Chapter 4

```
> restart;
> with(linalg):
Warning, the protected names norm and trace have been redefined and
unprotected
> with(plots):
Warning, the name changecoords has been redefined
> with(DEtools):
Warning, the name adjoint has been redefined
```

Question 1

(i)

We can readily write this

$$\frac{dy}{dx} = \frac{y}{2\,x}$$

$$\frac{dy}{y} = \frac{dx}{2\,x}$$

Integrating both sides

$$\int \frac{1}{y}\,dy = \int \frac{1}{2\,x}\,dx$$

$$\ln(y) = \frac{1\,\ln(x)}{2} + c_0$$

where c_0 is the constant of integration. Then

$$y = c\sqrt{x}$$

If $x(0) = 2$ and $y(0) = 3$, then $3 = c\sqrt{2}$ and

$$y = \frac{3\sqrt{x}}{\sqrt{2}}$$

(ii)

To verify this result

```
> dsolve(diff(y(x),x)=y(x)/(2*x),y(x));
```

$$y(x) = \_C1\,\sqrt{x}$$

```
> dsolve({diff(y(x),x)=y(x)/(2*x),y(2)=3},y(x));
```

$$y(x) = \frac{3}{2}\sqrt{2}\,\sqrt{x}$$

which is the same as

$$y(x) = \frac{3\sqrt{x}}{\sqrt{2}}$$

Question 2

(a)-(d)

```
> eqs1:={diff(x(t),t)=x(t)-3*y(t), diff(y(t),t)=-2*x(t)+y(t)};
```

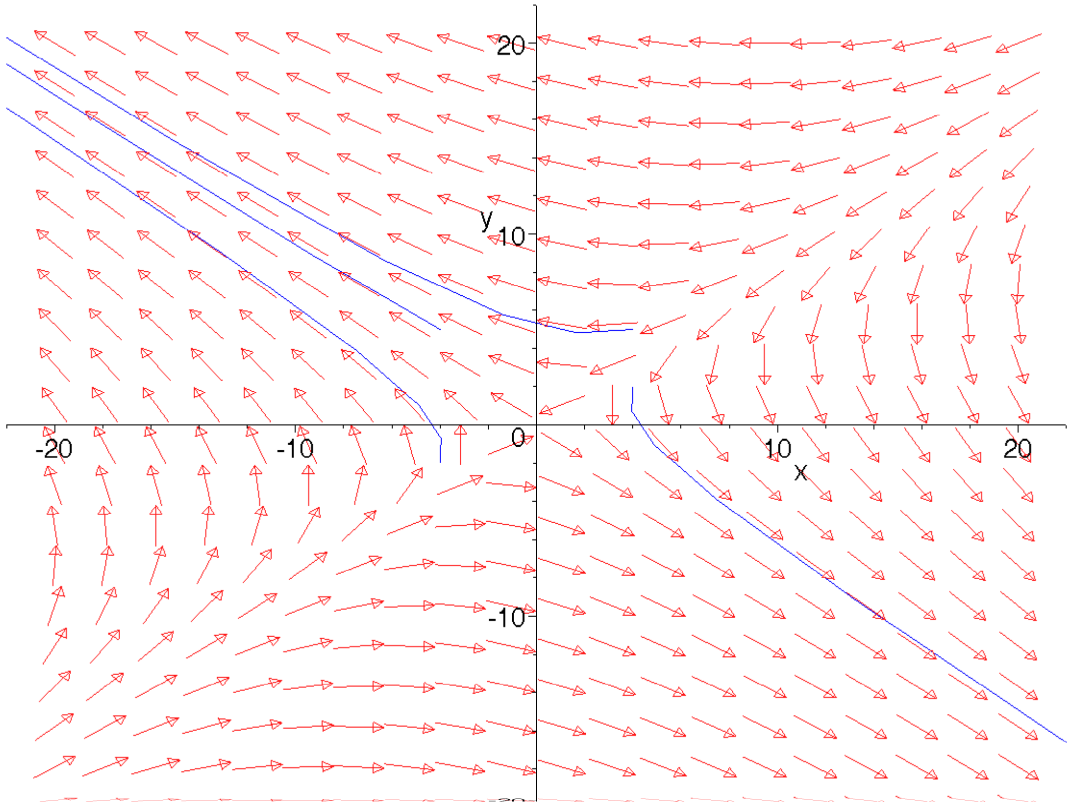
$$eqs1 := \{\frac{\partial}{\partial t}x(t) = x(t) - 3\,y(t), \frac{\partial}{\partial t}y(t) = -2\,x(t) + y(t)\}$$

```
> init1:=[[x(0)=4,y(0)=2],
  [x(0)=4,y(0)=5],[x(0)=-4,y(0)=-2],[x(0)=-4,y(0)=5] ];
```

$init1 := [$

$[x(0) = 4, y(0) = 2], [x(0) = 4, y(0) = 5], [x(0) = -4, y(0) = -2], [x(0) = -4, y(0) = 5]]$

```
> DEplot(eqs1,[x(t),y(t)],t=0..10,x=-20..20,y=-20..20,init1,ste
  psize=.2,arrows=medium, linecolour=blue,thickness=1);
```



Question 3

(i)

```
> x:='x':y:='y':
> eqs3:={diff(x(t),t)=-3*x(t)+y(t),diff(y(t),t)=x(t)-3*y(t)}
  ;
```

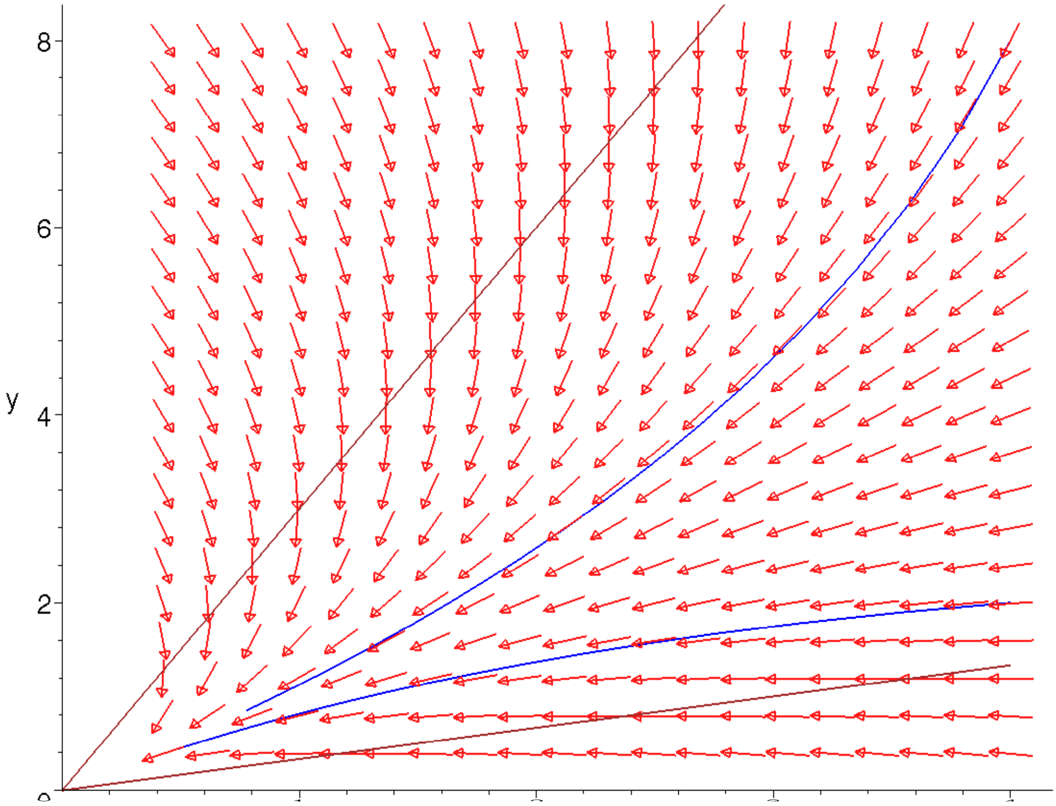
$$eqs3 := \{\frac{\partial}{\partial t}x(t) = -3\,x(t) + y(t), \frac{\partial}{\partial t}y(t) = x(t) - 3\,y(t)\}$$

```
> init31:=[[x(0)=4,y(0)=8],[x(0)=4,y(0)=2]];
```

$$init31 := [[x(0) = 4, y(0) = 8], [x(0) = 4, y(0) = 2]]$$

```
> curves31:=DEplot(eqs3,[x(t),y(t)],t=0..1,init31,stepsize=.
  2,arrows=medium, linecolour=blue,thickness=2):
> lines31:=plot({3*x,(1/3)*x},x=0..4,y=0..8,colour=brown,thi
```

```
ckness=2):
> display({curves31,lines31});
```

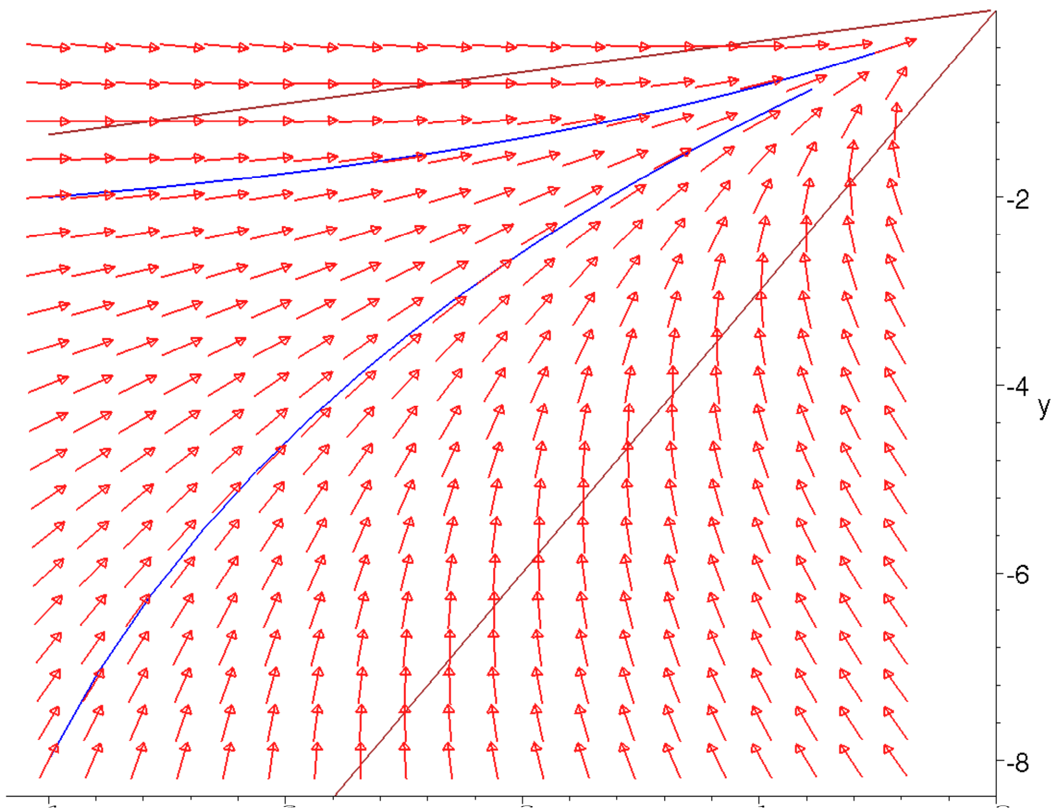


(ii)

```
> init32:=[[x(0)=-4,y(0)=-8],[x(0)=-4,y(0)=-2]];
```

init32 := [[x(0) = -4, y(0) = -8], [x(0) = -4, y(0) = -2]]

```
> curves32:=DEplot(eqs3,[x(t),y(t)],t=0..1,init32,stepsize=.
2,arrows=medium, linecolour=blue,thickness=2):
> lines32:=plot({3*x,(1/3)*x},x=-4..0,colour=brown,thickness
=2):
> display({curves32,lines32});
```

