A First Course in Digital Communications

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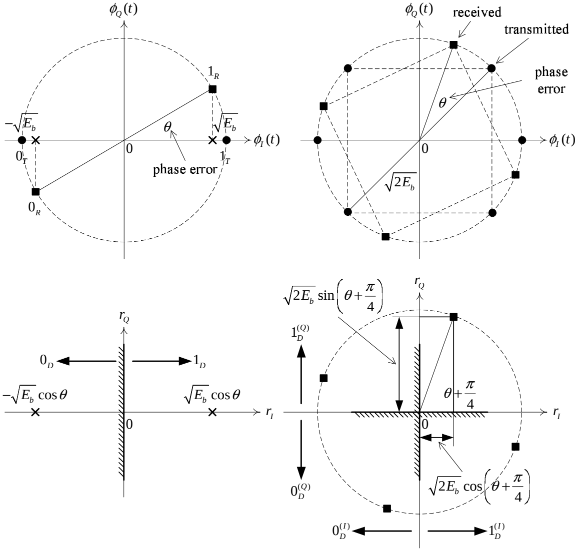
Introduction

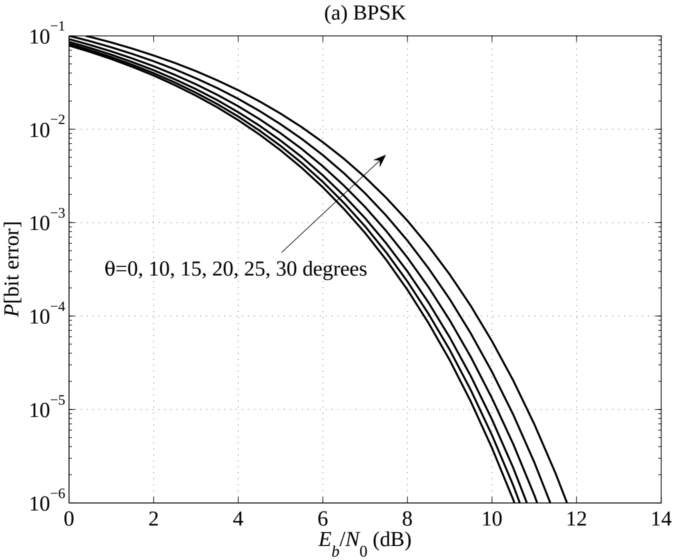
- Any successful communication system must establish synchronization.
- At the physical-layer level the receiver needs to estimate three parameters:
  (i) The incoming carrier frequency, f_c (Hz);
  (ii) Any phase shift or phase drift, $\theta(t)$ (radians), introduced during transmission (for coherent demodulation);
  (iii) The bit (symbol) timing, i.e., where on the time axis do the kT_b (or kT_s) (seconds) ticks occur.
- The concepts involved are illustrated by the discussion of the analysis and design of two basic circuits: the *phase-locked loop* (PLL) for f_c and $\theta(t)$ estimation and the *early-late gate synchronizer* for symbol timing.

Effect of Phase Error on BPSK and QPSK

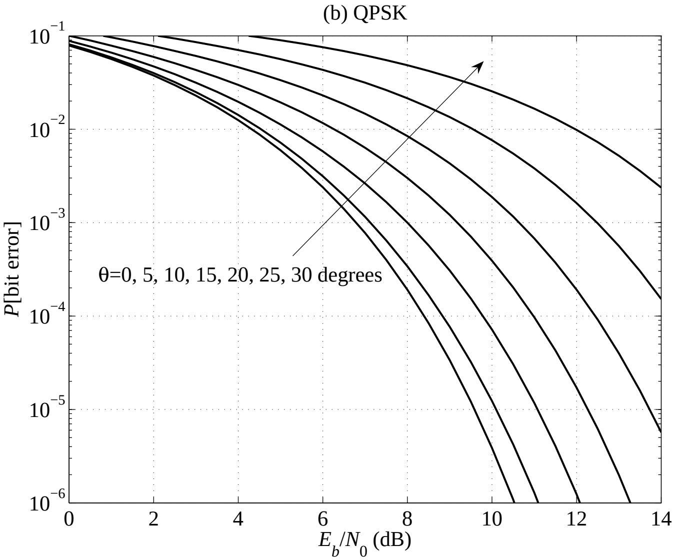


(a) BPSK (b) QPSK

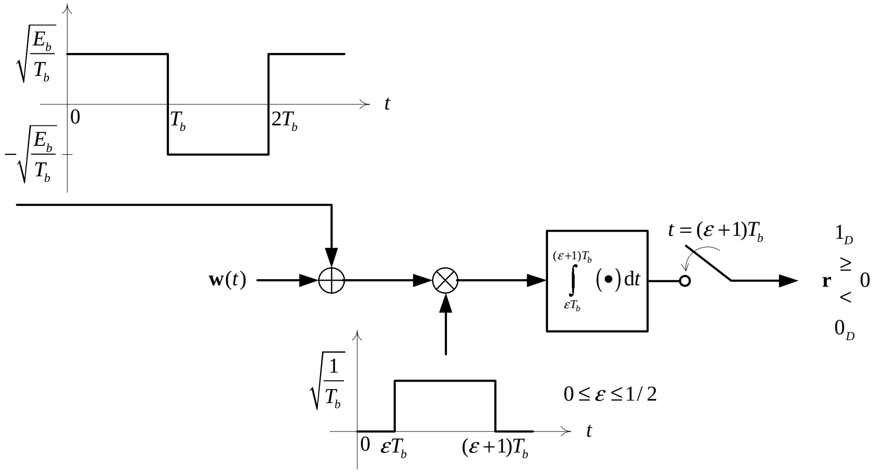
BPSK: $P\left[\text{bit error}\right] = Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\theta\right)$



$$\frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}(\cos\theta - \sin\theta)\right) + \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}(\cos\theta + \sin\theta)\right)$$

