)$ the system incurs a complete failure and moves to the down state 0.



The steady state balance equations can be written as:

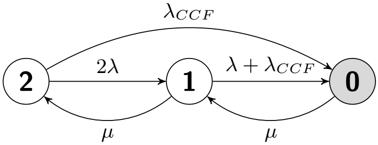
$$\begin{cases} 2\,\lambda\,\pi_2 & = & \mu\,\pi_1 \\ (\lambda+\mu)\,\pi_1 & = & 2\,\lambda c\,\pi_2 + \mu\,\pi_0 \\ \mu\,\pi_0 & = & 2\,\lambda(1-c)\,\pi_2 + \lambda\,\pi_1 \\ \pi_2+\pi_1+\pi_0 & = & 1 \end{cases}$$

from which, the steady state system unavailability $U^{(sh)}$ is obtained as:

$$U^{(sh)} = \pi_0 = \frac{2\lambda^2 + 2\lambda\mu(1-c)}{2\lambda^2 + 2\lambda\mu(2-c)+\mu^2}\,.$$

Two Repairable Components: Common Cause Failure

Common Cause Failures (CCF) are defined as the result of one or more events, causing the concurrent failure of one or more components.



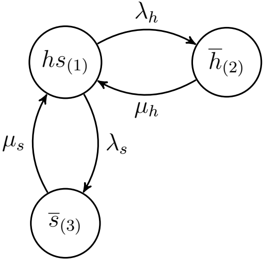
The steady state balance equations can be written as:

$$\begin{cases} (2\,\lambda + \lambda_{CCF})\,\pi_2 & = & \mu\,\pi_1 \\ (\lambda + \lambda_{CCF} + \mu)\,\pi_1 & = & 2\,\lambda\,\pi_2 + \mu\,\pi_0 \\ \mu\,\pi_0 & = & \lambda_{CCF}\,\pi_2 + (\lambda + \lambda_{CCF})\,\pi_1 \\ \pi_2 + \pi_1 + \pi_0 & = & 1 \end{cases}$$

from which, an expression for the steady state system unavailability $U^{(CCF)} = \pi_0$ can be easily derived.

Availability of Combined Hardware-Software system

When the system is down caused by the failure of any one of the two components the only possible action is the repair of the failed component. In other words, software cannot fail when the hardware is down and vice versa.



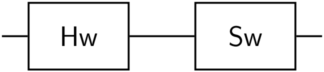
The balance equations in the steady state are:

$$
\begin{aligned}
\pi_1 \,(\lambda_h + \lambda_s) &= \pi_2 \,\mu_h + \pi_3 \,\mu_s \\
\pi_1 \,\lambda_h &= \pi_2 \,\mu_h \\
\pi_1 \,\lambda_s &= \pi_3 \,\mu_s
\end{aligned}
$$

$$
A_{CTMC} = \pi_1 = \frac{1}{1 + \dfrac{\lambda_h}{\mu_h} + \dfrac{\lambda_s}{\mu_s}} = \frac{\mu_h \mu_s}{\mu_h \mu_s + \lambda_h \mu_s} + \mu_h \lambda_s
$$

Availability of Combined Hardware-Software system

Consider the RBD model for the same (independent) system



The system steady state availability A_{RBD} is then:

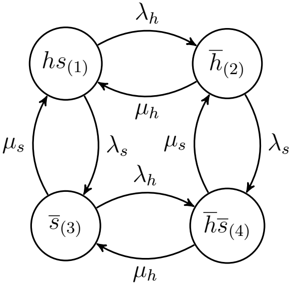
$$A_{RBD} = \frac{\mu_h}{(\lambda_h + \mu_h)} \frac{\mu_s}{(\lambda_s + \mu_s)} \quad (9)$$

We note that the independence assumption implies that the Sw may incur cycles of failure/repair even if the Hw is down, and viceversa.

Compare A_{RBD} vs A_{CTMC}

Availability of Combined Hardware-Software system

Comparing the results we see that $A_{CTMC} > A_{RBD}$.



This conclusion is not evident at the first sight. To give a full justification, we develop the CTMC for the independent case in Figure, where the only up state is still State 1.

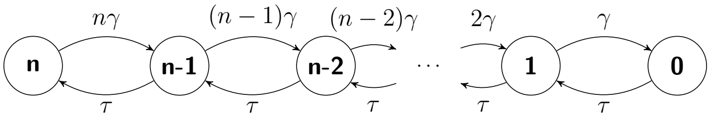
Solving the CTMC of the Figure would provide the same A_{RBD} expression.

The independent case has an additional failed state (State 4) that increases the system unavailability as compared with the dependent case.

Telecommunication Switching System Model

A telecommunications switching system has *n* trunks.

All trunks are subject to failure and repair. We assume the failure and repair times of each trunk are exponentially distributed with rates γ and τ, respectively.



Assume that a single repair facility is shared by all trunks in the system. Then, the pure availability model of the system is a homogeneous CTMC as shown in the Figure, where state *i* indicates that there are *i* non-failed trunks in the system.

Telecommunication Switching System Model

The steady state probability can be computed by solving the following balance equations:

$$\begin{aligned} n\gamma\pi_n &= \tau\pi_{n-1} \\ (k\gamma+\tau)\pi_k &= (k+1)\gamma\pi_{k+1}+\tau\pi_{k-1}, \quad k=1,2,...n-1 \\ \gamma\pi_1 &= \tau\pi_0. \end{aligned}$$

Solving the linear system of equations above, the steady state probability of being in state i is given by:

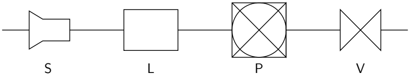
$$\pi_i = \frac{(\frac{\tau}{\gamma})^i/i!}{\sum_{k=0}^{n}(\frac{\tau}{\gamma})^k/k!}. \tag{10}$$

If we assume that the system is up as long as at least l trunks are functioning properly, then the system steady state availability is given by:

$$A(l) = \sum_{i=l}^{n} \pi_i.$$

A repairable fluid level controller - 1

Components are repairable but a single shared crew is available.



