Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014)
CHAPTER 13

PROBLEM 13.1. Isothermal Static Corona.
Find the pressure $p(r)$ and density $\rho(r)$ for a spherically symmetric isothermal static corona.

SOLUTION.
Writing $\rho = mn$ and taking account of the inverse square law fall-off of gravity ($g = GM_\odot/r^2$), where $M_\odot$ is the mass of the Sun, the equation of hydrostatic equilibrium for a spherically symmetric corona is

$$\frac{dp}{dr} = -\rho g = -\frac{GM_\odot \rho}{r^2}.$$

Coupling this with the equation of state ($p = \tilde{R}\rho T/\tilde{\mu}$) and putting $r_c = GM_\odot/(2v_c^2) = GM_\odot\tilde{\mu}/(2\tilde{R}T)$, we find

$$\frac{dp}{dr} = -\frac{2r_c p}{r^2}.$$

Integrating gives

$$\log_e p = \frac{2r_c}{r} + C,$$

where the condition $p = p_0$ at $r = r_0$ determines the constant of integration as $C = \log_e p_0 - 2r_c/r_0$.

Taking exponentials determines the pressure as

$$p = p_0 e^{-2r_c(1/r_0 - 1/r)}.$$

It can be seen that, as $r \to \infty$, so p tends to a constant value, but it transpires that this constant value far exceeds any reasonable inter-stellar pressure.

The corresponding density is

$$\rho = \rho_0 e^{-2r_c(1/r_0 - 1/r)},$$

were $r_c = GM_\odot\tilde{\mu}/(2\tilde{R}T)$.

PROBLEM 13.2. Effect of Depositing Heat at a Single Radius.

Find the temperature $T(r)$ for a corona with heat conduction inwards and outwards from a level (r_0) where heat is deposited.

SOLUTION (adapted from Goossens (2003, An Introduction to Plasma Astrophysics and MHD).

When energy transfer is by conduction alone, the heat flux across a sphere of radius r is constant, so that

$$r^2 \kappa \frac{dT}{dr} = \frac{2\kappa_0 C}{7}, \tag{1}$$

say, where $\kappa dT/dr$ is the heat flux density, $\kappa = \kappa_0 T^{5/2}$ is the coefficient of thermal conduction and C is constant.

For $r > r_0$, this equation may be integrated to give

$$T = T_0 \left(\frac{r_0}{r}\right)^{2/7}$$

after imposing the boundary conditions that $T = T_0$ at $r = r_0$ (the assumed level of discrete heat deposition) and that T vanishes at infinity.

For $r < r_0$ (i.e., below the location of heat deposition), Eq.(1) may be integrated to give

$$T^{7/2} = \frac{C}{r} + D.$$

Here the constants C and D are determined by the two conditions that $T(r_0)$=T_0 and $T(R_\odot)$=$T_\odot$, say, so that

$$T = \left[T_\odot^{7/2} \frac{r_0/r - 1}{r_0/R_\odot - 1} + T_0^{7/2} \frac{R_\odot/r - 1}{R_\odot/r_0 - 1} \right]^{2/7}.$$

PROBLEM 13.3. Properties of the Isothermal Solar Wind Solution.

Show that:

(a) in the low corona, the flow speed is 0.1–10 km s^{-1}, depending on the coronal temperature;

(b) below the critical point, the density variation is very similar to a static atmosphere;

(c) the mass-loss rate is of the order of $10^{-14} M_{\odot}$ per year.

SOLUTION

(a) **Flow Speed Low Down in Corona.**
For the isothermal solar wind solution we have from Eq.(13.6)

$$\left(\frac{v}{v_c}\right)^2 - \log_e \left(\frac{v}{v_c}\right)^2 = 4 \log_e \frac{r}{r_c} + \frac{4r_c}{r} - 3,$$

where, for $T_0 = 1$ MK, we find $v_c = 120$ km s^{-1} and $r_c = 7\ R_{\odot}$, while for, say, $T_0 = 2$ MK we find $v_c = 170$ km s^{-1} and $r_c = 3.5\ R_{\odot}$.

