Chapter 7

```
> with(plots):
Warning, the name changecoords has been redefined
> with(DEtools):
```

Question 1

Let

$$\delta = \frac{\lambda_1 - \lambda_0}{\lambda_2 - \lambda_1}$$

then we can solve for λ_2

```
> sold:=solve(delta=(lambda[1]-lambda[0])/(lambda[2]-lambda[1])
  ,lambda[2]);
```

$$sold := \frac{\delta\,\lambda_1 + \lambda_1 - \lambda_0}{\delta}$$

```
> subs({lambda[0]=3,lambda[1]=3.4495,delta=4.669},sold);
```

$$3.545773292$$

This approximation compares favourably with the computer programme which provides a value of 3.54409.

Question 2

We know from the previous question that

$$\lambda_2 = \frac{\lambda_1 - \lambda_0}{\delta} + \lambda_1$$

then λ_3 can be obtained by eliminating λ_2 in the expression

$$\lambda_3 = \frac{\lambda_2 - \lambda_1}{\delta} + \lambda_2$$

Thus

```
> subs(lambda[2] =
  (lambda[1]-lambda[0])/delta+lambda[1],(lambda[2]-lambda[1])/d
  elta+lambda[2]);
```

$$\frac{\lambda_1 - \lambda_0}{\delta^2} + \frac{\lambda_1 - \lambda_0}{\delta} + \lambda_1$$

So λ_3 can be expressed:

$$\lambda_3 = (\lambda_1 - \lambda_0)\left(\frac{1}{\delta^2} + \frac{1}{\delta}\right) + \lambda_1$$

Carrying out the same procedure, λ_4 down to λ_k, we obtain:

$$\lambda_4 = (\lambda_1 - \lambda_0)\left(\frac{1}{\delta^3} + \frac{1}{\delta^2} + \frac{1}{\delta}\right) + \lambda_1$$

:

$$\lambda_k = \frac{(\lambda_1 - \lambda_0)\left(\frac{1}{\delta^{(k-2)}} + \frac{1}{\delta^{(k-3)}} + ?\ ..\ 1\right)}{\delta} + \lambda_1$$

Let $\frac{1}{\delta} = r$, then the term in bracketsis $1 + r + r^2 + ?\ ..\ r^{(k-3)} + r^{(k-2)}$, whose limit is $\frac{1}{1-r}$.

Therefore we can take the limit as follows, were *lambdastar* denotes the limit.

```
> lambdastar:=(sum(((lambda[1]-lambda[0])/delta)*(1/delta^k),k=
  0..infinity)+lambda[1]);
>
```

$$lambdastar := \frac{\lambda_1 - \lambda_0}{-1 + \delta} + \lambda_1$$

```
> subs({delta=4.669, lambda[0]=3,
  lambda[1]=3.4495},lambdastar);
```

$$3.572012946$$

which is the approximate value at which chaos begins for the logistic equation.

Question 3

(i)

To show that $x_{n+1} = \lambda\, x_n\,(1 - x_n)$ has the same properties as $y_{n+1} = {y_n}^2 + c$, if

$$c = \frac{\lambda\,(2-\lambda)}{4} \quad \text{and} \quad y_n = \frac{\lambda}{2} - \lambda\, x_n$$

then we can substitute these values to obtain

```
> solve((lambda/2-lambda*x[n+1])=((lambda/2)-lambda*x[n])^2+
  lambda*(2-lambda)/4,x[n+1]);
```

$$-\lambda\, x_n\,(-1 + x_n)$$

Hence, $x_{n+1} = \lambda\, x_n\,(1 - x_n)$.

(ii)

In constructing the bifurcation diagram for $y_{n+1} = {y_n}^2 + c$ we need to know the range for the parameter c. Given $0 < \lambda < 4$, then $-2 < c < 0$. The resulting bifurcation diagram is then derived as follows.

```
> c:='c';b:='b';c:=-2+0.01*b; f:=x->x^2+c;
```

$$c := c$$

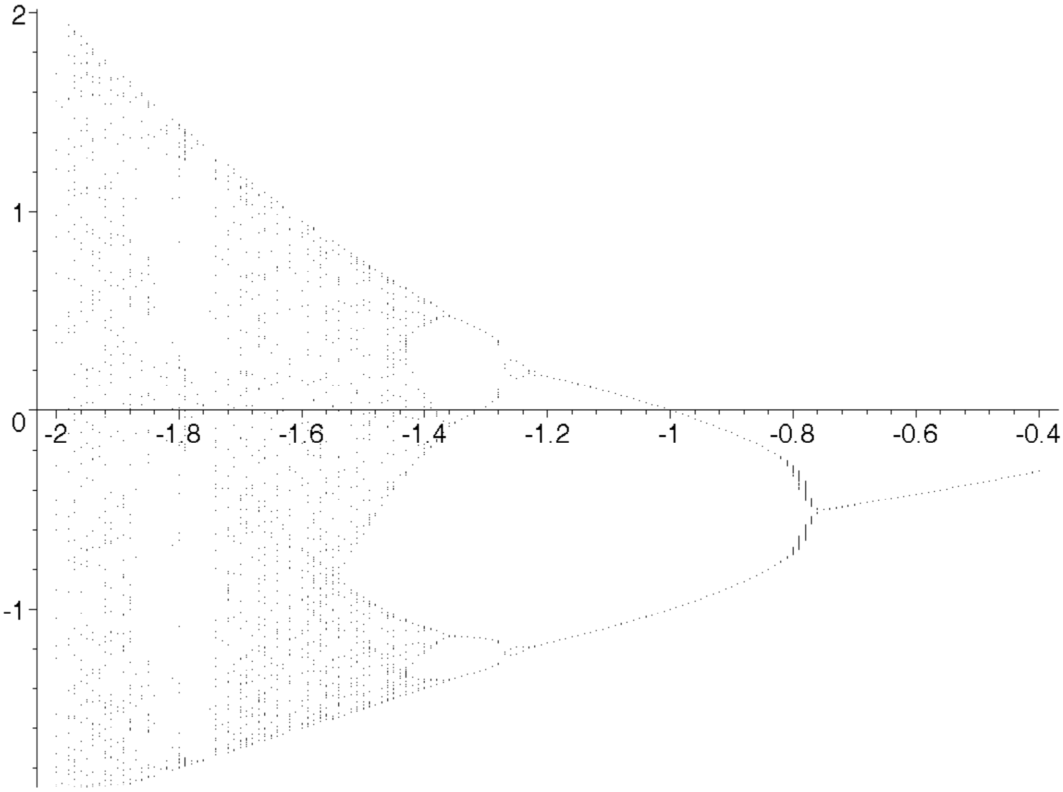
$$b := b$$

$$c := -2 + .01\, b$$

$$f := x \rightarrow x^2 + c$$

```
> points9:=evalf(seq(seq([c,(f@@k)((f@@50)(0.5))],k=0..50),b
```

```
=0..160),4):
> pointplot({points9},symbol=point);
```



c be from -2 to -0.4. This is because as *c* approaches zero, the exponents in the calculations get too large.