If more than one component is failed, the repair-person must decide which component to repair first.

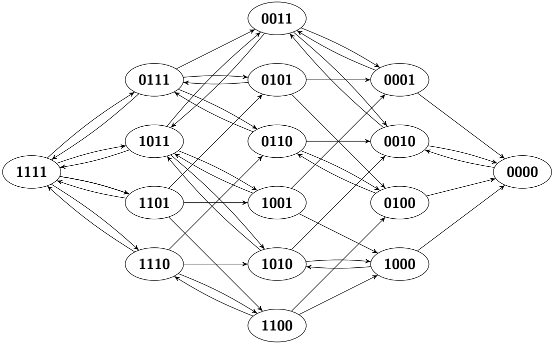
Different preemptive and non-preemptive repair policies can be envisaged and they can be easily represented as a CTMC on the whole state space.

By adopting a *preemptive repair priority policy*, a priority list is defined and the repair crew repairs the components according to the order defined in the priority list.

A repairable fluid level controller - 2

We assume that the priority list is ordered according to the decreasing values of the $MTTR$:

$$P \succ V \succ S \succ L$$



A repairable fluid level controller - 3

Given the failure and repair rates of the Table

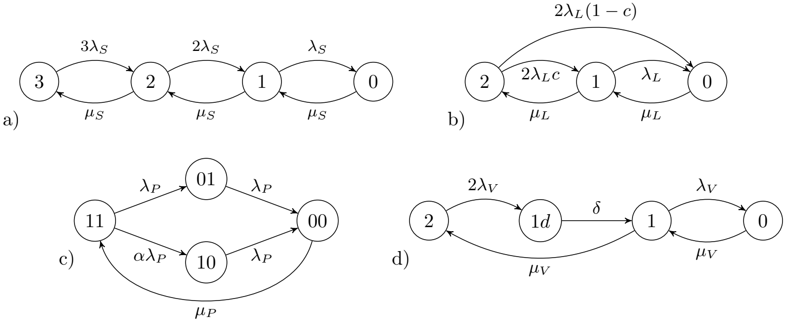
Table: Failure and repair rates for system of the Example

Component	$MTTF$ $\frac{1}{\lambda}$ (hours)	$MTTF^{-1}$ λ	$MTTR$ $\frac{1}{\mu}$ (hours)	$MTTR^{-1}$ μ
P	5000	2e-4	16	0.0625
V	10000	1e-4	8	0.125
S	2500	4e-4	4	0.25
L	2000	5e-4	2	0.5

From the CTMC, we can compute the exact steady state availability $A_{Sys} = 0.993414494$.

A redundant repairable fluid level controller - 4

We consider now redundant repairable subsystems as shown in Figure



A redundant repairable fluid level controller - 5

Subsystem S - There are 3 sensors that work in a 2-out-of-3 logic with a single repair-person. $A_S = \pi_3 + \pi_2$.

Subsystem L - The control logic is duplicated in parallel and the recovery mechanism has a coverage probability c. $A_L = \pi_2 + \pi_1$.

Subsystem P - The pump system is duplicated in warm stand-by configuration (α is the dormancy factor for the standby unit), and the unit is repaired at once upon system failure. $A_P = \pi_{11} + \pi_{01} + \pi_{10}$.

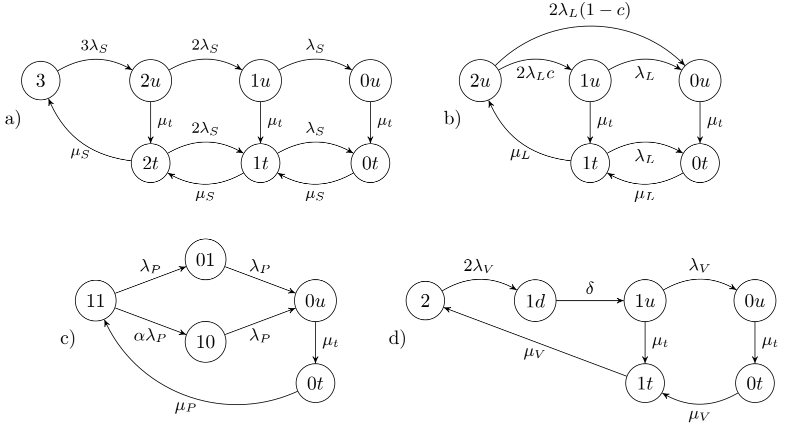
Subsystem V - The valve subsystem is duplicated but the reconfiguration upon failure of one component takes an exponentially distributed reconfiguration time with rate δ - $A_V = \pi_2 + \pi_1$.

The monolithic CTMC for the whole system has $n = 4 * 3 * 4 * 4 = 192$ states.

A redundant repairable fluid level controller - 6

In a more realistic model, if the repair person is not on site he/she must be notified and the *travel time* must be accounted for.

To include travel time, we duplicate all the states in which repair service is required, to distinguish whether the repair-person is on site or not.



A redundant repairable fluid level controller - 7

In the CTMCs for each subsystem we have denoted by a subscript u the states in which the repair-person is not on site and by a subscript t the states in which the repair-person is already on site.

From the states labeled t, when the repair person is on site, the repair is accomplished with the given repair rate, while from the states labeled u the travel transition with rate μ_t is accounted.