Now, close to the Sun at, say, $r = r_0$, where $v = v_0$, we have $v \ll v_c$, so that the first term on the left of the above equation is negligible and it reduces approximately to

$$-\log_e \left(\frac{v_0}{v_c}\right)^2 = 4 \log_e \frac{r_0}{r_c} + \frac{4r_c}{r_0} - 3,$$

or

$$\frac{v_0}{v_c} = \left(\frac{r_c}{r_0}\right)^2 \exp\left(\frac{3}{2} - \frac{2r_c}{r_0}\right).$$

Thus, if, for example, $T_0 = 1$ MK and $r_0 = 1.2 R_{\odot}$, we have

$$\frac{v_0}{v_c} = \left(\frac{6.89}{1.2}\right)^2 \exp\left(\frac{3}{2} - \frac{6.89}{0.6}\right) \approx 0.0015,$$

so that $v_0 = 0.2$ km s^{-1}. On the other hand, if $T_0 = 2$ MK, then $v_0 = 3$ km s^{-1}.

(b) **Density Variation**
First of all, we note that from PROBLEM 13.1 the density in an isothermal hydrostatic atmosphere is

$$\rho = \rho_0 e^{2r_c(1/r - 1/r_0)},$$

were $r_c = GM_{\odot}/(2RT_0)$.

Now, Eq.(13.4) is

$$\rho v \frac{dv}{dr} = -\frac{dp}{dr} - \frac{GM_{\odot}\rho}{r^2},$$

where $p = \tilde{R}\rho T/\tilde{\mu}$. It may be integrated to give Bernoulli's law, namely,

$$\frac{v^2}{2} - \frac{GM_\odot}{r} + v_c^2 \log_e \frac{\rho}{\rho_0} = E,$$

where $\rho_0 = \rho(r_0)$ and evaluating this at $r = r_0$ gives $E = \frac{1}{2}v_0^2 - GM_\odot/r_0$. Thus, this equation may be rewritten as an equation for $\rho(r)$ in terms of $v(r)$, namely,

$$\rho(r) = \rho_0 \exp\left[-2r_c\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] \exp -\frac{v^2 - v_0^2}{2v_c^2},$$

where $r_c = GM_\odot\tilde{\mu}/(2\tilde{R}T)$ and $v_c^2 = \tilde{R}T/\tilde{\mu}$. The first exponential is simply the hydrostatic expression for the density variation, whereas the second exponential represents a factor which is close to but less than unity since $v_0 \ll v_c$. Thus, the density in the subsonic region falls off with distance slightly more rapidly than in a hydrostatic atmosphere.

In particular, the density at the sonic point ($r = r_c$) is

$$\rho(r_c) = \rho_0 \exp\left[2\left(1 - \frac{r_c}{r_0}\right)\right] \exp -\frac{1}{2}\left(1 - \frac{v_0^2}{v_c^2}\right),$$

or, since $v_0 \ll v_c$,

$$\rho(r_c) \approx \rho_0 \exp\left(\frac{3}{2} - \frac{2r_c}{r_0}\right).$$

(c) **Mass Loss Rate**

The mass loss rate is

$$\frac{dM_0}{dt} = 4\pi r_0^2 \rho_0 v_0,$$

which may be evaluated directly by substituting $r_0 = 1.2\ R_\odot$, say, $\rho_0 = \tilde{\mu} n m_p$ with $\tilde{\mu} = 0.6$, $m_p = 1.673 \times 10^{-27}$ kg m^{-3} and a solar wind solution for v_0 such as 0.2 km s^{-1} for $T = 1$ MK or 3 km s^{-1} for $T = 2$ MK. This mass loss rate is expressed in kg s^{-1}, but it may be converted into some number of $M_\odot$ per year using the fact that $M_\odot = 1.99 \times 10^{30}$ kg and 1 year = 3.15×10^7 sec, to give a mass loss rate of $3 \times 10^{-15} M_\odot$ yr^{-1} for $T = 1$ MK or $4 \times 10^{-14} M_\odot$ yr^{-1} for $T = 2$ MK.

The alternative is to use mass conservation to rewrite the mass loss as

$$\frac{dM_0}{dt} = 4\pi r_c^2 \rho_c v_c,$$

and then to use values for r_c, ρ_c and v_c from the solar wind solutions.

PROBLEM 13.4. Maximum Temperature for an Isothermal Wind.

Show that, if $T > 5.8 \times 10^6$ K, an isothermal wind does not exist.

SOLUTION

For an isothermal wind solution, the flow starts out subsonically at the coronal base ($r = r_0 = 1.2R_\odot$, say) and accelerates through the critical point ($r = r_c = GM_\odot\tilde{\mu}/(2\tilde{R}T)$) to become supersonic.

