Appendix SA2.1 The Relationship between Approaches I and II

To examine the connection between the two alternative approaches to the numerical example in section 2.3, we consider a general two-sector economy with $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and let f_1 and f_2 represent values of the new final demands.[1]

A2.1.1 Approach I

Using the Leontief-inverse, we find $(\mathbf{I}-\mathbf{A}) = \begin{bmatrix} (1-a_{11}) & -a_{12} \\ -a_{21} & (1-a_{22}) \end{bmatrix}$ and, provided that $|\mathbf{I}-\mathbf{A}| \neq 0$, which means that (Appendix A)

$$(\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{|\mathbf{I}-\mathbf{A}|}\left[\operatorname{adj}(\mathbf{I}-\mathbf{A})\right] = \begin{bmatrix} \dfrac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|} & \dfrac{a_{12}}{|\mathbf{I}-\mathbf{A}|} \\ \dfrac{a_{21}}{|\mathbf{I}-\mathbf{A}|} & \dfrac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|} \end{bmatrix} \qquad \text{(A2.1.1)}$$

The associated gross outputs are found from $\mathbf{x} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$, namely

$$\begin{aligned} x_1 &= \left[\frac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|}\right] f_1 + \left[\frac{a_{12}}{|\mathbf{I}-\mathbf{A}|}\right] f_2 \\ x_2 &= \left[\frac{a_{21}}{|\mathbf{I}-\mathbf{A}|}\right] f_1 + \left[\frac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|}\right] f_2 \end{aligned} \qquad \text{(A2.1.2)}$$

A2.1.2 Approach II

The round-by-round calculation of total impacts requires only the elements of the **A** matrix. The first-round impact on sector 1 − in terms of what it must produce to satisfy its own and sector 2's needs for inputs – is $\underbrace{a_{11}f_1 + a_{12}f_2}_{\text{Sector 1, Round 1}}$. For sector 2, the first-round impact is $\underbrace{a_{21}f_1 + a_{22}f_2}_{\text{Sector 2, Round 1}}$. (These were \$465 and \$195 in the numerical example.)

The second-round impacts result from production that is required to take care of first-round needs. These are easily seen to be

[1] As elsewhere in Chapter 2, we ignore the "0" and "1" superscripts for notational simplicity when the intended meaning is clear from the context.

$$\text{For sector 1:} a_{11} \underbrace{\left(a_{11}f_1 + a_{12}f_2\right)}_{\text{Sector 1, Round 1}} + a_{12} \underbrace{\left(a_{21}f_1 + a_{22}f_2\right)}_{\text{Sector 2, Round 1}}$$

$$\text{For sector 2:} a_{21} \underbrace{\left(a_{11}f_1 + a_{12}f_2\right)}_{\text{Sector 1, Round 1}} + a_{22} \underbrace{\left(a_{21}f_1 + a_{22}f_2\right)}_{\text{Sector 2, Round 1}}$$

(These were \$118.50 and \$102.75 in the numerical example.)

The nature of the expansion is now clear. For sector 1 in round 3, we will have

$$a_{11} \underbrace{\left[a_{11}\left(a_{11}f_1 + a_{12}f_2\right) + a_{12}\left(a_{21}f_1 + a_{22}f_2\right)\right]}_{\text{Sector 1, Round 2}} + a_{12} \underbrace{\left[a_{21}\left(a_{11}f_1 + a_{12}f_2\right) + a_{22}\left(a_{21}f_1 + a_{22}f_2\right)\right]}_{\text{Sector 2, Round 2}}$$

and for sector 2 in round 3:

$$a_{21} \underbrace{\left[a_{11}\left(a_{11}f_1 + a_{12}f_2\right) + a_{12}\left(a_{21}f_1 + a_{22}f_2\right)\right]}_{\text{Sector 1, Round 2}} + a_{22} \underbrace{\left[a_{21}\left(a_{11}f_1 + a_{12}f_2\right) + a_{22}\left(a_{21}f_1 + a_{22}f_2\right)\right]}_{\text{Sector 2, Round 2}}$$

(These were \$43.46 and \$28.84 in the numerical example.)