If a single repair-person is shared among all the subsystems with the same preemptive priority order, the monolithic CTMC model has $5 * 6 * 5 * 7 = 1050$ states.

Markov Reward Models

- Markov Reward Models (MRM) are obtained by associating real valued reward rates with each state of a Markov chain.
- The reward rate often represents a performance level associated to a given state, or some property of the state.
- The CTMC will be referred to as the *structure-state* process, while the reward variables form the *reward structure*.
- Changing the reward structure on the same structure state process provides different views of the model and enables the computation of different (reward-based) measures for the CTMC.
- Two ways can be envisaged to assign rewards: *Reward rates* are non-negative real constants associated with states. *Impulse rewards* are non-negative real constants associated with states or transitions.

Markov Reward Models

Let r_i be the real valued reward rate attached to state i.

Define $\boldsymbol{r} = [r_i]$ to be the reward vector of dimension n defined over the state space Ω of a CTMC.

This definition implies that a reward $r_i \Delta t$ is accumulated when the process $Z(t)$ stays in state i for a time duration Δt.

The structure-state Markov process $Z(t)$, together with the reward rates attached to each state, form the Markov Reward Model (MRM).

Instantaneous as well as cumulative reward measures are defined in the following.

Expected instantaneous reward

Let $X(t)$ denote the instantaneous reward rate at time t. By definition:

$$X(t) = r_i \qquad \text{if} \qquad Z(t) = i.$$

The expected instantaneous reward rate at time t is computed as:

$$E[X(t)] = \sum_{i=1}^{n} r_i \, P\{Z(t) = i\} = \sum_{i=1}^{n} r_i \, \pi_i(t)$$

and expected reward rate in steady-state as:

$$E[X] = \sum_{i=1}^{n} r_i \, \pi_i$$

In matrix notation:

$$E[X(t)] = \boldsymbol{\pi}(t) \, \boldsymbol{r}^T \qquad\qquad E[X] = \boldsymbol{\pi} \, \boldsymbol{r}^T$$

The solution techniques for the reward Equations are the same as those for solving the standard Markov equations for the state probability vector, in transient or in steady state, respectively.

Expected accumulated reward

Cumulative measures are related to the accumulation of the reward during a finite time interval.
The total accumulated reward up to time t is the random variable:

$$Y(t) \; = \; \int_0^t X(u)\, du$$

The expected accumulated reward is given by:

$$E[Y(t)] \; = \; E\,[\int_0^t X(u)\, du\,] \; = \; \sum_i r_i \int_0^t \pi_i(u) du \; = \; \sum_i r_i\, b_i(t)$$

A related measure is the time averaged accumulated reward $E[W(t)] = E[Y(t)]/t$, which can also be seen as the average reward is accumulated in the interval $(0, t]$.

In matrix notation as:

$$E[Y(t)] \; = \; \boldsymbol{b}(t)\, \boldsymbol{r}^T \qquad\qquad E[W(t)] \; = \; \boldsymbol{g}(t)\, \boldsymbol{r}^T$$

Two repairable components: expected cost of repair

The non-shared repair policy, in which there are as many repair persons as failed components, provides a higher availability with respect to a shared repair policy with single repairperson.

Now we compare the expected cost of the repair with the same two policies by means of a MRM.

We assign to each state a reward rate equal to the cost of repairing the components that are failed in that state. Further, we assume that the cost of repair per unit time is c.

Two repairable components: expected cost of repair

In the non-shared repair the following reward rates are assigned:

$$r_2 = 0 \quad ; \quad r_1 = c \quad ; \quad r_0 = 2\,c\,.$$

The expected repair cost in steady state per unit time becomes:

$$E[C^{(ns)}] = c\,\pi_1 + 2\,c\,\pi_0 \;=\; \frac{2\lambda\,c\,(\lambda+\mu)}{(\lambda+\mu)^2}\,.$$

In the shared repair case the following reward rates are assigned:

$$r_2 = 0 \quad ; \quad r_1 = c \quad ; \quad r_0 = c\,.$$

The steady state expected repair cost per unit time becomes:

$$E[C^{(sh)}] = c\,\pi_1 + c\,\pi_0 \;=\; \frac{2\lambda\,c\,(\lambda+\mu)}{2\,\lambda^2 + 2\,\lambda\mu + \mu^2}\,.$$

Comparing the two costs it is easy to see that $C^{(ns)} > C^{(sh)}$. The non-shared repair policy provides a higher availability but at a higher expected repair cost.

Availability defined on a MRM

For a dependability model we define the following reward rates:

$$\begin{cases} r_i = 1 & \text{if} & i \in \Omega_u \\ r_i = 0 & \text{if} & i \in \Omega_d \,. \end{cases} \tag{11}$$

In vector form Equation (11) becomes:

$$\boldsymbol{r} = [\boldsymbol{r_u} \; \boldsymbol{r_d}] \quad \text{with} \quad \boldsymbol{r_u} = \boldsymbol{e} \quad \text{and} \quad \boldsymbol{r_d} = \boldsymbol{0} \,.$$

The transient availability $A(t)$ and the steady state availability A can be rewritten as:

$$A(t) \; = \boldsymbol{\pi}(t)\, \boldsymbol{r}^T \qquad\qquad A \; = \boldsymbol{\pi}\, \boldsymbol{r}^T \,.$$

