Chapter 5: Techniques of Integration

Part B: Partial Fractions, Improper Integrals, ODE

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Table of Contents

Integrating Rational Functions

We have seen several examples of integration of rational functions:

$$\int \frac{dx}{x-a} = \log|x-a|,$$
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right),$$
$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\log(x^2+a^2),$$

with the following holding when $n > 1$:

$$\int \frac{dx}{(x-a)^n} = \frac{-1}{n-1}\frac{1}{(x-a)^{n-1}},$$
$$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int \frac{dx}{(x^2+a^2)^{n-1}},$$
$$\int \frac{x\,dx}{(x^2+a^2)^n} = \frac{-1}{2(n-1)(x^2+a^2)^{n-1}}.$$

Integrating Rational Functions

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Our approach is based on the fact that every real polynomial can be expressed as a product of factors, each of which is either linear or quadratic with no real roots.

We can focus on rational functions $p(x)/q(x)$ with $\deg p < \deg q$. For we can always divide p by q to get $p(x) = s(x)q(x) + r(x)$ with r either being 0 or satisfying $\deg r < \deg q$. Then,

$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}.$$

Now $s(x)$ is a polynomial and easy to integrate, so we need only analyze $r(x)/q(x)$.

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A pair of polynomials q_1 and q_2 is called **coprime** if the only polynomials which are their common factors are constants.

Partial Fractions

Theorem 1

Suppose $\deg p < \deg q$ *and* $q = q_1 q_2$ *is a factoring into non-constant coprime polynomials. Then there are polynomials* r_1 *and* r_2 *such that* $\deg r_i < \deg q_i$ *for* $i = 1, 2$, *and*

$$\frac{p(x)}{q(x)} = \frac{r_1(x)}{q_1(x)} + \frac{r_2(x)}{q_2(x)}.$$

Partial Fractions

Proof. We need to show that $p = r_1 q_2 + r_2 q_1$.
Let $\mathcal{I}$ be the collection of all $p_1 q_2 + p_2 q_1$, where p_1, p_2 are polynomials.

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Let $\mathcal{I}$ be the collection of all $p_1 q_2 + p_2 q_1$, where p_1, p_2 are polynomials.
Let $\ell \in \mathcal{I}$ be a non-zero polynomial of least degree. Then $q_i = t_i \ell + s_i$ where $\deg s_i < \deg \ell$ or $s_i = 0$. Now,

$$q_i = t_i(p_1 q_2 + p_2 q_1) + s_i \implies s_i \in \mathcal{I} \implies s_i = 0 \implies \ell \text{ divides } q_i.$$

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Therefore $p = p_1 q_2 + p_2 q_1$. Now $p_1 = p_1' q_1 + r_1$ and $p_2 = p_2' q_2 + r_2$, with $r_i = 0$ or $\deg r_i < \deg q_i$. This gives

$$p = (p_1' q_1 + r_1) q_2 + (p_2' q_2 + r_2) q_1 = (p_1' + p_2') q + r_2 q_1 + r_1 q_2.$$

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$$p = (p_1' q_1 + r_1) q_2 + (p_2' q_2 + r_2) q_1 = (p_1' + p_2') q + r_2 q_1 + r_1 q_2.$$

On matching degrees, we see we must have $p_1' + p_2' = 0$ and hence $p = r_2 q_1 + r_1 q_2$. □

Example

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We have noted that every polynomial is a product of linear and quadratic factors, with the quadratic factors having no real roots. For example, the factoring may look as follows:

$$q(x) = (x+1)^2(x^2+1)^2.$$

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$$q(x) = (x+1)^2(x^2+1)^2.$$

Further, let $p(x) = x^5 + 7x^4 + 5x^3 + 12x^2 + 4x + 7$. By application of the previous Theorem, we see that

$$\frac{x^5 + 7x^4 + 5x^3 + 12x^2 + 4x + 7}{(x+1)^2(x^2+1)^2} = \frac{Ax+B}{(x+1)^2} + \frac{Cx^3 + Dx^2 + Ex + F}{(x^2+1)^2}.$$

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From the preceding theorem and example, we see that we need to work on two fronts:

1. Methods to find the unknown constants on the right hand side.
2. Methods to integrate rational functions of the form $p(x)/q(x)^n$ where $q(x)$ is either linear or quadratic.

Case of $p(x)/q(x)^n$

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Theorem 2

Consider a rational function $p(x)/q(x)^n$ with $\deg p < n(\deg q)$. It can be expressed as

$$\frac{p(x)}{q(x)^n} = \frac{r_1(x)}{q(x)^n} + \frac{r_2(x)}{q(x)^{n-1}} + \cdots + \frac{r_n(x)}{q(x)},$$

with each r_i satisfying either $r_i = 0$ or $\deg r_i < \deg q$.

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with each r_i satisfying either $r_i = 0$ or $\deg r_i < \deg q$.