However, if $r_c < r_0$, there is no longer a critical point above the coronal base, and so any solution that starts subsonically will decelerate and remain subsonic. This condition may be written

$$\frac{GM_\odot\tilde{\mu}}{2\tilde{R}T} < 1.2R_\odot,$$

or

$$T > \frac{GM_\odot\tilde{\mu}}{2.4R_\odot\tilde{R}}.$$

By substituting $G = 6.67\times10^{-11}$, $M_\odot = 1.99\times10^{30}$, $\tilde{\mu} = 0.6$, $R_\odot = 6.96\times10^8$ and $\tilde{R} = 8.3 \times 10^3$ in MKS units, this becomes

$$T > 5.8 \times 10^6 \text{ K},$$

as required.

PROBLEM 13.5. Polytropic Solar Wind.

For a spherically symmetric polytropic solar wind:

(a) find the critical point location and deduce that

$$\tfrac{1}{2}v^2(r) + \frac{c_{s\alpha}^2(r)}{\alpha - 1} - \frac{GM_\odot}{r} = \text{constant};$$

(b) find the condition that $p \to 0$ as $r \to \infty$ for breeze solutions.

SOLUTION.

(a) (adapted from lecture notes of Clare Parnell)

Consider a spherically symmetric polytropic solar wind, whose velocity [$\mathbf{v} = v(r)\hat{\mathbf{r}}$], pressure [$p(r)$] and density [$\rho(r)$] satisfy

$$\rho r^2 v = D,$$

$$\rho v \frac{dv}{dr} = -\frac{dp}{dr} - \frac{\rho G M_\odot}{r^2},$$

and

$$\frac{p}{\rho^\alpha} = K,$$

where D and K are constant.

Eliminating p and ρ from these equations gives

$$\left(v - \frac{c_{s\alpha}^2(r)}{v} \right) \frac{dv}{dr} = \frac{2c_{s\alpha}^2(r)}{r} - \frac{GM_\odot}{r^2}, \tag{2}$$

where

$$c_{s\alpha}^2(r) = \frac{\alpha p}{\rho}$$

is the polytropic sound speed.

The location of the critical point (r_c, v_c) is given by $v_c = c_{s\alpha}(r_c)$ and

$$2c_{s\alpha}^2(r_c) r_c = GM_\odot,$$

where

$$c_{s\alpha}^2(r_c) = \frac{\alpha p}{\rho} = \alpha K \rho^{\alpha-1} = \alpha K \left(\frac{D}{r_c^2 c_{s\alpha}(r_c)} \right)^{\alpha-1}.$$

Thus, solving for $c_{s\alpha}(r_c)$, we have

$$c_{s\alpha}^2(r_c) = \left(\frac{\alpha K D^{\alpha-1}}{r_c^{2\alpha-2}} \right)^{2/(\alpha+1)}.$$

After substituting this into $2c_{s\alpha}^2(r_c) r_c = GM_\odot$, the critical radius is, finally,

$$r_c = \left(\frac{GM_\odot}{2(\alpha K)^{2/(\alpha+1)} D^{(2\alpha-2)/(\alpha+1)}} \right)^{(\alpha+1)/(5-3\alpha)}.$$

In order to solve the differential equation (2) for $v(r)$, note that pressure gradient term in the equation of motion can be written as

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{K}{\rho}\frac{d\rho^\alpha}{dr} = K\alpha\rho^{\alpha-2}\frac{d\rho}{dr} = \frac{\alpha K}{\alpha-1}\frac{d\rho^{\alpha-1}}{dr},$$

and so the equation of motion can be written as

$$v\frac{dv}{dr} = -\frac{\alpha K}{\alpha-1}\frac{d\rho^{\alpha-1}}{dr} - \frac{GM_\odot}{r^2}.$$

Since this is now separable, it can be integrated to give

$$\tfrac{1}{2}v^2(r) + \frac{c_{s\alpha}^2(r)}{\alpha - 1} - \frac{GM_\odot}{r} = \text{constant}, \tag{3}$$

as required.

(b) We have

$$c_s^2 \sim T(r) \sim \rho^{\alpha-1}.$$

Thus, for a breeze in which $v \ll c_s$ and $c_s(r) \to 0$ as $r \to \infty$, Eq.(3) implies

$$c_s^2 \sim \frac{1}{r} \qquad \text{or} \qquad T \sim \frac{1}{r}.$$

Therefore

$$\rho \sim c_s^{2/(\alpha-1)} \sim \frac{1}{r^{1/(\alpha-1)}}.$$

Thus, as $r \to \infty$,

$$p \sim \rho T \sim \frac{1}{r^{\alpha/(\alpha-1)}} \to 0,$$

provided $\alpha > 1$.

PROBLEM 13.6. Condition for Existence of a Polytropic Solar Wind.

Show that a polytropic solar wind with $\alpha = 1.1$ will exist if $T_0 > 1.1$ MK, but that an adiabatic wind needs $T_0 > 4.6$ MK.