Using the reward rates as in Equation (11), the expected interval availability in the interval $(0, t]$ becomes:

$$A_I(t) \; = \boldsymbol{g}(t)\, \boldsymbol{r}^T \,.$$

Expected Uptime and Expected Downtime

Adopting the reward rates as in Equation (11), the expected uptime becomes:

$$U_I(t) = \boldsymbol{b}(t)\, \boldsymbol{r}^T .$$

In steady state, the expected uptime over any interval T_I will then turn out to be,

$$U_I = T_I \boldsymbol{\pi}\, \boldsymbol{r}^T = T_I\, A .$$

Similarly, adopting the following reward rates:

$$\boldsymbol{r} = [\boldsymbol{r_u}\ \boldsymbol{r_d}] \qquad \text{with} \qquad \boldsymbol{r_u} = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{r_d} = \boldsymbol{e} ,$$

the expected downtime $D_I(t)$ in the interval $(0, t]$ is given by:

$$D_I(t) = \boldsymbol{b}(t)\, \boldsymbol{r}^T .$$

In steady state, the expected downtime over any interval T_I will then turn out to be,

$$D_I = T_I \boldsymbol{\pi}\, \boldsymbol{r}^T = T_I\, U .$$

Expected number of transitions

Let q_{ij} $(i \neq j)$ be a generic non-zero, non-diagonal entry of the infinitesimal generator $\boldsymbol{Q}$. State i is directly connected to state j by means of the transition rate q_{ij}. The expected number of transitions $E[N_{ij}(t)]$ from state i to state j in the interval $(0, t]$ is given by:

$$E[N_{ij}(t)] \;=\; q_{ij} \int_0^t \pi_i(u)\, du \;=\; q_{ij}\, b_i(t)\,. \tag{12}$$

In steady state the expected number of transitions $i \to j$ per unit time is:

$$\eta_{ij} \;=\; \lim_{t\to\infty} E[N_{ij}(t)]/t \;=\; \lim_{t\to\infty} q_{ij}\, b_i(t)/t \;=\; q_{ij}\, \pi_i\,. \tag{13}$$

Defining the reward structure, as:

$$\begin{cases} r_k = q_{ij} & \text{if} \qquad k = i \\ r_k = 0 & \text{if} \qquad k \neq i \end{cases}$$

the measures in (12) and (13) are examples of the expected accumulated reward in the interval $(0, t]$ and the expected reward rate in steady state.

Expected Number of Visits

When computing the expected number of visits into a state we need to distinguish whether the visits occur entering or exiting the state.

The expected number of input visits to state j in the interval $(0, t]$ (denoted as $E[N_j(t)]$) can be computed as the sum of the expected number of transitions entering state j.

The transition rates of the transitions entering state j are located on the j-th column of the infinitesimal generator $\boldsymbol{Q}$ (excluding the diagonal element), hence,

$$E[N_j(t)] = \sum_{i=1, i \neq j}^{n} q_{ij} \int_0^t \pi_i(u)\, du = \sum_{i=1, i \neq j}^{n} q_{ij}\, b_i(t),$$

in steady state the expected number of visits to state j per unit time is:

$$\eta_j = \sum_{i=1, i \neq j}^{n} q_{ij}\, \pi_i \, .$$

Expected Number of Visits

Alternatively, the number of visits out of state j can be evaluated as:

$$E[N_j(t)] = \sum_{i=1, i \neq j}^{n} q_{ji} \int_0^t \pi_j(u)\, du = \sum_{i=1, i \neq j}^{n} q_{ji}\, b_j(t) = q_j b_j(t)\,,$$

in steady state the expected number of visits to state j per unit time is:

$$\eta_j = \sum_{i=1, i \neq j}^{n} q_{ji}\, \pi_j = q_j \pi_j\,.$$

Expected number of system failures/repairs

The expected number of system failures in the interval $(0, t]$, $E[N_F(t)]$, is given by the total expected number of transitions from any state in Ω_u to any state in Ω_d.

The transitions that contribute to the expected number of system failures are those that connects an up state with a down state, and correspond to the non-zero entries in the partition $\boldsymbol{Q}_{ud}$. Hence:

$$E[N_F(t)] = \boldsymbol{b}_u(t)\,\boldsymbol{Q}_{ud}\,\boldsymbol{e}_d^T$$

where $\boldsymbol{Q}_{ud}\,\boldsymbol{e}_d^T$ is the row sum of matrix $\boldsymbol{Q}_{ud}$.

The expected number of system failures in the interval $(0, t]$ can be incorporated into a MRM by defining the following reward structure

$$\begin{cases} \boldsymbol{r}_u = \boldsymbol{Q}_{ud}\,\boldsymbol{e}_d^T \\ \boldsymbol{r}_d = \boldsymbol{0} \end{cases}$$

Expected number of system failures/repairs

To compute the expected number of system repairs $E[N_R(t)]$ we proceed in a similar way.

If the matrix $\boldsymbol{Q}_{du}$ has non-zero entries, then

$$E[N_R(t)] = \boldsymbol{b}_d(t)\, \boldsymbol{Q}_{du}\, \boldsymbol{e}_u^T$$

Taking the limit (after dividing by t) as $t \to \infty$ we obtain the expected number of system failures η_F and of system repairs η_R, respectively, per unit time in steady state.

The quantities η_F and η_R can also be seen as the unconditional failure intensity (failure frequency) and the unconditional repair intensity (repair frequency).

$$\eta_F = \boldsymbol{\pi}_u\, \boldsymbol{Q}_{ud}\, \boldsymbol{e}_d^T \qquad \eta_R = \boldsymbol{\pi}_d\, \boldsymbol{Q}_{du}\, \boldsymbol{e}_u^T .$$

One repairable component

Consider a system with one repairable component; we obtain

$$\begin{aligned} E[N_F(t)] = b_1(t)\,\lambda &= \lambda \int_0^t \left[\frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu}\, e^{-(\lambda+\mu)\,u} \right] du \\ &= \frac{\lambda\,\mu}{\lambda + \mu}\, t + \frac{\lambda^2}{(\lambda + \mu)^2} (1 - e^{-(\lambda+\mu)\,t}) \end{aligned}$$

As $t \to \infty$ the expected number of system failures grows linearly with slope $\lambda\,\mu/(\lambda + \mu)$.