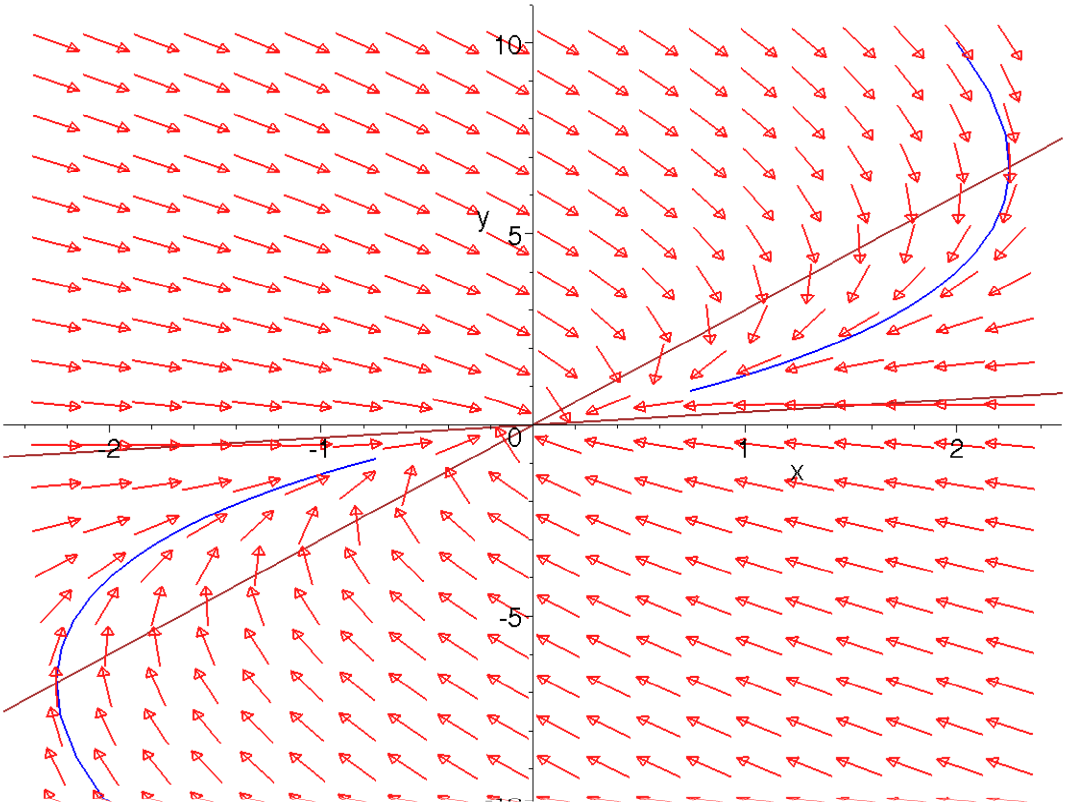


(iii)

```
> init33:=[[x(0)=2,y(0)=10],[x(0)=-2,y(0)=-10]];
```

$init33 := [[x(0) = 2, y(0) = 10], [x(0) = -2, y(0) = -10]]$

```
> curves33:=DEplot(eqs3,[x(t),y(t)],t=0..1,init33,stepsize=.
  2,arrows=medium, linecolour=blue,thickness=2):
> lines33:=plot({3*x,(1/3)*x},x=-2.5..2.5,colour=brown,thick
  ness=2):
> display({curves33,lines33});
```

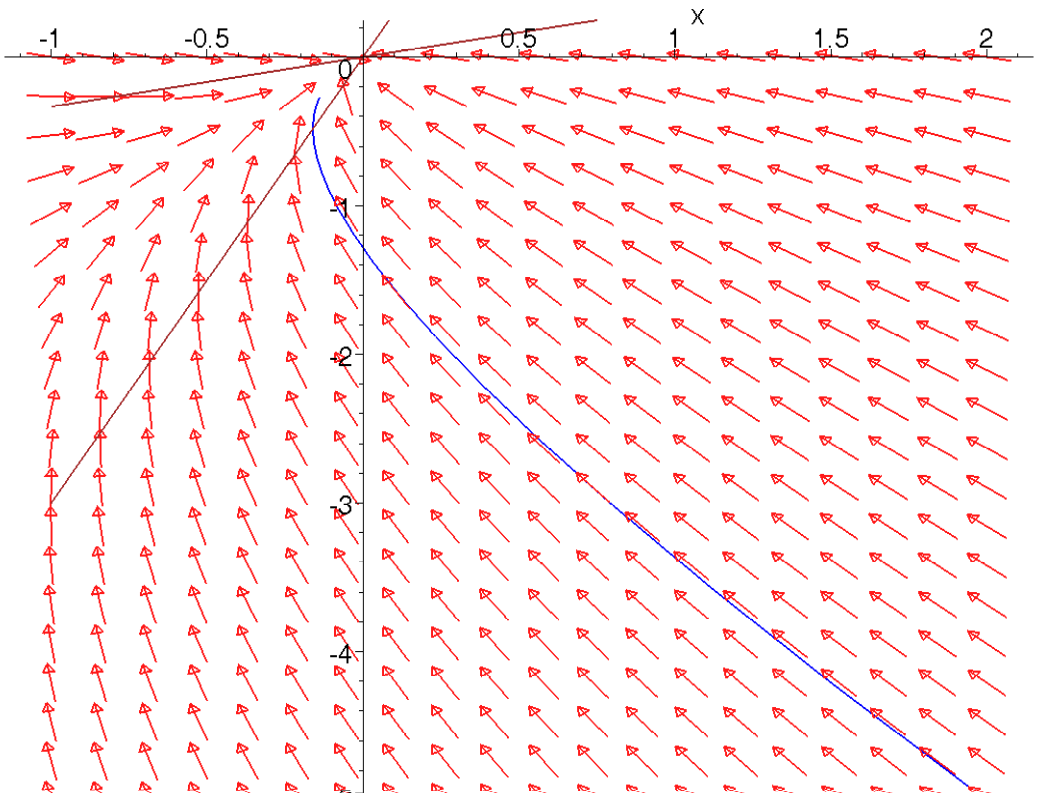


(iv)

```
> init34:=[[x(0)=2,y(0)=-5]];
```

$$init34 := [[x(0) = 2, y(0) = -5]]$$

```
> curves34:=DEplot(eqs3,[x(t),y(t)],t=0..1,x=-1..2,y=-5..0,i
nit34,stepsize=.2,arrows=medium,
linecolour=blue,thickness=2):
> lines34:=plot({3*x,(1/3)*x},x=-1..2,colour=brown,thickness
=2):
> display({curves34,lines34});
```



As can be seen from the resulting figure, the trajectory passes into another quadrant before converging on the fixed point.

Question 4

(i)-(iii)

```
> x:='x': y:='y':
> solve({-2*x-y+9=0,-y+x+3=0},{x,y});
```

$$\{x=2, y=5\}$$

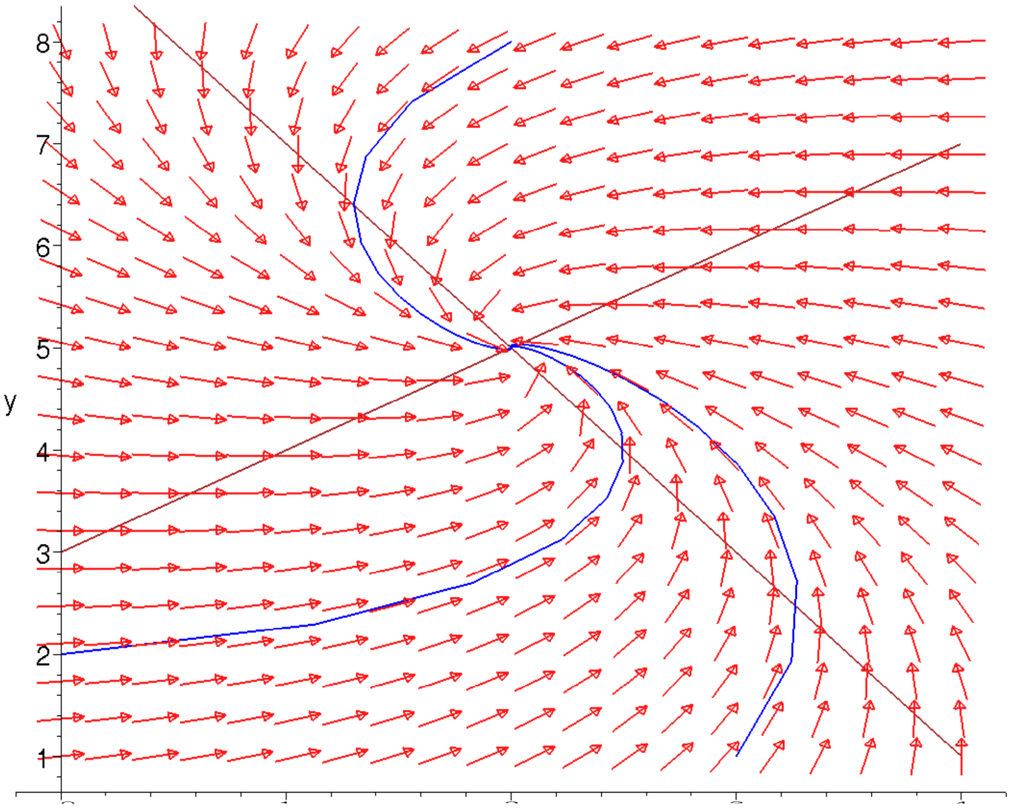
```
> eqs4:={diff(x(t),t)=-2*x(t)-y(t)+9,diff(y(t),t)=-y(t)+x(t)+3}
  ;
```

$$eqs4 := \{\frac{\partial}{\partial t}\mathrm{y}(t) = -\mathrm{y}(t) + \mathrm{x}(t) + 3, \frac{\partial}{\partial t}\mathrm{x}(t) = -2\,\mathrm{x}(t) - \mathrm{y}(t) + 9\}$$

```
> init4:=[[x(0)=0,y(0)=2],[x(0)=2,y(0)=8],[x(0)=3,y(0)=1]];
```

$$init4 := [[\mathrm{x}(0)=0, \mathrm{y}(0)=2], [\mathrm{x}(0)=2, \mathrm{y}(0)=8], [\mathrm{x}(0)=3, \mathrm{y}(0)=1]]$$

```
> curves41:=DEplot(eqs4,[x(t),y(t)],t=0..20,x=0..4,init4,stepsi
  ze=.2,arrows=medium, linecolour=blue,thickness=2):
> lines41:=plot({-2*x+9,x+3},x=0..4,colour=brown,thickness=2):
> display({curves41,lines41});
```



The resulting diagram shows quite clearly that all trajectories follow a counter-clockwise spiral towards the fixed point. However, it also shows that the spiral motion involves a direct *repeated* over- and under-shooting.