What we immediately see is that this is the same bifurcation diagram as $x_{n+1} = \lambda\, x_n\, (1 - x_n)$.

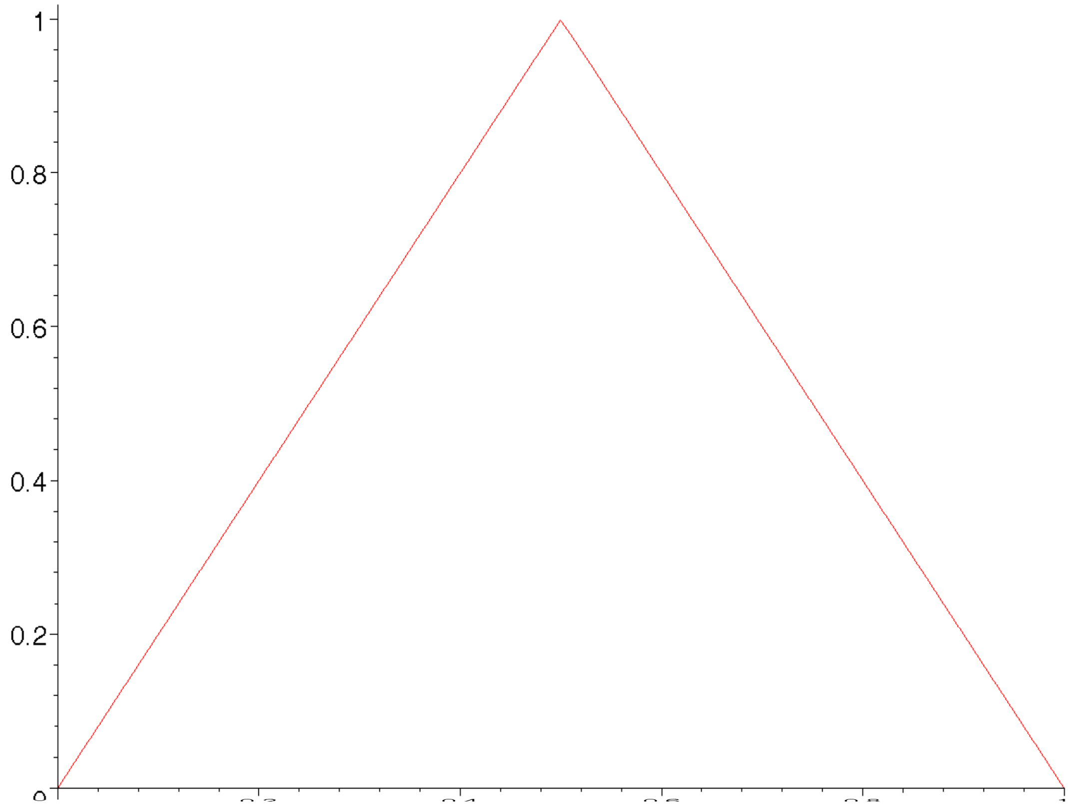
Question 4

First we plot the tent function.

```
> g:=x->piecewise( x<1/2,2*x,x<1,2*(1-x));
```

$$g := x \to \text{piecewise}\left(x < \frac{1}{2}, 2\,x, x < 1, 2 - 2\,x\right)$$

```
> plot(g(x),x=0..1);
```




```
> a:='a'; b:='b'; f:='f';
```

$$a := a$$

$$b := b$$

$$f := f$$

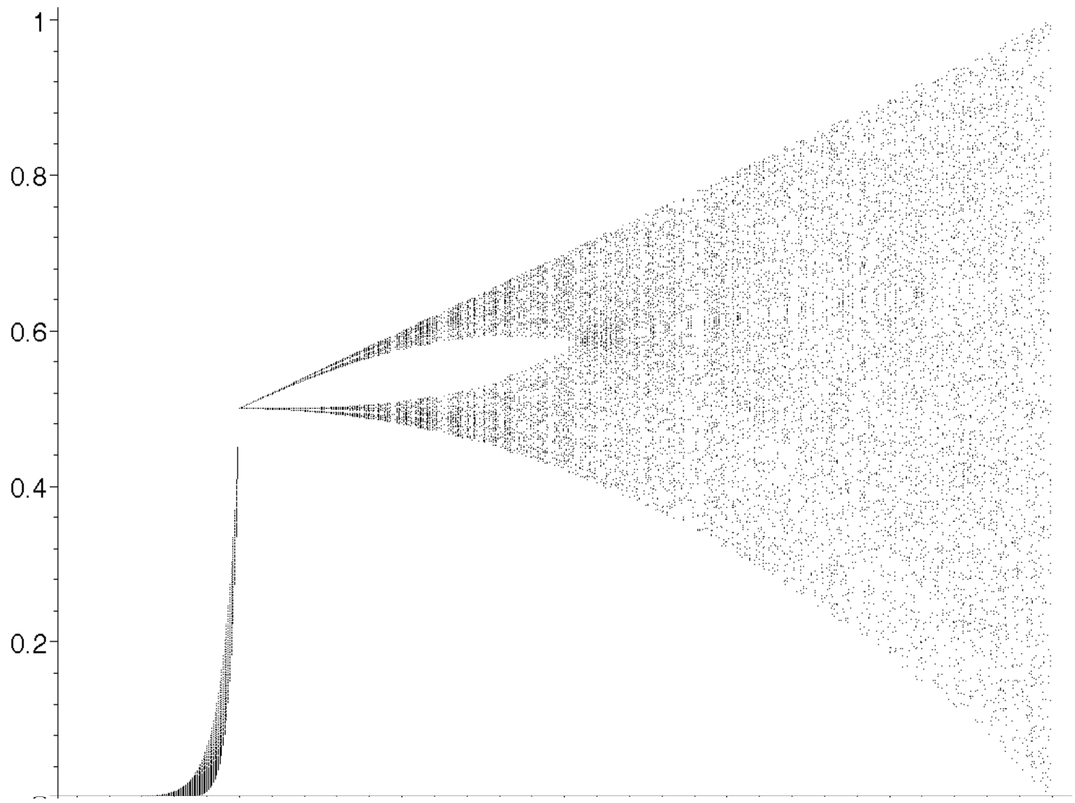
```
> f:=x->piecewise(x<1/2,2*a*x,x<1,2*a*(1-x));
```

$$f := x \rightarrow \mathrm{piecewise}\left(x < \frac{1}{2}, 2\,a\,x, x < 1, 2\,a\,(1-x)\right)$$

```
> a:=(0.001)*b;
```

$$a := .001\,b$$

```
> points10:=evalf(seq(seq([a,(f@@k)((f@@50)(0.5))],k=0..50),b=4
00..1000),4):
> pointplot({points10},symbol=point);
```