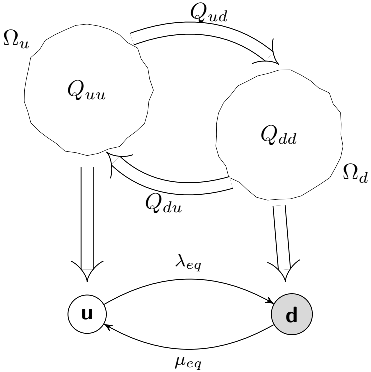
In the limit the expected number of failure per unit time in steady state, that results to be

$$\eta_F = \lambda\,\mu/(\lambda + \mu)$$

Equivalent failure and repair rate

In steady state availability modeling it is sometimes useful to have a high-level view of the system, and reduce the system to a two state model.

The single up state and the single down state communicate via an *equivalent failure rate* and an *equivalent repair rate*.



Equivalent failure and repair rate

For this aggregation, we define an equivalent failure rate λ_{eq} and an equivalent repair rate μ_{eq} such that the steady state availability of the original model can be written as:

$$A = \frac{\mu_{eq}}{\lambda_{eq} + \mu_{eq}} = \frac{\text{MTTF}_{eq}}{\text{MTTF}_{eq} + \text{MTTR}_{eq}}$$

with $\text{MTTF}_{eq} = 1/\lambda_{eq}$ and $\text{MTTR}_{eq} = 1/\mu_{eq}$.

λ_{eq} and μ_{eq} are defined as:

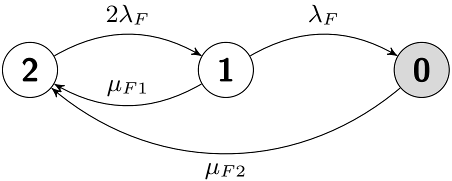
$$\lambda_{eq} = P\{\text{transition } u \rightarrow d \text{ takes place} \,|\, \text{system is up}\} = \frac{\eta_F}{A}$$

$$\mu_{eq} = P\{\text{transition } d \rightarrow u \text{ takes place} \,|\, \text{system is down}\} = \frac{\eta_R}{1 - A}$$

where η_F is the unconditional failure intensity and A is the steady state availability; similarly, η_R is the unconditional repair intensity.

High Availability Platform from Sun Microsystem - 1

In the present example, we consider the detailed lower-level CTMC model of the fan subsystem.



States 2 and 1 are up states while state 0 is a down state. For this model, writing out the balance equations, we get

$$\begin{cases} 2\,\lambda_F\,\pi_2 & = & \mu_{F1}\,\pi_1 + \mu_{F2}\,\pi_0 \\ (\lambda_F + \mu_{F1})\,\pi_1 & = & 2\,\lambda_F\,\pi_2 \\ \mu_{F2}\,\pi_0 & = & \lambda_F\,\pi_1 \\ \pi_0 + \pi_1 + \pi_2 & = & 1\,. \end{cases}$$

High Availability Platform from Sun Microsystem - 2

Solving the balance equations, we get

$$\begin{cases} \pi_2 &= \dfrac{(\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})}{(2\lambda_F^2 + 3\,\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})} \\ \pi_1 &= \dfrac{2\,\lambda_F\, \mu_{F2}}{(2\lambda_F^2 + 3\,\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})} \\ \pi_0 &= \dfrac{2\,\lambda_F^2}{(2\lambda_F^2 + 3\,\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})} \end{cases}.$$

The steady state unavailability U_{ss} and availability A_{ss} are given by:

$$U_{ss} = \pi_0 = \frac{2\,\lambda_F^2}{(2\lambda_F^2 + 3\,\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})}$$

$$A_{ss} = 1 - U_{ss} = \frac{(3\,\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})}{(2\lambda_F^2 + 3\,\lambda_F\, \mu_{F2} + \mu_{F1}\, \mu_{F2})}$$

The expected downtime D_I of the system in minutes per year is given by:

$$D_I = 60 \times 24 \times 365 \times (1 - A_{ss})\,.$$

High Availability Platform from Sun Microsystem - 3

Substituting,

$$\lambda_F = 3.4905147 \times 10^{-7} \ \ f/h$$
$$\mu_{F1} = 0.24 \ \ r/h$$
$$\mu_{F2} = 0.226415 \ \ r/h$$

we get the steady state availability of the cooling subsystem, the steady state unavailability and the expected steady state downtime, respectively,

$$A_{ss} = 0.9999999999955$$
$$U_{ss} = 4.48425628 \times 10^{-12}$$
$$D_I = 60 \times 24 \times 365 \times U_{ss} = 2.35692510 \times 10^{-6} \ \ min/yr$$

Defects Per Million

Defects per million (DPM) is a commonly used service (un)reliability measure for telecommunication systems.

The metric counts the number of unsuccessful attempts per million attempts at obtaining a service.

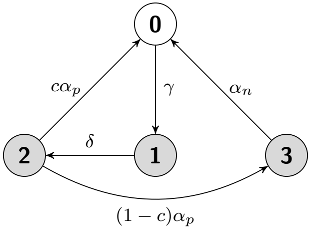
A simple approach to calculate the DPM is to multiply the unavailability by a constant of proportionality, because if the system is unavailable at the moment the service is requested, then the request can be considered lost or unsuccessful.

Given the system unreliability U_{ss} in steady state, the DPM metric can be approximated as,

$$DPM = U_{ss} \times 10^6 \, .$$

Defects Per Million for an application server

This example, presents the modeling of an application server that services Session Initiation Protocol (SIP) request messages.