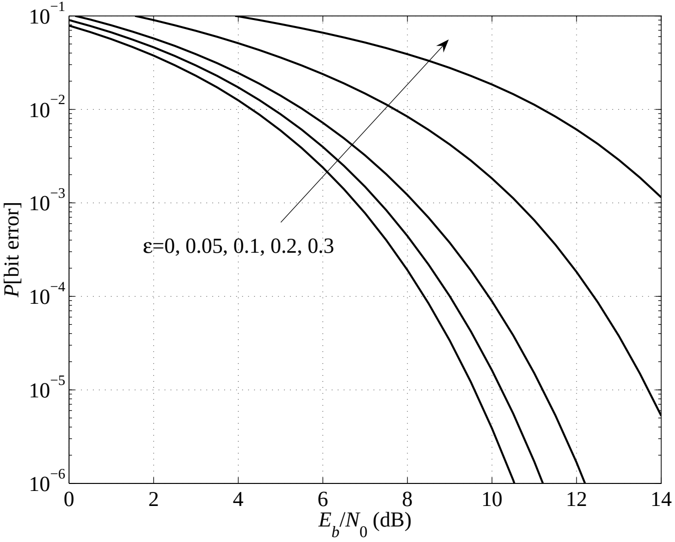


Effect of Symbol Timing Error on NRZ-L Modulation



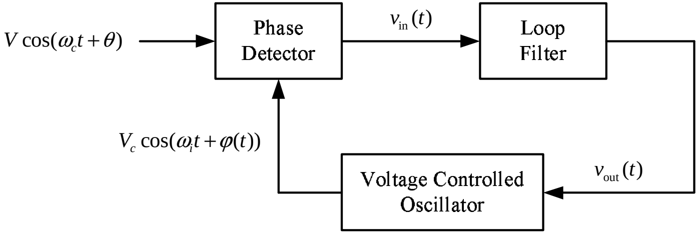
$\mathbf{r} = \pm\sqrt{E_b} + \mathbf{w}$ when two consecutive bits agree and
$\mathbf{r} = \pm\sqrt{E_b}(1-2\epsilon) + \mathbf{w}$ if they disagree.

$P\left[\text{bit error}\right] = \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}(1-2\epsilon)\right)$

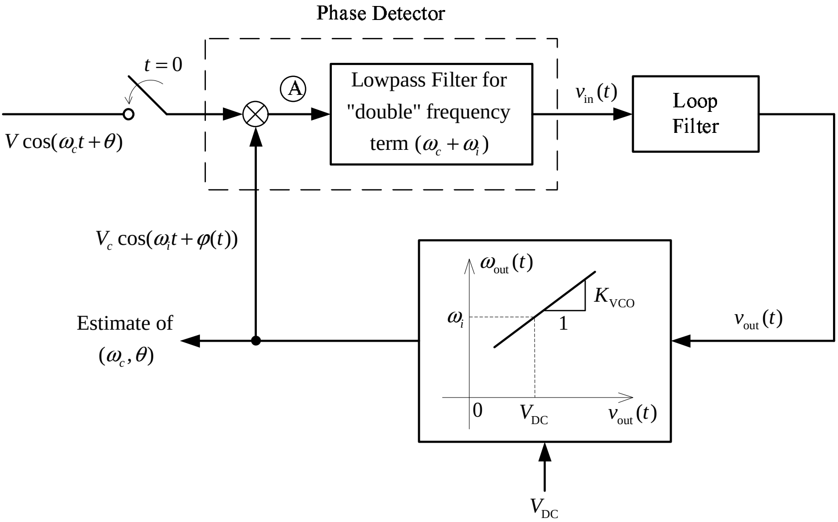


Phase-Locked Loop (PLL)

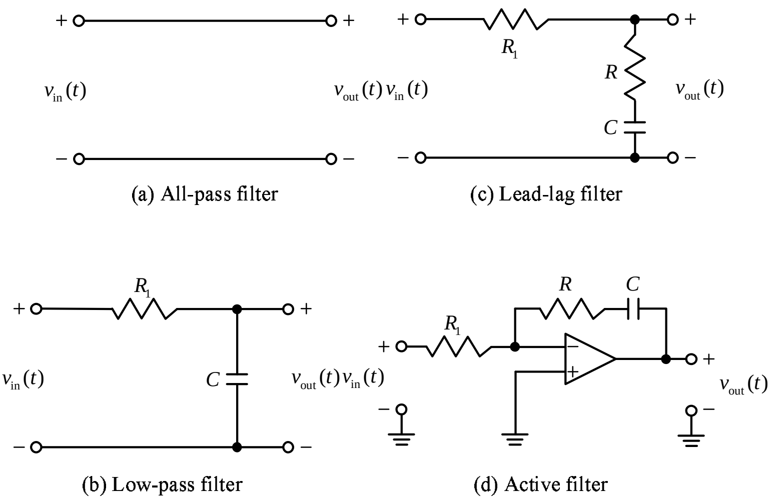
- PLL is a circuit that locks onto the frequency ω_c (rad/sec) of a received sinusoid, $V\cos(\omega_c t + \theta)$ and estimates the phase offset, θ (radians).
- Loop implies a feedback circuit and this is precisely what a PLL is: It is a feedback control system whose function is to track the frequency and phase of an input sinusoid.



A Sinusoid Phase Detector



Four Possible Loop Filters



Analysis of the First-Order PLL Circuit

- Consider the simplest case where the loop filter is an “allpass” filter.
- The signal at point Ⓐ of the loop is

$$\frac{K_m V V_c}{2}\left\{\cos\left[(\omega_c+\omega_i)+\theta+\varphi(t)\right]+\cos\left[(\omega_c-\omega_i)t+\theta-\varphi(t)\right]\right\},$$

where K_m is a constant, ω_i is the nominal frequency of the VCO (reasonably close to ω_c), and $\varphi(t)$ is a slow phase variation in the output of the VCO.

$$v_{\text{in}}(t)=\frac{K_m V V_c}{2}\cos\left[(\omega_c-\omega_i)\,t+\theta-\varphi\,(t)\right]=\frac{K_m V V_c}{2}\cos\psi\,(t)\,,$$

where $\psi(t)\equiv(\omega_c-\omega_i)t+\theta-\varphi(t)=(\omega_c t+\theta)-(\omega_i t+\varphi(t))$ is the *instantaneous phase error*.

$$\begin{aligned}\omega_{\text{out}}(t) &= \omega_i+K_{\text{vco}}v_{\text{out}}(t)=\omega_i+K_{\text{vco}}v_{\text{in}}(t)\\ &= \omega_i+\frac{K_{\text{vco}}K_m V V_c}{2}\cos\psi(t)=\omega_i+K_{\text{loop}}\cos\psi(t),\end{aligned}$$

where $K_{\text{loop}}\equiv\frac{K_{\text{vco}}K_m V V_c}{2}$ is called the *loop gain*.

