3 Chapter 3 Solution Set

Problems

3.1 Determine the Cauchy initial conditions such that the solution to the initial value problem has only positive frequency components. Discuss the consequences of this. HINT: Example 3.1.

The plane wave amplitude of the negative frequencies was found in Example 3.1 to be

$$A(\mathbf{K}, -cK) = \frac{1}{2}[\widetilde{u}_{t_0}(\mathbf{K}) - \frac{i}{cK}\widetilde{u}'_{t_0}(\mathbf{K})]e^{-icKt_0}$$

where $\widetilde{u}_{t_0}(\mathbf{K})$ and $\widetilde{u}'_{t_0}(\mathbf{K})$ are the spatial Fourier transforms of the Cauchy conditions at time t_0. In order for this amplitude to vanish we then require that

$$\widetilde{u}_{t_0}(\mathbf{K}) = \frac{i}{cK}\widetilde{u}'_{t_0}(\mathbf{K}) \rightarrow \widetilde{u}'_{t_0}(\mathbf{K}) = -icK\widetilde{u}_{t_0}(\mathbf{K})$$

which, upon inverse Fourier transformation yields

$$\frac{\partial}{\partial t}u(\mathbf{r}, t)|_{t=t_0} = -ic\int d^3K\, K\widetilde{u}_{t_0}(\mathbf{K})e^{i\mathbf{K}\cdot r} = -c\sqrt{\nabla^2}u(\mathbf{r}, t)|_{t=t_0},$$

where $\sqrt{\nabla^2}$ is the integral (convolutional) operator

$$\sqrt{\nabla^2} = \frac{1}{(2\pi)^3}\int d^3K\, Ke^{i\mathbf{K}\cdot\mathbf{r}}.$$

3.2 Compute the plane wave expansion of the free field propagator $g_f(\mathbf{R}, \tau) = g_+(\mathbf{R}, \tau) - g_-(\mathbf{R}, \tau)$ by employing the general procedure described in Example 3.1. [Hint: See problem 1.12.]

The Cauchy initial conditions for the free space propagator were computed in Problem 1.12 where they were found to be

$$g_f(\mathbf{R}, \tau)|_{\tau=0}, \quad \frac{\partial}{\partial\tau}g_f(\mathbf{R}, \tau)|_{\tau=0} = c^2\frac{\delta'(R)}{2\pi R},$$

where $\delta'(R)$ denotes the derivative of the delta function. We then find using the development employed in Example 3.1 that

$$A(\mathbf{K}, cK) = \frac{i}{2cK}\widetilde{u}'_{t_0}(\mathbf{K}), \quad A(\mathbf{K}, -cK) = -\frac{i}{2cK}\widetilde{u}'_{t_0}(\mathbf{K})$$

where

$$\tilde{u}'_{t_0}(\mathbf{K}) = \int d^3R\, c^2 \frac{\delta'(R)}{2\pi R} e^{-i\mathbf{K}\cdot\mathbf{R}}.$$

The simplest way to proceed is to to use the easily proven identity

$$\delta(\mathbf{R}) = -\frac{\delta'(R)}{2\pi R}$$

which yields

$$\tilde{u}'_{t_0}(\mathbf{K}) = -c^2 \int d^3R\, \delta(\mathbf{R}) e^{-i\mathbf{K}\cdot\mathbf{R}} = -c^2.$$

The plane wave amplitudes are then found to be

$$A(\mathbf{K}, cK) = -\frac{ic}{2K}, \quad A(\mathbf{K}, -cK) = \frac{ic}{2K}.$$

The plane wave expansion of the free field propagator is then found from Eq.(3.11) to be

$$\begin{aligned} g_f(\mathbf{r}, t) &= \frac{1}{(2\pi)^3} \int d^3K\, [-\frac{ic}{2K}] e^{i(\mathbf{K}\cdot\mathbf{r} - cKt)} + \frac{1}{(2\pi)^3} \int d^3K\, \frac{ic}{2K} e^{i(\mathbf{K}\cdot\mathbf{r} + cKt)} \\ &= -\frac{ic}{2(2\pi)^3} \int_{-\infty}^{\infty} kdk \int_{4\pi} d\Omega_s\, e^{ik(\mathbf{s}\cdot\mathbf{r} - ct)}, \end{aligned}$$

where we have set $K = k$.