Proof. Divide by q repeatedly. This gives

$$\begin{aligned} p(x) &= p_1(x)q(x) + r_1(x) = (p_2(x)q(x) + r_2(x))q(x) + r_1(x) \\ &= ((p_3(x)q(x) + r_3(x))q(x) + r_2(x))q(x) + r_1(x) \\ &= \cdots = \sum_{i=1}^{n} r_i(x)q(x)^{i-1}, \end{aligned}$$

with each $r_i(x)$ satisfying either $r_i = 0$ or $\deg r_i < \deg q$. Now divide by $q(x)^n$ to get the result. □

Case of $p(x)/q(x)^n$

Corollary 3

Consider $p(x)/q(x)^n$ *with* $\deg p < n(\deg q)$.

1. *If* $q(x) = x - a$*, the function can be expressed as*

$$\frac{p(x)}{q(x)^n} = \frac{A_1}{(x-a)^n} + \frac{A_2}{(x-a)^{n-1}} + \cdots + \frac{A_n}{(x-a)}$$

with $A_i \in \mathbb{R}$.

2. *If* $q(x) = x^2 + \alpha x + \beta$*, the function can be expressed as*

$$\frac{p(x)}{q(x)^n} = \frac{B_1 x + C_1}{(x^2+\alpha x+\beta)^n} + \frac{B_2 x + C_2}{(x^2+\alpha x+\beta)^{n-1}} + \cdots + \frac{B_n x + C_n}{(x^2+\alpha x+\beta)}$$

with $B_i, C_i \in \mathbb{R}$.

Anti-derivatives

On combining Theorem 1 with Corollary 3 we see that any rational function $p(x)/q(x)$ with $\deg p < \deg q$ can be expressed as a sum of terms of the form $A/(x-a)^k$ or $(Bx+C)/(x^2+\alpha x+\beta)^k$, which we shall call its **partial fractions decomposition**.

Example 4

$$\frac{x^5+7x^4+5x^3+12x^2+4x+7}{(x+1)^2(x^2+1)^2} = \sum_{i=1}^{2} \frac{A_i}{(x+1)^i} + \sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}.$$

Example

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$$x^2+2x+2 = A + B(x-1) + C(x-1)^2.$$

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Put $x=1$ to get $A=5$. If we substitute this in the last expression and also move the A term to the left hand side, we see that both sides must be divisible by $x-1$. Dividing by $x-1$ gives

$$x+3 = B + C(x-1).$$

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Again, $x=1$ gives $B=4$, and then $C=1$. So the partial fractions decomposition is

$$\frac{x^2+2x+2}{(x-1)^3} = \frac{5}{(x-1)^3} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)}.$$

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This is easy to integrate:

$$\int \frac{x^2+2x+2}{(x-1)^3}\,dx = -\frac{5/2}{(x-1)^2} - \frac{4}{(x-1)} + 2\log|x-1| + C.$$

Example

Consider $\dfrac{x^3+9x^2+8}{(x-1)^2(x+2)^2} = \dfrac{A}{(x-1)^2} + \dfrac{B}{(x-1)} + \dfrac{C}{(x+2)^2} + \dfrac{D}{(x+2)}$.

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Multiplying by $(x-1)^2$ and then evaluating at $x = 1$ gives $A = 2$.
Substitute this and simplify to get

$$\frac{x^2+8x}{(x-1)(x+2)^2} = \frac{B}{(x-1)} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)}.$$

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$$\frac{4}{(x+2)^2} = \frac{C}{(x+2)^2} + \frac{D}{(x+2)}.$$

This immediately gives $D=0$ and $C=4$. Therefore,

$$\frac{x^3+9x^2+8}{(x-1)^2(x+2)^2} = \frac{2}{(x-1)^2} + \frac{1}{(x-1)} + \frac{4}{(x+2)^2}.$$

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Hence,

$$\int \frac{x^3+9x^2+8}{(x-1)^2(x+2)^2}\,dx = \frac{-2}{x-1} + \log|x-1| - \frac{4}{x+2} + C.$$

Example

Consider $\frac{x^5 + 4x^3 - x^2 + 3x}{(x^2+1)^3}$. Apply the proof of Theorem 2:

$$\begin{aligned} x^5 + 4x^3 - x^2 + 3x &= (x^3 + 3x - 1)(x^2+1) + 1 \\ &= (x(x^2+1) + 2x - 1)(x^2+1) + 1 \\ &= x(x^2+1)^2 + (2x-1)(x^2+1) + 1. \end{aligned}$$

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Hence,

$$\frac{x^5 + 4x^3 - x^2 + 3x}{(x^2+1)^3} = \frac{x}{x^2+1} + \frac{2x-1}{(x^2+1)^2} + \frac{1}{(x^2+1)^3}.$$

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Hence,

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We carry out the integration using the results stated at the start of this section. The result is

$$\begin{aligned} \int \frac{x^5+4x^3-x^2+3x}{(x^2+1)^3}\,dx = \frac{1}{2}\log(x^2+1) - \frac{\arctan x}{8} - \frac{2+x}{2(x^2+1)} \\ + \frac{3x^3+5x}{8(x^2+1)^2} + C. \end{aligned}$$

Example

Consider $\dfrac{1}{(x^2+1)^2(x^2+x+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2} + \dfrac{Ex+F}{x^2+x+1}$.