- As $\omega_{\text{out}}(t)$ is the instantaneous frequency of the VCO output, one has

$$\omega_{\text{out}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\omega_i t + \varphi(t)\right] = \omega_i + \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \omega_i + K_{\text{loop}}\cos\psi(t)$$

$$\Rightarrow \qquad \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = K_{\text{loop}}\cos\psi(t).$$

- Let $\Delta\omega \equiv (\omega_c - \omega_i)$. Then $\psi(t) = \Delta\omega t + \theta - \varphi(t)$ and

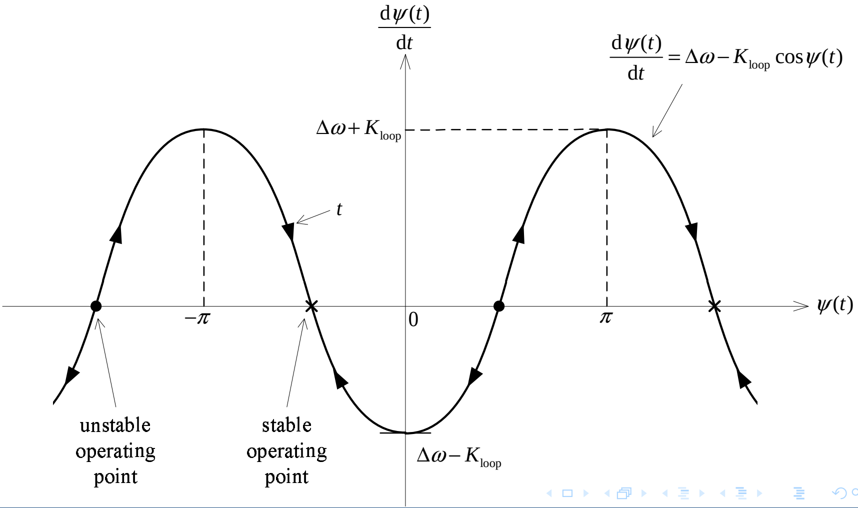
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\varphi(t) - \theta - \Delta\omega t) + \Delta\omega t\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[-\psi(t) + \Delta\omega t\right] = -\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + \Delta\omega.$$

- The differential equation governing the phase error, $\psi(t)$, is:

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + K_{\text{loop}}\cos\psi(t) = \Delta\omega, \quad t \geq 0.$$

Phase Plane of the First-Order PLL Circuit

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + K_{\text{loop}} \cos\psi(t) = \Delta\omega, \quad t \geq 0.$$



Observations about the Phase Plane Plot

1) The phase error, $\psi(t)$, at all times, has to be a point on the trajectory. Where one starts depends on $\psi(0)$, the initial condition.
2) The intersection points represent equilibrium or solution points in the steady state, i.e., as $t \to \infty$ and the transient response has died out. However, some are stable and others are unstable. Note that $\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} > 0$ implies that $\psi(t)$ increases with time while $\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} < 0$ means $\psi(t)$ decreases.
3) The stable operating points are given by:

$$\psi(t)\Big|_{t\to\infty} = -\cos^{-1}\left(\frac{\Delta\omega}{K_{\text{loop}}}\right) + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

4) Consider the stable operating point of $-\cos^{-1}\left(\frac{\Delta\omega}{K_{\text{loop}}}\right)$. Since $\psi(t) = (\omega_c - \omega_i)t + \theta - \varphi(t)$ approaches a constant as $t \to \infty$, we conclude that the angular frequency ω_i approaches ω_c as $t \to \infty$, i.e., the loop locks onto the incoming frequency. The phase $\varphi(t)$, however, tends to $\left[\theta + \cos^{-1}\left(\frac{\Delta\omega}{K_{\text{loop}}}\right)\right]$. Therefore there is a phase error of magnitude, $\left|\cos^{-1}\left(\frac{\Delta\omega}{K_{\text{loop}}}\right)\right|$.
5) The stable operating point and achievement of lock occurs only if $-K_{\text{loop}} < \Delta\omega < K_{\text{loop}}$, i.e., the trajectory must intersect the ψ axis. Otherwise the phase error either keeps increasing if $\Delta\omega > K_{\text{loop}}$, or keeps decreasing if $\Delta\omega < -K_{\text{loop}}$, with time.
6) Observe that if $\Delta\omega = 0$ at $t = 0$, i.e., initially the incoming frequency and VCO frequency are the same, the steady-state phase error is $-\frac{\pi}{2}$, i.e., the VCO signal is in quadrature with the incoming signal.

To understand the PLL's transient performance (how does the phase error behaves over time), need to solve the differential equation:

$$\int_{\psi(t=0)}^{\psi(t)} \frac{\mathrm{d}\psi}{\Delta\omega - K_{\text{loop}} \cos\psi} = \int_{t=0}^{t} \mathrm{d}t = t.$$

$$\left[\frac{1}{\sqrt{K_{\text{loop}}^2 - (\Delta\omega)^2}} \ln \left\{ \frac{\sqrt{K_{\text{loop}}^2 - (\Delta\omega)^2} \tan\frac{\psi}{2} + \Delta\omega - K_{\text{loop}}}{\sqrt{K_{\text{loop}}^2 - (\Delta\omega)^2} \tan\frac{\psi}{2} - \Delta\omega + K_{\text{loop}}} \right\} \right]_{\psi=\psi(0)}^{\psi=\psi(t)} = t,$$

where $|\Delta\omega| < K_{\text{loop}}$ (the lock-in region). Define $\omega_d \equiv \frac{\Delta\omega}{K_{\text{loop}}}$, we get

$$\ln \left\{ \frac{\sqrt{1-\omega_d^2} \tan\frac{\psi(t)}{2} - (1-\omega_d)}{\sqrt{1-\omega_d^2} \tan\frac{\psi(t)}{2} + (1-\omega_d)} \right\} =$$

$$K_{\text{loop}} \left(\sqrt{1-\omega_d^2} \right) t + \ln \left\{ \frac{\sqrt{1-\omega_d^2} \tan\frac{\psi(0)}{2} - (1-\omega_d)}{\sqrt{1-\omega_d^2} \tan\frac{\psi(0)}{2} + (1-\omega_d)} \right\}.$$