We also note that this leads directly to the following expansion for the frequency domain free field propagator $G_f(\mathbf{r}, \omega)$

$$G_f(\mathbf{r}, \omega) = -\frac{ik}{8\pi^2} \int_{4\pi} d\Omega_s\, e^{ik\mathbf{s}\cdot\mathbf{r}}.$$

an expansion that will be used frequently in subsequent chapters.

3.3 Compute the plane wave expansion Eq.(3.11) of the field radiated by a source $q(\mathbf{r}, t)$ for times t exceeding the turn-off time $t = T_0$ of the source. Hint: Express the field for $t > T_0$ in terms of the free field propagator.

If $t > T_0$ we can express the primary field solution to the radiation problem in terms of the free field propagator in the form

$$u(\mathbf{r}, t) = \int_{-\infty}^{\infty} dt' \int_{\tau_0} d^3r'\, q(\mathbf{r}', t') g_f(\mathbf{r} - \mathbf{r}', t - t').$$

Using the plane wave expansion of the free field propagator from the previous problem we obtain

$$\begin{aligned} u(\mathbf{r}, t) &= \int_{-\infty}^{\infty} dt' \int_{\tau_0} d^3r'\, q(\mathbf{r}', t') \{-\frac{ic}{2(2\pi)^3} \int_{-\infty}^{\infty} kdk \int_{4\pi} d\Omega_s\, e^{ik(\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}') - c(t-t'))}\} \\ &= -\frac{ic}{2(2\pi)^3} \int_{-\infty}^{\infty} kdk \int_{4\pi} d\Omega_s\, \tilde{q}(k\mathbf{s}, ck) e^{ik(\mathbf{s}\cdot\mathbf{r} - ct)}, \end{aligned}$$

where

$$\tilde{q}(k\mathbf{s}, ck) = \int_{-\infty}^{\infty} dt' \int_{\tau_0} d^3r'\, q(\mathbf{r}', t') e^{-ik(\mathbf{s}\cdot\mathbf{r}' + ct')}$$

is the space-time Fourier transform of the source $\tilde{q}(\mathbf{K},\omega)$ evaluated on the surface of the Ewald sphere $\mathbf{K}=k\mathbf{s}$ with $k=\omega/c$.

3.4 Determine a source $q(\mathbf{r},t)$ supported on the space-time boundary $t=t_0$ that radiates a field for $t>t_0$ which has prescribed Cauchy conditions at $t=t_0>0$.

On equating the primary field solution of a field radiated by a 3D source to the solution to the initial value problem given in Eq.(1.40a) we obtain

$$u(\mathbf{r},t)=\int_0^{T_0}dt'\int_{\tau_0}d^3r'\,q(\mathbf{r}',t')g_+(\mathbf{r}-\mathbf{r}',t-t')$$
$$=\frac{1}{c^2}\int d^3r'\,[u_{t_0}(\mathbf{r}')\frac{\partial}{\partial t_0}g_+(\mathbf{r}-\mathbf{r}',t-t_0)-g_+(\mathbf{r}-\mathbf{r}',t-t_0)u'_{t_0}(\mathbf{r}')].$$

If we now require the source to be supported on the boundary $t=t_0$ we obtain

$$\int_{\tau_0}d^3r'\,q_{t_0}(\mathbf{r}')g_+(\mathbf{r}-\mathbf{r}',t-t_0)$$
$$=\frac{1}{c^2}\int d^3r'\,[u_{t_0}(\mathbf{r}')\frac{\partial}{\partial t_0}g_+(\mathbf{r}-\mathbf{r}',t-t_0)-g_+(\mathbf{r}-\mathbf{r}',t-t_0)u'_{t_0}(\mathbf{r}')].$$

which requires that

$$q_{t_0}(\mathbf{r}')=\frac{1}{c^2}[-u_{t_0}(\mathbf{r}')\frac{\partial}{\partial t'}\delta(t'-t_0)-u'_{t_0}(\mathbf{r}')\delta(t'-t_0)]$$

3.5 Find the plane wave expansion in the form of Eq.(3.16a) of a wave field that satisfies the homogeneous Helmholtz equation over all of space and whose Dirichlet and Neumann conditions on the plane $z=0$ are $U_0(x,y,\omega)$ and $U'_0(x,y,\omega)$, respectively. What must be true of these boundary conditions if the field is to be finite over all of space?