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Multiply through by $(x^2+1)^2(x^2+x+1)$:

$$1 = (Ax+B)(x^2+1)(x^2+x+1) + (Cx+D)(x^2+x+1) + (Ex+F)(x^2+1)^2.$$

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Each side must have remainder 1 if we divide by x^2+1. This gives $1 = Dx - C$ and hence $C = -1$, $D = 0$. The decomposition becomes

$$\frac{x+1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Ex+F}{x^2+x+1}.$$

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This gives $x+1 = (Ax+B)(x^2+x+1) + (Ex+F)(x^2+1)$.
Comparing remainders on dividing by x^2+1 gives $x+1 = Bx - A$, so $A = -1$ and $B = 1$. The decomposition now reduces to

$$\frac{x}{x^2+x+1} = \frac{Ex+F}{x^2+x+1}.$$

Hence $E = 1$ and $F = 0$, and the partial fraction decomposition is completely worked out.

Example

Consider the partial fractions decomposition

$$\frac{x^5 + 7x^4 + 5x^3 + 12x^2 + 4x + 7}{(x+1)^2(x^2+1)^2} = \sum_{i=1}^{2} \frac{A_i}{(x+1)^i} + \sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}.$$

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We deal with the linear factors first. Multiply both sides by $(x+1)^2$:

$$\frac{x^5+7x^4+5x^3+12x^2+4x+7}{(x^2+1)^2} = A_1(x+1)+A_2+\sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}(x+1)^2.$$

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Put $x = -1$ to get $A_2 = 4$. Substitute this to get

$$\frac{x^4 + 2x^3 + 3x^2 + x + 3}{(x^2+1)^2} = A_1 + \sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}(x+1).$$

Example

Consider the partial fractions decomposition

$$\frac{x^5+7x^4+5x^3+12x^2+4x+7}{(x+1)^2(x^2+1)^2} = \sum_{i=1}^{2} \frac{A_i}{(x+1)^i} + \sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}.$$

We deal with the linear factors first. Multiply both sides by $(x+1)^2$:

$$\frac{x^5+7x^4+5x^3+12x^2+4x+7}{(x^2+1)^2} = A_1(x+1)+A_2+\sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}(x+1)^2.$$

Put $x=-1$ to get $A_2=4$. Substitute this to get

$$\frac{x^4+2x^3+3x^2+x+3}{(x^2+1)^2} = A_1 + \sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}(x+1).$$

Again put $x=-1$ and get $A_1=1$. Substitute and simplify:

$$\frac{2x^2-x+2}{(x^2+1)^2} = \sum_{i=1}^{2} \frac{B_i x + C_i}{(x^2+1)^i}.$$

(continued)

Example - continued

Multiply both sides by $(x^2+1)^2$:

$$2x^2 - x + 2 = (B_1x + C_1)(x^2+1) + (B_2x + C_2).$$

Example - continued

Multiply both sides by $(x^2+1)^2$:

$$2x^2 - x + 2 = (B_1x + C_1)(x^2+1) + (B_2x + C_2).$$

At this stage, we only have a few terms to deal with, and we can read off the coefficients easily: $B_1 = 0$, $C_1 = 2$, $B_2 = -1$, $C_2 = 0$.

Example - continued

Multiply both sides by $(x^2+1)^2$:

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At this stage, we only have a few terms to deal with, and we can read off the coefficients easily: $B_1 = 0$, $C_1 = 2$, $B_2 = -1$, $C_2 = 0$.

We have finally reached our goal:

$$\frac{x^5 + 7x^4 + 5x^3 + 12x^2 + 4x + 7}{(x+1)^2(x^2+1)^2} = \frac{1}{x+1} + \frac{4}{(x+1)^2} + \frac{2}{x^2+1} - \frac{x}{(x^2+1)^2}.$$

The integration is left to you!

Table of Contents

Improper Integrals

Our definition of definite integrals requires a bounded function f over a closed and bounded domain $[a, b]$.

Applications of integration often involve situations where these requirements are not met, and either the function or the domain is unbounded. Such integrals are called **improper**. We shall evaluate them by considering them as limits of 'proper' ones.

Improper Integrals

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Applications of integration often involve situations where these requirements are not met, and either the function or the domain is unbounded. Such integrals are called **improper**. We shall evaluate them by considering them as limits of 'proper' ones.

On the other hand, the requirement of taking a closed interval is not important. Suppose f is bounded on $[a, b)$. One can define $f(b) = 0$ and consider $\int_a^b f(x)\,dx$. You can easily check that the result is independent of the number assigned to $f(b)$. Further, the result equals $\lim_{t \to b-} \int_a^t f(x)\,dx$.

Improper integrals of the first kind

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If the integrand f is bounded but the interval of integration is not, we have an **improper integral of the first kind**. These integrals are defined via limits as follows:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx,$$

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If the defining limit exists and is finite, we say the improper integral **converges**. Else, we say it **diverges**.

Anti-derivatives

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In the definition of $\int_{-\infty}^{\infty} f(x)\,dx$ we can use any convenient a, and then **both** $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ need to converge for $\int_{-\infty}^{\infty} f(x)\,dx$ to be defined.

Example 5

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b = \lim_{b\to\infty}(1-e^{-b}) = 1.$$

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If $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}\frac{1}{x^{p-1}}\Big|_1^b$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right) = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p < 1. \end{cases}$$

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Overall,

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p \leq 1. \end{cases}$$

Thus the integral converges when $p > 1$ and diverges when $p \leq 1$.