- Without loss of generality we let $0 < \Delta\omega < K_{\text{loop}}$, which means $0 < \omega_d < 1$. Therefore the numerator of the LHS is less than denominator of the LHS. Multiply both sides of the above equation by -1 and then take the exponential of both sides, one has

$$\frac{a\tan\frac{\psi(t)}{2}+b}{a\tan\frac{\psi(t)}{2}-b} = \left[\frac{a\tan\frac{\psi(0)}{2}+b}{a\tan\frac{\psi(0)}{2}-b}\right]\mathrm{e}^{-a\,K_{\text{loop}}\,t}, \quad t \geq 0.$$

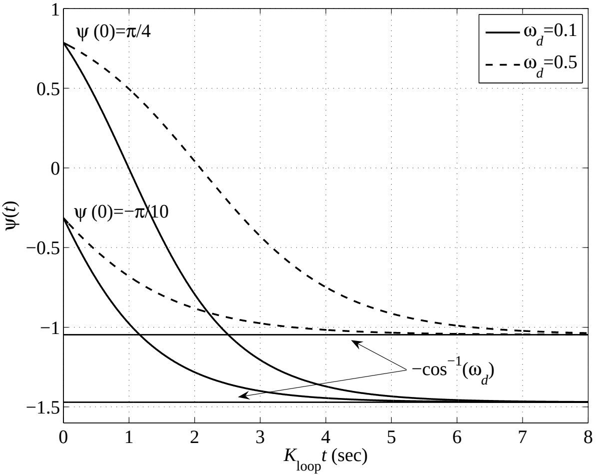
  where $a \equiv \sqrt{1-\omega_d^2}$; $b \equiv (1-\omega_d)$.

- From the above equation we would infer that $\psi(t)$ approximately decays exponentially to its steady-state value with a time constant of $\frac{1}{a\,K_{\text{loop}}}$.

- Letting $A \equiv \left[\frac{a\tan\frac{\psi(0)}{2}+b}{a\tan\frac{\psi(0)}{2}-b}\right]$, the explicit expression for $\psi(t)$ is

$$\psi(t) = 2\tan^{-1}\left\{\frac{-b\left(1+A\mathrm{e}^{-a\,K_{\text{loop}}\,t}\right)}{a\left(1-A\mathrm{e}^{-a\,K_{\text{loop}}\,t}\right)}\right\}.$$

$\psi(t) = 2\tan^{-1}\left\{\dfrac{-b\left(1+Ae^{-aK_{\text{loop}}t}\right)}{a\left(1-Ae^{-aK_{\text{loop}}t}\right)}\right\}$

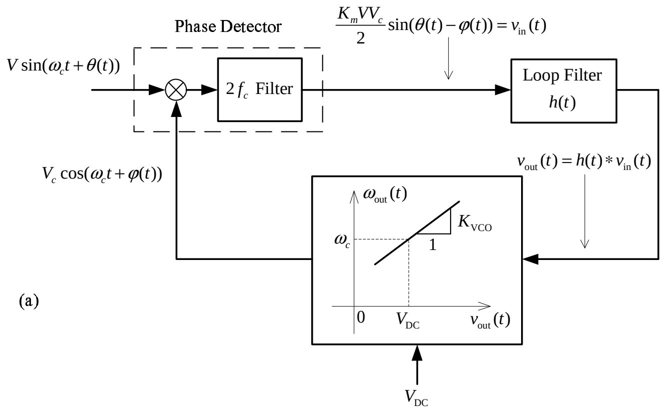


The steady-state is reached within $\frac{3}{aK_{\text{loop}}}$ to $\frac{5}{aK_{\text{loop}}}$ seconds.

Equivalent Model of a Sinusoidal PLL

- Practical loop filter is chosen so that the PLL become a second-order PLL, meaning that the differential equation for the phase error is a second-order, nonlinear differential equation.
- The loop filter in this case is chosen from one of those shown in Figures (b), (c) or (d) of page 10.
- A second-order PLL extends its lock range, has better performance in noise and can achieve a steady-state phase error of zero, i.e., not only is frequency lock achieved but also phase lock as well.
- We now develop a somewhat more general model for the sinusoidal PLL. It could be termed the equivalent baseband model or the incremental model because the frequency ω_c is suppressed.
- The VCO output is assumed to have an angular frequency of ω_c, any deviation from this, say $\Delta\omega t$, is subsumed in the instantaneous phase, $\varphi(t)$, i.e., $\varphi(t) = \Delta\omega t + \varphi'(t)$.

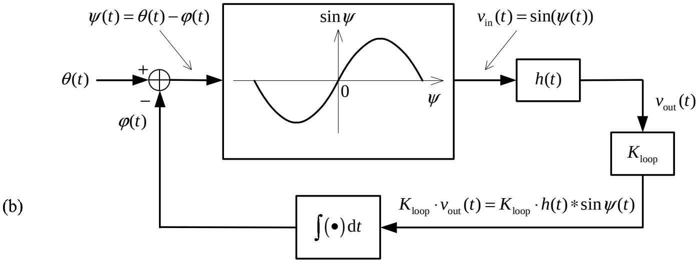
Sinusoidal PLL



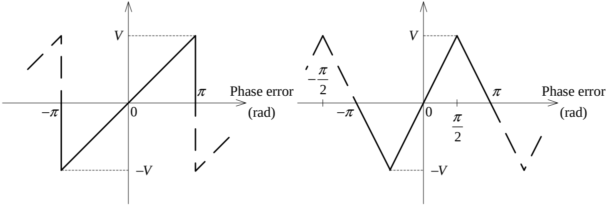
$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = K_{\text{loop}} h(t) * \sin(\theta(t) - \varphi(t)),$$

where $\psi(t) \equiv \theta(t) - \varphi(t)$ is the phase error. It follows that $\varphi(t) = K_{\text{loop}} \int h(t) * \sin(\theta(t) - \varphi(t))\mathrm{d}t = K_{\text{loop}} \int v_{\text{out}}(t)\mathrm{d}t$. This leads to an equivalent model of the PLL.