SOLUTION For polytropic flow the Bernouilli equation is

$$\tfrac{1}{2}v^2(r) + \frac{c_{s\alpha}^2(r)}{\alpha - 1} - \frac{GM_\odot}{r} = \text{constant},$$

which may be evaluated at the coronal base (r_0) (where $v \ll v_\infty$) and at infinity (where $T \to 0$) to give

$$\frac{\alpha \tilde{R} T_0/\tilde{\mu}}{(\alpha - 1)} - \frac{GM_\odot}{r_0} = \tfrac{1}{2}v_\infty^2.$$

Thus, the corona will be able to expand (i.e., possess a positive v_∞^2) provided it is hot enough that the thermal energy exceeds the gravitational contribution, namely, if

$$\frac{\alpha \tilde{R} T_0/\tilde{\mu}}{(\alpha - 1)} > \frac{GM_\odot}{r_0},$$

or

$$T_0 > \frac{\alpha - 1}{(\alpha}\frac{\tilde{\mu} G M_\odot}{\tilde{R} r_0}.$$

For $r_0 = 1.2\ R_\odot$ and $\alpha = 1.1$, say, inserting the standard values for $G = 6.67 \times 10^{-11}$, $M_\odot = 1.99 \times 10^{30}$, $\tilde{\mu} = 0.6$, $R_\odot = 6.96 \times 10^8$ and $\tilde{R} = 8.3 \times 10^3$ in MKS units gives $T_0 > 1.1\ 10^6$ K, as required.

On the other hand, for $\alpha = 5/3$, we find $T_0 > 4.6\ 10^6$ K. Since the actual coronal temperature is significantly less than this, we deduce that the solar corona would not expand as the solar wind if it were adiabatic.

PROBLEM 13.7. Properties of a Polytropic Solar Wind.
Show that:

(a) if $1 < \alpha < 5/3$, the critical point is a saddle point;

and (b) if $1 < \alpha < 3/2$, there is a solar wind solution whose pressure force always dominates gravity and that changes from subsonic to supersonic as it passes through the critical point.

SOLUTION (adapted from lecture notes of Clare Parnell).

(a) **The Critical Point**.

First, normalise the above solution with respect to values at the critical point $(r_c, c_{s\alpha})$ by writing

$$\bar{v}(\bar{r}) = \frac{v(r)}{c_{s\alpha}(r_c)}, \qquad \bar{r} = \frac{r}{r_c}, \qquad \bar{c}_{s\alpha}(\bar{r}) = \frac{c_{s\alpha}(r)}{c_{s\alpha}(r_c)}.$$

Then the equation for $v(r)$ that was derived at the end of the last example becomes

$$f(\bar{r}, \bar{v}) \equiv \tfrac{1}{2}\bar{v}^2(\bar{r}) + \frac{\bar{c}^2_{s\alpha}(\bar{r})}{\alpha - 1} - \frac{2}{\bar{r}} = C, \tag{4}$$

say, where

$$c^2_{s\alpha}(r) = \frac{\alpha p}{\rho} = \alpha K \rho^{\alpha - 1} = \alpha K \left(\frac{D}{r^2 v}\right)^{\alpha - 1} \quad \text{and} \quad c^2_{s\alpha}(r_c) = \alpha K \left(\frac{D}{r_c^2 v_c}\right)^{\alpha - 1},$$

so

$$\bar{c}^2_{s\alpha}(\bar{r}) = (\bar{r}^2 \bar{v})^{1-\alpha}. \tag{5}$$

Clearly the solutions depend on the value of the polytropic constant α, and the value of C for the solution through the critical point $(\bar{r}, \bar{v}) = (1, 1)$ is given by

$$C = \frac{1}{\alpha - 1} - \frac{3}{2}.$$

Now, a critical point occurs where the first derivatives of $f(\bar{r}, \bar{v})$ vanish, namely, when

$$\frac{\partial f}{\partial \bar{r}} \equiv -2\bar{r}^{1-2\alpha}\bar{v}^{1-\alpha} + \frac{2}{\bar{r}^2} = 0, \qquad \frac{\partial f}{\partial \bar{v}} \equiv \bar{v} - \bar{r}^{2(1-\alpha)}\bar{v}^{-\alpha} = 0,$$

namely, when $(\bar{r}, \bar{v}) = (1, 1)$.