Comparison Theorem

Theorem 6

Suppose $f, g \colon [a, \infty) \to \mathbb{R}$ *are continuous functions and* $0 \leq f(x) \leq g(x)$ *for every* $x \in [a, \infty)$. *If* $\int_a^\infty g(x)\,dx$ *converges, then* $\int_a^\infty f(x)\,dx$ *converges and* $\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$.

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Proof. Let $F(t) = \int_a^t f(x)\,dx$ and $G(t) = \int_a^t g(x)\,dx$.

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Suppose $f, g\colon [a,\infty) \to \mathbb{R}$ are continuous functions and $0 \le f(x) \le g(x)$ for every $x \in [a,\infty)$. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges and $\int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$.

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Since $f, g \ge 0$, the functions F, G are increasing.

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By the Monotone Convergence Theorem,

$$\int_a^\infty g(x)\,dx = \lim_{b\to\infty} G(b) \ge G(t) \ge F(t) \quad \text{for every } t \ge a.$$

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Hence F is bounded, and by the Monotone Convergence Theorem again, we have the convergence of $\lim_{b\to\infty} F(b) = \displaystyle\int_a^\infty f(x)\,dx$. □

Gaussian Integral

Consider the improper integral $\int_0^\infty e^{-x^2}\,dx$. We have

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

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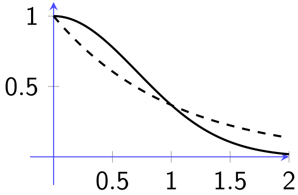

For $x \geq 1$ we have $x^2 \geq x$ and hence $0 \leq e^{-x^2} \leq e^{-x}$. Since $\int_1^\infty e^{-x}\,dx$ converges, so does $\int_1^\infty e^{-x^2}\,dx$. Therefore $\int_0^\infty e^{-x^2}\,dx$ converges. Similarly, $\int_{-\infty}^0 e^{-x^2}\,dx$ converges.

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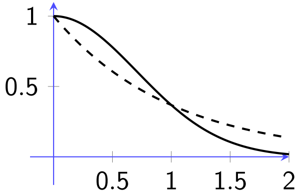

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Hence $\int_{-\infty}^\infty e^{-x^2}\,dx = \int_{-\infty}^0 e^{-x^2}\,dx + \int_0^\infty e^{-x^2}\,dx$ converges.

Gaussian Integral – estimates

We can find numerical approximations for $\int_0^\infty e^{-x^2}\,dx$ as follows. First truncate the interval of integration to some $[0, b]$. Then use a step function to get an approximate value of $\int_0^b e^{-x^2}\,dx$.

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As an example let us first set $b = 3$. Next we set $n = 6$ and partition $[0, 3]$ into 6 equal subintervals. On the i^{th} subinterval we approximate $f(x) = e^{-x^2}$ by its value at the midpoint c_i.

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c_i	0.25	0.75	1.25	1.75	2.25	2.75
$f(c_i)$	0.9394	0.5698	0.2096	0.0468	0.0063	0.0005

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Now,

$$\int_0^\infty e^{-x^2}\,dx \approx \int_0^3 e^{-x^2}\,dx \approx \sum_{i=1}^{6} f(c_i) \times 0.5 = 0.886213\ldots.$$

The exact value of the integral is $\sqrt{\pi}/2 = 0.886226\ldots$. With these few calculations we already have accuracy to 4 decimal places!

Improper integrals of the second kind

Improper integrals of the second kind occur when f has a vertical asymptote, such as when we try to integrate $1/\sqrt{x}$ over $[0,1]$.

These are defined by taking limits at the points where f has a vertical asymptote.

Example 7

The function $1/\sqrt{x}$ is unbounded on $(0,1]$. On the other hand, it is continuous on $[a,1]$ for every $a \in (0,1)$. Therefore we can define its improper integral on $[0,1]$ by integrating on $[a,1]$ and then letting $a \to 0+$:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0+} 2\sqrt{x}\Big|_a^1 = 2\lim_{a\to 0+}(1-\sqrt{a}) = 2.$$

Task 1

Show that $\int_0^1 x^\alpha\,dx$ *converges for* $-1 < \alpha < 0$ *and diverges for* $\alpha \leq -1$.

Comparison Theorem

Theorem 8

Suppose $f, g\colon (a, b] \to \mathbb{R}$ *are continuous functions that are unbounded on* $(a, b]$ *but bounded on every* $[x, b]$ *with* $a < x < b$*, and* $0 \le f(x) \le g(x)$ *for every* $x \in (a, b]$*. If* $\int_a^b g(x)\,dx$ *converges, then* $\int_a^b f(x)\,dx$ *converges and* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$*.*

Proof. Exercise. □

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Suppose $f, g\colon (a,b] \to \mathbb{R}$ are continuous functions that are unbounded on $(a,b]$ but bounded on every $[x,b]$ with $a < x < b$, and $0 \le f(x) \le g(x)$ for every $x \in (a,b]$. If $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges and $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Proof. Exercise. □

Example 9

Consider the improper integral $\int_0^1 e^{-x}x^\alpha\,dx$ with $\alpha < 0$. It is improper because $\lim\limits_{x\to 0+} e^{-x}x^\alpha = \infty$. We compute as follows for $0 < x \le 1$:

$$-1 < \alpha < 0 \implies 0 < e^{-x}x^\alpha \le x^\alpha \text{ and } \int_0^1 x^\alpha\,dx \text{ converges.}$$

$$\alpha \le -1 \implies e^{-x}x^\alpha \ge e^{-1}x^\alpha > 0 \text{ and } \int_0^1 x^\alpha\,dx \text{ diverges.}$$

Hence, by Comparison Theorem, the integral converges for $-1 < \alpha < 0$.

