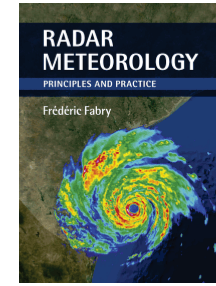
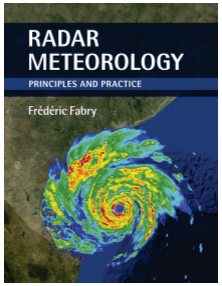



# Supplement e03.1:
# The radar equation

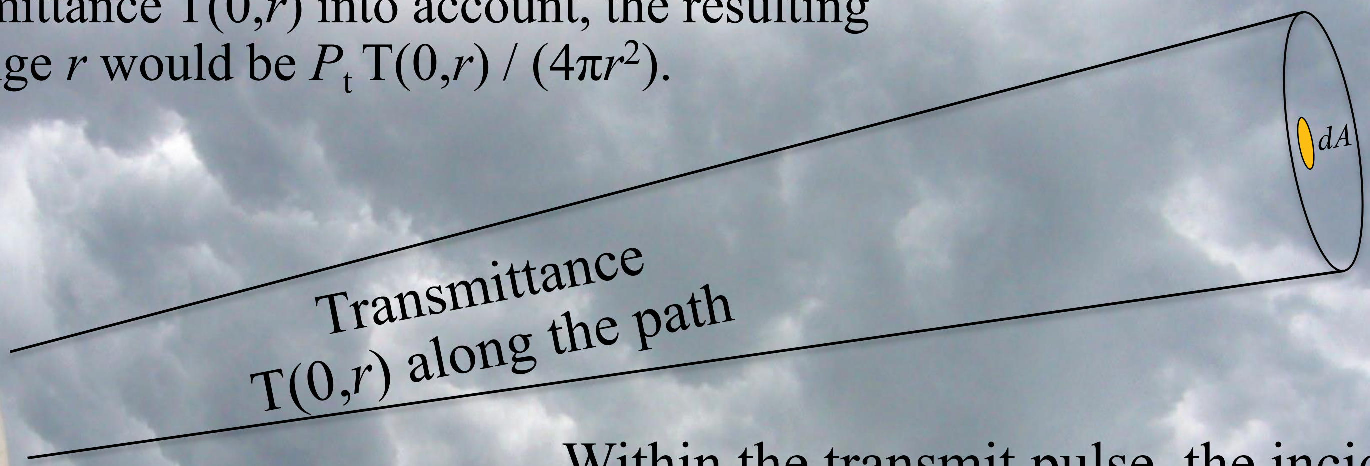
## Many versions of the radar equation for meteorological radars and their derivation

# Transmit power density at range $r$ for an isotropic antenna



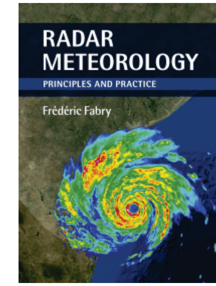

If the antenna was isotropic (sending the same energy flux in all directions), the power flux at range $r$ would be uniform over the area of a sphere of radius $r$ centered on the antenna. Taking the transmittance $T(0,r)$ into account, the resulting power flux at range $r$ would be $P_t\, T(0,r) / (4\pi r^2)$.

Power $P_t$ transmitted over a time interval $\tau$




Within the transmit pulse, the incident power density $dP_i$ per unit area dA would be: $dP_i = [P_t\, T(0,r) / (4\pi r^2)]\, dA$.

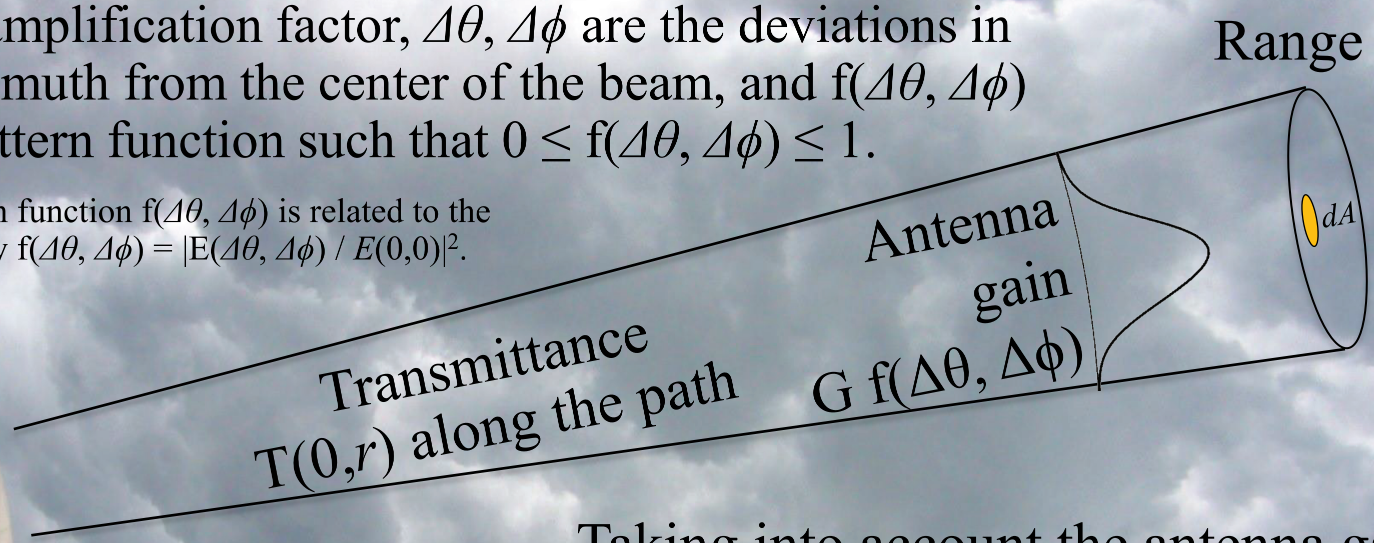
# Transmit power density at range $r$ for an antenna with gain