Question 5

(i)

The characteristic roots are obtained by defining the matrix of the system and using the **eigenvalues** command.

```
> A:=matrix([[2, 3], [3, 2]]);
```

$$A := \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

```
> eigenvalues(A);
```

$$5, -1$$

(ii)

```
> eigenvectors(A);
```

$$[5, 1, \{[1, 1]\}], [-1, 1, \{[1, -1]\}]$$

(iii)

```
> eq1:=diff(x(t),t)=2*x(t)+3*y(t);
```

$$eq1 := \frac{\partial}{\partial t} \mathrm{x}(t) = 2\,\mathrm{x}(t) + 3\,\mathrm{y}(t)$$

```
> eq2:=diff(y(t),t)=3*x(t)+2*y(t);
```

$$eq2 := \frac{\partial}{\partial t} \mathrm{y}(t) = 3\,\mathrm{x}(t) + 2\,\mathrm{y}(t)$$

```
> sol:=dsolve({eq1,eq2},{x(t),y(t)});
```

$$sol := \{y(t) = \_C1\, e^{(5t)} - \_C2\, e^{(-t)}, x(t) = \_C1\, e^{(5t)} + \_C2\, e^{(-t)}\}$$

```
> collect(sol,{exp(-t),exp(5*t)});
```

$$\{y(t) = \_C1\, e^{(5t)} - \_C2\, e^{(-t)}, x(t) = \_C1\, e^{(5t)} + \_C2\, e^{(-t)}\}$$

which can be expressed:

$$y(t) = c1\, e^{(5t)} + c2\, e^{(-t)}, x(t) = c1\, e^{(5t)} + c2\, e^{(-t)}$$

(iv)

```
> dsolve({eq1,eq2,x(0)=1,y(0)=0},{x(t),y(t)});
```

$$\{y(t) = \frac{1}{2} e^{(5t)} - \frac{1}{2} e^{(-t)}, x(t) = \frac{1}{2} e^{(5t)} + \frac{1}{2} e^{(-t)}\}$$

Question 6

```
> A:='A':eq1:='eq1':eq2:='eq2':
> A:=matrix([[1, 3], [5, 3]]);
```

$$A := \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$$

(i)

```
> eigenvalues(A);
```

6, -2

(ii)

```
> eigenvectors(A);
```

$$\left[6, 1, \{\left[1, \frac{5}{3}\right]\}\right], [-2, 1, \{[-1, 1]\}]$$

(iii)

```
> eq1:=diff(x(t),t)=x(t)+3*y(t);
```

$$eq1 := \frac{\partial}{\partial t} x(t) = x(t) + 3\, y(t)$$

```
> eq2:=diff(y(t),t)=5*x(t)+3*y(t);
```

$$eq2 := \frac{\partial}{\partial t} y(t) = 5\, x(t) + 3\, y(t)$$

```
> dsolve({eq1,eq2,x(0)=1,y(0)=3},{x(t),y(t)});
```

$$\{x(t) = \frac{3}{2} e^{(6t)} - \frac{1}{2} e^{(-2t)}, y(t) = \frac{5}{2} e^{(6t)} + \frac{1}{2} e^{(-2t)}\}$$

Question 7

```
> A:='A':
> V:=matrix([[1, 1], [-2, 2]]);
```

$$V := \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$$

```
> W:=det(V);
```

$$W := 4$$

```
> A:=matrix([[1, 1], [-2, 4]]);
```

$$A := \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

```
> eigenvalues(A);
```

$$3, 2$$

```
> eigenvectors(A);
```

$$[2, 1, \{[1, 1]\}], [3, 1, \{[1, 2]\}]$$

The Wronksian matrix, here denoted W1, is formed from the eigenvectors of the system. If det(W1) is non-zero, then the eigenvectors are linearly independent.

```
> V1:=matrix([[1, 1], [1, 2]]);
```

$$V1 := \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

```
> W1:=det(V1);
```

$$\mathrm{W1} := 1$$

NOTE: It does not matter which way the eigenvectors are listed. If in reverse order then

```
> V2:=matrix([[1, 1], [2, 1]]);
```

$$V2 := \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

```
> W2:=det(V2);
```

$$\mathrm{W2} := -1$$

which is still non-zero. All that matters for proving linear independence is that the Wronksian is non-zero.

Question 8

```
> x:='x':y:='y':A:='A':W:='W':
> A:=matrix([[1, 0, 0], [2, 3, 1], [0, 2, 4]]);
```

$$A := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

(i)

The eigenvalues and eigenvectors are as follows:

```
> eigenvalues(A);
```

$$1, 5, 2$$

```
> eigenvectors(A);
```

$$\left[1, 1, \left\{\left[1, \frac{-3}{2}, 1\right]\right\}\right], [5, 1, \{[0, 1, 2]\}], [2, 1, \{[0, -1, 1]\}]$$

```
>
```

(ii)

The general solution is derived by solving the following differential equation system:

```
> eq81:=diff(x(t),t)=x(t);
  eq82:=diff(y(t),t)=2*x(t)+3*y(t)+z(t);
  eq83:=diff(z(t),t)=2*y(t)+4*z(t);
```

$$eq81 := \frac{\partial}{\partial t} x(t) = x(t)$$

$$eq82 := \frac{\partial}{\partial t} y(t) = 2\,x(t) + 3\,y(t) + z(t)$$

$$eq83 := \frac{\partial}{\partial t} z(t) = 2\,y(t) + 4\,z(t)$$

```
> sol8:=dsolve({eq81,eq82,eq83},{x(t),y(t),z(t)});
```

$$sol8 := \{z(t) = \_C3\,e^{t} + \_C1\,e^{(2t)} + \_C2\,e^{(5t)},\ x(t) = \_C3\,e^{t},\ y(t) = -\frac{3}{2}\_C3\,e^{t} - \_C1\,e^{(2t)} + \frac{1}{2}\_C2\,e^{(5t)}\}$$

```
> collect(sol8,{exp(t),exp(2*t),exp(5*t)});
```

$$\{z(t) = \_C3\,e^{t} + \_C1\,e^{(2t)} + \_C2\,e^{(5t)},\ x(t) = \_C3\,e^{t},\ y(t) = -\frac{3}{2}\_C3\,e^{t} - \_C1\,e^{(2t)} + \frac{1}{2}\_C2\,e^{(5t)}\}$$

```
>
```

(iii)

The Wronksian is given by:

```
> V:=matrix([[0, 1, 0], [1, -3/2, 1], [-1, 1, 2]]);
```

$$V := \begin{bmatrix} 0 & 1 & 0 \\ 1 & \frac{-3}{2} & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

```
> W:=det(V);
```

$$W := -3$$

Question 9

```
> x:='x':y:='y':A:='A':
```

(i)

(a)

```
> A:=matrix([[-3, 1], [1, -3]]);
```

$$A := \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

```
> eigenvalues(A);
```

$$-2, -4$$

```
> eigenvectors(A);
```

$$[-2, 1, \{[1, 1]\}], [-4, 1, \{[-1, 1]\}]$$

(b)

```
> sol91:=dsolve({diff(x(t),t)=-3*x(t)+y(t),diff(y(t),t)=x
(t)-3*y(t)},{x(t),y(t)});
```

$$sol91 := \{x(t) = \_C1\,e^{(-4t)} + \_C2\,e^{(-2t)},\ y(t) = -\_C1\,e^{(-4t)} + \_C2\,e^{(-2t)}\}$$

```
> collect(sol91,{exp(-4*t),exp(-2*t)});
```

$$\{x(t) = \_C1\, e^{(-4\,t)} + \_C2\, e^{(-2\,t)},\ y(t) = -\_C1\, e^{(-4\,t)} + \_C2\, e^{(-2\,t)}\}$$

(c)

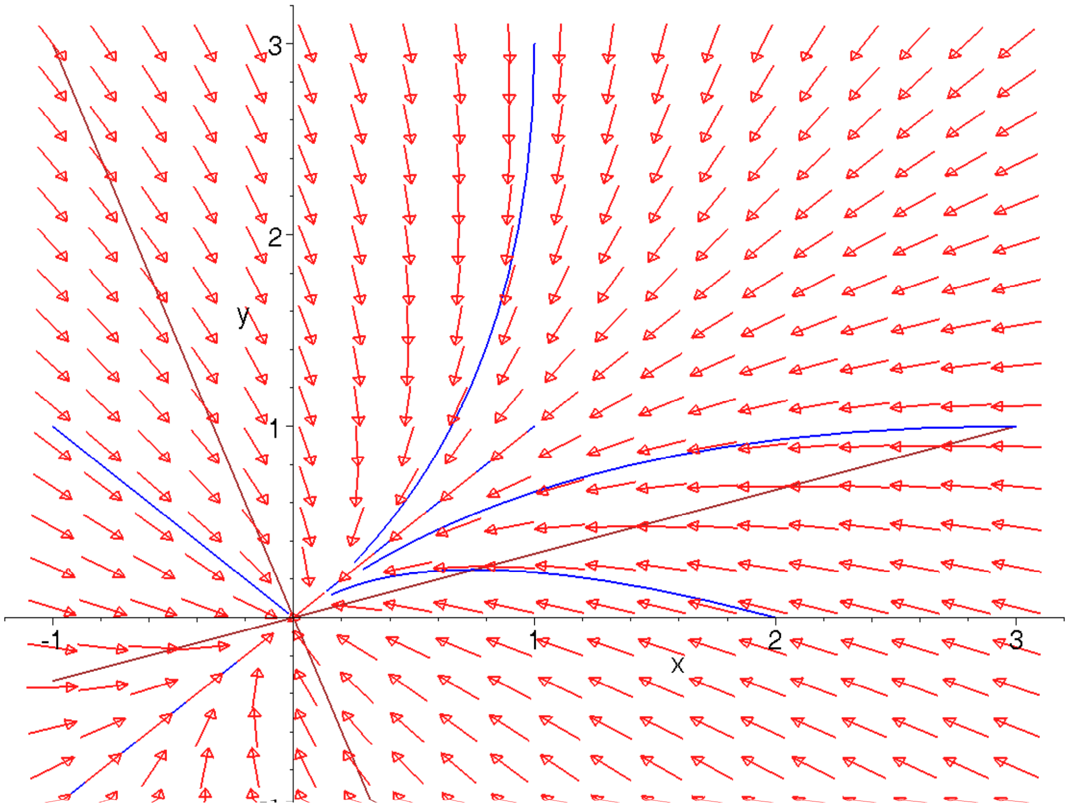
```
> eqs91:={diff(x(t),t)=-3*x(t)+y(t),diff(y(t),t)=x(t)-3*y(t)};
```

$$eqs91 := \{\frac{\partial}{\partial t} x(t) = -3\, x(t) + y(t),\ \frac{\partial}{\partial t} y(t) = x(t) - 3\, y(t)\}$$

```
> init91:=[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=1],[x(0)=-1,y(0)=-1],[x(0)=2,y(0)=0],[x(0)=3,y(0)=1],[x(0)=1,y(0)=3]];
```

$$init91 := [[x(0) = 1,\ y(0) = 1],\ [x(0) = -1,\ y(0) = 1],\ [x(0) = -1,\ y(0) = -1],\ [x(0) = 2,\ y(0) = 0],\ [x(0) = 3,\ y(0) = 1],\ [x(0) = 1,\ y(0) = 3]]$$

```
> curves91:=DEplot(eqs91,[x(t),y(t)],t=0..1,x=-1..3,y=-1..3,init91,stepsize=.2,arrows=medium,linecolour=blue,thickness=2):
> lines91:=plot({-3*x,(1/3)*x},x=-1..3,y=-1..3,colour=brown,thickness=2):
> display({curves91,lines91});
```



(d)

All trajectories converge on the fixed point and the system shows an improper node.