Question 5

Given

$$\frac{\partial}{\partial t} x = 4\,x - \frac{\lambda\,x}{1 + 4\,x^2}$$

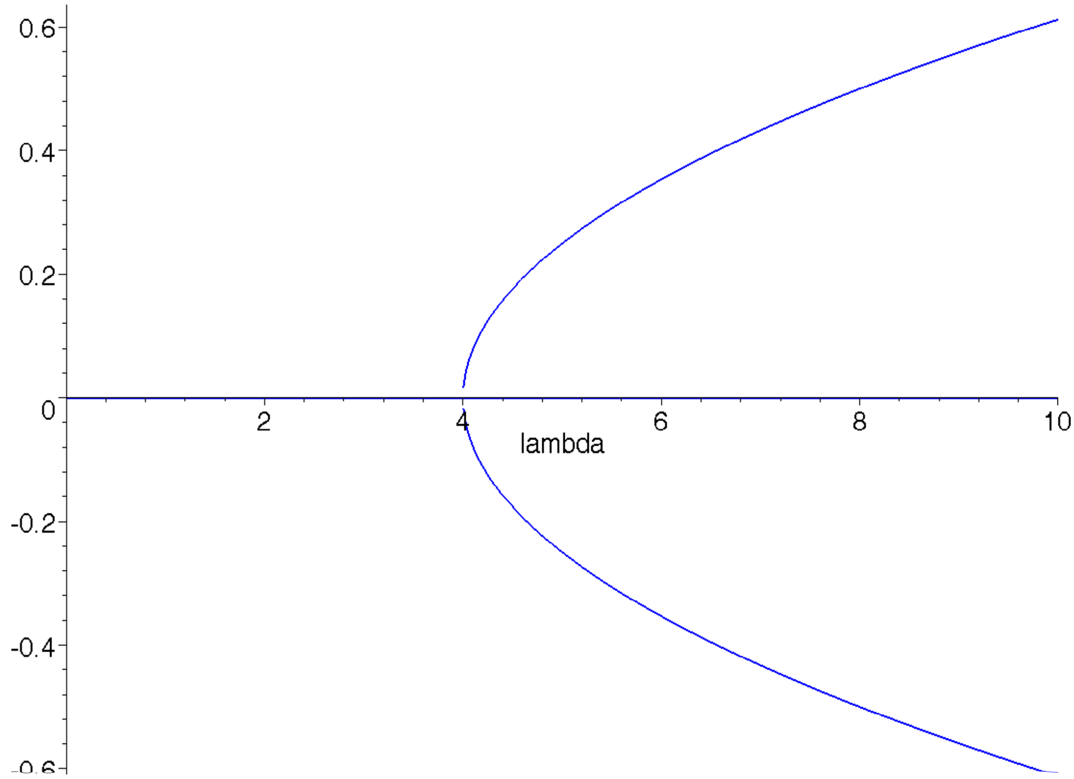
then in equilibrium

$$4\,x = \frac{\lambda\,x}{1 + 4\,x^2}$$

```
> solve(4*x=lambda*x/(1+4*x^2),x);
```

$$0, \frac{1}{4}\sqrt{-4+\lambda}, -\frac{1}{4}\sqrt{-4+\lambda}$$

```
> plot({0,1/4*sqrt(-4+lambda),
  -1/4*sqrt(-4+lambda)},lambda=0..10,thickness=2,colour=blue);
```



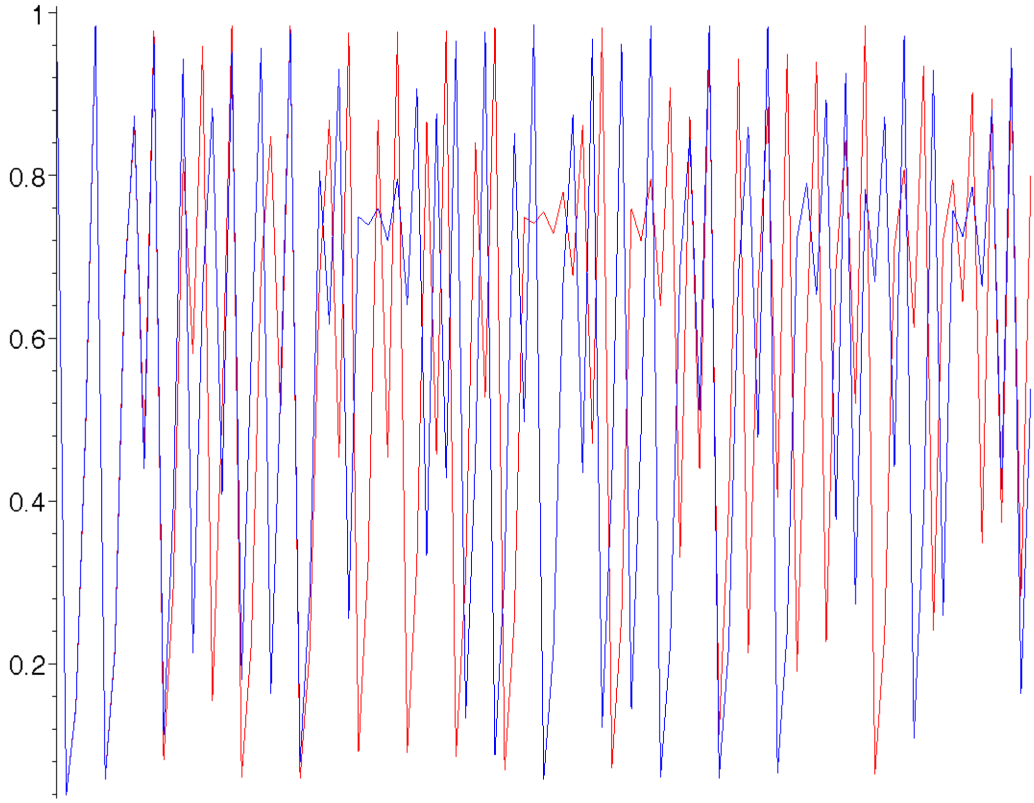
What this illustrates is a (subcritical) pitchfork bifurcation.