On taking the spatial Fourier transform of both sides of Eq.(3.16a) we find that

$$\tilde{U}_0(\mathbf{K}_\rho,\omega)=\frac{2\pi i}{\gamma}[A^{(+)}(\mathbf{k}^+,\omega)+A^{(-)}(\mathbf{k}^-,\omega)],$$
$$\tilde{U}'_0(\mathbf{K}_\rho,\omega)=-2\pi[A^{(+)}(\mathbf{k}^+,\omega)-A^{(-)}(\mathbf{k}^-,\omega)].$$

Solving these two coupled equations for the plane wave amplitudes $A^{(\pm)}$ we obtain

$$A^{(+)}(\mathbf{k}^+,\omega)=-\frac{1}{4\pi}[i\gamma\tilde{U}_0(\mathbf{K}_\rho)+\tilde{U}'_0(\mathbf{K}_\rho)],$$
$$A^{(-)}(\mathbf{k}^+,\omega)=-\frac{1}{4\pi}[i\gamma\tilde{U}_0(\mathbf{K}_\rho)-\tilde{U}'_0(\mathbf{K}_\rho)].$$

The required plane wave expansion is then given by

$$U(\mathbf{r},\omega)=-\frac{i}{8\pi^2}\int\frac{d^2K_\rho}{\gamma}[i\gamma\tilde{U}_0(\mathbf{K}_\rho)+\tilde{U}'_0(\mathbf{K}_\rho)]e^{i\mathbf{k}^+\cdot\mathbf{r}}$$
$$-\frac{i}{8\pi^2}\int\frac{d^2K_\rho}{\gamma}[i\gamma\tilde{U}_0(\mathbf{K}_\rho)-\tilde{U}'_0(\mathbf{K}_\rho)]e^{i\mathbf{k}^-\cdot\mathbf{r}}.$$

In order for the field to be finite over all of space requires that the evanescent plane waves in the above expansion vanish and this requires that

$$i\gamma\tilde{U}_0(\mathbf{K}_\rho) \pm \tilde{U}_0'(\mathbf{K}_\rho) = 0, \quad \forall|\mathbf{K}_\rho| > k$$

which will occur if and only if both $\tilde{U}_0(\mathbf{K}_\rho)$ and $\tilde{U}_0'(\mathbf{K}_\rho)$ vanish over the evanescent region.

3.6 Compute the plane wave expansion found in Problem 3.5 for a single plane wave propagating along the positive z axis; i.e., $U(\mathbf{r},\omega) = \exp(ikz)$.

For this case we have that

$$\tilde{U}_0(\mathbf{K}_\rho,\omega) = (2\pi)^2\delta(\mathbf{K}_\rho), \quad \tilde{U}_0'(\mathbf{K}_\rho,\omega) = (2\pi)^2 ik\delta(\mathbf{K}_\rho)$$

from which we find that

$$A^{(+)}(\mathbf{k}^+,\omega) = -\frac{1}{4\pi}[i\gamma(2\pi)^2\delta(\mathbf{K}_\rho) + (2\pi)^2 ik\delta(\mathbf{K}_\rho)] = -\pi i(\gamma+k)\delta(\mathbf{K}_\rho)$$
$$A^{(-)}(\mathbf{k}^+,\omega) = -\frac{1}{4\pi}[i\gamma(2\pi)^2\delta(\mathbf{K}_\rho) - (2\pi)^2 ik\delta(\mathbf{K}_\rho)] = -\pi i(\gamma-k)\delta(\mathbf{K}_\rho).$$

The required plane wave expansion is then given by

$$U(\mathbf{r},\omega) = \frac{i}{2\pi}\int \frac{d^2K_\rho}{\gamma}[-\pi i(\gamma+k)\delta(\mathbf{K}_\rho)]e^{i\mathbf{k}^+\cdot\mathbf{r}}$$
$$+\frac{i}{2\pi}\int \frac{d^2K_\rho}{\gamma}[-\pi i(\gamma-k)\delta(\mathbf{K}_\rho)]e^{i\mathbf{k}^-\cdot\mathbf{r}},$$

which, of course, reduces to $\exp(ikz)$ as it must.