(ii)

```
> x:='x':y:='y':A:='A':
```

(a)

```
> A:=matrix([[2, -4], [1, -3]]);
```

$$A := \begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}$$

```
> eigenvalues(A);
```

$$1, -2$$

```
> eigenvectors(A);
```

$$[1, 1, \{[4, 1]\}], [-2, 1, \{[1, 1]\}]$$

(b)

```
> sol92:=dsolve({diff(x(t),t)=2*x(t)-4*y(t),diff(y(t),t)=
x(t)-3*y(t)},{x(t),y(t)});
```

$$sol92 := \{x(t) = \_C1\, e^t + \_C2\, e^{(-2\,t)}, y(t) = \frac{1}{4}\_C1\, e^t + \_C2\, e^{(-2\,t)}\}$$

```
> collect(sol92,{exp(-2*t),exp(t)});
```

$$\{x(t) = \_C1\, e^t + \_C2\, e^{(-2\,t)}, y(t) = \frac{1}{4}\_C1\, e^t + \_C2\, e^{(-2\,t)}\}$$

(c)

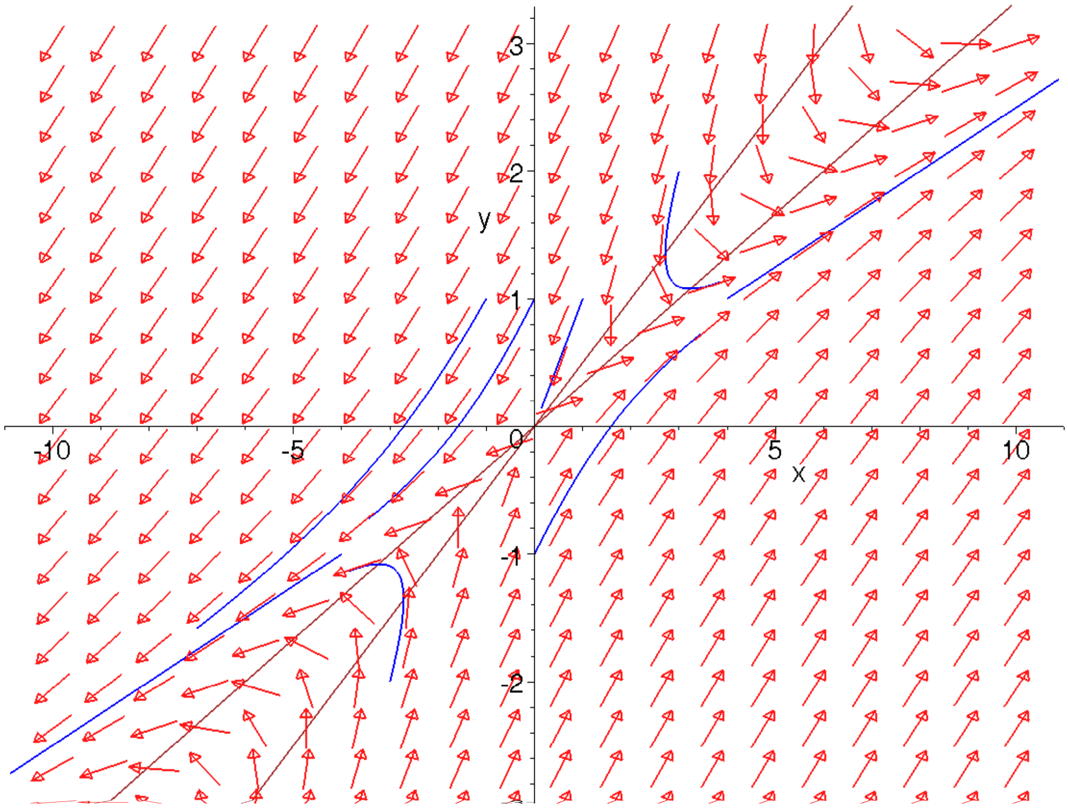
```
> eqs92:={diff(x(t),t)=2*x(t)-4*y(t),diff(y(t),t)=x(t)-3*
y(t)};
```

$$eqs92 := \{\frac{\partial}{\partial t}y(t) = x(t) - 3\,y(t), \frac{\partial}{\partial t}x(t) = 2\,x(t) - 4\,y(t)\}$$

```
> init92:=[[x(0)=1,y(0)=1],[x(0)=-1,y(0)=1],[x(0)=4,y(0)=
1],[x(0)=-4,y(0)=-1],[x(0)=0,y(0)=1],[x(0)=0,y(0)=-1],[
x(0)=3,y(0)=2],[x(0)=-3,y(0)=-2]];
```

$$init92 := [[x(0) = 1, y(0) = 1], [x(0) = -1, y(0) = 1], [x(0) = 4, y(0) = 1], [x(0) = -4, y(0) = -1], [x(0) = 0, y(0) = 1], [x(0) = 0, y(0) = -1], [x(0) = 3, y(0) = 2], [x(0) = -3, y(0) = -2]]$$

```
> curves92:=DEplot(eqs92,[x(t),y(t)],t=0..1,x=-10..10,y=-
3..3,init92,stepsize=.2,arrows=medium,linecolour=blue,t
hickness=2):
> lines92:=plot({(1/2)*x,(1/3)*x},x=-10..10,y=-3..3,colou
r=brown,thickness=2):
> display({curves92,lines92});
```



(d)

This diagram shows a saddle-point solution.

(iii)

```
> x:='x':y:='y':A:='A':
```

(a)

```
> A:=matrix([[0, 1], [-4, 0]]);
```

$$A := \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

```
> eigenvalues(A);
```

$$2\,I,\ -2\,I$$

```
> eigenvectors(A);
```

$$[2\,I, 1, \{[1, 2\,I]\}], [-2\,I, 1, \{[1, -2\,I]\}]$$

(b)

```
> sol93:=dsolve({diff(x(t),t)=y(t),diff(y(t),t)=-4*x(t)},
{x(t),y(t)});
```

sol93 :=

$$\{y(t) = 2\ \_C1\cos(2\,t) - 2\ \_C2\sin(2\,t),\ x(t) = \_C1\sin(2\,t) + \_C2\cos(2\,t)\}$$

(c)

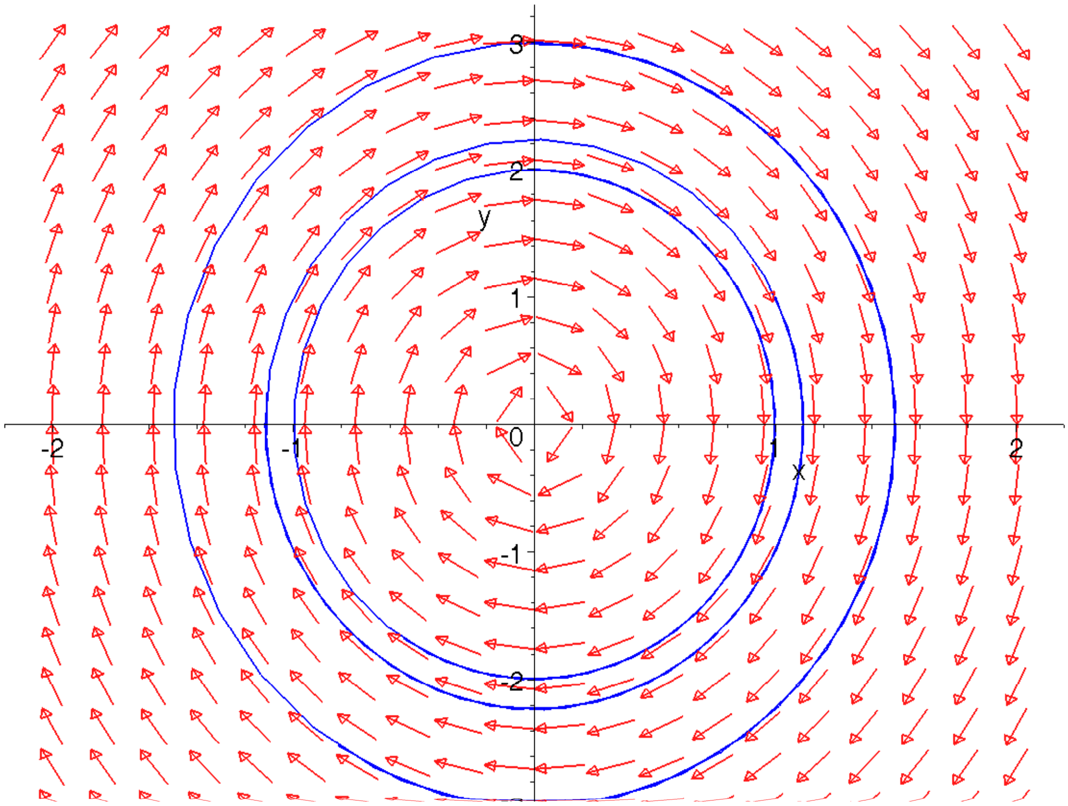
```
> eqs93:={diff(x(t),t)=y(t),diff(y(t),t)=-4*x(t)};
```

$$eqs93 := \{\frac{\partial}{\partial t}\,y(t) = -4\,x(t),\ \frac{\partial}{\partial t}\,x(t) = y(t)\}$$

```
> init93:=[[x(0)=1,y(0)=1],[x(0)=0,y(0)=2],[x(0)=0,y(0)=3
]];
```

$$init93 := [[x(0) = 1, y(0) = 1], [x(0) = 0, y(0) = 2], [x(0) = 0, y(0) = 3]]$$

```
> curves93:=DEplot(eqs93,[x(t),y(t)],t=0..5,x=-2..2,y=-3..3,init93,stepsize=.1,arrows=medium,linecolour=blue,thickness=2):
> display(curves93);
```



(d)

This diagram shows a centre node, with a clear cyclical pattern.

(iv)

```
> x:='x':y:='y':A:='A':
```

(a)

```
> A:=matrix([[-1, 1], [-1, -1]]);
```

$$A := \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

```
> eigenvalues(A);
```

$$-1 + I, -1 - I$$

```
> eigenvectors(A);
```

$$[-1 + I, 1, \{[1, I]\}], [-1 - I, 1, \{[1, -I]\}]$$

(b)

```
> sol94:=dsolve({diff(x(t),t)=-x(t)+y(t),diff(y(t),t)=-x(t)-y(t)},{x(t),y(t)});
```

$sol94 := \{$

$$y(t) = -e^{(-t)}(-\_C1\cos(t) + \_C2\sin(t)), x(t) = e^{(-t)}(\_C1\sin(t) + \_C2\cos(t))\}$$

(c)

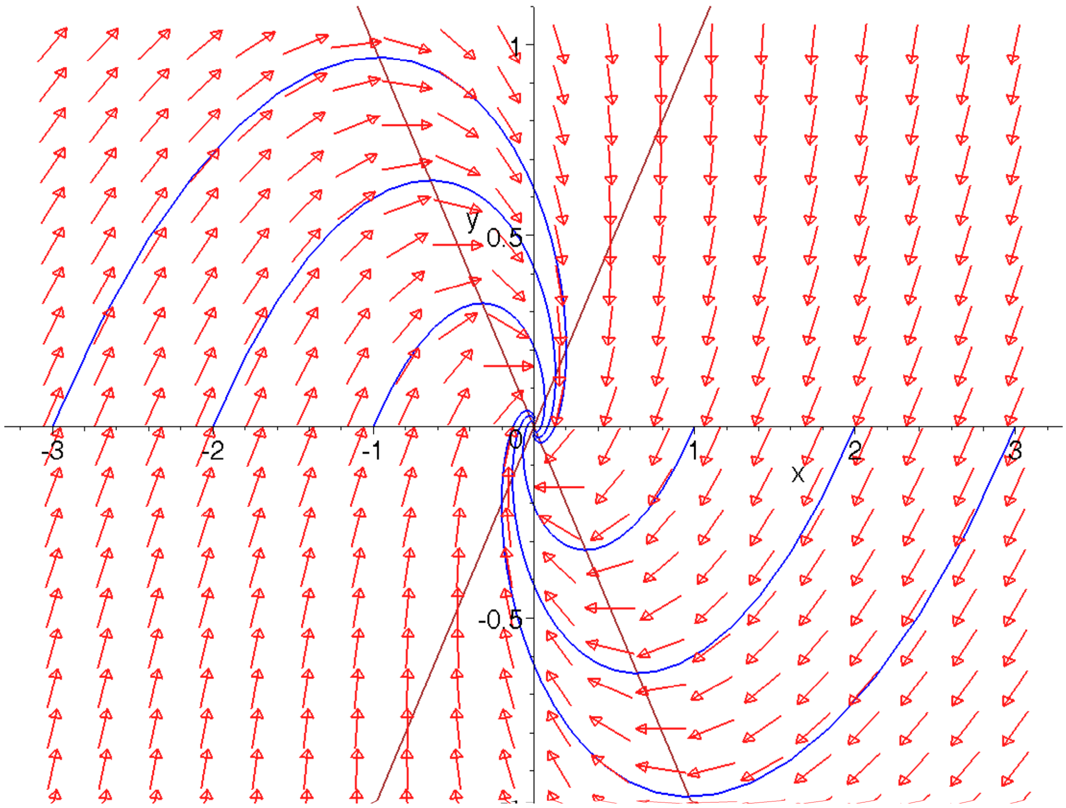
```
> eqs94:={diff(x(t),t)=-x(t)+y(t),diff(y(t),t)=-x(t)-y(t)};
```

$$eqs94 := \{\frac{\partial}{\partial t}x(t) = -x(t) + y(t), \frac{\partial}{\partial t}y(t) = -x(t) - y(t)\}$$

```
> init94:=[[x(0)=1,y(0)=0],[x(0)=2,y(0)=0],[x(0)=3,y(0)=0
],[x(0)=-1,y(0)=0],[x(0)=-2,y(0)=0],[x(0)=-3,y(0)=0]];
```

