Solutions to exercises in chapter 5

1. **Interference**

   Maxima are at $sin(\theta) = m\lambda/d$, for the third order maximum we have $m = 3$. Using the small angle approximation this gives $\theta = 3\lambda/d = 3 \cdot 2\pi \cdot 0.1\mu m/(18\mu m) = 6\pi/180$ or $\theta = 6°$

2. **Interference 2**

   We are looking in the direct back direction, so we have constructive interference in the case of $(m_1 + 1/2)\lambda_1 = 2nD$ and destructive interference at $m_2\lambda_2 = 2nD$. As there is no maximum between 450 and 600 nm, these are neighbouring orders. Therefore $(m_2 - 1/2)\lambda_1 = m_2\lambda_2$ oder $2m_2 = \lambda_1/(\lambda_1 - \lambda_2)$ and hence $D = 1/(4n) \cdot \lambda_2\lambda_1/(\lambda_1 - \lambda_2) = 1/6 \cdot 600nm \cdot 450nm/150nm = 1/6 \cdot 600nm \cdot 3 = 300nm$.

3. **Diffraction**

   The first minimum is at $\sin(\theta) = m \cdot \lambda/a$ with $m = 1$. The first side-maximum is at $\sin(\theta) = 1.5 \cdot \lambda/a = 1.5 \cdot 0.5/2 = 0.375$. This gives a distance of $d = D \cdot \tan(\theta) \simeq 0.4m$.

4. **Reading glasses**

   If the eye is able to image object far away (at g = ∞), the focal length needs to be equal to the object distance. Therefore we have $f_{glass} = 25cm = 1/4m$ or in other words 4 dioptries.

5. **Magnifying glass**

   a)

   b) A virtual image.

   c) This image equation says: $\frac{1}{f} = \frac{1}{b} + \frac{1}{g}$. We want a 10 times magnification, therefore $|B| = 10G$, hence $|b| = 10g$. The image is supposed to be behind the lens, such that the sign of b needs to be negative, i.e. $b = -10g = -25$ cm. Inserting this we obtain: $\frac{1}{f} = -\frac{1}{25\text{cm}} + \frac{1}{2.5\text{cm}} = \frac{10-1}{25\text{cm}} = \frac{9}{25\text{cm}}$ or $f = 25/9$ cm $\simeq 2.8$ cm.

6. **Lens**

   The magnification is: $M = f/(f - g)$, hence we need: $\partial M/\partial f$ and $\partial M/\partial g$.

   $\partial M/\partial f = 1/(f - g) - f/(f - g)^2 = -g/(f - g)^2$

   $\partial M/\partial g = f/(f - g)^2$

   Inserting this in error propagation:

   $\sigma_M^2 = g^2/(f - g)^4\sigma_f^2 + f^2/(f - g)^4\sigma_g^2$.

   Numerically:

   $f/(f - g)^2 = 10/25cm^{-1} = 0.4cm^{-1}$ and $g/(f - g)^2 = 15/25cm^{-1} = 0.6cm^{-1}$.

   With $\sigma_g = 0.5$ cm and $\sigma_f = 0.1$ cm we obtain: $sigma_M^2 = (0.6 \cdot 0.1)^2 + (0.4 \cdot 0.5)^2 = (36 + 400) \cdot 10^{-4} \simeq 4.4 \cdot 10^{-2}$. Hence $\sigma_M = 0.22$ with $M = 2$.

7. **Lens 2**

   The focal length of such a lens is given by $f = n_2 \cdot r/(2(n_2 - n_1))$. The lenses refractive index is always $n_2 = 1.45$. For the three different cases we have: $n_1^{air} = 1$, $n_1^{water} = 1.33$, $n_1^{oil} = 1.5$. And hence: $n_2/(n_2 - n_1)^{air} = 1.45/(1.45 - 1) = 3.22$, $n_2/(n_2 - n_1)^{water} = 1.45/(1.45 - 1.33)) = 12.08$, $n_2/(n_2 - n_1)^{oil} = 1.45/(1.45 - 1.5) = -29$. With r=5mm, this gives the respective focal lengths: $f^{air} = 1.6r = 8$ mm, $f^{water} = 6r = 30$ mm, $f^{oil} = -14.5r =$ -72.5 mm. Hence the lens is a diverging lens in oil!