3.7 Compute the plane wave amplitudes and plane wave expansion for a monochromatic wave field that propagates into the r.h.s. $z > 0$ and whose Dirichlet conditions over the plane $z = 0$ is the Rect function

$$\mathrm{Rect}(x) = \begin{cases} 1 & -X_0 \le x \le +X_0 \\ 0 & \text{else.} \end{cases}$$

Again we make use of the general plane wave expansion given in Eq.(3.16a) which, for a wavefield propagating into the r.h.s. $z > 0$, must have

$$A^{(-)}(\mathbf{k}^+,\omega) = -\frac{1}{4\pi}[i\gamma\tilde{U}_0(\mathbf{K}_\rho) - \tilde{U}_0'(\mathbf{K}_\rho)] = 0 \to \tilde{U}_0'(\mathbf{K}_\rho) = i\gamma\tilde{U}_0(\mathbf{K}_\rho),$$

so that

$$A^{(+)}(\mathbf{k}^+,\omega) = -\frac{1}{4\pi}[i\gamma\tilde{U}_0(\mathbf{K}_\rho) + \tilde{U}_0'(\mathbf{K}_\rho)] = -\frac{i}{2\pi}\gamma\tilde{U}_0(\mathbf{K}_\rho).$$

The transform $\tilde{U}_0(\mathbf{K}_\rho)$ is readily computed and we find that

$$A^{(+)}(\mathbf{k}^+,\omega) = -2i\gamma\frac{\sin K_x X_0}{K_x}\delta(K_y)$$

where $\mathbf{K}_\rho = (K_x, K_y)$. The plane wave expansion is then found to be

$$U(\mathbf{r}, \omega) = \frac{i}{2\pi} \int \frac{d^2 K_\rho}{\gamma} [-2i\gamma \frac{\sin K_x X_0}{K_x} \delta(K_y)] e^{i\mathbf{k}^+ \cdot \mathbf{r}}$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dK_x \frac{\sin K_x X_0}{K_x} e^{i(K_x x + \sqrt{k^2 - K_x^2} z)}.$$

3.8 Use the method of stationary phase to derive Eq.(3.17) from the plane wave expansions in Eqs.(3.16a).

The method of stationary phase is treated in some depth in Born and Wolf and its specific use in evaluating the asymptotic expansion of angular spectrum integrals is covered in depth in Mandel and Wolf both given in the bibliography of the book. Here we will use the results for the general case of 2D integrals treated in Appendix III of Born and Wolf and refer the reader to Mandel and Wolf for a more complete treatment.

For simplicity we will only consider $U^{(+)}$ which is the first term in Eq.(3.16a) and which we can write the in the form

$$U^{(+)}(\mathbf{r}, \omega) = \frac{ik}{2\pi} \int dp dq \frac{A(kp, kq, \omega)}{m} e^{ik(px+qy+mz)},$$

where $K_x = kp$, $K_y = kq$, $\mathbf{k}^+ = kp\hat{\mathbf{x}} + kq\hat{\mathbf{y}} + km\hat{\mathbf{z}}$ and $m = \sqrt{1 - p^2 - q^2}$. It is shown in Appendix III of Born and Wolf that the asymptotic approximation of an integral of the form given above is given by

$$U^{(+)}(x, y, z, \omega) \sim -\frac{\sigma}{\sqrt{|\alpha\beta - \gamma^2|}} \frac{A(kp_0, kq_0, \omega)}{m_0} e^{ik(p_0 x + q_0 y + m_0 z)} \tag{3.1}$$