Equivalent Model of the Sinusoidal PLL



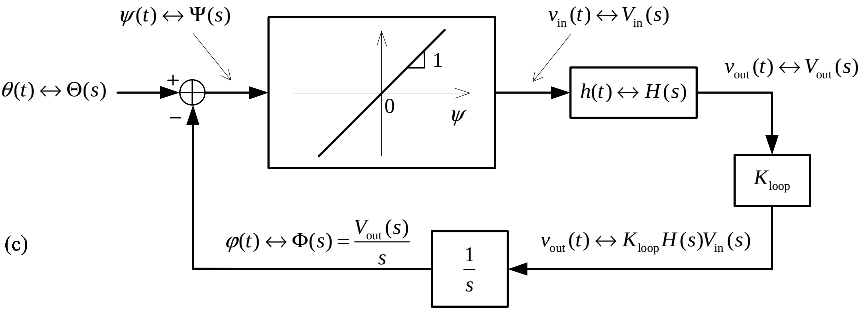
Other nonlinearities for the phase detector are shown below.



(a) Sawtooth (b) Triangular

Linear Model of the Sinusoidal PLL

If the phase error is small enough (a situation that occurs once the PLL has locked onto the frequency of the incoming signal and now is tracking slow phase changes in it) then $\sin\psi \approx \psi$ $(\psi \ll 1)$ and the PLL can be represented by a linear model:



Second-Order Phase-Locked Loop Dynamics

- Consider the filter of Figure (d) on page 10 with $H(s) = 1 + \frac{a}{s}$.

$$V_{\text{out}}(s) = \frac{s+a}{s} V_{\text{in}}(s) \;\leftrightarrow\; \frac{\mathrm{d}v_{\text{out}}(t)}{\mathrm{d}t} = a v_{\text{in}}(t) + \frac{\mathrm{d}v_{\text{in}}(t)}{\mathrm{d}t}$$

- Since $\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = K_{\text{loop}} v_{\text{out}}(t)$ and $v_{\text{in}}(t) = \sin\psi(t)$, one has

$$\frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} = K_{\text{loop}} \frac{\mathrm{d}v_{\text{out}}(t)}{\mathrm{d}t} = aK_{\text{loop}} \sin\psi(t) + K_{\text{loop}} \cos\psi(t) \frac{\mathrm{d}\psi(t)}{\mathrm{d}t}.$$

- Because $\frac{\mathrm{d}^2\varphi(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\varphi(t) - \theta(t) + \theta(t)\right] = -\frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}$, the differential equation for the phase error is:

$$\frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} + K_{\text{loop}} \cos\psi(t) \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + aK_{\text{loop}} \sin\psi(t) = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}.$$

- Consider a constant frequency input, $\theta(t) = \Delta\omega t$. Then

$$\frac{\mathrm{d}^2\psi(t)}{\mathrm{d}t^2} + K_{\text{loop}} \cos\psi(t) \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} + aK_{\text{loop}} \sin\psi(t) = 0.$$

- Unfortunately, no solution is available for the above equation.

Phase Plane Plot for Second-Order Phase-Locked Loop

- Normalize the time axis by letting $\tau \equiv K_{\text{loop}} t$, then

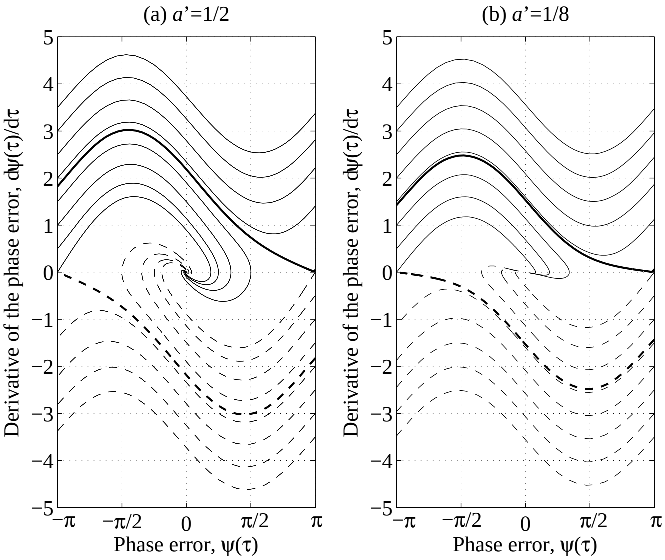
$$K_{\text{loop}}^2 \left[\frac{\mathrm{d}^2\psi(\tau)}{\mathrm{d}\tau^2} + \cos\psi(\tau)\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau} + \frac{a}{K_{\text{loop}}}\sin\psi(\tau) \right] = 0.$$

$$\frac{\mathrm{d}^2\psi(\tau)}{\mathrm{d}\tau^2} + \cos\psi(\tau)\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau} + a'\sin\psi(\tau) = 0 \text{where } a' \equiv \frac{a}{K_{\text{loop}}}.$$

- Now $\frac{\mathrm{d}^2\psi(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}\right]$. Divide the previous equation by $\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}$ and realize that $\frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}\right]}{\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}} = \frac{\mathrm{d}\left[\frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}\right]}{\mathrm{d}\psi(\tau)}$, we have

$$\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}\psi} = -\cos\psi - a'\frac{\sin\psi}{\dot{\psi}}, \text{ where } \dot{\psi} = \frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}.$$

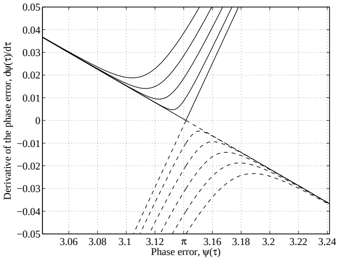
- The phase plane is a plot of $\dot{\psi}$ versus ψ and the above equation gives the slope of the trajectories at each point $(\dot{\psi}, \psi)$ in the phase plane.