8. **Microscope**

a) The total magnification is given by the product of the magnifications of the two lenses making up the microscope. For the first lens we have: $M_1 = b_1/g_1$ and $1/g_1 = -1/b_1 + 1/f_1$ or $b_1 = -f_1 \cdot g_1/(f_1 - g_1)$, and hence $M_1 = f_1/(f_1 - g_1)$. For the second lens we have: $M_2 = b_2/g_2$, where $g_2 = d - b_1$ and in the same way as before: $b_2 = f_2 \cdot g_2/(f_2 - g_2)$ and hence $M_2 == f_2/(f_2 - g_2) = f_2/(f_2 - d + b_1) = f_2/(f_2 - d + f_1 g_1/(f_1 - g_1))$. The total magnification hence becomes: $M = f_1 f_2/((f_1 - g_1) \cdot (f_2 - d + f_1 g_1/(f_1 - g_1))) = f_1 f_2/(f_1 f_2 + g_1 d - f_2 g_1 - f_1 d + f_1 g_1))$ or numerically: $M_1 = 5mm/(5-6)mm = -5$ and $b_1 = 30mm = 3cm$, giving $g_2 = 20 - 3cm = 17cm$ and $M_2 = 18cm/(18-17)cm = 18$, which finally yields $M = -18 \cdot 5 = -90$.

b) We need $\frac{\partial M}{\partial f_1}$; $\frac{\partial M}{\partial f_2}$; $\frac{\partial M}{\partial g_1}$; and $\frac{\partial M}{\partial d}$

$\frac{\partial M}{\partial f_1} = \frac{f_2 g_1 (d - f_2)}{(f_1 f_2 + g_1 d - f_2 g_1 - f_1 d + f_1 g_1)^2}$

$\frac{\partial M}{\partial f_2} = \frac{f_1 (g_1 d - f_1 d + f_1 g_1)}{(f_1 f_2 + g_1 d - f_2 g_1 - f_1 d + f_1 g_1)^2}$

$\frac{\partial M}{\partial g_1} = \frac{f_1 f_2 (d - f_2 + f_1)}{(f_1 f_2 + g_1 d - f_2 g_1 - f_1 d + f_1 g_1)^2}$

$\frac{\partial M}{\partial d} = \frac{f_1 f_2 g_1}{(f_1 f_2 + g_1 d - f_2 g_1 - f_1 d + f_1 g_1)^2}$

$\sigma_M = \frac{\sqrt{(f_2 g_1 (d - f_2))^2 \sigma_{f_1}^2 + (f_1 (g_1 d - f_1 d + f_1 g_1))^2 \sigma_{f_2}^2 + (f_1 f_2 (d - f_2 + f_1))^2 \sigma_{g_1}^2 + (f_1 f_2 g_1)^2 \sigma_d^2}}{(f_1 f_2 + g_1 d - f_2 g_1 - f_1 d + f_1 g_1)^2}$

9. **Spherical mirror**

To get an upright image in a spherical mirror, the object needs to be inside the focal length, such that a virtual image is formed. The image position should be 50 cm from the object, hence $g - b = 50$cm. In addition, we want a twofold magnification, i.e. $\frac{B}{G} = -\frac{b}{g} = 2$. Combining these two criteria gives $3g = 50$cm. Finally we use the image equation: $\frac{1}{f} = \frac{1}{g} + \frac{1}{b} = \frac{1}{g} - \frac{1}{2g} = \frac{1}{2g}$, and find $f = 2g$. Using $3g = 50$cm, we thus get $f = \frac{1}{3}$m and hence the radius of curvature is $r = 2f = \frac{2}{3}$m.