where p_0, q_0 are the so-called critical or stationary points satisfying

$$\frac{\partial}{\partial p}(px + qy + mz) = 0, \quad \frac{\partial}{\partial q}(px + qy + mz) = 0$$

with $m = \sqrt{1 - p^2 - q^2}$. The various quantities appearing in Eq.(3.1) are defined as

$$\alpha = \frac{\partial^2}{\partial p^2}(px+qy+mz), \quad \beta = \frac{\partial^2}{\partial q^2}(px+qy+mz), \quad \gamma = \frac{\partial^2}{\partial p \partial q}(px+qy+mz)$$

and

$$\sigma = \begin{cases} +1 & \alpha\beta > \gamma^2,\, \alpha > 0 \\ -1 & \alpha\beta > \gamma^2,\, \alpha < 0 \\ -i & \alpha\beta < \gamma^2. \end{cases}$$

We find that

$$\frac{\partial}{\partial p}(px + qy + mz) = x - \frac{p}{m} z, \quad \frac{\partial}{\partial q}(px + qy + mz) = y - \frac{q}{m} z,$$

yielding the following equations for the stationary points

$$x - \frac{p_0}{m_0} z = 0, \quad y - \frac{q_0}{m_0} z = 0. \tag{3.2}$$

If we now couple the above set of equations with $m_0 = \sqrt{1-p_0^2-q_0^2}$ and $r = \sqrt{x^2+y^2+z^2}$ one finds that the solution to Eqs.(3.2) are given by $p_0 = x/r$ and $q_0 = y/r$. We also find that

$$\alpha = -z\frac{p_0^2+m_0^2}{m_o^3}, \quad \beta = -z\frac{q_0^2+m_0^2}{m_o^3}, \quad \gamma = p_0q_0\frac{z}{m_0^3}.$$

This then yields

$$\alpha\beta-\gamma^2 = z^2\left[[\frac{(x/r)^2+(z/r)^2}{(z/r)^3}][\frac{(y/r)^2+(z/r)^2}{(z/r)^3}]\right]-(x/r)^2(y/r)^2\frac{z^2}{(z/r)^6} = \frac{r^4}{z^2},$$

and $\alpha\beta > \gamma$ and $\alpha < 0$ since $z > 0$ so that $\sigma < 0$. Putting the above together yields

$$U^{(+)}(x,y,z,\omega) \sim \overbrace{\frac{z}{r^2}}^{-\frac{\sigma}{\sqrt{|\alpha\beta-\gamma^2|}}} \overbrace{\frac{r}{z}A(k\frac{x}{r},k\frac{y}{r},\omega)}^{A(kp_0,kq_0,\omega)/m_0} \overbrace{e^{ikr}}^{e^{ik(p_0x+q_0y+m_0z)}} = A^{(+)}(\mathbf{k}^+,\omega)\frac{e^{ikr}}{r}.$$

3.9 Use the multipole expansion of the plane wave given in Eq.(3.35) of Example 3.4 in the plane wave expansion Eq.(3.9a) to obtain a multipole expansion of the field represented by this plane wave expansion. Determine the multipole moments in terms of the plane wave amplitude $A(k\mathbf{s},\omega)$.

On substituting the multipole expansion of the plane wave from Eq.(3.35) into the plane wave expansion Eq.(3.9a) we obtain

$$U(\mathbf{r},\omega) = \int d\Omega_{\mathbf{s}}\, A(k\mathbf{s},\omega)\, \overbrace{4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(kr)Y_l^m(\hat{\mathbf{r}})Y^{*m}_{\ l}(\mathbf{s})}^{e^{ik\mathbf{s}\cdot\mathbf{r}}}$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\{4\pi i^l\int d\Omega_{\mathbf{s}}\, A(k\mathbf{s},\omega)Y^{*m}_{\ l}(\mathbf{s})\}j_l(kr)Y_l^m(\hat{\mathbf{r}})$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^m j_l(kr)Y_l^m(\hat{\mathbf{r}})$$

where

$$a_l^m = 4\pi i^l\int d\Omega_{\mathbf{s}}\, A(k\mathbf{s},\omega)Y^{*m}_{\ l}(\mathbf{s}).$$

3.10 Use the plane wave expansion of the multipole field $j_l(kr)Y_l^m(\hat{\mathbf{r}})$ given in Eq.(3.36) of Example 3.4 in the multipole expansion of the solution to the interior boundary value problem for Dirichlet conditions over a sphere given in Example 3.7 to obtain a plane wave expansion of the solution to this problem.