One method of providing fault-tolerance is to use application server replication in cold-standby mode, that acts as the backup for the primary server.

The server is available for handling incoming requests in State 0. The unavailability in State 1 is not observable until the failure is detected. After detection the system enters State 2 in which recovery is initiated.

If the threshold of the number of failures within a certain interval has not been reached, the process is restarted on the same node otherwise reaches State 3.

Defects Per Million for an application server

Writing the balance equations for this model, we get:

$$\begin{cases} \gamma\,\pi_0 & = & c\alpha_p\,\pi_2 + \alpha_n\,\pi_3 \\ \delta\,\pi_1 & = & \gamma\,\pi_0 \\ \alpha_p\,\pi_2 & = & \delta\,\pi_1 \\ \alpha_n\,\pi_3 & = & (1-c)\,\alpha_p\,\pi_2 \\ \pi_0+\pi_1+\pi_2+\pi_3 & = & 1\,. \end{cases}$$

Solving the balance equations, we get for the steady state availability:

$$A_{ss} = \pi_0 = \frac{1}{\gamma}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}.$$

For this system, the DPM can be approximated as,

$$DPM = (1-A_{ss})\times 10^6\,.$$

Substituting $1/\gamma = 10$ days, $1/\delta = 1$ sec, $1/\alpha_p = 30$ sec, $1/\alpha_n = 15$ sec, and $c = 0.95$, we get $A_{ss} = 0.99996325$, and $DPM = 36.75$.

IBM Blade Server System availability

Using a CTMC to model the availability of the whole system will be too complex.

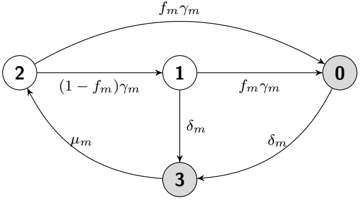
Instead, we will illustrate the use of CTMC for several subsystems.

The high level FT will be combined with the CTMC models of the subsystems presented here as a hierarchical modeling case study.

This example and the CTMC models are derived from:

W. E. Smith, K. S. Trivedi, L. Tomek, and J. Ackaret, "Availability analysis of blade server systems," *IBM Systems Journal*, vol. 47, no. 4, pp. 621–640, 2008.

Midplane Availability Model of the IBM Blade Server



Balance equations,

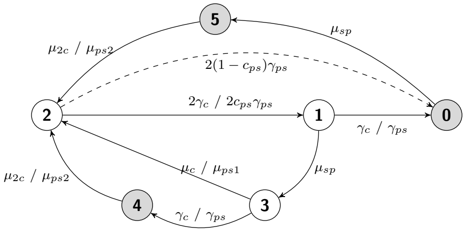
$$\begin{cases} \gamma_m\,\pi_2 & = & \mu_m\,\pi_3 \\ (f_m\gamma_m + \delta_m)\,\pi_1 & = & (1-f_m)\,\gamma_m\,\pi_2 \\ \delta_m\,\pi_0 & = & f_m\gamma_m\,(\pi_2+\pi_1) \\ \mu_m\,\pi_3 & = & \delta_m\,(\pi_0+\pi_1) \\ \pi_0+\pi_1+\pi_2+\pi_3 & = & 1\,. \end{cases}$$

Solving the above balance equations, the steady state availability is given as $A_{Md} = \pi_2 + \pi_1$.

A closed-form expression for the midplane steady state availability is:

$$A_{Md} = \frac{\delta_m\mu_m(\gamma_m+\delta_m)}{(\delta_m+f_m\gamma_m)(\gamma_m\mu_m+\delta_m\mu_m+\gamma_m\delta_m)}\,.$$

Cooling and Power Domain of the IBM Blade Server



Let d_C represent the denominator for the closed-form expression for steady state availability of this submodel.

$$d_C = \frac{1}{\gamma_c^2}\left(\frac{1}{\gamma_c} + \frac{1}{\mu_c} + \frac{3}{\mu_{sp}}\right) + \frac{1}{\gamma_c}\left(\frac{2}{\mu_{sp}^2} + \frac{2}{\mu_{2c}}\left(\frac{1}{\mu_c} + \frac{1}{\mu_{sp}}\right) + \frac{3}{\mu_c \mu_{sp}}\right) + \frac{2}{\mu_c \mu_{sp}}\left(\frac{1}{\mu_{2c}} + \frac{1}{\mu_{sp}}\right)$$

Then, the steady state availability of the cooling subsystem, A_C, is:

$$A_C = \frac{1}{\gamma_c \, d_C}\left(\frac{2}{\mu_c \gamma_c} + \left(\frac{1}{\gamma_c} + \frac{1}{\mu_c}\right)\left(\frac{1}{\gamma_c} + \frac{1}{\mu_{sp}}\right) + \frac{2}{\mu_{sp}}\left(\frac{1}{\gamma_c} + \frac{1}{\mu_c}\right)\right).$$

Cooling and Power Domain of the IBM Blade Server

The BladeCenter contains two identical power domain subsystems.