But radar antenna focus as much energy as possible in a narrow beam. The antenna has a *gain* function $G\,\mathrm{f}(\Delta\theta, \Delta\phi)$ where $G$ is the peak power amplification factor, $\Delta\theta$, $\Delta\phi$ are the deviations in elevation and azimuth from the center of the beam, and $\mathrm{f}(\Delta\theta, \Delta\phi)$ is the antenna pattern function such that $0 \le \mathrm{f}(\Delta\theta, \Delta\phi) \le 1$.

Technical note: the pattern function $\mathrm{f}(\Delta\theta, \Delta\phi)$ is related to the electric field strength $E$ by $\mathrm{f}(\Delta\theta, \Delta\phi) = |E(\Delta\theta, \Delta\phi) / E(0,0)|^2$.

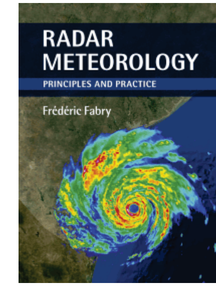
Power $P_t$ transmitted over a time interval $\tau$



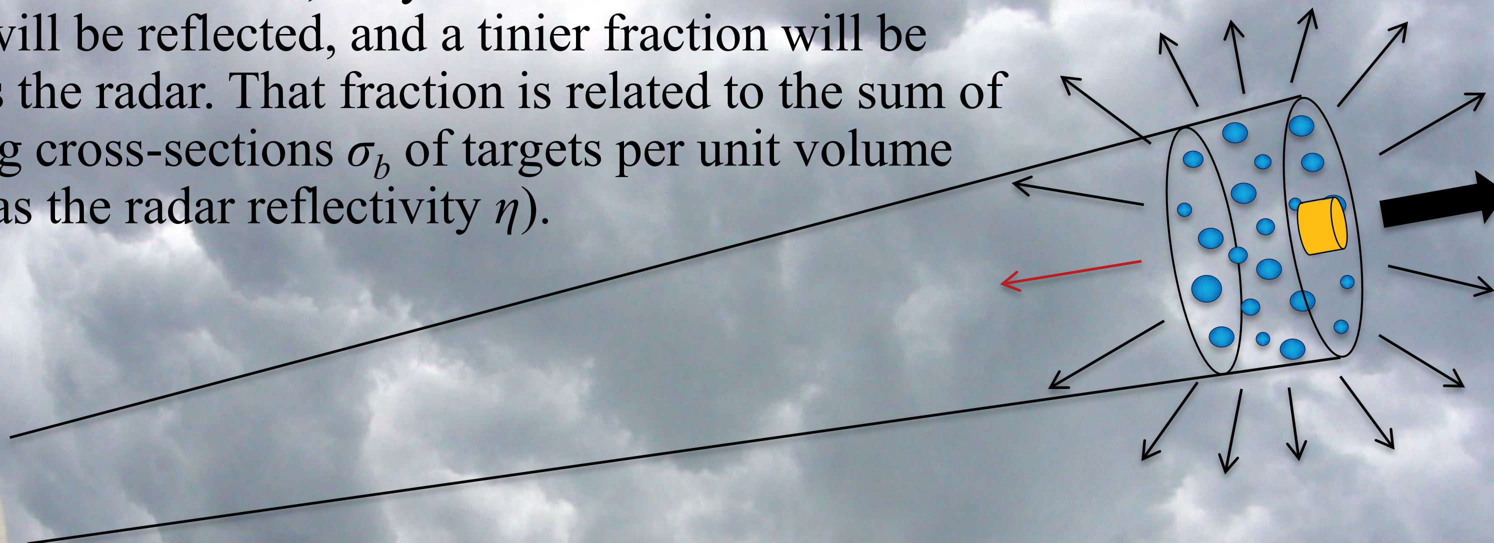

Taking into account the antenna gain, the incident power density becomes:

$$dP_i = [P_t\, G\, \mathrm{f}(\Delta\theta, \Delta\phi)\, \mathrm{T}(0,r) \,/\, (4\pi r^2)]\, \mathrm{d}A.$$

# Reflected power at range $r$



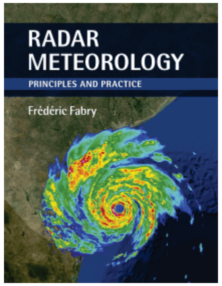

Scatterers only make a small proportion of the volume illuminated by the radar. Hence, only a small fraction of the incident energy will be reflected, and a tinier fraction will be reflected towards the radar. That fraction is related to the sum of the backscattering cross-sections $\sigma_b$ of targets per unit volume (also referred to as the radar reflectivity $\eta$).