We showed in Example 3.7 that the solution to the interior Dirichlet problem for a sphere is given by

$$U(\mathbf{r},\omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{u_l^m}{j_l(ka)}j_l(kr)Y_l^m(\hat{\mathbf{r}}),$$

where

$$u_l^m = \int d\Omega' \, Y_l^{m*}(\hat{\mathbf{r}}') U(\mathbf{r}', \omega)|_{r'=a}.$$

On substituting the plane wave expansion from Eq.(3.36) we then obtain

$$U(\mathbf{r}, \omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{u_l^m}{j_l(ka)} \overbrace{\frac{(-i)^l}{4\pi} \int d\Omega_s \, Y_l^m(\mathbf{s}) e^{ik\mathbf{s}\cdot\mathbf{r}}}^{j_l(kr)Y_l^m(\hat{\mathbf{r}})}$$

$$= \int d\Omega_s \, \{\frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l \frac{u_l^m}{j_l(ka)} Y_l^m(\mathbf{s})\} e^{ik\mathbf{s}\cdot\mathbf{r}}$$

$$= \int d\Omega_s \, A(\mathbf{s}) e^{ik\mathbf{s}\cdot\mathbf{r}}$$

which is a homogeneous plane wave expansion with plane wave amplitude

$$A(\mathbf{s}) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l \frac{u_l^m}{j_l(ka)} Y_l^m(\mathbf{s}).$$

3.11 Use the multipole expansion of the Dirichlet Green function in Section 3.4 and the solution of the exterior boundary value problem in Section 2.8.2 to show that the radiation pattern of a field radiated by a source confined to a sphere of radius a_0 centered at the origin admits the expansion

$$f(\mathbf{s}) = \sum_{l,m} f_l^m(\omega) Y_l^m(\mathbf{s}), \tag{3.3a}$$

where the expansion coefficients are given in terms of Dirichlet conditions over the sphere by

$$f_l^m(\omega) = \frac{(-i)^{(l+1)}}{k h_l^+(ka_0)} \int_{4\pi} d\Omega_r U(a_0 \hat{\mathbf{r}}) Y_l^{m*}(\hat{\mathbf{r}}), \tag{3.3b}$$

where Ω_r is the solid angle on the unit sphere.

The solution to the exterior Dirichlet problem for a sphere can be expressed using the results of Section 2.8.2 in the form

$$U(\mathbf{r}, \omega) = \int_{\partial\tau} dS' \, U(\mathbf{r}', \omega) \frac{\partial}{\partial n'} G_D(\mathbf{r}, \mathbf{r}', \omega),$$

where G_D is the Dirichlet Green function that vanishes over the surface $\partial\tau$ of the sphere and obeys the SRC at infinity and where the normal derivative is directed *inward* into the interior of the sphere. On making use of this Green function from Eq.(3.46a) we find that

$$\frac{\partial}{\partial n'} G_D(\mathbf{r}, \mathbf{r}', \omega) = -\frac{\partial}{\partial r'} G_D(\mathbf{r}, \mathbf{r}', \omega) = ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \{k j_l'(kr') h_l^+(kr)$$

$$- \frac{j_l(ka_0)}{h_l^+(ka_0)} k h_l^+(kr) h_l^{+\prime}(kr')\} Y_l^m(\hat{\mathbf{r}}) Y_l^{m*}(\hat{\mathbf{r}}'),$$

where j_l' and $h_l^{+'}$ denote the derivatives of these quantities with respect to their arguments. On setting $r' = a_0$ the above expression simplifies to