Gamma Function

The **Gamma function** is an instance of an improper integral that involves both an unbounded interval as well as an unbounded function.

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt \qquad (x > 0).$$

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1. The integral from 0 to 1 is improper when $0 < x < 1$ and we have already established its convergence in the previous Example.

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1. The integral from 0 to 1 is improper when $0 < x < 1$ and we have already established its convergence in the previous Example.
2. For any fixed $x > 0$, $e^{-t/2} t^{x-1} \to 0$ as $t \to \infty$. Hence there is an a such that $t \geq a$ implies $e^{-t/2} t^{x-1} \leq 1$. Therefore, for $x \geq a$, $0 \leq e^{-t} t^{x-1} \leq e^{-t/2}$. Again, the Comparison Theorem gives the convergence of $\int_a^\infty e^{-t} t^{x-1}\, dt$ and hence of $\int_1^\infty e^{-t} t^{x-1}\, dt$.

Gamma Function – Properties

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We now apply integration by parts to obtain a relationship between different values of $\Gamma(x)$. Let $0 < a < b$. Then,

$$\int_a^b e^{-t} t^x \, dt = -e^{-t} t^x \Big|_a^b + x \int_a^b e^{-t} t^{x-1} \, dt.$$

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It is easy to compute that $\Gamma(1) = 1$. Hence

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1, \quad \Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1, \quad \Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1, \; \ldots .$$

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Task 2

Show that $\Gamma(1/2)$ equals the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} \, dx$.

Table of Contents

Ordinary Differential Equations

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An **Ordinary Differential Equation** or ODE is an equation involving a function $f(x)$ and some of its derivatives, as well as the variable x. The task is to solve for f.

The order of the highest derivative of f that occurs in the ODE is called the **order** of the ODE. Here are some ODEs for an unknown function $y = f(x)$.

1. $y' = \cos x$ (first-order).
2. $y' = 5xy$ (first-order).
3. $y'' = -3y + 1$ (second-order).
4. $(y''')^2 + y\sec(y'') + y'/y + \tan x = 0$ (third-order).

Of course, one may use different variable and function names. For example, the variable may be time t, the unknown function may be position $x(t)$, and the ODE may be $x'' = -5x$.

Existence and Uniqueness of Solutions

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A typical ODE will have multiple solutions. The reason is that a differential equation has information about how a quantity changes. The final value of the quantity depends on how it changes (described by the ODE) as well as its starting state. If we know the starting state, we may be able to narrow down to exactly one solution.

Existence and Uniqueness of Solutions

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Example 10

We have seen that every solution of $y' = y$ has the form $y = Ae^x$. Each value of A leads to a different solution. If we know that $y(0) = 5$, we can solve for A and get a unique solution,

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A collection of data of the form $y^{(k)}(a) = 0$, with $k = 0, 1, \ldots, n-1$, for an n^{th}-order ODE is called its **initial conditions**.

Separable First-Order ODE

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This gives an equation involving y. If we are fortunate, we can solve it to obtain an explicit formula for y.

Exponential Growth and Decay

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$$\int \frac{dy}{y} = \int k\,dx.$$

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Task 3

Show that the solutions $y(t)$ of $y' = M - ky$ have the form $y = (M - Ae^{-kt})/k$ if $k \neq 0$.

Logistic Growth

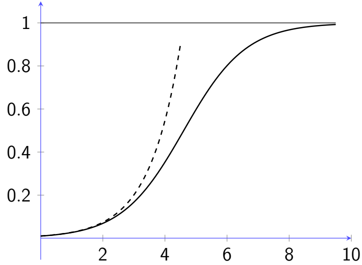
Consider a population $y(t)$ governed by the separable ODE $y' = ky(M-y)$ and with values in $[0, M]$. We have the following implications:

$$
\begin{aligned}
y' = ky(M-y) &\implies \int \frac{dy}{y(M-y)} = \int k\,dt \\
&\implies \frac{1}{M}\int \left(\frac{1}{y} + \frac{1}{M-y}\right) dy = \int k\,dt \\
&\implies \log\left(\frac{y}{M-y}\right) = kMt + d \\
&\implies \frac{y}{M-y} = Ae^{kMt} \\
&\implies y = \frac{AMe^{kMt}}{1+Ae^{kMt}} = \frac{AM}{e^{-kMt}+A}.
\end{aligned}
$$

This model describes a population whose growth is initially exponential but then tapers off as it approaches a maximum sustainable value of M.