The second derivatives determine the nature of the critical point, as follows.

$$\frac{\partial^2 f}{\partial \bar{r}^2} \equiv -2(1-2\alpha)\bar{r}^{-2\alpha}\bar{v}^{1-\alpha} - \frac{4}{\bar{r}^3} = -6 + 4\alpha \quad \text{at} \quad (\bar{r}, \bar{v}) = (1, 1),$$

$$\frac{\partial^2 f}{\partial \bar{r}\partial \bar{v}} \equiv -2(1-\alpha)\bar{r}^{1-2\alpha}\bar{v}^{-\alpha} = -2(1-\alpha) \quad \text{at} \quad (\bar{r}, \bar{v}) = (1, 1),$$

$$\frac{\partial^2 f}{\partial \bar{v}^2} \equiv 1 + \alpha\bar{r}^{2(1-\alpha)}\bar{v}^{-(1+\alpha)} = 1 + \alpha \quad \text{at} \quad (\bar{r}, \bar{v}) = (1, 1).$$

Then the value of

$$F \equiv \frac{\partial^2 f}{\partial \bar{r}^2}\frac{\partial^2 f}{\partial \bar{v}^2} - \left(\frac{\partial^2 f}{\partial \bar{r}\partial \bar{v}}\right)^2 = 2(3\alpha - 5)$$

determines the type of critical point.

When $1 < \alpha < 5/3$, we find $F < 0$ and so we have a saddle point. But when $\alpha > 5/3$, $F > 0$ and $\partial^2 f/\partial \bar{r}^2 > 0$, so that the critical point is a local minimum.

(b) **The Solar Wind Solution**.

As well as passing through a saddle point, a solar wind velocity needs to increase with distance and go from subsonic to supersonic as it passes through the critical point. The Mach number $M = v(r)/c_{s\alpha}(r)$ may be nondimensionalised to give

$$\bar{M}(\bar{r}, \bar{v}) = \frac{\bar{v}}{\bar{c}_{s\alpha}} = \frac{\bar{v}}{(\bar{r}^2\bar{v})^{(1-\alpha)/2}} = \bar{v}^{(1+\alpha)/2}\bar{r}^{\alpha-1}.$$

In the $\bar{v}$-$\bar{r}$ phase plane, the solutions therefore go from subsonic to supersonic on the curve $\bar{M}(\bar{r}, \bar{v}) = 1$, namely,

$$\bar{v} = \bar{r}^{2(1-\alpha)/(1+\alpha)}, \tag{6}$$

which decreases with $\bar{r}$ when $\alpha > 1$.

We next need to determine whether, when $1 < \alpha < 5/3$, there is a solution through the critical point that is initially subsonic when $\bar{r} < 1$ and then becomes supersonic when $\bar{r} > 1$. The critical point for such a solution lies on $\bar{M}(\bar{r}, \bar{v}) = 1$ or Eq.(6). Since then $d\bar{v}/d\bar{r} = -2[(\alpha-1)(\alpha+1)]\bar{r}^{-(3\alpha-1)(\alpha+1)}$, the gradient of this Mach 1 curve through the critical point $(\bar{r}, \bar{v}) = (1,1)$ is $-2(\alpha-1)(\alpha+1)$.

We now need to compare this with the gradients of the flow solutions through the critical point, which may be calculated as follows. By differentiating the Eq.(4) for $f(\bar{r}, \bar{v})$ with $\bar{c}_{s\alpha}(\bar{r})$ given by (5), we find

$$[\bar{v} - \bar{r}^{2(1-\alpha)}\bar{v}^{-\alpha}]\frac{d\bar{v}}{d\bar{r}} = 2\bar{v}^{1-\alpha}\bar{r}^{1-2\alpha} + 2\bar{r}^{-2}.$$

Since we are interested in the gradients of the curves through the critical point $(\bar{r}, \bar{v}) = (1,1)$, we linearise about this point by putting $\bar{r} = 1 + R$ and $\bar{v} = 1 + V$ and supposing the curves are locally of the form $V = \alpha R$, so that the above equation approximates to

$$\alpha = \frac{dV}{dR} = 2\frac{(3-2\alpha) + (1-\alpha)\alpha}{2(\alpha-1) + (1+\alpha)\alpha}$$

This may in turn be solved for α to give

$$\alpha_{\pm} = -2\frac{\alpha-1}{\alpha+1} \pm \frac{\sqrt{2}}{\alpha+1}\sqrt{5-3\alpha}.$$

When $\alpha > 5/3$ the roots of α are complex since the saddle point no longer exists. When $\alpha = 5/3$ there is one root and the gradient is the same as that of the Mach 1 curve, since $\bar{M} \equiv 1$ is now a solution with the flow remaining sonic for all r.