$init94 := [[x(0) = 1, y(0) = 0], [x(0) = 2, y(0) = 0], [x(0) = 3, y(0) = 0], [x(0) = -1, y(0) = 0], [x(0) = -2, y(0) = 0], [x(0) = -3, y(0) = 0]]$

```
> curves94:=DEplot(eqs94,[x(t),y(t)],t=0..5,x=-3..3,y=-1..1,init94,stepsize=.1,arrows=medium,linecolour=blue,thickness=2):
> lines94:=plot({x,-x},x=-3..3,y=-1..1,colour=brown,thickness=2):
> display({curves94,lines94});
```



(d)

Question 10

(i)

Fixed points are

```
> solve({0=x*(1-x/6)-6*x*y/(8+8*x),0=0.2*y*(1-0.4*y/x)},{x,y});
```

$\{y = 0., x = 6.\}, \{x = -7.095595044, y = -17.73898761\}, \{x = .8455950436, y = 2.113987609\}$

We rule out the negative values of x and y, so that there are two fixed points to consider, P1 = (0,6) and P2 = (0.8456, 2.1140)

(ii)

```
> fx:=diff(x*(1-x/6)-6*x*y/(8+8*x),x);
```

$$fx := 1 - \frac{1}{3}x - \frac{6\,y}{8+8\,x} + \frac{48\,x\,y}{(8+8\,x)^2}$$

```
> fy:=diff(x*(1-x/6)-6*x*y/(8+8*x),y);
```

$$fy := -6\,\frac{x}{8+8\,x}$$

```
> gx:=diff(0.2*y*(1-0.4*y/x),x);
```

$$gx := .08\,\frac{y^2}{x^2}$$

```
> gy:=diff(0.2*y*(1-0.4*y/x),y);
```

$$gy := .2 - \frac{.16\,y}{x}$$

```
> fx1=subs({x=6,y=0},fx);
```

$$fx1 = -1$$

```
> fy1=subs({x=6,y=0},fy);
```

$$fy1 = \frac{-9}{14}$$

```
> gx1=subs({x=6,y=0},gx);
```

$$gx1 = 0.$$

```
> gy1=subs({x=6,y=0},gy);
```

$$gy1 = .2$$

```
> mA:=Matrix([[-1, -9/14], [0, 0.2]]);
```

$$mA := \begin{bmatrix} -1 & \frac{-9}{14} \\ 0 & .2 \end{bmatrix}$$

```
> eigenvals(mA);
```

$$-1., .2000000000$$

```
> fx2=subs({x=0.8456,y=2.1140},fx);
```

$$fx2 = .2526639893$$

```
> fy2=subs({x=0.8456,y=2.1140},fy);
```

$$fy2 = -.3436280884$$

```
> gx2=subs({x=0.8456,y=2.1140},gx);
```

$$gx2 = .5000000000$$

```
> gy2=subs({x=0.8456,y=2.1140},gy);
```

$$gy2 = -.2000000000$$

```
> mB:=Matrix([[0.2527, -0.3436], [0.5, -0.2]]);
```

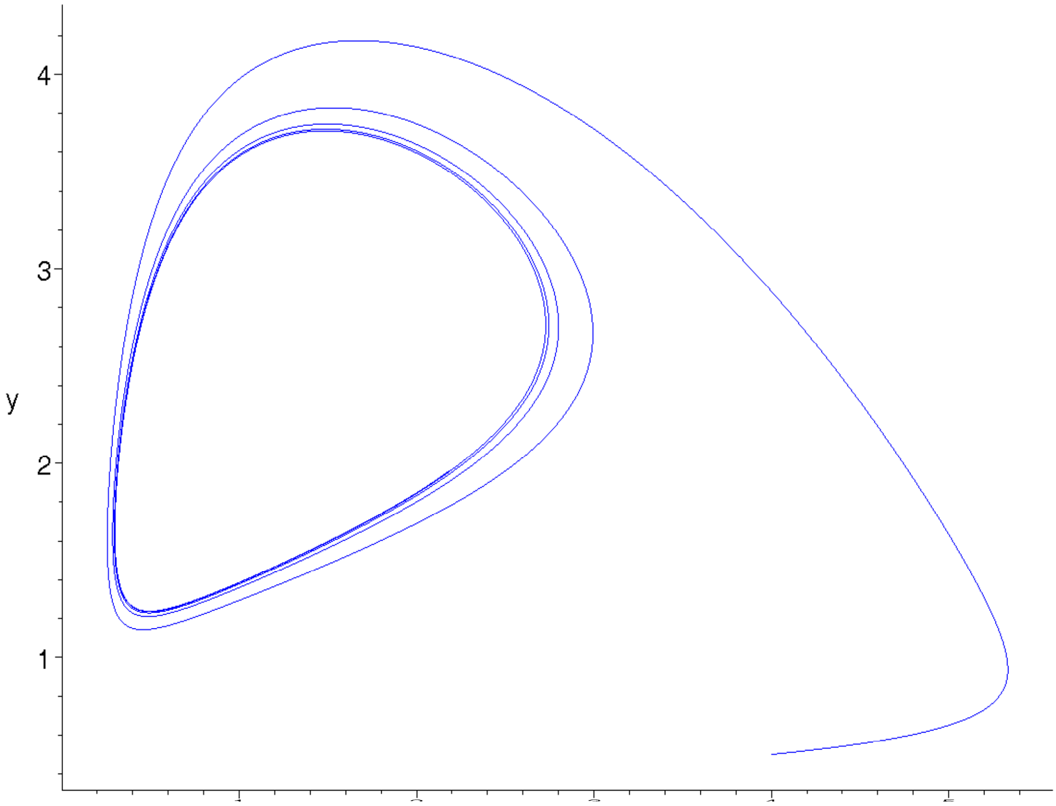
$$mB := \begin{bmatrix} .2527 & -.3436 \\ .5 & -.2 \end{bmatrix}$$

```
> eigenvals(mB);
```

$$.02635000000 + .3472256867\, I,\ .02635000000 - .3472256867\, I$$

Since only the eigenvalues of matrix mB are complex conjugate, then point P2 is a limit cycle. We can verify this by plotting the phase portrait. We take point (4,0.5) as an initial point.

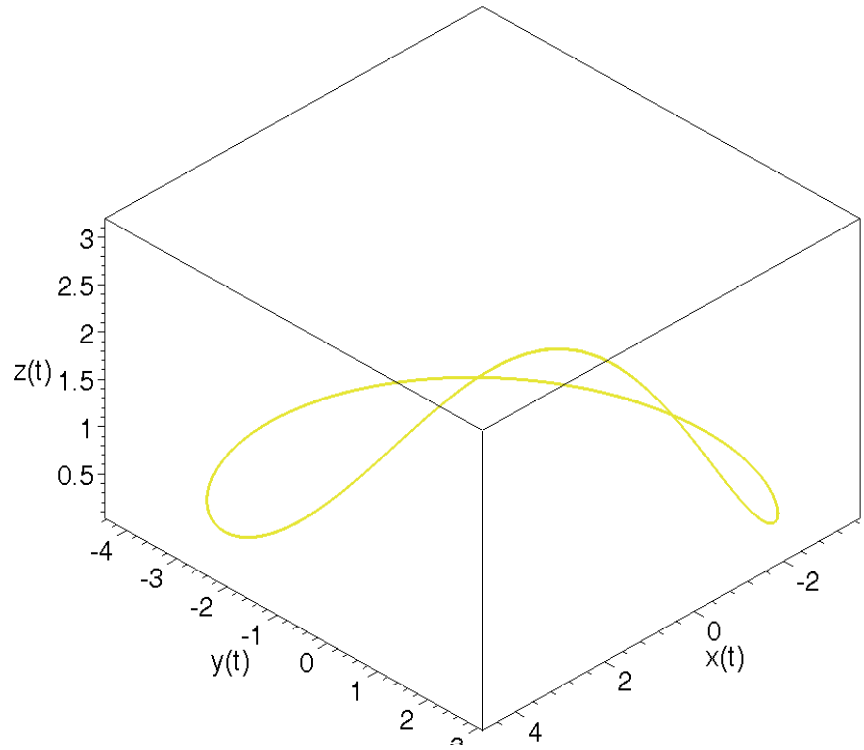
```
> phaseportrait(
  [D(x)(t)=x(t)*(1-x(t)/6)-6*x(t)*y(t)/(8+8*x(t)),D(y)(t)=0.
  2*y(t)*(1-0.4*y(t)/x(t))],
  [x(t),y(t)],t=0..100,
  [[x(0)=4,y(0)=0.5]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```



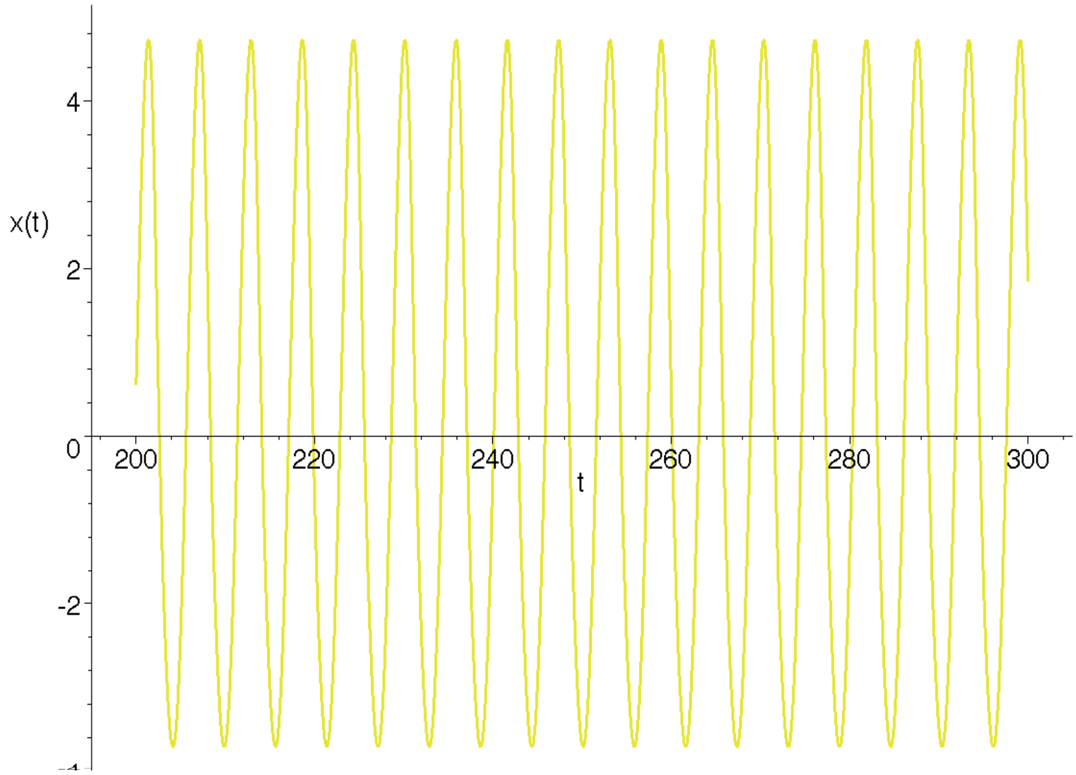

```
>
```

Question 11

```
> DEplot3d({D(x)(t)=-y-z,D(y)(t)=x+0.2*y,D(z)(t)=0.2+z*(x-2.5)}
  ,{x(t),y(t),z(t)},t=200..300,[[x(0)=1,y(0)=1,z(0)=1]],scene=[
  x(t),y(t),z(t)],stepsize=0.05);
```