Logistic Growth

Here is a graph of the solution with $k = M = 1$ and $A = 0.01$:



The solid curve shows the logistic growth. In its early stages it resembles the exponential growth corresponding to $y' = y$ and $y(0) = 0.01$. The parameter A in the logistic growth solution can be determined if we know the initial value $y(0)$.

$$y(0) = \frac{AM}{1+A} \implies (1+A)y(0) = AM \implies A = \frac{y(0)}{M - y(0)}.$$

General and Particular Solutions

When we solve a first-order ODE we typically get a family of solutions, generated by one parameter. The common formula for this family is called a **general solution** of the ODE.

ODE	General Solution
$y' = ky$	$y = Ae^{kt}$
$y' = M - ky$	$y = \frac{1}{k}(M - Ae^{-kt})$
$y' = ky(M - y)$	$y = \frac{AMe^{kMt}}{1 + Ae^{kMt}}$

When the parameter A is given a specific value, we get an individual solution, which is called a **particular solution**.

Example

Consider the equation $y' = -xy$. Separating variables gives $y'/y = -x$ and then $\log|y| = -\frac{x^2}{2} + C$. So the general solution is

$$y(x) = Ae^{-x^2/2}.$$

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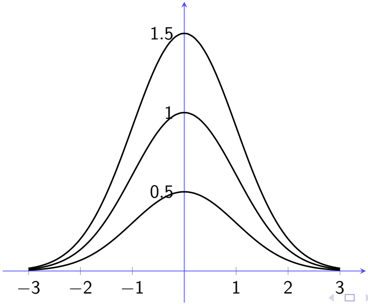
Various particular solutions of $y' = -xy$ are shown below.



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We can obtain a general solution by separation of variables.

$$\begin{aligned} y' = y^{1/3} &\implies \int y^{-1/3}\, dy = \int 1\, dx \\ &\implies \frac{3}{2} y^{2/3} = x + c \\ &\implies y = (\frac{2}{3}x + A)^{3/2}. \end{aligned}$$

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This shows that a general solution may not catch *all* solutions, and an initial value problem may have multiple solutions.

Linear First Order ODE

A first-order ODE is called **linear** if it has the form

$$y' + P(x)y = Q(x). \tag{3}$$

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This is a separable ODE and we have learned how to solve it. We have,

$$\begin{aligned}
\frac{y'}{y} = -P(x) &\implies \int \frac{dy}{y} = -\int P(x)\,dx \\
&\implies \log|y(x)| = -R(x) + C \\
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However, due to the preceding example, we are concerned whether we have really found all solutions. The next theorem gives a positive answer.

Homogeneous Case

Theorem 11

Suppose that the function $P(x)$ in (4) is continuous on an interval I. Then every solution of (4) has the form $y(x) = Ae^{-R(x)}$, with $A \in \mathbb{R}$ and $R'(x) = P(x)$.

Proof. Since $P(x)$ is continuous, it has an anti-derivative $R(x)$, and we can easily verify that $y(x) = Ae^{-R(x)}$ is a solution.

Conversely, let y be a solution. Consider the ratio of y and $e^{-R(x)}$:

$$\left(\frac{y}{e^{-R(x)}}\right)' = (ye^{R(x)})' = (y' + P(x)y)e^{R(x)} = 0 \implies \frac{y}{e^{-R(x)}} = A.$$

□

Example

Consider the ODE $xy' + (1-x)y = 0$. Put it in standard form:

$$y' + \underbrace{\left(\frac{1}{x} - 1\right)}_{P(x)} y = 0.$$

Now, $\displaystyle\int \left(\frac{1}{x} - 1\right)\, dx = \log(x) - x.$

So the general solution is

$$y = Ae^{x-\log(x)} = A\frac{e^x}{x}.$$

Non-homogeneous Case

The ODE (3) is called **non-homogeneous** if $Q(x)$ is not identically zero.

Theorem 12

Consider a non-homogeneous linear ODE of the form (3). Let y_p *be a particular solution of this ODE and let* y_h *be the general solution of the corresponding homogeneous ODE (4). Then* $y_h + y_p$ *is the general solution of (3).*

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Proof. It is trivial to check that $y_h + y_p$ is a solution of $y' + P(x)y = Q(x)$. Now let y be any solution of $y' + P(x)y = Q(x)$. Then

$$(y - y_p)' + P(x)(y - y_p) = Q(x) - Q(x) = 0,$$

hence $y - y_p$ solves the homogeneous ODE and equals one of the members of the family y_h. □

Variation of Parameters

Theorem 13

Suppose that the functions $P(x)$ and $Q(x)$ in (3) are continuous. Then a particular solution y_p of (3) can be obtained by

$$y_p = \left(\int Q(x)\, e^{R(x)}\, dx \right) e^{-R(x)},$$

where $R'(x) = P(x)$.