Question 6

```
> f:=x->3.94*x*(1-x);
```

$$f := x \rightarrow 3.94\, x\, (1 - x)$$

```
> data1:=[seq([k,(f@@k)(0.99)],k=0..100)]:
> plot1:=listplot(data1,colour=blue):
> data2:=[seq([k,(f@@k)(0.9901)],k=0..100)]:
> plot2:=listplot(data2,colour=red):
> display(plot1,plot2);
```



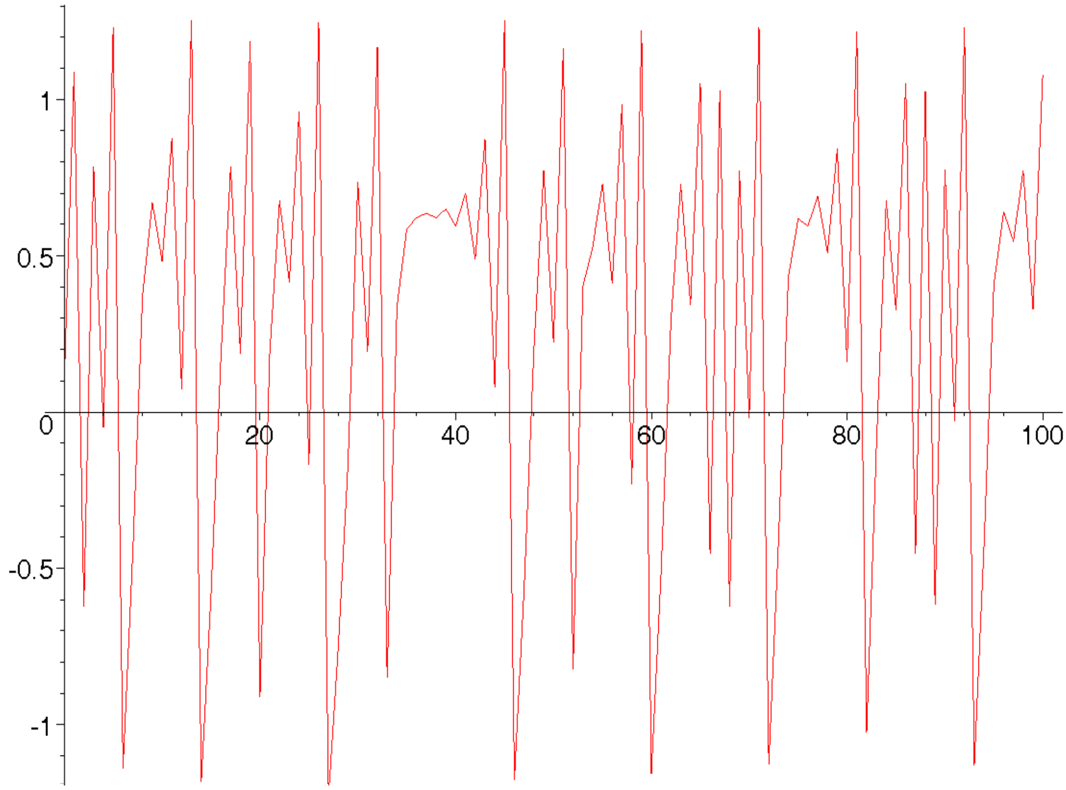
As can be seen from the graph, with just a minor change in the initial condition the two series soon begin to differ.

```
>
```

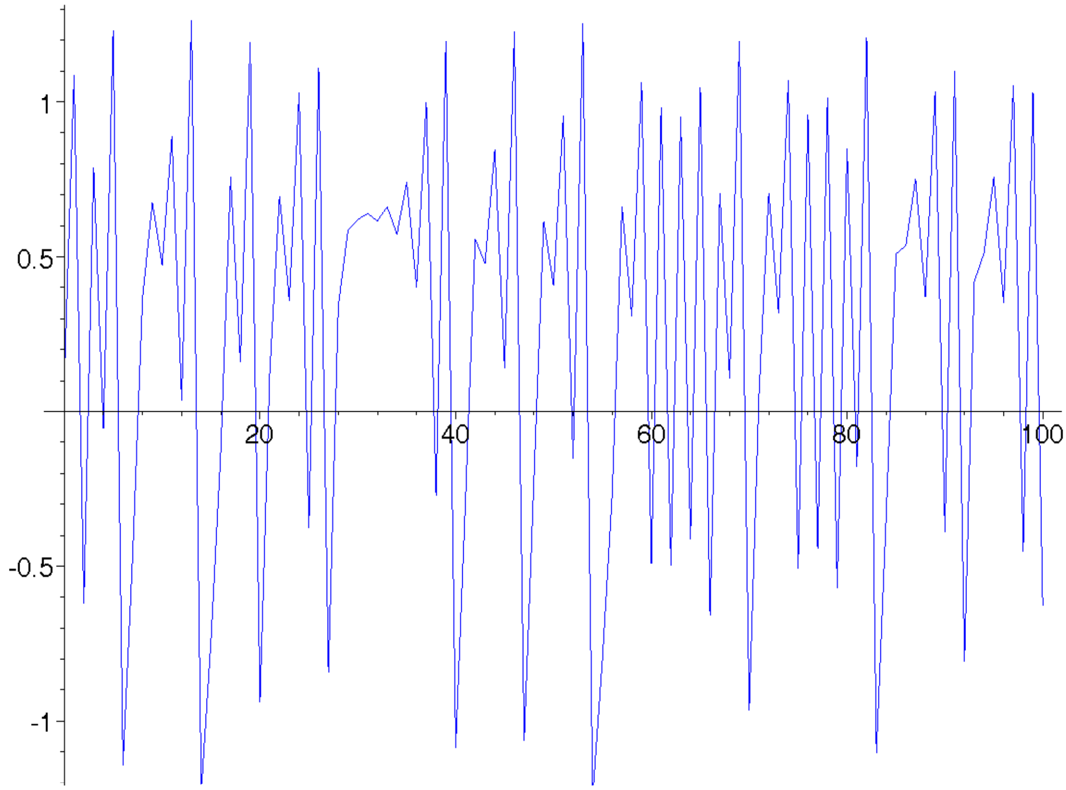
Question 7

Although this question is for a spreadsheet, we shall deal with it within *Maple*.