When $1 < \alpha < 5/3$ there are two real roots. The first, namely,

$$\alpha_{-} = -2\frac{\alpha-1}{\alpha+1} - \frac{\sqrt{2}}{\alpha+1}\sqrt{5-3\alpha}$$

has a slope that is steeper than the Mach 1 curve and so it represents the solution that goes from supersonic to subsonic. On the other hand, the second,

$$\alpha_{+} = -2\frac{\alpha-1}{\alpha+1} + \frac{\sqrt{2}}{\alpha+1}\sqrt{5-3\alpha},$$

has a slope that is shallower than the Mach 1 curve and so it represents the solution that goes from supersonic to subsonic.

However, α_+ may be positive or negative. The change from positive to negative occurs when

$$\sqrt{2}\sqrt{5-3\alpha} = 2(\alpha - 1)$$

or

$$-2\alpha^2 + \alpha + 3 = -(2\alpha - 3)(\alpha + 1) = 0.$$

Thus, when $3/2 < \alpha < 5/3$, then $\alpha_+ < 0$ and the solution is unphysical, since it starts with supersonic speed near the Sun (just like the α_- solution).

However, when $1 < \alpha < 3/2$, then $\alpha_+ > 0$ and we have physical solution that starts with subsonic speed near the Sun and is continuously accelerated as it moves from the Sun to cross the critical point and become supersonic. When $\alpha = 3/2$, the solution through the critical point has a constant wind speed.

What is happening physically can be deduced by comparing terms in the momentum equation. When $1 < \alpha < 3/2$, then $dv/dr > 0$ at all radii and

$$-\frac{dp}{dr} > \frac{\rho G M_\odot}{r^2},$$

so that the pressure falls off with r and the outwards pressure force always dominates gravity. On the other hand, when $\alpha = 3/2$, then $dv/dr > 0$, the pressure force is proportional to r^{-2} and the pressure gradient balances gravity. Finally, when $3/2 < \alpha < 5/3$, then $dv/dr < 0$ and

$$-\frac{dp}{dr} < \frac{\rho G M_\odot}{r^2},$$

so that the pressure falls off so slowly with r that gravity always dominates the pressure force.

PROBLEM 13.8. Properties of a Rotating Wind with Angular Speed $\Omega_\odot$.

Show that in the equatorial plane the addition of rotation:

(a) changes Eq.(13.5) to

$$\left(v_r - \frac{v_c^2}{v_r}\right)\frac{dv_r}{dr} = \frac{2v_c^2}{r^3}(r^2 - r_{c0}r + \tfrac{1}{2}\tau^2 r_0^2),$$

where r_{c0} is the critical point radius in the absence of rotation and $\tau = r_0\Omega_\odot/v_c$;

(b) gives one critical point for $r_{c0} > r_0$ when $0 < \tau < \tau_1$, two when $\tau_1 < \tau < \tau_2$ and none when $\tau > \tau_2$, where $\tau_1 = \sqrt{2}(r_{c0}/r_0 - 1)^{1/2}$ and $\tau_2 = r_{c0}/(\sqrt{2}r_0)$;

(c) and changes the solar wind density by a small amount to be estimated.

SOLUTION (adapted from book by Marcel Goossens "An Introduction to Plasma Astrophysics and MHD").

(a) **Effect on Equation for v_r**

Consider a steady-state thermally-driven wind rotating with angular speed $\Omega_\odot$ in spherical polar coordinates (r, θ, ϕ) by assuming azimuthal symmetry $(\partial/\partial\phi = 0)$ and considering only the flow components (v_r, v_ϕ) in the equatorial plane.

The equation of axial momentum is

$$v_r \frac{dv_\phi}{dr} + \frac{v_r v_\phi}{r} = 0,$$

which may be integrated to give $rv_\phi = L$ or

$$v_\phi = \frac{L}{r} = \frac{r_0^2\Omega_\odot}{r},$$

where L is a constant, namely, the angular momentum per unit mass.

The equation of radial momentum is

$$v_r \frac{dv_r}{dr} - \frac{v_\phi^2}{r} = \frac{GM_\odot}{r^2} - \frac{1}{r}\frac{dp}{dr}, \qquad (7)$$

where $v_\phi^2/r = L^2/r^3 = r_0^4\Omega_\odot^2/r^3$. Integrating this gives in general Bernoulli's equation, namely,

$$\frac{v_r^2 + v_\phi^2}{2} - \frac{GM_\odot}{r} + \int \frac{dp}{\rho} = E,$$

where E is constant.