```
> DEplot({D(x)(t)=-y-z,D(y)(t)=x+0.2*y,D(z)(t)=0.2+z*(x-2.5)},{
  x(t),y(t),z(t)},t=200..300,[[x(0)=1,y(0)=1,z(0)=1]],scene=[t,
  x(t)],stepsize=0.05);
```



Question 12

(i)

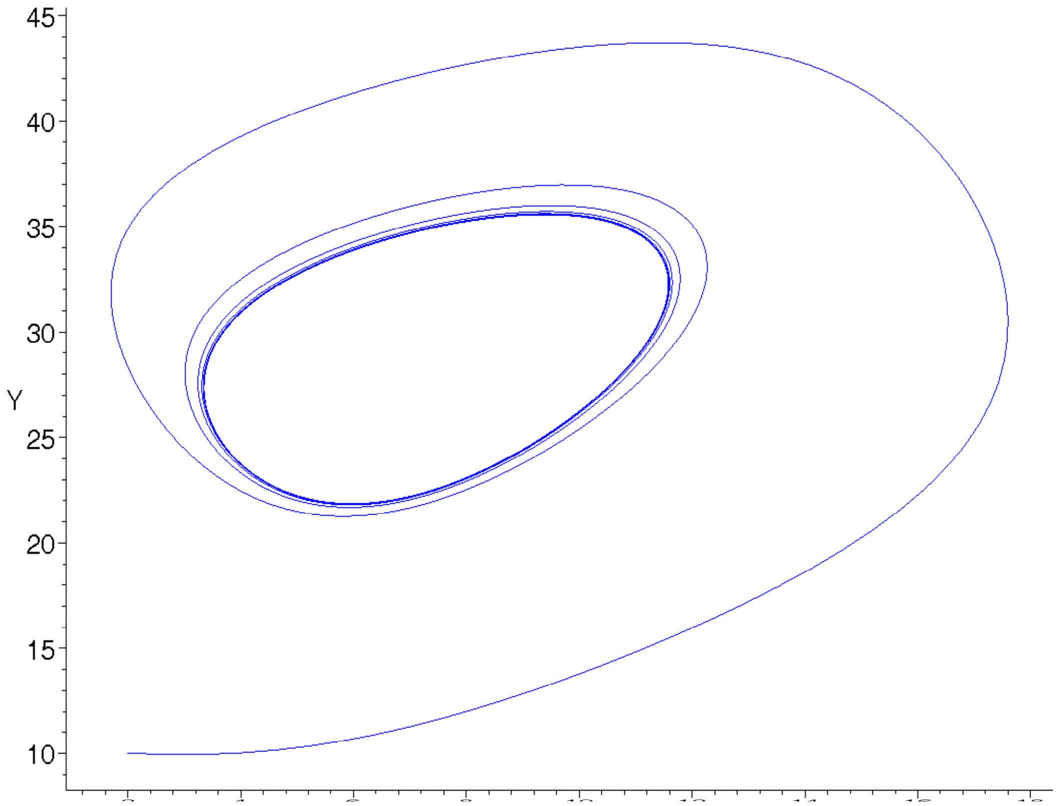
```
> solve({p=0.5+0.25*Y, Y=-0.025*p^3+0.75*p^2-6*p+40},{p,Y});
```

$$\{Y = 26.22133502, p = 7.055333756\},$$
$$\{p = 11.47233312 - 10.32004334\, I, Y = 43.88933249 - 41.28017335\, I\},$$
$$\{Y = 43.88933249 + 41.28017335\, I, p = 11.47233312 + 10.32004334\, I\}$$

So the economically meaningful fixed point is $(p^*,Y^*) = (7.0553, 26.2213)$.

(ii)

```
> phaseportrait(
  [D(p)(t)=0.75*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=2*(p-
  0.5-0.25*Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=2,Y(0)=10]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```

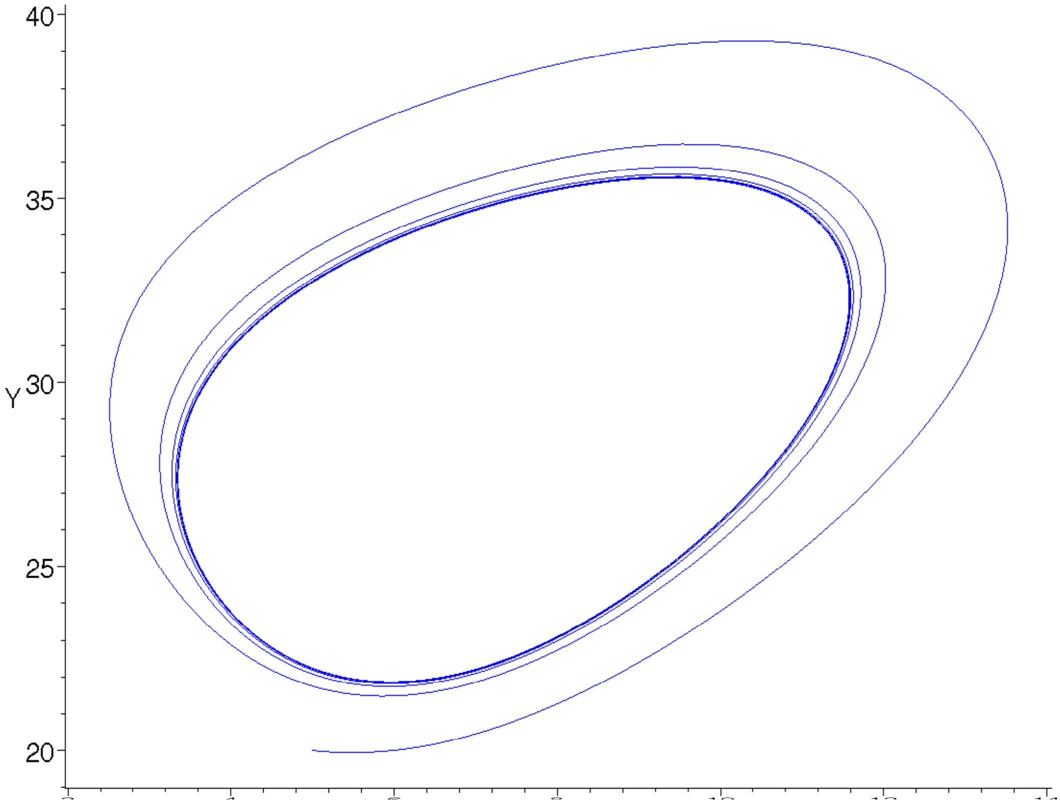


It can be seen that the system has a stable limit cycle.

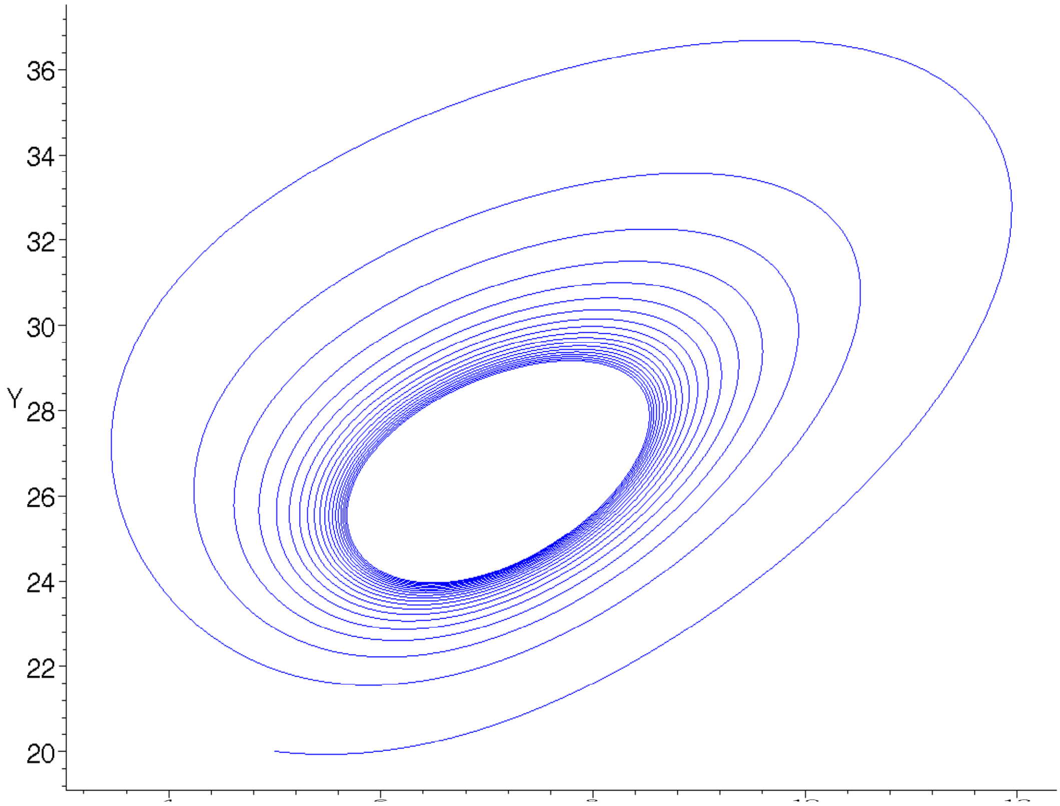
Question 13

```
> phaseportrait(
  [D(p)(t)=0.75*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=2*(p-0.5
  -0.25*Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=5,Y(0)=20]],
  stepsize=.05,
  linecolour=blue,
```