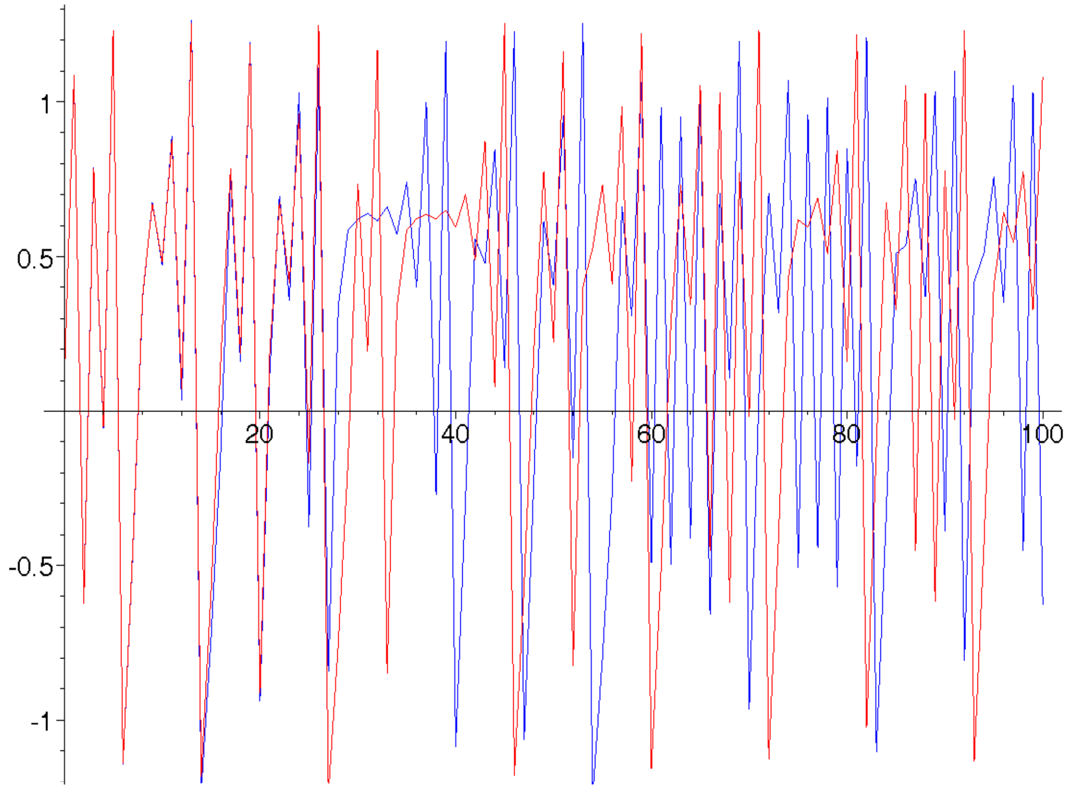
```
> x1:=array(0..100):y1:=array(0..100):
  a:=1.4:b:=0.3:imax:=99:
  x1[0]:=0.1:y1[0]:=0.1:
  for i from 0 to imax do
  x1[i+1]:=1+y1[i]-a*(x1[i])^2:
  y1[i+1]:=b*x1[i]:
  od:

> x1points:=[seq([t,x1[t]],t=0..100)]:
> x1plot:=plot(x1points):
> display(x1plot);
```




```
> x2:=array(0..100):y2:=array(0..100):
  a:=1.4:b:=0.3:imax:=99:
  x2[0]:=0.101:y2[0]:=0.1:
  for i from 0 to imax do
  x2[i+1]:=1+y2[i]-a*(x2[i])^2:
  y2[i+1]:=b*x2[i]:
  od:
> x2points:=[seq([t,x2[t]],t=0..100)]:
> x2plot:=plot(x2points,colour=blue):
> display(x2plot);
```




```
> display(x1plot,x2plot);
```

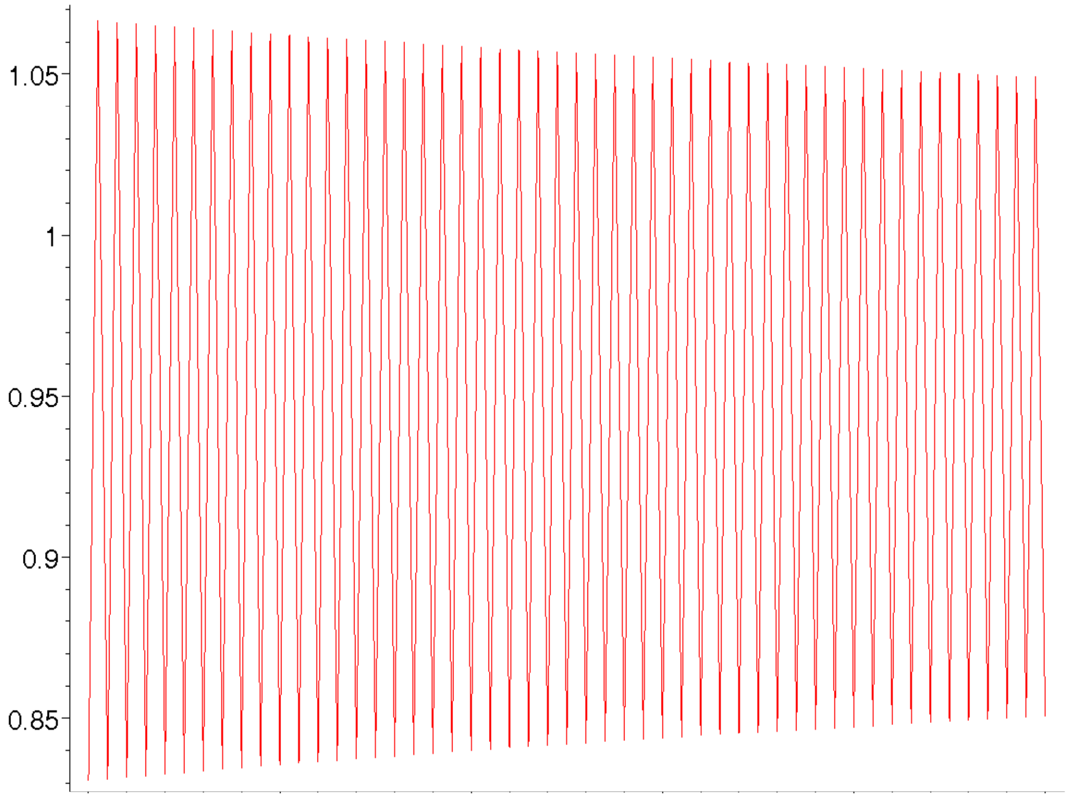


The plot of the variable *x* indicates that the two series begin to converge on each other. In other words, after the initial iterations, the Henon map is not sensitive to initial conditions.

```
>
```