Without going further, we can develop an expression for an approximation to x_1 in terms of f_1 and f_2 and the technical coefficients on the basis of only three rounds of effects. Collecting the terms for round-by-round effects on sector 1, we have

$$\begin{aligned} x_1 \cong\ & f_1 + a_{11}f_1 + a_{11}^2 f_1 + a_{12}a_{21}f_1 + a_{11}^3 f_1 + a_{11}a_{12}a_{21}f_1 \\ & + a_{12}a_{21}a_{11}f_1 + a_{12}f_2 + a_{11}a_{12}f_2 + a_{12}a_{22}f_2 + a_{11}a_{11}a_{12}f_2 \\ & + a_{11}a_{12}a_{22}f_2 + a_{12}a_{21}a_{12}f_2 + a_{12}a_{22}a_{22}f_2 \end{aligned}$$

or

$$\begin{aligned} x_1 \cong\ & (1 + a_{11} + a_{11}^2 + a_{12}a_{21} + a_{11}^3 + a_{11}a_{12}a_{21} + a_{12}a_{21}a_{11})f_1 \\ & + (a_{12} + a_{11}a_{12} + a_{12}a_{22} + a_{11}a_{11}a_{12} + a_{11}a_{12}a_{22} \\ & + a_{12}a_{21}a_{12} + a_{12}a_{22}a_{22})f_2 \end{aligned} \qquad \text{(A2.1.3)}$$

A similar expression can be derived for x_2.

The object of this algebra is to make clear that in round 2, the effect is found in products of *pairs* of coefficients (e.g., a_{11}^2 and $a_{11}a_{12}$); in round 3, the effect comes from products of *triples* of coefficients (e.g., a_{11}^3 and $a_{11}a_{12}a_{21}$). Similarly, in round 4, sets of four coefficients will be multiplied together, . . . and in round *n*, sets of *n* coefficients will be multiplied. In monetary terms, all $a_{jj} < 1$ and $a_{ij} < 1$ since producer *j* must buy, from himself and each supplier *i*, less than one dollar's worth of inputs per dollar's worth of output. Therefore it is clear that eventually the effects in the "next" round will be essentially negligible. Mathematically, the expression for x_1 has the form

$$\begin{aligned} x_1 = (1 + \text{infinite series of terms involving products of pairs, triples,} \ldots, \text{ of } a_{ij}) f_1 \\ + (\text{similar infinite series}) f_2 \end{aligned} \quad \text{(A2.1.4)}$$

There would be a parallel expression for x_2. If we denote these two parenthetical series terms for x_1 by s_{11} and s_{12}, and in the similar expression for x_2 by s_{21} and s_{22}, we have gross outputs related to final demands by

$$\begin{aligned} x_1 &= s_{11} f_1 + s_{12} f_2 \\ x_2 &= s_{21} f_1 + s_{22} f_2 \end{aligned} \quad \text{(A2.1.5)}$$

The evaluation of the *s* terms as four different infinite series would be a difficult and tedious task.

Alternatively, we could think of the new total output x_1 as composed of two parts: (a) the new final demands for sector 1's output, f_1, and (b) all direct and indirect effects on sector 1 generated by f_1 and f_2. (This approach was suggested in Dorfman, Samuelson and Solow, 1958, section 9.3.) To this end, define $F_1 = a_{11} f_1 + a_{12} f_2$, the first-round response from sector 1, and, similarly, let $F_2 = a_{21} f_1 + a_{22} f_2$ for sector 2. These first-round outputs will similarly generate second-round outputs, and so on, exactly as did f_1 and f_2 above. The suggestion is that the final outputs can be looked at as (1) a series of round-by-round effects on f_1 and f_2 *or* as (2) f_1 and f_2, plus a series of round-by-round effects on F_1 and F_2. In this alternative view, a complete derivation similar to that preceding (A2.1.5) would lead to

$$\begin{aligned} x_1 &= f_1 + s_{11} F_1 + s_{12} F_2 \\ x_2 &= f_2 + s_{21} F_1 + s_{22} F_2 \end{aligned} \quad \text{(A2.1.6)}$$

Substituting $F_1 = a_{11} f_1 + a_{12} f_2$ and $F_2 = a_{21} f_1 + a_{22} f_2$ and collecting terms,

$$\begin{aligned} x_1 &= \left(1 + s_{11} a_{11} + s_{12} a_{21}\right) f_1 + \left(s_{11} a_{12} + s_{12} a_{22}\right) f_2 \\ x_2 &= \left(s_{21} a_{11} + s_{22} a_{21}\right) f_1 + \left(1 + s_{21} a_{12} + s_{22} a_{22}\right) f_2 \end{aligned} \quad \text{(A2.1.7)}$$