For an isothermal plasma, $p = v_c^2\rho$, where $v_c^2 = \tilde{R}T/\tilde{\mu}$ is the square of the *isothermal sound speed*, and so Bernouilli's equation becomes

$$\frac{v_r^2 + v_\phi^2}{2} - \frac{GM_\odot}{r} + v_c^2 \log_e \frac{\rho(r)}{\rho(r_0)} = E. \qquad (8)$$

Now, substitute into the radial equation of motion (7) for $p(r) = v_c^2 \rho(r)$ and for $\rho(r)$ from the equation of mass conservation,

$$\rho \; v_r \; r^2 = \text{constant}$$

to give, as required,

$$\left(v_r - \frac{v_c^2}{v_r} \right) \frac{dv_r}{dr} = \frac{2v_c^2}{r^3}(r^2 - r_{c0} r + \tfrac{1}{2}\tau^2 r_0^2),$$

where $r_{c0} = GM_\odot / v_c^2$ is the critical point radius in the absence of rotation and $\tau = r_0 \Omega_\odot / v_c = v_\phi(r_0)/v_c$ measures the importance of rotation.

(b) **Critical Points**

The critical points of the above equation, where dv_r/dr is undefined, occur where the brackets on both sides vanish, namely, when $v_r = v_c$ and $r = r_c$ such that

$$f(r_c) \equiv r_c^2 - r_{c0} r_c + \tfrac{1}{2}\tau^2 r_0^2 = 0. \tag{9}$$

Let us assume that $r_{c0} > r_0$ so that the critical point in the case of no rotation lies above the coronal base, as in the solar case. For the Sun, $\tau \approx 0.01$ and so rotation has a small effect on the wind solution. However, for rapidly rotating objects, two critical values of τ arise, namely,

$$\tau_1 = \sqrt{2} \left(\frac{r_{c0}}{r_0} - 1 \right)^{1/2}, \qquad \tau_2 = \frac{r_{c0}}{\sqrt{2} r_0}.$$

Note that $f(r_c)$ is a quadratic function of r_c with either 0, 1 or 2 roots.

When $\tau = 0$, we have either $r = r_{c0}$, the previous non-rotating result or $r = 0$, which is unphysical, so let us consider what happens as we increase the value of the parameter τ. First of all, when $\tau \ll 1$, we may expand about these two solutions to find one physical solution, namely, $r \approx r_{c0}(1 - \frac{1}{2}\tau^2)$, and one unphysical solution, namely, $r \approx r_0^2 \tau^2/(2r_{c0})$, which lies below the coronal base $r = r_0$.

The presence of exactly one physical solution to (9) persists for higher values of τ in the range $0 < \tau < \tau_1$, where τ_1 is the value of τ that makes $r_c = r_0$, so that the second solution is at the coronal base. Putting $r_c = r_0$ in the equation (9), we find

$$r_0^2 - r_{c0} r_0 + \tfrac{1}{2}\tau^2 r_0^2 = 0,$$

or

$$\tau^2 = 2\left(\frac{r_{c0}}{r_0} - 1\right),$$

as required.

When $\tau > \tau_1$, the second solution moves above $r = r_0$ and so we have two physical solutions. However, this situation does not continue indefinitely as we increase τ. The minimum in the quadratic function $f(r_c)$ given by (9) is located at $r_c = \frac{1}{2}r_{c0}$, and the value of $f(r_c)$ at this point is

$$f(\tfrac{1}{2}r_{c0}) = -\frac{r_{c0}^2}{4} + \frac{\tau^2 r_0^2}{2}.$$

Thus, $f(\frac{1}{2}r_{c0}) > 0$ and so there are no solutions, when

$$\tau > \tau_2 \equiv \frac{r_{c0}}{\sqrt{2}\, r_0}.$$

In other words, we have shown that there are two critical points when $\tau_1 < \tau < \tau_2$ and none when $\tau > \tau_2$, as required.

Density

From Eq.(8) the density can be written in terms of the velocity as

$$\frac{\rho(r)}{\rho(r_0)} = \exp\frac{1}{2v_c^2}\left[2E - (v_r^2 + v_\phi^2) + \frac{2r_{c0}}{r}\right],$$

where $r_{c0} = GM_\odot/v_c^2$ and, by putting $r = r_0$, we have $E = \frac{1}{2}(v_{r0}^2 + v_{\phi 0}^2) - r_{c0}/r_0$. Thus, the density can be rewritten,

$$\frac{\rho(r)}{\rho(r_0)} = \exp\frac{1}{2v_c^2}\left[v_{r0}^2 - v_r^2 + v_{\phi 0}^2 - v_\phi^2 + 2r_{c0}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right].$$