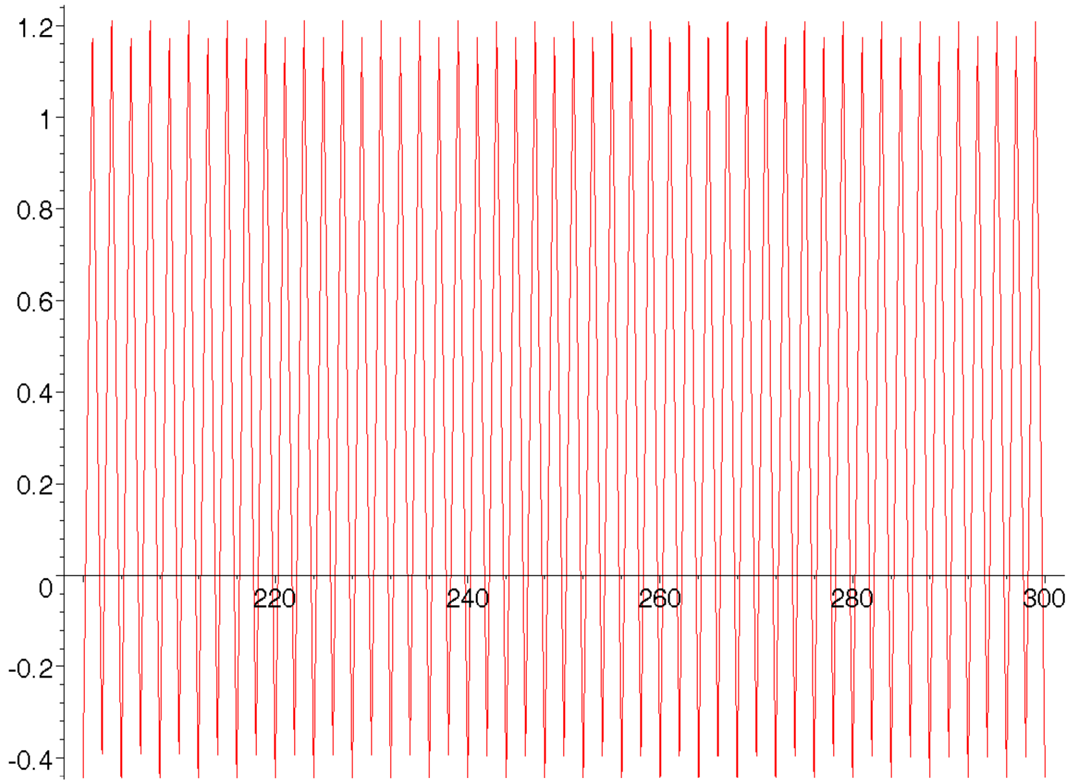
Question 8

```
> x:=array(0..300):y:=array(0..300):
  a:=0.3675:b:=0.3:imax:=299:
  x[0]:=0.1:y[0]:=0.1:
  for i from 0 to imax do
  x[i+1]:=1+y[i]-a*(x[i])^2:
  y[i+1]:=b*x[i]:
  od:
> xpoints:=[seq([t,x[t]],t=200..300)]:
> xplot:=plot(xpoints):
> display(xplot);
```



The plot shows a clear two-cycle.

```
> x:=array(0..300):y:=array(0..300):
  a:=0.9125:b:=0.3:imax:=299:
  x[0]:=0.1:y[0]:=0.1:
  for i from 0 to imax do
  x[i+1]:=1+y[i]-a*(x[i])^2:
  y[i+1]:=b*x[i]:
  od:
> xpoints:=[seq([t,x[t]],t=200..300)]:
> xplot:=plot(xpoints):
> display(xplot);
```



This only just illustrates that a four-cycle results.

Question 9

The system under investigation is

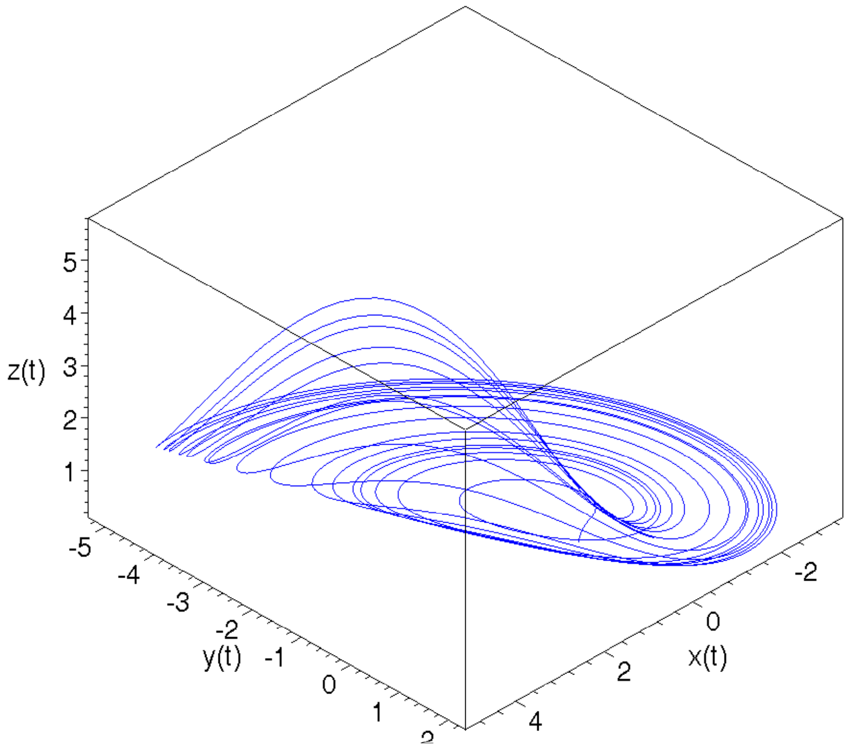
$$\frac{\partial}{\partial t} x = -y - z$$

$$\frac{\partial}{\partial t} y = x + .398\, y \qquad (\mathrm{x}(0), \mathrm{y}(0), \mathrm{z}(0)) = (.1, .1, 1.)$$

$$\frac{\partial}{\partial t} z = 2 + z\,(x - 4)$$

(i)