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Proof. Let $R'(x) = P(x)$. We have seen that the general solution of (4) is $y_h = Ae^{-R(x)}$. We substitute a function $h(x)$ for the parameter A to obtain a candidate solution of (3),

$$y = h(x)\, e^{-R(x)}.$$

Then $y' = h'(x)\, e^{-R(x)} - h(x)P(x)\, e^{-R(x)} = h'(x)\, e^{-R(x)} - P(x)y$, hence $y' + P(x)y = h'(x)\, e^{-R(x)}$. Therefore, we need $h'(x)\, e^{-R(x)} = Q(x)$, or $h(x) = \int Q(x)\, e^{R(x)}\, dx$. □

Example

Consider $xy' + (1-x)y = e^{2x}$. First we put it in the standard form,

$$y' + \left(\frac{1}{x} - 1\right) y = \frac{e^{2x}}{x}.$$

We have already worked out the general solution of the homogeneous part as $y_h = A\frac{e^x}{x}$. By variation of parameters, a particular solution is calculated as follows:

$$h(x) = \int \frac{e^{2x}}{x} e^{R(x)}\, dx = \int \frac{e^{2x}}{x} e^{\log(x)-x}\, dx = e^x \implies y_p = \frac{e^{2x}}{x}.$$

Therefore the general solution of this non-homogeneous equation is,

$$y = y_h + y_p = A\frac{e^x}{x} + \frac{e^{2x}}{x}.$$

Existence and Uniqueness

Theorem 14

Suppose that $P(x)$ and $Q(x)$ in (3) are continuous on an interval I. Consider an initial condition $y(x_0) = y_0$ with $x_0 \in I$ and $y_0 \in \mathbb{R}$. Then (3) has a unique solution which satisfies this initial condition.

Proof. We already know that the general solution of (3) is

$$y = \left(\int Q(x)\, e^{R(x)}\, dx \right) e^{-R(x)} + A e^{-R(x)},$$

with $R'(x) = P(x)$. We can take any choice of anti-derivative for $\int Q(x)\, e^{R(x)}\, dx$. Let us take $\int_{x_0}^{x} Q(t)\, e^{R(t)}\, dt$. Then:

$$y_0 = \left(\int_{x_0}^{x_0} Q(t)\, e^{R(t)}\, dt \right) e^{-R(x_0)} + A e^{-R(x_0)} = A e^{-R(x_0)}.$$

Choose $R(x) = \int_{x_0}^{x} P(t)\, dt$. Then $R(x_0) = 0$ and we get $A = y_0$.
(continued)

Existence and Uniqueness – continued

We have reached the following solution that also satisfies the initial condition:

$$y = \left(\int_{x_0}^{x} Q(t)\, e^{R(t)}\, dt \right) e^{-R(x)} + y_0 e^{-R(x)}, \text{ with } R(x) = \int_{x_0}^{x} P(t)\, dt.$$

As for uniqueness, let y_1 be another solution of (3). Then $y_1 - y$ solves (4), hence we have $y_1 - y = Ae^{-R(x)}$. The common initial condition then gives $0 = Ae^{-R(x_0)}$, so $A = 0$ and $y_1 - y = 0$. □

Task 4

Find a solution of the initial value problem $xy' + (1 - x)y = e^{2x}$ *and* $y(1) = 0$.

Autonomous ODE

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In this section we shall see that we can obtain a qualitative description of the solutions of an autonomous ODE without actually solving it.

Autonomous ODE

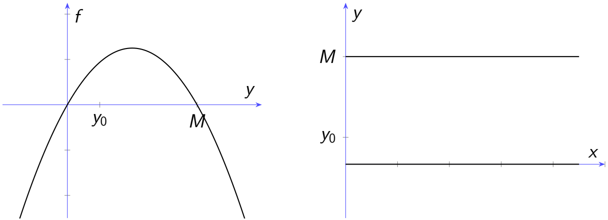
We begin with two key observations about an autonomous ODE $y' = f(y)$.

1. If $f(y_0) = 0$ for some y_0, then the constant function $y = y_0$ is a solution. Such a constant solution is called an **equilibrium solution** and the value y_0 is called a **critical value**.
2. If $y(x)$ is a solution then so is the shift $y_c(x) = y(x + c)$:

$$y_c'(x) = y'(x + c) = f(y(x + c)) = f(y_c(x)).$$

Example – Logistic equation

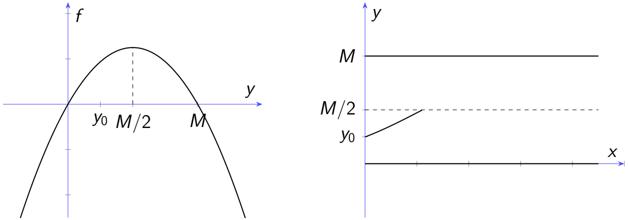
The logistic equation $y' = ky(M - y)$ is an autonomous equation with critical values 0 and M. Its equilibrium solutions are $y = 0$ and $y = M$. First, we plot $f(y) = ky(M - y)$ and the two equilibrium solutions. We have also marked an initial value $y(0) = y_0$ for a particular solution.