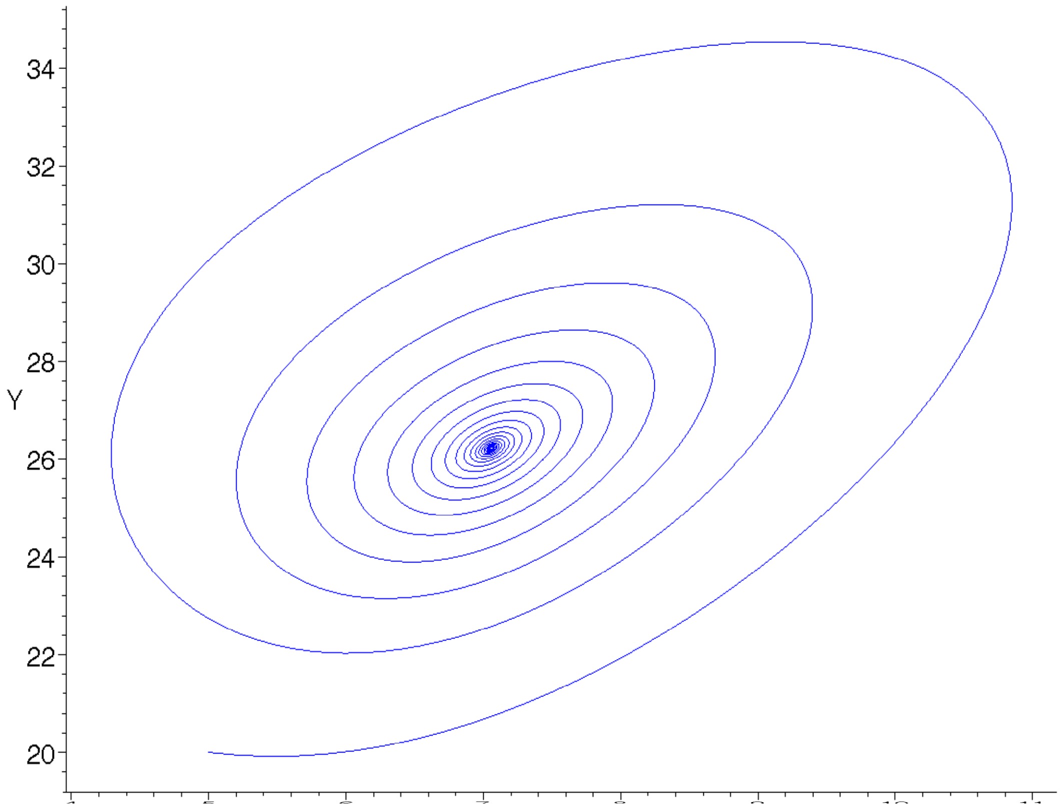
```
arrows=none,
thickness=1);
```



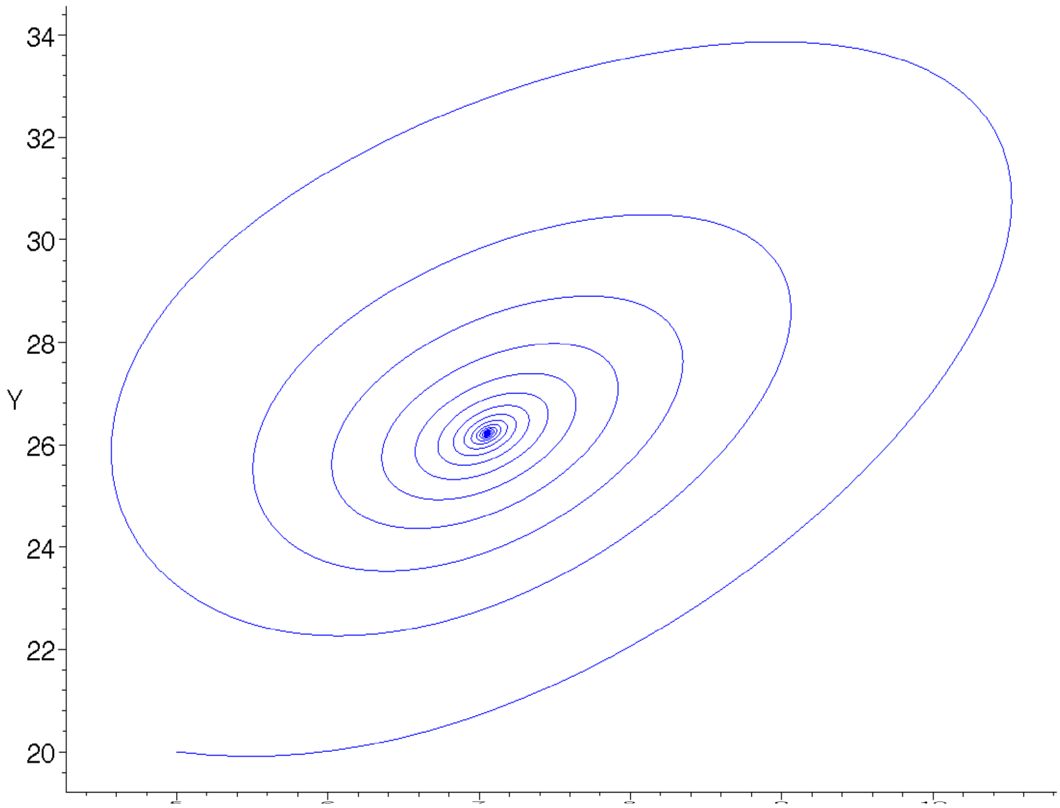

```
> phaseportrait(
[D(p)(t)=0.75*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=2.5*(p-0
.5-0.25*Y)],
[p(t),Y(t)],t=0..100,
[[p(0)=5,Y(0)=20]],
stepsize=.05,
linecolour=blue,
arrows=none,
thickness=1);
```




```
> phaseportrait(
  [D(p)(t)=0.75*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=3*(p-0.5
  -0.25*Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=5,Y(0)=20]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```




```
> phaseportrait(
  [D(p)(t)=0.75*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=3.2*(p-0
  .5-0.25*Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=5,Y(0)=20]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```




```
>
```

Question 14

```
> solve({p=0.5+0.25*Y, Y=-0.025*p^3+0.75*p^2-8*p+40},{p,Y});
```

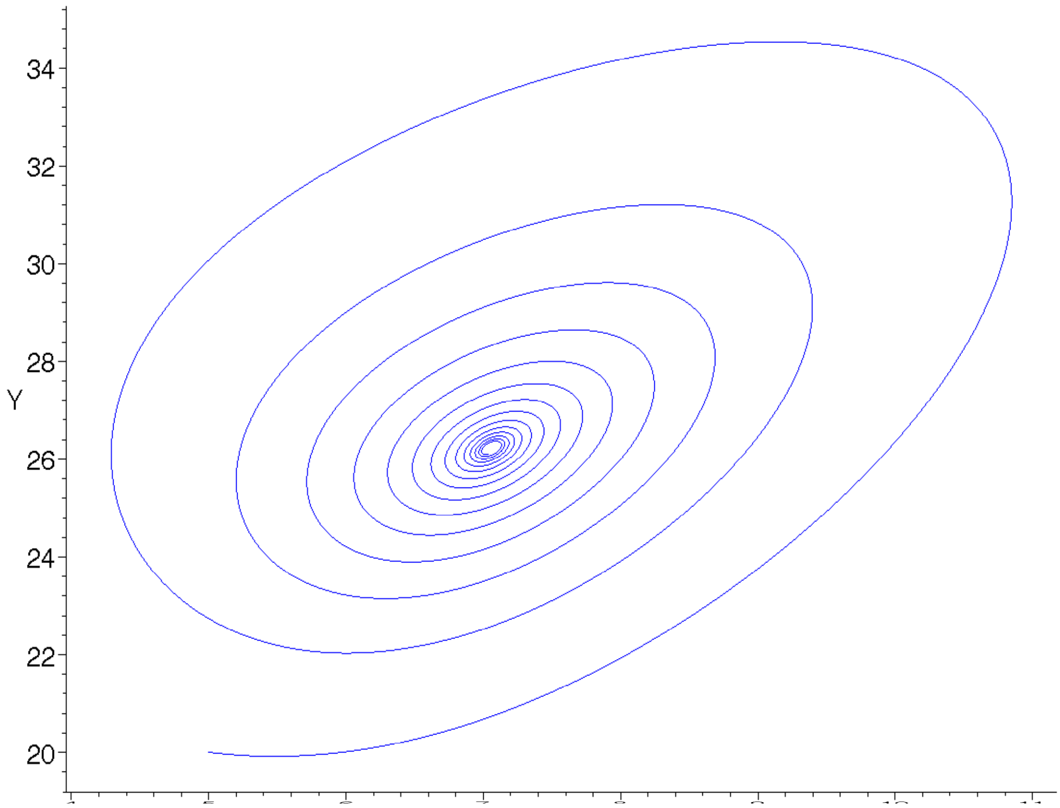
$\{Y = 16.54183226, p = 4.635458065\},$

$\{p = 12.68227097 - 14.19801862\, I, Y = 48.72908387 - 56.79207447\, I\},$

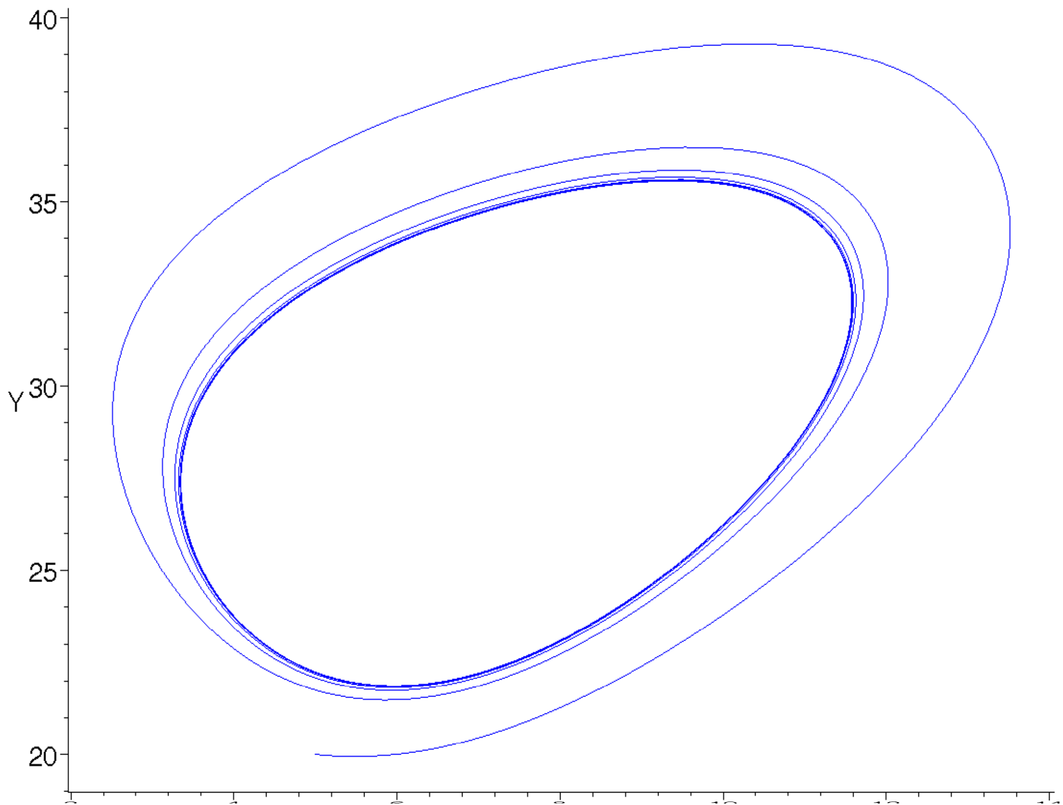
$\{Y = 48.72908387 + 56.79207447\, I, p = 12.68227097 + 14.19801862\, I\}$

Economically meaningful fixed point is then $(p^*, Y^*) = (4.6355, 16.5418)$.

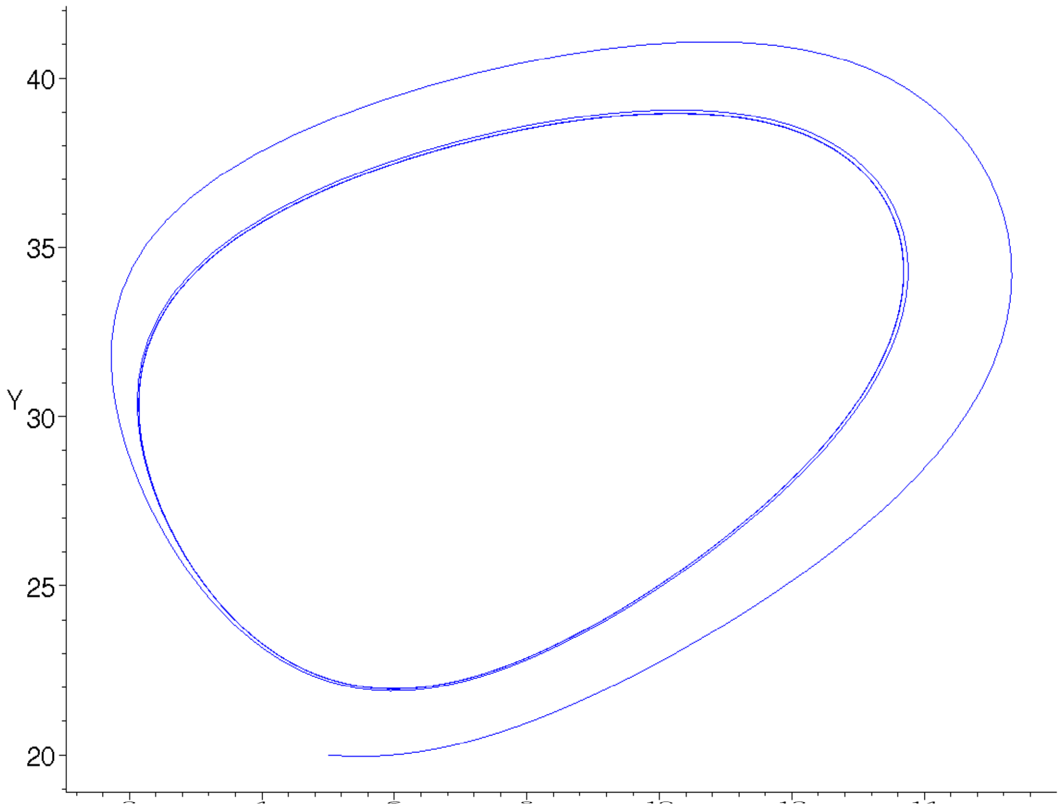
```
> phaseportrait(
  [D(p)(t)=0.5*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=2*(p-0.5-
  0.25*Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=5,Y(0)=20]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```