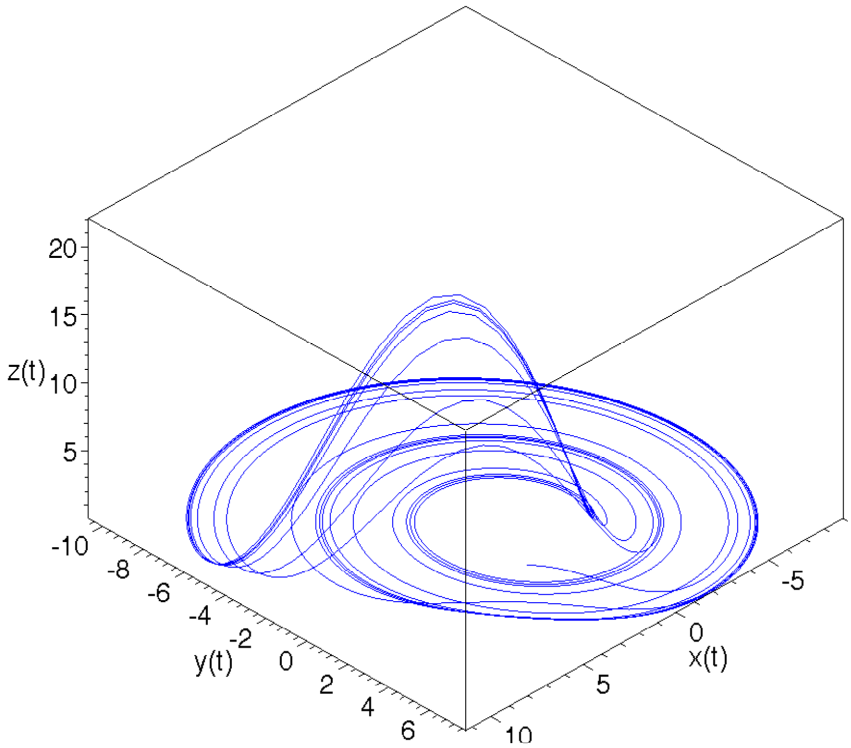
```
> x:='x':y:='y':z:='z':
> DEplot3d(
  {diff(x(t),t)=-y(t)-z(t),diff(y(t),t)=x(t)+0.398*y(t),diff
  (z(t),t)=2+z(t)*(x(t)-4)},
  {x(t),y(t),z(t)},t=0..100,
  [[x(0)=0.1,y(0)=0.1,z(0)=0.1]],
  scene=[x(t),y(t),z(t)],
  stepsize=.05,
  linecolour=blue,
  arrows=none,thickness=1);
```



>

(ii)

```
> x:='x':y:='y':z:='z':
> DEplot3d(
  {diff(x(t),t)=-y(t)-z(t),diff(y(t),t)=x(t)+0.2*y(t),diff(z
  (t),t)=0.2+z(t)*(x(t)-5.7)},
  {x(t),y(t),z(t)},t=0..100, [[x(0)=5,y(0)=5,z(0)=5]],
  scene=[x(t),y(t),z(t)],
  stepsize=.05,
  linecolour=blue,
  arrows=none,thickness=1);
```



A similar chaotic fold also occurs.

Question 10

The system under investigation is

$$\frac{\partial}{\partial t} x = -y - z$$

$$\frac{\partial}{\partial t} y = x + .2\, y$$

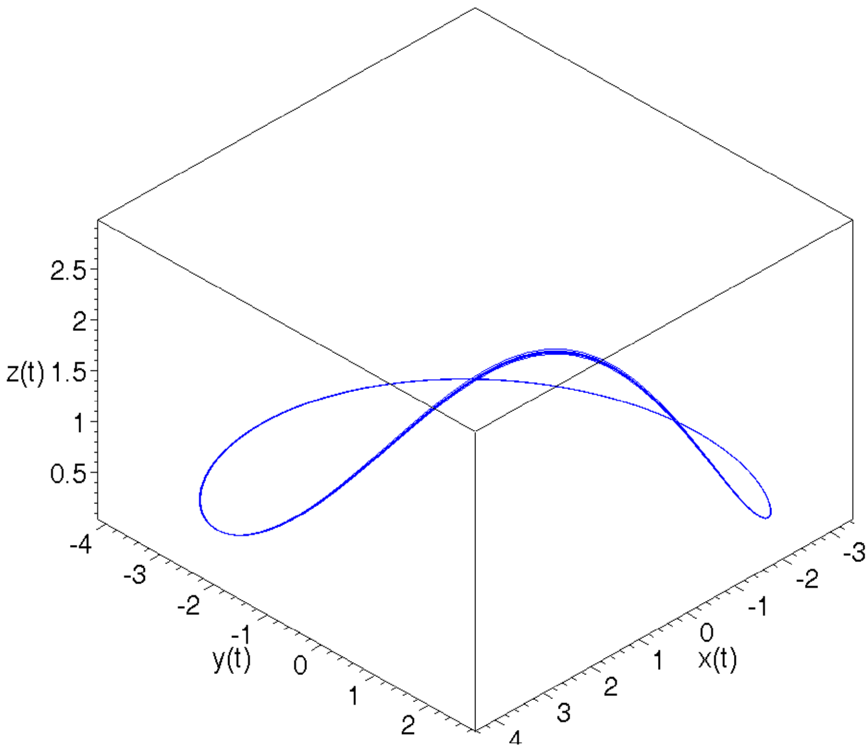
$$\frac{\partial}{\partial t} z = .2 + z\,(x - c)$$

In each plot we ignore the initial part of the trajectory and plot only for $t = 50$ to 100. In each case we assume the intial point is $(\mathrm{x}(0), \mathrm{y}(0), \mathrm{z}(0)) = (1, 1, 1)$.

(i) $c = 2.3$

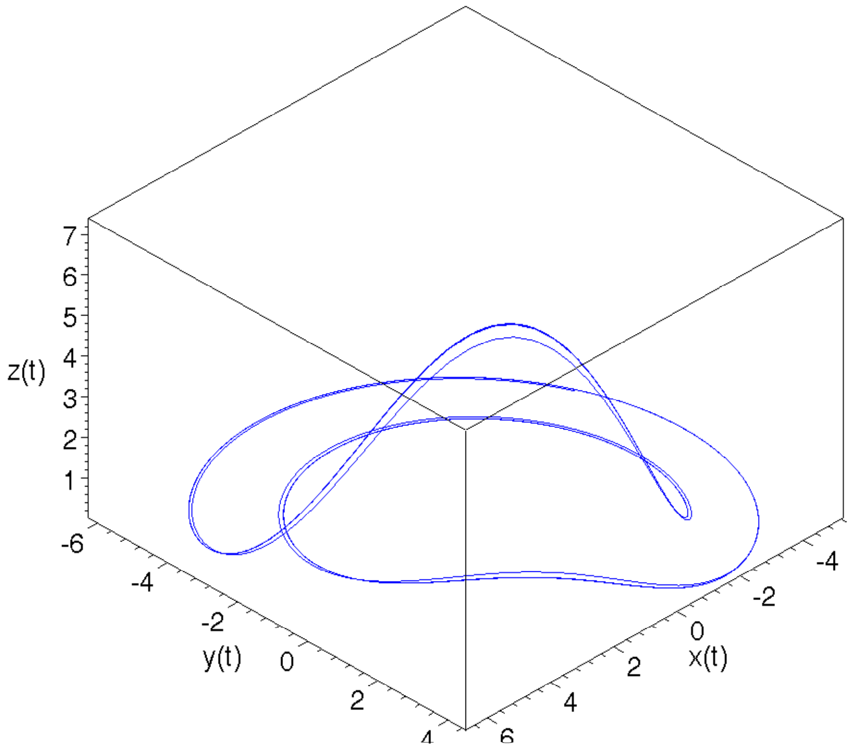
```
> x:='x':y:='y':z:='z':
> DEplot3d(
  {diff(x(t),t)=-y(t)-z(t),diff(y(t),t)=x(t)+0.2*y(t),diff(z
  (t),t)=0.2+z(t)*(x(t)-2.3)},
  {x(t),y(t),z(t)},t=50..100, [[x(0)=1,y(0)=1,z(0)=1]],
  scene=[x(t),y(t),z(t)],
  stepsize=.05,
  linecolour=blue,
```

```
arrows=none,thickness=1);
```