Since $y'(0) = f(y_0)$ is positive, the solution is initially an increasing one. As y increases from y_0, so does $y' = f(y)$ and so the graph of $y(x)$ is initially convex. It stays convex until y reaches $M/2$.
(continued)

Example – Logistic equation

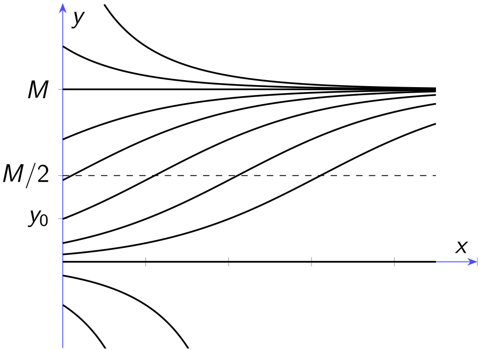
At this stage we have the following picture:



As y increases past $M/2$, y' becomes decreasing. Hence the graph of y becomes concave and flattens out.
(continued)

Example – Logistic equation

The completed graph is shown below, along with a few shifts corresponding to different initial conditions between 0 and M. We have also shown examples of solutions with initial conditions that are either negative or more than 1. For these solutions, y' is always negative and so they are decreasing.

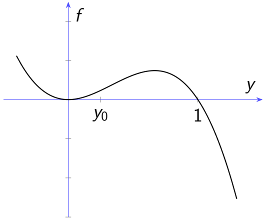


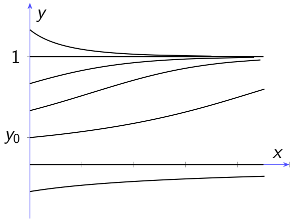
Types of Equilibria

In the logistic equation, solutions starting near $y = 0$ move away from it and so $y = 0$ is called an **unstable equilibrium**. Solutions starting near $y = M$ approach it asymptotically, hence $y = M$ is called a **stable equilibrium**. We may also have equilibrium points with mixed behaviour, these are called **semistable**.

Example 15

Consider the autonomous ODE $y' = y^2(1 - y)$.





There is a stable equilibrium at $y = 1$ and a semistable one at $y = 0$.

Asymptotic Behaviour

Theorem 16

Consider an initial value problem $y' = f(y)$, $y(0) = y_0$, *where* $f : (a,b) \to \mathbb{R}$ *is positive and continuous, and* $y_0 \in (a,b)$. *This initial value problem has a unique solution* $y\colon (\alpha,\beta) \to (a,b)$ *which is a strictly increasing bijection. In particular,* $\lim_{x\to\beta-} y(x) = b$. *(We may have* $\beta = \infty$.*)*

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Theorem 16

Consider an initial value problem $y' = f(y)$, $y(0) = y_0$, where $f : (a, b) \to \mathbb{R}$ is positive and continuous, and $y_0 \in (a, b)$. This initial value problem has a unique solution $y\colon (\alpha, \beta) \to (a, b)$ which is a strictly increasing bijection. In particular, $\lim_{x \to \beta-} y(x) = b$. (We may have $\beta = \infty$.)

Proof. Since f is positive and continuous on (a, b), so is $1/f$. Therefore $1/f$ has a strictly increasing and surjective anti-derivative $F\colon (a, b) \to (\alpha, \beta)$. We may assume $F(y_0) = 0$. Define $y(x) = F^{-1}(x)$.

Asymptotic Behaviour

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$$y'(x) = \frac{1}{F'(F^{-1}(x))} = f(F^{-1}(x)) = f(y(x)) \quad \text{and} \quad y'(0) = F^{-1}(0) = y_0.$$

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Asymptotic Behaviour

Task 5

State and prove a version of Theorem 16 in which the hypothesis $f > 0$ is replaced by $f < 0$.

Task 6

Consider an initial value problem $y' = f(y)$, $y(0) = y_0$, where f is continuous and y_0 belongs to the domain of f. Will there be a solution? Will it be unique?

Classification of Equilibria

Theorem 17

Consider an autonomous ODE $y' = f(y)$, with f being differentiable. Let c be a critical value.

1. *If $f'(c) < 0$ then $y = c$ is a stable equilibrium solution.*
2. *If $f'(c) > 0$ then $y = c$ is an unstable equilibrium solution.*

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We have $\delta > 0$ such that $0 < |y - c| < \delta$ implies

$$-\epsilon = \frac{3}{2}f'(c) < \frac{f(y)}{y - c} < \frac{1}{2}f'(c) = -\epsilon'.$$

Classification of Equilibria

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Note that $c < y < c + \delta \implies f(y) < 0$ and
$c - \delta < y < c \implies f(y) > 0$.
(continued)

Classification of Equilibria – continued

Now consider the case $c - \delta < y_0 < c$. From Theorem 16 we know there is a unique strictly increasing solution $y(x)$ with $y(0) = y_0$. Take any y in (y_0, c). Then

$$-\epsilon < -\frac{f(y)}{c-y} \implies \epsilon(c-y) > f(y) \implies \frac{1}{f(y)} > \frac{1}{\epsilon(c-y)}.$$

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$$F(y) = \int_{y_0}^{y} \frac{dt}{f(t)} \geq \int_{y_0}^{y} \frac{dt}{\epsilon(c-t)} = -\frac{1}{\epsilon} \log\left(\frac{c-y}{c-y_0}\right) \to \infty \text{ as } y \to c- .$$

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Hence $f(y) > 0$ if $c < y < c + \delta$ and $f(y) < 0$ if $c - \delta < y < c$.
Now apply Theorem 16 and Task 5 to $(c, c+\delta)$ and $(c - \delta, c)$. □