```
> phaseportrait(
  [D(p)(t)=0.75*(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=2*(p-0.5
  -0.25*Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=5,Y(0)=20]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```




```
> phaseportrait(
  [D(p)(t)=(-0.025*p^3+0.75*p^2-6*p+40-Y),D(Y)(t)=2*(p-0.5-0.25
  *Y)],
  [p(t),Y(t)],t=0..100,
  [[p(0)=5,Y(0)=20]],
  stepsize=.05,
  linecolour=blue,
  arrows=none,
  thickness=1);
```

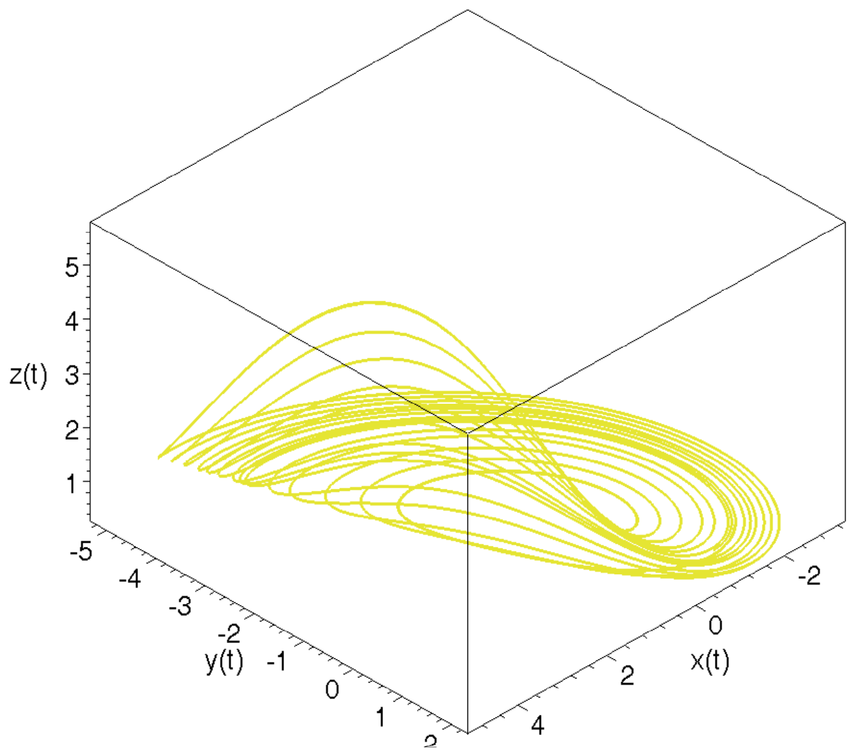


The system exhibits a limit cycle, but the limit cycle gets larger the larger the value of α.

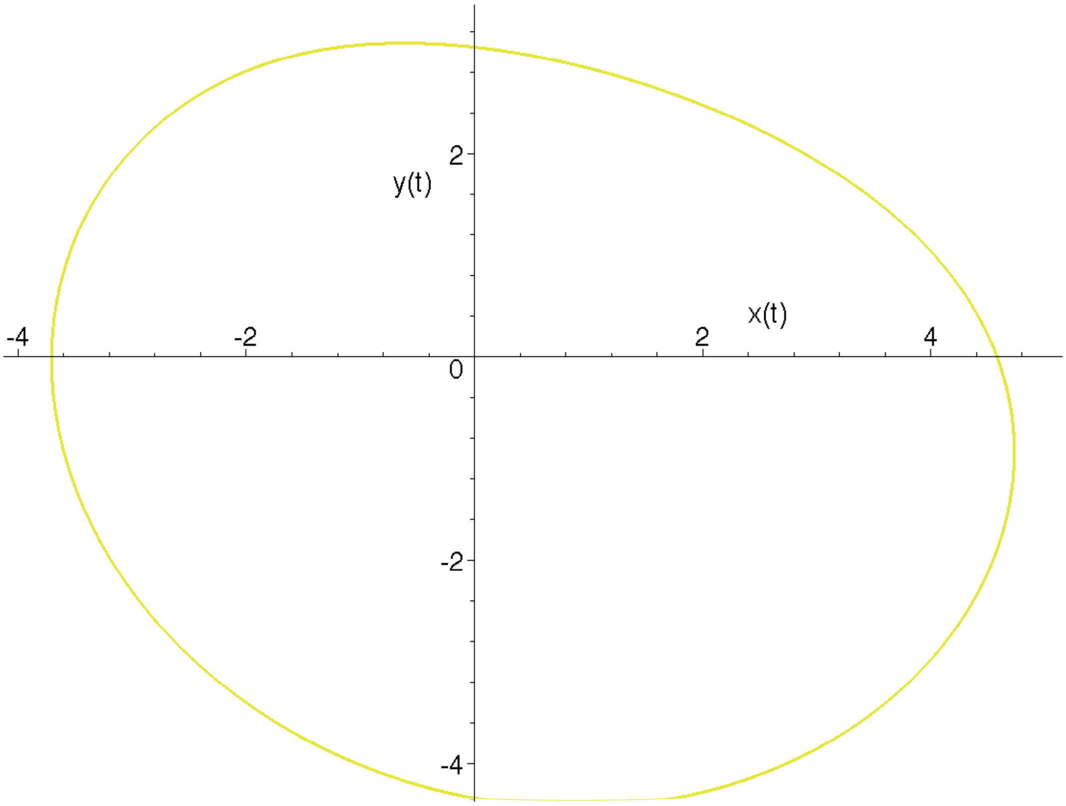
Question 15

Although requested to perform this on a spreadsheet, we shall display the result here in *Maple*.

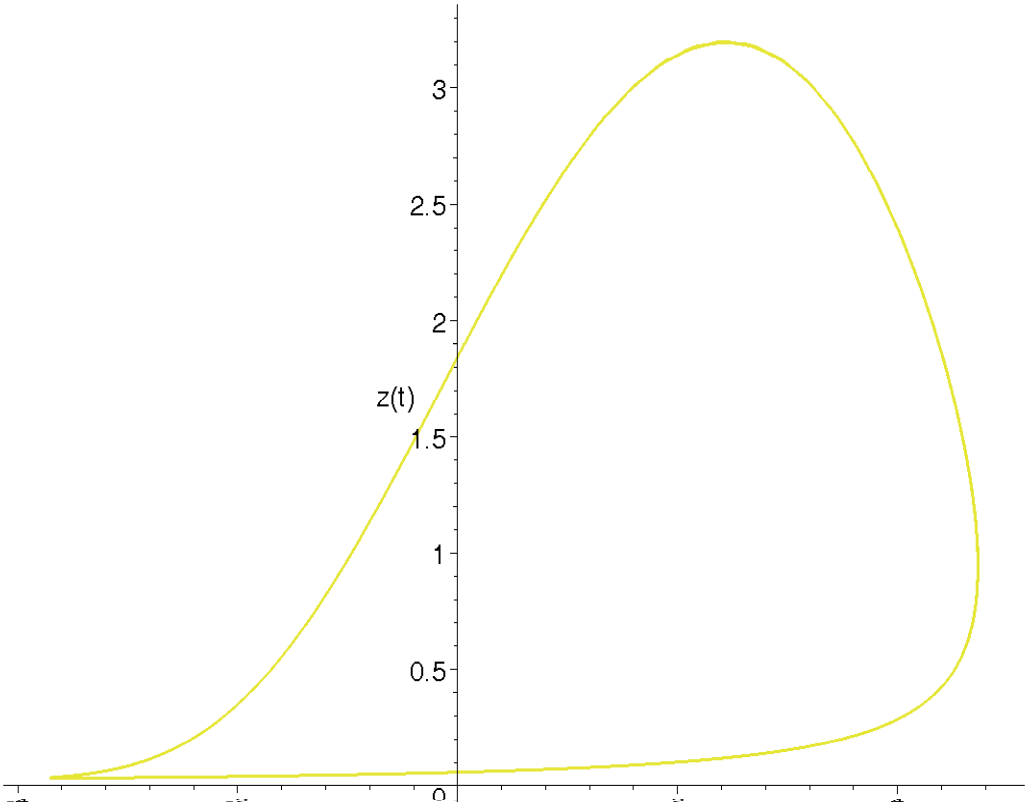
```
> DEplot3d({D(x)(t)=-y-z,D(y)(t)=x+0.4*y,D(z)(t)=2+z*(x-4)},{x(
  t),y(t),z(t)},t=200..300,[[x(0)=0.1,y(0)=0.1,z(0)=0.1]],scene
  =[x(t),y(t),z(t)],stepsize=0.05);
```



```
> DEplot({D(x)(t)=-y-z,D(y)(t)=x+0.2*y,D(z)(t)=0.2+z*(x-2.5)},{
x(t),y(t),z(t)},t=200..300,[[x(0)=0.1,y(0)=0.1,z(0)=0.1]],sce
ne=[x(t),y(t)],stepsize=0.05);
```

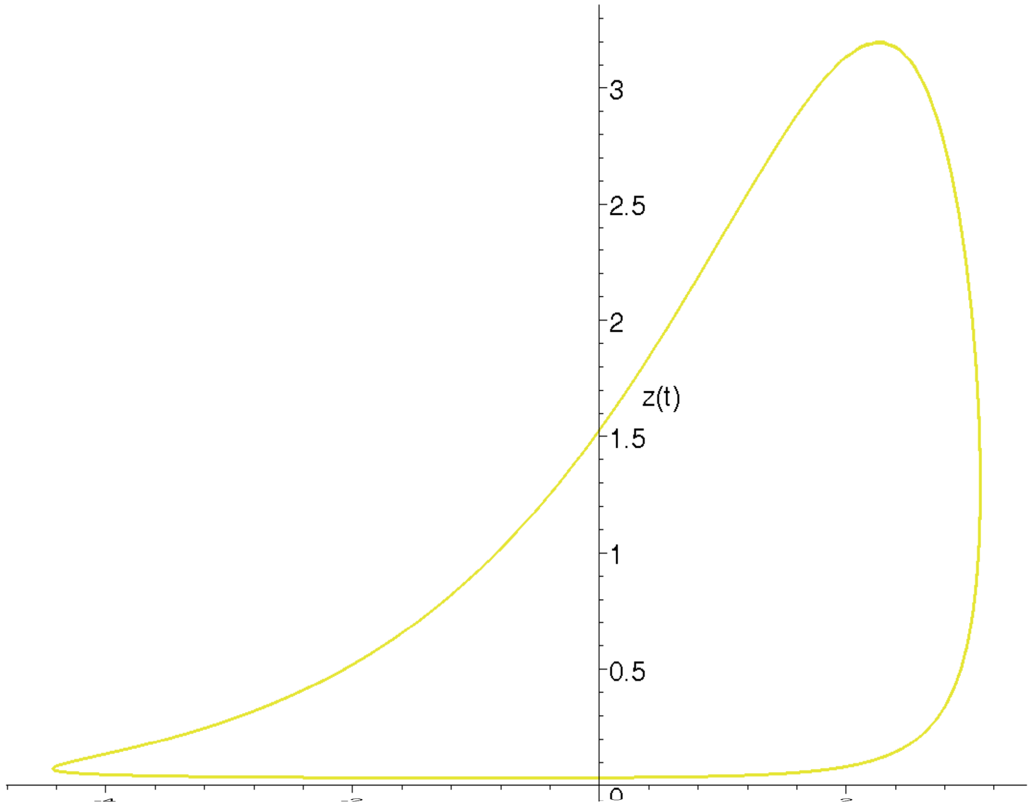



```
> DEplot({D(x)(t)=-y-z,D(y)(t)=x+0.2*y,D(z)(t)=0.2+z*(x-2.5)},{
x(t),y(t),z(t)},t=200..300,[[x(0)=0.1,y(0)=0.1,z(0)=0.1]],sce
ne=[x(t),z(t)],stepsize=0.05);
```




```
> DEplot({D(x)(t)=-y-z,D(y)(t)=x+0.2*y,D(z)(t)=0.2+z*(x-2.5)},{
```

```
x(t),y(t),z(t)},t=200..300,[[x(0)=0.1,y(0)=0.1,z(0)=0.1]],sce
ne=[y(t),z(t)],stepsize=0.05);
```




```
>
```