(a) a'=1/2
(b) a'=1/8
Derivative of the phase error, dψ(τ)/dτ
5
4
3
2
1
0
−1
−2
−3
−4
−5
−π
−π/2
0
π/2
π
Phase error, ψ(τ)

Some Observations of the Phase Plane

- There is a region in the $(\dot{\psi}, \psi)$ plane where the phase error converges to 0 without the phase exceeding the $[-\pi, \pi]$ range. Outside this region the PLL still converges to a stable point, but one that is a multiple of 2π. This phenomena is known as *cycle skipping*.
- The phase plane plot is shown only in the range $[-\pi, \pi]$ since the plot is periodic with period 2π.
- There are singular points at $(\dot{\psi} = 0, \psi = k2\pi)$ which are stable operating points. There are also singular points at $(\dot{\psi} = 0, \dot{\psi} = (2k+1)\pi)$, which are unstable and called *saddle points*.



Lock-in Range of Second-Order PLL

Multiply the previous equation by $\dot{\psi}$ and integrate over $[-\pi, \pi]$:

$$\frac{1}{2}\left[\dot{\psi}^2(\pi) - \dot{\psi}^2(-\pi)\right] = -\int_{-\pi}^{\pi} \dot{\psi}\cos\psi \mathrm{d}\psi - a' \int_{-\pi}^{\pi} \sin\psi \mathrm{d}\psi.$$

The second term is zero. Integrating the first term gives:

$$\frac{1}{2}\left[\dot{\psi}^2(\pi) - \dot{\psi}^2(-\pi)\right] = \int_{-\pi}^{\pi} \sin\psi \mathrm{d}\dot{\psi}.$$

Substituting $\mathrm{d}\dot{\psi} = \left[-\cos\psi - a'\frac{\sin\psi}{\dot{\psi}}\right]\mathrm{d}\psi$ yields

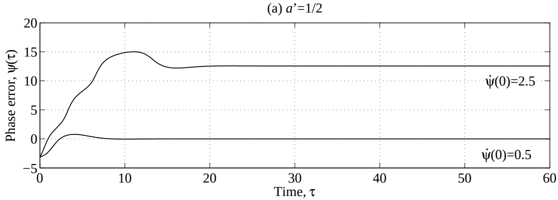
$$\frac{1}{2}\left[\dot{\psi}^2(\pi) - \psi^2(-\pi)\right] = -\int_{-\pi}^{\pi} \sin\psi\cos\psi \mathrm{d}\psi - a' \int_{-\pi}^{\pi} \frac{\sin^2\psi}{\dot{\psi}}\mathrm{d}\psi = -a' \int_{-\pi}^{\pi} \frac{1-\cos(2\psi)}{\dot{\psi}}\mathrm{d}\psi$$

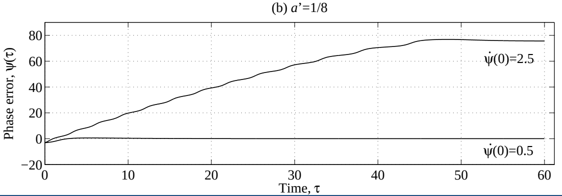
If $\dot{\psi} = \frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau} > 0$, the RHS < 0 and if $\dot{\psi} < 0$ the RHS > 0 $\Rightarrow$ For any cycle of width 2π, $|\dot{\psi}|$ must decrease regardless of the initial value of $\dot{\psi}$ $\Rightarrow$ the lock range is infinite.

State-Space Approach to The Phase Plane Method

- A high-order differential equation is changed into a set of first-order differential equations which can then be solved numerically.
- Define two states $y = \frac{\mathrm{d}\psi(\tau)}{\mathrm{d}\tau}$ and $x = \psi(\tau)$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y\cos x - a'\sin x, \quad \frac{\mathrm{d}x}{\mathrm{d}\tau} = y.$$





Phase and Carrier Frequency Acquisition

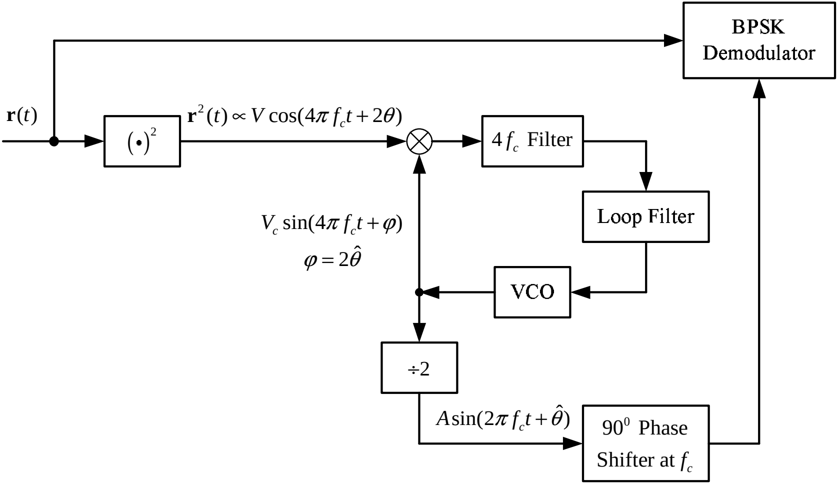
- PLL needs a spectral component at the carrier frequency, f_c.
- Suppressed carrier modulations such as BPSK, QPSK, and M-QAM do not have the spectral component at f_c.
- How to obtain a spectral component for these modulations?
- For BPSK, the received signal can be written as

$$\mathbf{r}(t) = \left\{ \sum_{k=-\infty}^{\infty} b_k \left[u(kT_b) - u((k+1)T_b) \right] \right\} \sqrt{E_b} \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t + \theta) + \mathbf{w}(t),$$

$$\begin{aligned} \mathbf{r}^2(t) &= I^2(t) E_b \left(\frac{2}{T_b} \right) \cos^2(2\pi f_c t + \theta) + \textbf{noise terms} \\ &= \frac{E_b}{T_b} + \cos(2\pi(2f_c)t + 2\theta) + \textbf{noise terms}. \end{aligned}$$

$\mathbf{r}^2(t)$ has a spectral component at $2f_c$ for the PLL to lock on!