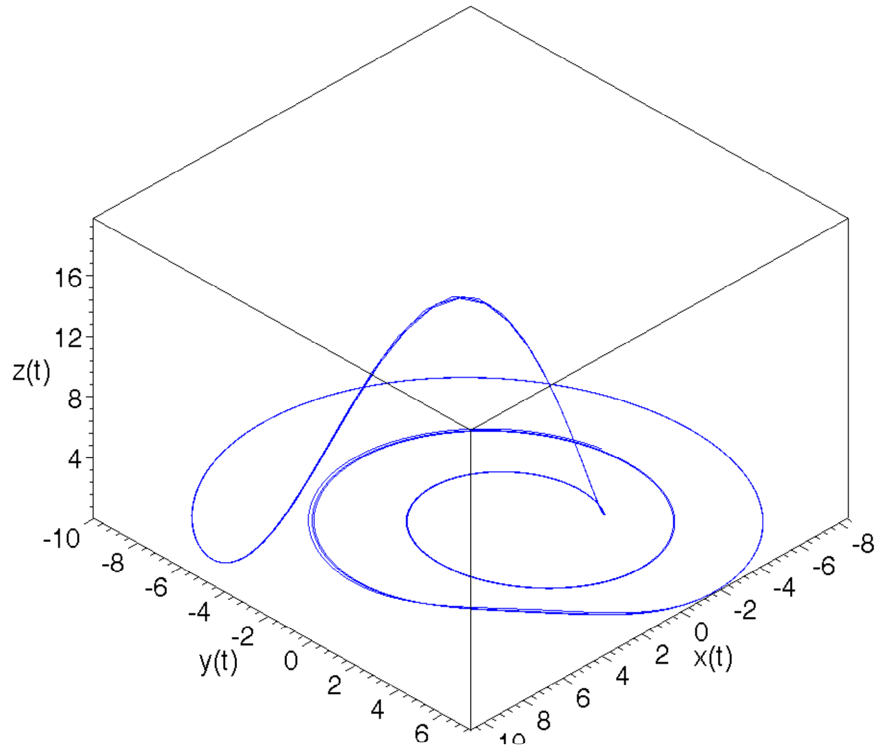
(ii) $c = 3.3$

```
> x:='x':y:='y':z:='z':
> DEplot3d(
  {diff(x(t),t)=-y(t)-z(t),diff(y(t),t)=x(t)+0.2*y(t),diff(z
  (t),t)=0.2+z(t)*(x(t)-3.3)},
  {x(t),y(t),z(t)},t=50..100, [[x(0)=1,y(0)=1,z(0)=1]],
  scene=[x(t),y(t),z(t)],
  stepsize=.05,
  linecolour=blue,
  arrows=none,thickness=1);
```



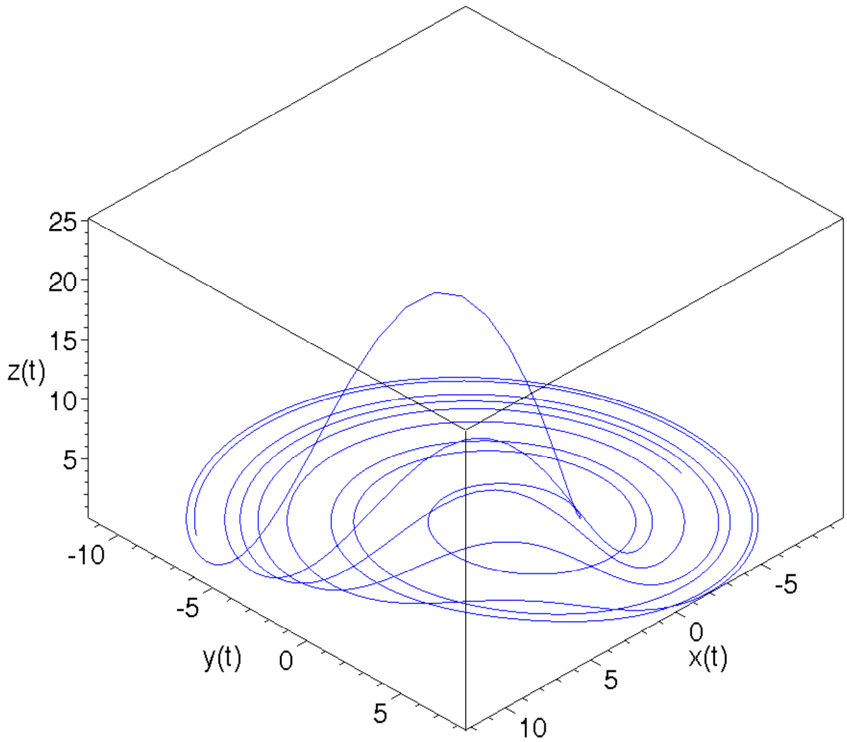
(iii) $c = 5.3$

```
> x:='x':y:='y':z:='z':
> DEplot3d(
  {diff(x(t),t)=-y(t)-z(t),diff(y(t),t)=x(t)+0.2*y(t),diff(z
  (t),t)=0.2+z(t)*(x(t)-5.3)},
  {x(t),y(t),z(t)},t=50..100, [[x(0)=1,y(0)=1,z(0)=1]],
  scene=[x(t),y(t),z(t)],
  stepsize=.05,
  linecolour=blue,
  arrows=none,thickness=1);
```



(iv) $c = 6.3$

```
> x:='x':y:='y':z:='z':
> DEplot3d(
  {diff(x(t),t)=-y(t)-z(t),diff(y(t),t)=x(t)+0.2*y(t),diff(z
  (t),t)=0.2+z(t)*(x(t)-6.3)},
  {x(t),y(t),z(t)},t=50..100, [[x(0)=1,y(0)=1,z(0)=1]],
  scene=[x(t),y(t),z(t)],
  stepsize=.05,
  linecolour=blue,
  arrows=none,thickness=1);
```


25
20
15
z(t)
10
5
-10
-5
y(t)
0
5
-5
0
x(t)
5
10