10. **Resolution limit**

a) Diffraction at the lens: the angle of the first diffraction minimum of the first source must be smaller than the angle at which the second source is located. This means that the smallest possible angle separation of the sources is $\Delta\alpha \simeq \lambda/D$, where D is the diameter of the objective lens. For a circular opening, the exact result is (the previous result would actually be for a square opening): $\Delta\alpha \simeq 1.22\lambda/D$.

b) If the object is positioned at the focal length, the angle between two objects is in the small-angle approximation: $\alpha \simeq d/f$, where d is the distance between the objects and f is the focal length. This is the angle limited by diffraction according to a), i.e. we obtain for the smallest distance being resolved: $d = 1.22\frac{f\lambda}{D}$.

c) The ratio of D and f of a lens determines its numerical aperture. For $D = 2f$, $NA = 1$, so we guess $NA = \frac{D}{2f}$. Insert this in the result in b) we obtain Abbe's resolution limit: $d = \frac{1.22\lambda}{2NA}$ or for the given NA: $d \simeq 0.6\lambda$. This means that the resolution is basically limited by the wave-length of the light used.

d) Due to the change in speed of light in materials, the wave-length is actually materials dependent: $\lambda_n = \lambda/n$. This means that the resolution limit in a different material can be somewhat lower than in air (e.g. immersion oil). Using this, the resolution limit can be improved by a factor of $n_{\text{oil}} \simeq 1.5$ compared to an air objective.

11. **Resolution limit 2**

a) Diffraction limit: $D \cdot \sin(\theta) = 1.22\lambda$, where D is the size of the iris, i.e. D=2 mm. Therefore the eye's angular resolution is $\sin(\theta) = 1.22\lambda/D = 1.22 \cdot 2\pi \cdot 10^{-7}\text{m}/(2 \cdot 10^{-3}\text{m}) = 1.22\pi \cdot 10^{-4}$ or measured in degrees:

$\theta = 1.22 \cdot 10^{-4} \cdot 180° = 10^{-2} 2.2° \simeq 1/45°$.

In order to be able to distinguish two point on an image, they need to have at least this angle separation, therefore $\Delta L/L = 1.22\pi \cdot 10^{-4}$ or $L = 10^4 \Delta L/(1.22\pi) = 10^4 \cdot 1.5 \cdot 10^{-3}/(1.22\pi)$ m $=$ $15/(1.22\pi)$ m $= 15/3.8$ m $\simeq 4$ m.

b) The ratio of image to object *size* is equal to the ratio of image and object *distance*. With $b = 2.5$ cm, $g = 4$ m and $G = 1.5$ mm we get the image size $B = G \cdot b/g = 2.5 \cdot 1.5 \cdot 10^{-5}\text{m}^2/4\text{m} = 3.75/4$ 10^{-5} m $\simeq 10\mu$ m.

12. **Resolution limit 3**

    a) From the previous exercise we know that the distance between the headlights, $d = 1.8m$ has to be larger than an arc minute to be resolved. Therefore the distance to the car D needs to be smaller than $d/D = \pi/180 \cdot 1/60$ or $D = d \cdot 60 \cdot 180/\pi = 1.8 \cdot 3 \cdot 3.6 \cdot 10^3/\pi m \simeq 6.2km$.

13. **Radio protection**

    a) The definition of the shielding length says that after a thickness of $d = 1mm$ the intensity of radiation has fallen to 1/e of its initial value. With $\ln(10) = 2.3$, we need to have $6 \cdot 2.3 \simeq 14$ shielding lengths for an intensity reduction of a factor of 10^6. This means we need 14 mm of lead.

    b) We can obtain the absorption rate in 10 m of water from the particle current density and the detector volume. At a current density of 10^{11} particles per second and cm^2, there are 10^{17} particles impinging on the detector (surface 100 m^2) every second. A year has about $\pi \cdot 10^7$ s, so there are $\pi \cdot 10^{24}$ particles hitting the detector every year. From all of these, there are 10^5 actually detected, so the absorption rate is $\frac{1}{\pi \cdot 10^{19}} = \pi \cdot 10^{-20}$. This rate corresponds to $(I_0 - I_0 \exp(-d/\lambda))/I_0 simeq d/\lambda$ for small values of d/λ, which is certainly the case here. Given the detector size, we know $d = 10m$, such that we can directly obtain $\lambda \simeq \pi \cdot 10^{20} m$. This is actually about the distance to the centre of the milky way or 30'000 light years!