$$\frac{\partial}{\partial n'}G_D(\mathbf{r},\mathbf{r}',\omega)|_{r'=a_0} = ik^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\{j_l'(ka_0) - \frac{j_l(ka_0)}{h_l^+(ka_0)}h_l^{+'}(ka_0)\}h_l^+(kr)Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}'})$$

$$= ik^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{j_l'(ka_0)h_l^+(ka_0) - j_l(ka_0)h_l^{+'}(ka_0)}{h_l^+(ka_0)}h_l^+(kr)Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}'})$$

$$= \frac{1}{a_0^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{h_l^+(kr)}{h_l^+(ka_0)}Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}'})$$

where we have used the Wronskian

$$j_l'(ka_0)h_l^+(ka_0) - j_l(ka_0)h_l^{+'}(ka_0) = -\frac{i}{(ka_0)^2}.$$

On making use of the above expression for the normal derivative of the exterior Dirichlet Green function we then find that

$$U(\mathbf{r},\omega) = \int_{4\pi} d\Omega'\, U(a_0\hat{\mathbf{r}'},\omega)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{h_l^+(kr)}{h_l^+(ka_0)}Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}'})$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\{\frac{1}{h_l^+(ka_0)}\int_{4\pi} d\Omega'\, U(a_0\hat{\mathbf{r}'},\omega)Y_l^{m*}(\hat{\mathbf{r}'})\}h_l^+(kr)Y_l^m(\hat{\mathbf{r}})$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^m h_l^+(kr)Y_l^m(\hat{\mathbf{r}})$$

where

$$a_l^m = \frac{1}{h_l^+(ka_0)}\int_{4\pi} d\Omega'\, U(a_0\hat{\mathbf{r}'},\omega)Y_l^{m*}(\hat{\mathbf{r}'})$$

are the multipole moments of the radiated field.

The final step is to make use of the far field expression for the spherical Hankel functions

$$h_l^+(kr) \sim (-i)^{(l+1)}\frac{e^{ikr}}{kr}$$

to find that

$$U(\mathbf{r},\omega) \sim \{\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(-i)^{(l+1)}a_l^m}{k}Y_l^m(\hat{\mathbf{r}})\}\frac{e^{ikr}}{r}$$

which then yields the expression given in Eq.(3.3a) for the radiation pattern.

3.12 Compute the radiation pattern of a field radiated by a source confined to a sphere of radius a_0 centered at the origin from the solution of the exterior Dirichlet problem for a sphere presented in Example 3.5. Verify that the solution you obtained is identical to that given in the previous problem.

We have from Example 3.5 that the field radiated by a source confined to a sphere of radius a_0 centered at the origin can be expressed in the form

$$U_+(\mathbf{r},\omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^m(\omega) h_l^+(kr) Y_l^m(\theta,\phi),$$

where the expansion coefficients (multipole moments) are given by

$$a_l^m(\omega) = \frac{1}{h_l^+(ka)} \int d\Omega\, Y_l^{m*}(\theta,\phi) U_+(r=a_0,\theta,\phi,\omega),$$

where $U_+(r=a_0,\theta,\phi,\omega)$ is the Dirichlet boundary condition on the sphere $r=a_0$. On making use of the far field expression for the spherical Hankel functions

$$h_l^+(kr) \sim (-i)^{(l+1)} \frac{e^{ikr}}{kr}$$

we then find that

$$U_+(\mathbf{r},\omega) \sim \{\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{(-i)^{(l+1)} a_l^m(\omega)}{k} Y_l^m(\theta,\phi)\} \frac{e^{ikr}}{r}$$

which yields that radiation pattern

$$f(\mathbf{s}) = \sum_{l,m} f_l^m(\omega) Y_l^m(\theta,\phi),$$

where the expansion coefficients are given in terms of Dirichlet conditions over the sphere by

$$f_l^m(\omega) = \frac{(-i)^{(l+1)} a_l^m(\omega)}{k} = \frac{(-i)^{(l+1)}}{k h_l^+(ka_0)} \int_{4\pi} d\Omega_r U(a_0\hat{\mathbf{r}}) Y_l^{m*}(\hat{\mathbf{r}}),$$

and which are identical to the expressions given in the previous problem.