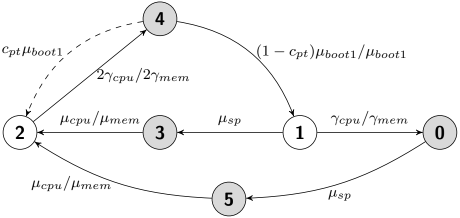
Let d_P represents the expression for the denominator.

$$
\begin{aligned}
d_P \quad = \quad & \frac{1}{\gamma_{ps}^2}\Big(\frac{1}{\gamma_{ps}} + \frac{2c_{ps}}{(\mu_{ps1} - \mu_{ps2})} + \frac{1}{\mu_{ps1}} + \frac{2}{\mu_{ps2}} + \frac{3}{\mu_{sp}}\Big) + \\
& \frac{1}{\gamma_{ps}}\Big(\frac{2}{\mu_{sp}^2} + \frac{2}{\mu_{ps2}}\big(\frac{1}{\mu_{ps2}} + \frac{1}{\mu_{sp}}\big) + \frac{3}{\mu_{ps1}\mu_{sp}}\Big) + \\
& \frac{2}{\mu_{ps1}\mu_{sp}}\Big(\frac{1}{\mu_{ps2}} + \frac{1}{\mu_{sp}}\Big) .
\end{aligned}
$$

Then, a closed-form expression for the steady state availability of the power supply subsystem, A_P, is given by:

$$
A_P = \frac{1}{\gamma_{ps}\ d_P}\Big(\frac{2c_{ps}}{\mu_{ps1}\gamma_{ps}} + \big(\frac{1}{\gamma_{ps}} + \frac{1}{\mu_{ps1}}\big)\big(\frac{1}{\gamma_{ps}} + \frac{1}{\mu_{sp}}\big) + \frac{2c_{ps}}{\mu_{sp}}\big(\frac{1}{\gamma_{ps}} + \frac{1}{\mu_{ps1}}\big)\Big) .
$$

Processor and Memory Model of the IBM Blade Server



Let d_{CPU} denote the denominator for the closed-form expression of the steady state availability,

$$d_{CPU} = \left(\frac{2}{\mu_{boot1}} + \frac{1}{\gamma_{cpu}} + \frac{2(1-c_{pt})}{\mu_{cpu}} + \frac{2(1-c_{pt})}{\mu_{sp}} \right).$$

A closed-form expression for the steady state availability of the processor subsystem, A_{CPU} is:

$$A_{CPU} = \frac{2(1-c_{pt})}{(\gamma_{cpu}+\mu_{sp})d_{CPU}} + \frac{1}{\gamma_{cpu}\ d_{CPU}}$$

Processor and Memory Model of the IBM Blade Server

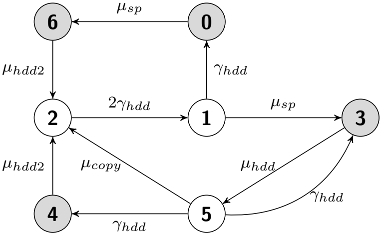
Each blade server has two banks of memory.

Let d_M denote the denominator for the closed-form expression of the steady state memory subsystem availability,

$$d_M = \Big(\frac{2}{\mu_{boot1}} + \frac{1}{\gamma_{mem}} + \frac{2}{\mu_{mem}} + \frac{2}{\mu_{sp}}\Big).$$

A closed-form expression for the steady state availability of the memory subsystem, A_M is:

$$A_M = \frac{2}{(\gamma_{mem} + \mu_{sp})\ d_M} + \frac{1}{\gamma_{mem}\ d_M}.$$



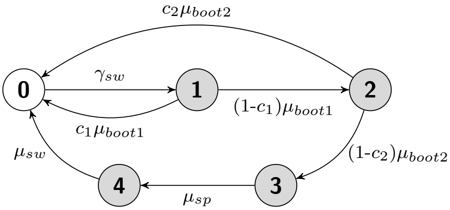
Let d_{RAID} denote the denominator for a closed-form expression for the steady state availability of the disc subsystem utilizing RAID1:

$$
\begin{aligned}
d_{RAID} = \; & \frac{1}{\gamma_{hdd}^2}\left(\frac{1}{\gamma_{hdd}} + \frac{3}{\mu_{copy}} + \frac{2}{\mu_{hdd}} + \frac{3}{\mu_{sp}}\right) + \\
& \frac{1}{\gamma_{hdd}}\left(\frac{2}{\mu_{sp}^2} + \frac{2}{\mu_{copy}\,\mu_{hdd2}} + \frac{4}{\mu_{copy}\,\mu_{hdd}} + \frac{3}{\mu_{copy}\,\mu_{sp}} + \frac{2}{\mu_{hdd2}\,\mu_{sp}}\right) + \\
& \frac{2}{\mu_{copy}\,\mu_{sp}}\left(\frac{1}{\mu_{sp}} + \frac{1}{\mu_{hdd2}}\right).
\end{aligned}
$$

Then the steady state RAID availability, A_{RAID}, is given by:

$$
A_{RAID} = \frac{1}{\gamma_{hdd} d_{RAID}}\left(\frac{2}{\mu_{copy}\gamma_{hdd}} + \left(\frac{1}{\mu_{sp}} + \frac{1}{\gamma_{hdd}}\right)\left(\frac{1}{\gamma_{hdd}} + \frac{1}{\mu_{copy}}\right) + 2\left(\frac{1}{\mu_{sp}\gamma_{hdd}} + \frac{1}{\mu_{copy}\mu_{sp}}\right)\right).
$$

Software Availability Model of the IBM Blade Server



A closed-form expression for the steady state availability of the software subsystem is:

$$A_{Sw} = \frac{1}{\gamma_{sw}}\left(\frac{1}{\mu_{boot1}}+(1-c_1)\frac{1}{\mu_{boot2}}+\frac{1}{\gamma_{sw}}+(1-c_1+c_1c_2-c_2)\left(\frac{1}{\mu_{sw}}+\frac{1}{\mu_{sp}}\right)\right)^{-1}.$$

Parametric Sensitivity Analysis

Sensitivity analysis is performed by computing the partial derivatives of the output metric of interest with respect to each input parameter.

The derivatives are referred to as sensitivity functions.