Both (A2.1.5) and (A2.1.7) show x_1 and x_2 as linear functions of f_1 and f_2, so the coefficients in corresponding positions must be equal. That is,

$$\begin{aligned} s_{11} &= 1 + s_{11} a_{11} + s_{12} a_{21} \quad & s_{12} &= s_{11} a_{12} + s_{12} a_{22} \\ s_{12} &= s_{21} a_{11} + s_{22} a_{21} \quad & s_{22} &= 1 + s_{21} a_{12} + s_{22} a_{22} \end{aligned}$$

The top two are linear equations in the unknowns s_{11} and s_{12}, and the bottom two are linear equations in s_{21} and s_{22}. Rearranging to emphasize that the *s* are unknowns and the *a* are known coefficients,

$$\begin{aligned}(1-a_{11})s_{11}-a_{21}s_{12}&=1\\-a_{12}s_{11}+(1-a_{22})s_{12}&=0\\(1-a_{11})s_{21}-a_{21}s_{22}&=0\\-a_{12}s_{21}+(1-a_{22})s_{22}&=1\end{aligned}$$

or

$$\begin{bmatrix}(1-a_{11}) & -a_{21}\\ -a_{12} & (1-a_{22})\end{bmatrix}\begin{bmatrix}s_{11}\\ s_{12}\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$
$$\begin{bmatrix}(1-a_{11}) & -a_{21}\\ -a_{12} & (1-a_{22})\end{bmatrix}\begin{bmatrix}s_{21}\\ s_{22}\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix} \tag{A2.1.8}$$

Both sets of equations have the same coefficient matrix. Since

$$\begin{bmatrix}(1-a_{11}) & -a_{21}\\ -a_{12} & (1-a_{22})\end{bmatrix}^{-1}=\frac{1}{(1-a_{11})(1-a_{22})-a_{12}a_{21}}\begin{bmatrix}(1-a_{22}) & a_{21}\\ a_{12} & (1-a_{11})\end{bmatrix}$$

and since $(1-a_{11})(1-a_{22})-a_{12}a_{21}=|\mathbf{I}-\mathbf{A}|$ [in (A2.1.1) and (A2.1.2)], the solutions to the two pairs of linear equations in (A2.1.8) are

$$\begin{bmatrix}s_{11}\\ s_{12}\end{bmatrix}=\begin{bmatrix}\dfrac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|} & \dfrac{a_{21}}{|\mathbf{I}-\mathbf{A}|}\\ \dfrac{a_{12}}{|\mathbf{I}-\mathbf{A}|} & \dfrac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} \text{ and}$$

$$\begin{bmatrix}s_{21}\\ s_{22}\end{bmatrix}=\begin{bmatrix}\dfrac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|} & \dfrac{a_{21}}{|\mathbf{I}-\mathbf{A}|}\\ \dfrac{a_{12}}{|\mathbf{I}-\mathbf{A}|} & \dfrac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$$

That is,

$$s_{11} = \frac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|},\ s_{12} = \frac{a_{12}}{|\mathbf{I}-\mathbf{A}|},\ s_{21} = \frac{a_{21}}{|\mathbf{I}-\mathbf{A}|},\ s_{22} = \frac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|}$$

These algebraic expressions equate the four infinite series terms, whose complex form was suggested in (A2.1.3) and (A2.1.4), to very simple functions of the elements of **A**. Moreover, these four simple functions are precisely the four elements of the Leontief inverse, as found in (A2.1.1). In economic terms, the $(\mathbf{I}-\mathbf{A})^{-1}$ matrix captures in each of its elements all of the infinite series of round-by-round direct and indirect effects that the new final demands have on the outputs of the two sectors. (A demonstration along these lines is much more complex for a three-sector input–output model and unwieldy for more than three sectors.)

The elements of this Leontief inverse matrix are often termed *multipliers.* With $(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{L} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$ and forecasts for f_1 and f_2, the total effect on x_1 is given by $l_{11}f_1 + l_{12}f_2$, the sum of the multiplied effects of each of the individual final demands. And similarly for x_2. Input–output multipliers are explored in Chapter 6.

References

Dorfman, Robert, Paul A. Samuelson and Robert Solow. 1958. *Linear Programming and Economic Analysts.* New York: McGraw-Hill.