2nd-Power Synchronizer for Carrier Recovery for BPSK



Carrier Recovery for QPSK

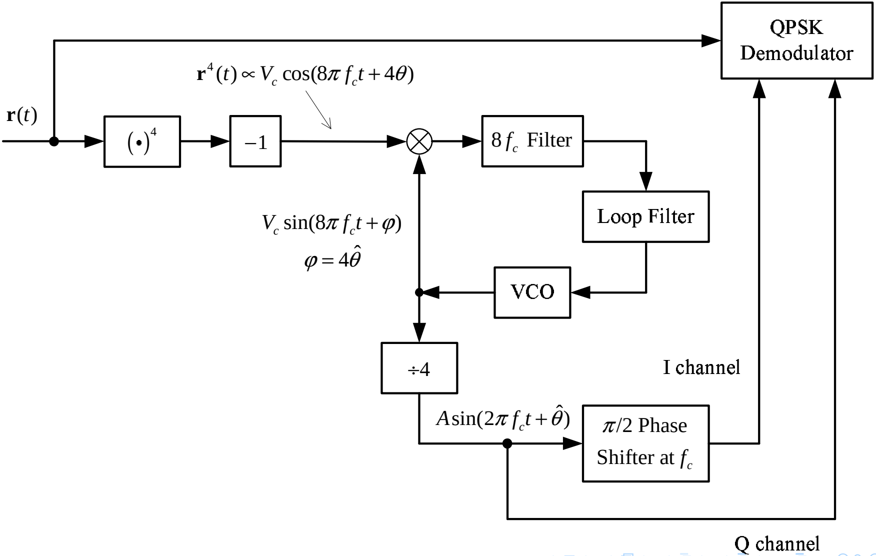
$$\mathbf{r}(t) = I(t)\sqrt{E_b}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t+\theta)+Q(t)\sqrt{E_b}\sqrt{\frac{2}{T_s}}\sin(2\pi f_c t+\theta)+\mathbf{w}(t),$$

$$I(t) \equiv \sum_{k=-\infty}^{\infty} b_k^{(I)}\left[u(kT_s)-u((k+1)T_s)\right],\ Q(t) \equiv \sum_{k=-\infty}^{\infty} b_k^{(Q)}\left[u(kT_s)-u((k+1)T_s)\right],$$

$$\begin{aligned}
\mathbf{r}^2(t) &= I^2(t)E_b\left(\frac{2}{T_s}\right)\cos^2(2\pi f_c t+\theta)+Q^2(t)E_b\left(\frac{2}{T_s}\right)\sin^2(2\pi f_c t+\theta)\\
&+ 2I(t)Q(t)E_b\left(\frac{2}{T_s}\right)\cos(2\pi f_c t+\theta)\sin(2\pi f_c t+\theta)+\textbf{noise terms}\\
&= \frac{2E_b}{T_s}+\frac{2E_b}{T_s}I(t)Q(t)\sin(2\pi(2f_c)t+2\theta)+\textbf{noise terms}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}^4(t) &= \frac{4E_b^2}{T_s^2}+\frac{8E_b^2}{T_s}I(t)Q(t)\sin(2\pi(2f_c)t+2\theta)\\
&+ \frac{4E_b^2}{T_s^2}I^2(t)Q^2(t)\sin^2(2\pi(2f_c)t+2\theta)+\textbf{even more noise terms}.
\end{aligned}$$

4th-Power Synchronizer for Carrier Recovery for QPSK



Carrier Recovery for M-PSK and M-QAM

- M-PSK needs an Mth-power block to produce a spectral component at Mf_c for the PLL circuit.
- Consider QAM with signals on a rectangular grid:

$$s(t) = \underbrace{\left\{ \sum_{k=-\infty}^{\infty} I_{i,k} \left[u(kT_s) - u\left((k+1)T_s\right) \right] \right\}}_{\equiv I(t)} \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t + \theta)$$

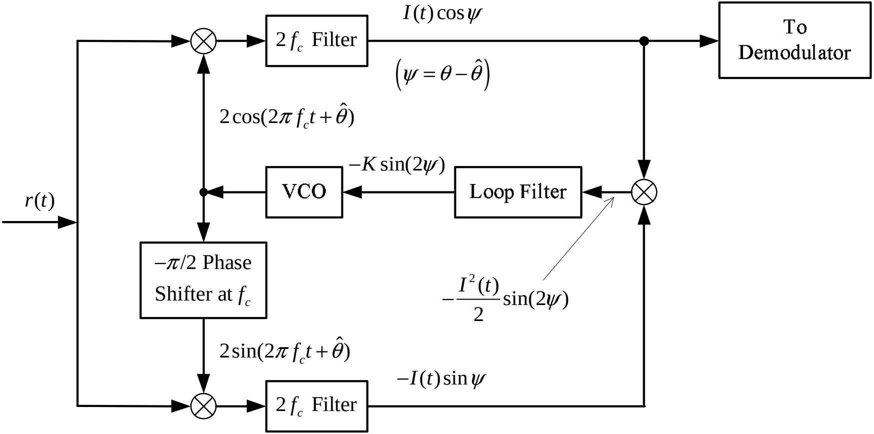
$$+ \underbrace{\left\{ \sum_{k=-\infty}^{\infty} Q_{j,k} \left[u(kT_s) - u\left((k+1)T_s\right) \right] \right\}}_{\equiv Q(t)} \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t + \theta),$$

where $I_{i,k}$ is one of the amplitudes $(\pm\Delta, \pm 3\Delta, \ldots)$ on the inphase axis, $Q_{j,k}$ is one of the amplitudes $(\pm\Delta, \pm 3\Delta, \ldots)$ on the quadrature axis, $i = 1, 2, \ldots, 2^{\lambda_I}$, $j = 1, 2, \ldots, 2^{\lambda_Q}$.