The sensitivity function of a given measure Y, which depends on a parameter θ, is computed as in the following Equations for unscaled and scaled sensitivity.

$$
\begin{aligned}
S_\theta(Y) &= \frac{\partial Y}{\partial \theta} \\
SS_\theta(Y) &= \frac{\partial Y}{\partial \theta}\left(\frac{\theta}{Y}\right)
\end{aligned}
$$

The absolute values of sensitivity functions provide the ranking that is used to compare the degree of influence of each input parameter on the output metric of interest.

DPM for an application server: Sensitivity analysis

Returning to the application server Example, the steady state availability is expressed as,

$$A_{ss}(\gamma,\delta,\alpha_p,\alpha_n,c)=\pi_0=\frac{1}{\gamma}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}.$$

The unscaled and scaled sensitivities of A_{ss} are shown in the Table.

Parameter θ	$S_\theta(A_{ss})$	$SS_\theta(A_{ss})$
γ	$-\frac{1}{\gamma^2}\left[\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]\times \left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-2}$	$-\left[\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]\times \left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}$
δ	$\frac{1}{\gamma\delta^2}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-2}$	$\frac{1}{\delta}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}$
α_p	$\frac{1}{\gamma\alpha_p^2}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-2}$	$\frac{1}{\alpha_p}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}$
α_n	$\frac{1-c}{\gamma\alpha_n^2}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-2}$	$\frac{1-c}{\alpha_n}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}$
c	$\frac{1}{\gamma\alpha_n}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-2}$	$\frac{c}{\alpha_n}\left[\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\alpha_p}+\frac{(1-c)}{\alpha_n}\right]^{-1}$

DPM for an application server: Sensitivity analysis

Substituting the numerical values taken from the DPM Example, (i.e., $1/\gamma = 10$ days, $1/\delta = 1$ sec, $1/\alpha_p = 30$ sec, $1/\alpha_n = 15$ sec, and $c = 0.95$), we get the numerical values, for the scaled sensitivity only:

Parameter, θ	$SS_\theta(A_{ss})$
γ	-3.6746e-005
α_p	3.4721e-005
c	1.6492e-005
δ	1.1574e-006
α_n	8.6802e-007

Numerical Methods for Steady State Analysis of CTMC

A CTMC is completely specified once its infinitesimal generator $\boldsymbol{Q}$ and the initial probability vector $\boldsymbol{\pi}(0)$ are specified.

Instead, the steady state probability vector $\boldsymbol{\pi}$ does not depend on $\boldsymbol{\pi}(0)$ and is obtained by solving Equation (4).

Numerical techniques for solving Equation (4) are illustrated and discussed in [*].

A brief survey is given in the following.

[*] W. Stewart, *Introduction to the Numerical Solution of Markov Chains*. Princeton University Press, 1994.

Power method

The equation for steady state probabilities (4) may be rewritten as

$$\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{I} + \boldsymbol{Q}/q) = \boldsymbol{\pi}\,\boldsymbol{Q}^{\star},$$

where $q \geq \max_i |q_{ii}|$ and $\boldsymbol{Q}^{\star} = \boldsymbol{I} + \boldsymbol{Q}/q$.

The derivation and meaning of matrix $\boldsymbol{Q}^{\star}$ is extensively discussed in Chapter 10.

The entries $q^{\star}_{ij}$ represent the ultimate probability of jumping into state j when a transition out of state i occurs and $t \to \infty$.

Note that all the entries of matrix $\boldsymbol{Q}^{\star}$ are probabilities, i.e. real numbers

$\boldsymbol{Q}^{\star}$ is a stochastic matrix and is called the generator matrix of the Discrete Time Markov Chain (DTMC) embedded into the uniformized CTMC generated by matrix $\boldsymbol{Q}$.

Power method

In iterative form the above Equation can be written as:

$$\boldsymbol{\pi}^{(i)} = \boldsymbol{\pi}^{(i-1)} \boldsymbol{Q}^{\star} \ ,$$

where $\boldsymbol{\pi}^{(i)}$ is the value of the iterate at the end of the i-th step.

We start off the iteration by initializing $\boldsymbol{\pi}^{(0)}$ to an initial guess (whose values does not influence the final solution), as f.i.

$$\boldsymbol{\pi}^{(0)} = \boldsymbol{\pi}(0),$$

Since $q > \max_i |q_{ii}|$ the embedded DTMC is aperiodic and the iteration converges to a fixed point.

The number of iterations, k, needed for convergence is governed by the second largest eigenvalue of $\boldsymbol{Q}^{\star}$.

Successive Over Relaxation (SOR)

The equation for steady state probabilities defines the linear system:

$$\boldsymbol{\pi} \boldsymbol{Q} = \mathbf{0} \,.$$

Standard numerical techniques are applicable in this case. Iterative methods such as successive over-relaxation (SOR) are usually more convenient.
The matrix $\boldsymbol{Q}$ is split into three summands:

$$\boldsymbol{Q} = \boldsymbol{D} - \boldsymbol{L} - \boldsymbol{U} \,,$$

where $\boldsymbol{L}$ and $\boldsymbol{U}$ are strictly upper triangular and lower triangular respectively and $\boldsymbol{D}$ is a diagonal matrix.

Then the SOR iteration equation can be written as:

$$\boldsymbol{\pi}^{(k+1)} = \boldsymbol{\pi}^{(k)}[(1-\omega) + \omega \boldsymbol{L}\boldsymbol{D}^{-1}] + \omega \boldsymbol{\pi}^{(k+1)} \boldsymbol{U}\boldsymbol{D}^{-1} \,,$$

where $\boldsymbol{\pi}^{(k)}$ is the k-th iterate for $\boldsymbol{\pi}$, and ω, is the relaxation parameter.