After putting $v_\phi = r_0^2\Omega_\odot/r$, $v_{\phi 0} = r_0\Omega_\odot$ and $\tau = r_0\Omega_\odot/v_c$, this becomes

$$\frac{\rho(r)}{\rho(r_0)} = \exp\left[-\frac{r_{c0}}{r_0}\left(1 - \frac{r_0}{r}\right)\right] \times \exp\left[-\frac{v_r^2 - v_{r0}^2}{2v_c^2}\right] \times \exp\left[\frac{\tau^2}{2}\left(1 - \frac{r_0^2}{r^2}\right)\right].$$

The first exponential represents a hydrostatic atmosphere, while the second gives the effect of flow and the third shows the effect of rotation.

Since the final exponential is close to but in excess of unity (since $\tau \ll 1$ and $r > r_0$) the effect of rotation is to increase slightly the density and so to make it decrease less rapidly with distance. Correspondingly, since $\rho\, v_r\, r^2 =$ constant, the outflow speed decreases slightly and so increases less rapidly with distance.

PROBLEM 13.9. Angular Momentum Loss
Show that the time-scale for angular momentum loss is of order the Sun's age.

SOLUTION. (adapted from book by Marcel Goossens "An Introduction to Plasma Astrophysics and MHD").

From PROBLEM 13.3, we have

$$\frac{dM_\odot}{dt} \approx 10^{-14} M_\odot / yr.$$

On the other hand, the rate of loss of angular momentum is

$$\frac{dJ_\odot}{dt} = \frac{dM_\odot}{dt} r_A^2 \Omega_\odot \approx 10^{-14} M_\odot r_A^2 \Omega_\odot / \mathrm{yr}$$

Now, assume that the Sun rotates as a solid body and that its moment of inertia is the same as that of a uniform-density sphere of the same mass and radius. Then

$$J_\odot = \Omega_\odot I_\odot = \Omega_\odot \frac{2 M_\odot R_\odot^2}{5}$$

The time-scale for angular momentum loss follows (with $r_A = 12\ R_\odot$) as

$$\tau_J = \frac{J_\odot}{dJ_\odot/dt} \approx \frac{0.4}{144 \times 10^{-14}} \mathrm{yr} \approx 10^{11} \mathrm{yr}$$

which is ten times the Sun's age of 10^{10} yr.

This rough estimate is likely to underestimate the much faster rate of angular momentum loss in the Sun's youth, when its angular velocity and magnetic field are likely to have been much larger.

PROBLEM 13.10. Isothermal Coronal Hole.
For an isothermal coronal hole with area function $A(r) = ar^n$, show how the value of n affects the location of the critical point and find the behaviour of the velocity and pressure at large distances for the solar wind solution.

SOLUTION. For a nonradial area expansion $A(r) = ar^n$, the equation of mass continuity becomes

$$r^n \rho v = K,$$

say, where K is a constant. This equation determines the density $\rho(r)$ in terms of the velocity $v(r)$.

The momentum equation with the usual inverse-square law for gravity is unchanged as

$$\rho v \frac{dv}{dr} = -\frac{dp}{dr} - \frac{\rho G M_\odot}{r^2},$$

and, since the plasma is isothermal, the equation of state is

$$p = v_c^2 \rho,$$

where $v_c^2 = RT$ is constant.

These equations may be combined to give the basic equation for $v(r)$ with a critical sonic point at $(v, r) = (v_c, r_c)$, namely,

$$\left(v - \frac{v_c^2}{v}\right) \frac{dv}{dr} = \frac{n v_c^2}{r^2}(r - r_c),$$

where $r_c = GM_\odot/(nc_c^2)$. Thus, the critical radius (r_c) is located closer to the Sun if the open magnetic field expands more rapidly (i.e., the value of n is larger).

The solution of this separable equation that passes through the critical point is

$$\tfrac{1}{2}\left(\frac{v^2}{v_c^2} - 1\right) - \log_e\left(\frac{v}{v_c}\right) = n \log_e\left(\frac{r}{r_c}\right) + n\left(\frac{r_c}{r} - 1\right).$$

When r is large, the first term on the left and the first on the right dominate and so

$$v \sim \sqrt{2 n v_c^2 \log_e \frac{r}{r_c}}.$$

The asymptotic behaviour of the density follows from the continuity equation as

$$\rho \sim \frac{K}{r^n \sqrt{[2 n v_c^2 \log_e(r/r_c)]}}.$$

Since the temperature is constant, this implies that $p \to 0$ as $r \to \infty$, as expected for the solar wind.