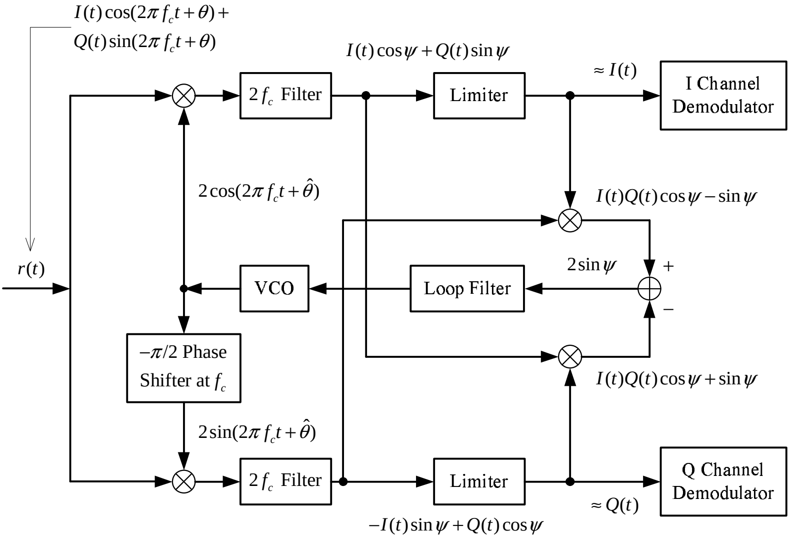
- Treat $I(t)$, $Q(t)$ as two random processes and note that $E\{\mathbf{I}(t)\} = E\{\mathbf{Q}(t)\} = 0$; $E\{\mathbf{I}(t)\mathbf{Q}(t)\} = E\{\mathbf{I}(t)\}E\{\mathbf{Q}(t)\} = 0$; $E\{\mathbf{I}^2(t)\} = \sigma_I^2$; $E\{\mathbf{Q}^2(t)\} = \sigma_Q^2$.
- For *nonsymmetrical* QAM constellations ($\sigma_I^2 \neq \sigma_Q^2$) it can be shown that $E\left\{\mathbf{r}^2(t)\right\}$ has a spectral component at $2f_c$ and one can use the same synchronizer circuit as for BPSK.
- For *symmetrical* QAM constellations ($\sigma_I^2 = \sigma_Q^2 = \sigma^2$), $E\left\{\mathbf{r}^4(t)\right\}$ has a spectral component at $4f_c$. Therefore the QPSK synchronizer block diagram can be used for symmetrical QAM.

Costas Loop for Carrier Recovery for BPSK

Costas loop avoids the squaring or fourth-power block, which is difficult to implement at high frequencies.

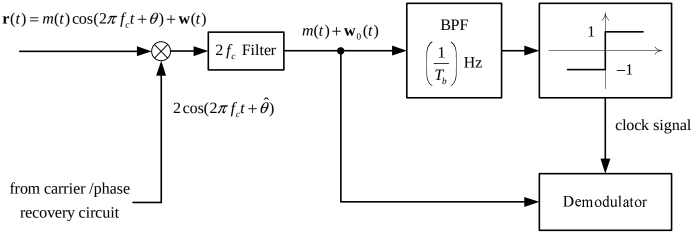


Costas Loop for Carrier Recovery for QPSK

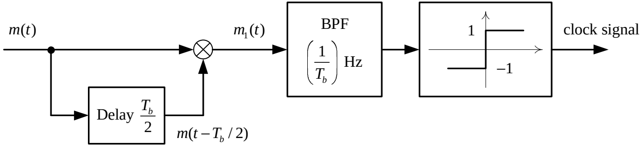


Estimation of Symbol Timing

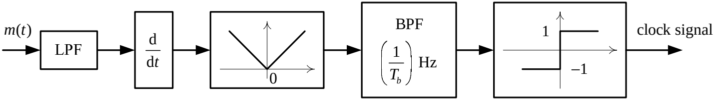
- Symbol timing recovery circuits can be broadly classified into open-loop and closed-loop.
- They may be further classified into non-data-aided (NDA) and data-aided (DA) synchronizers
- Shall consider NDA symbol synchronizers and restrict to binary baseband signals.
- The following circuit works if a baseband modulation $m(t)$ has a spectral component at $f = \frac{1}{T_b}$:



- Most popular baseband modulations (NRZ-L, Miller, bi-phase, etc.) do not have a spectral component at $\frac{1}{T_b}$: it needs to be created.
- In the first circuit, $m_1(t)$ is always positive in the second half of every bit period, T_b. It will be negative in the first half if two successive bits disagree. This produces a spectral component at the data rate $r_b = \frac{1}{T_b}$ as well as at the harmonics.

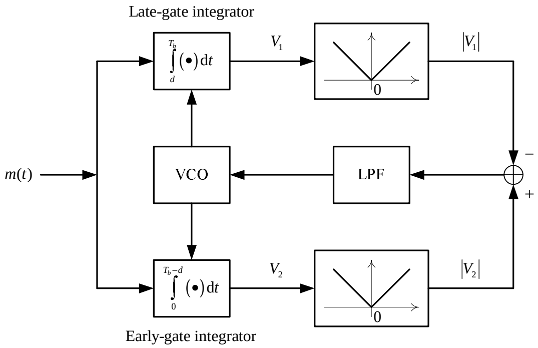


- The second circuit is an edge detector where the differentiator produces positive or negative spikes at symbol transitions.



Early-Late Gate Clock Synchronizer

- Open-loop synchronizer has an unavoidable nonzero average tracking error (though small for large SNR, it cannot be made zero).
- A closed-loop symbol synchronizer circumvents this problem.
- *Early-late gate synchronizer* is the most popular.



- When the VCO's square-wave clock and the incoming data, $m(t)$, are in perfect synchronization, both integrators accumulate the same amount of signal energy over $(T_b - d) \Rightarrow$ the error signal is zero.
- If $m(t)$ is delayed by $\Delta < d$, the early-gate integrator accumulates over $(T_b - \Delta - d)$, while the late-gate integrator still accumulates over $(T_b - d)$ seconds $\Rightarrow$ the error signal is proportional to $-\Delta$ which would delay VCO's timing. The opposite happens when the data timing is in advance of the VCO timing.