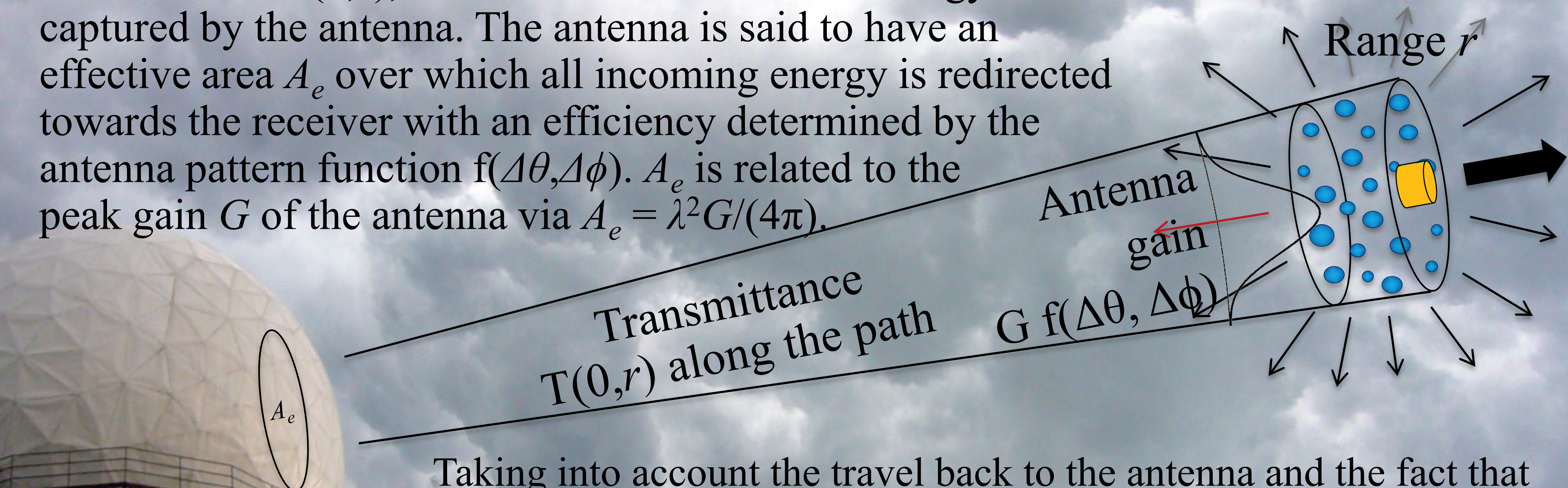
Radar reflectivity being the equivalent surface area per unit volume that will scatter energy, we must introduce a range depth $dr$ into the equation. The scattered energy $dP_s$ per unit volume is hence

$$dP_s = [P_t G \, f(\Delta\theta, \Delta\phi) \, T(0,r) / (4\pi r^2)] \, \eta \, dA \, dr.$$

# Reflected power at range *r* captured by the antenna



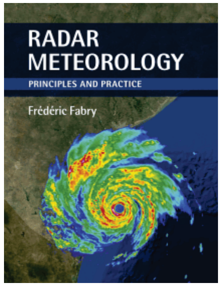

After travelling back another distance $r$ through a medium of transmittance $T(0,r)$, some of the backscattered energy is captured by the antenna. The antenna is said to have an effective area $A_e$ over which all incoming energy is redirected towards the receiver with an efficiency determined by the antenna pattern function $f(\Delta\theta,\Delta\phi)$. $A_e$ is related to the peak gain $G$ of the antenna via $A_e = \lambda^2 G/(4\pi)$.



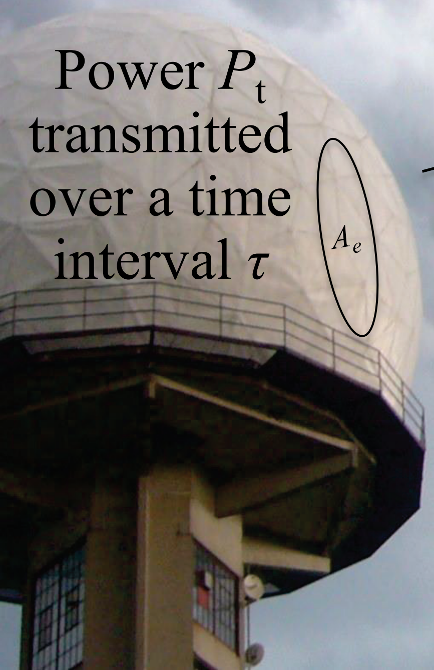

Taking into account the travel back to the antenna and the fact that the scattered energy is dispersed in all directions over an area $4\pi r^2$, the energy per unit volume received on the antenna effective area is:

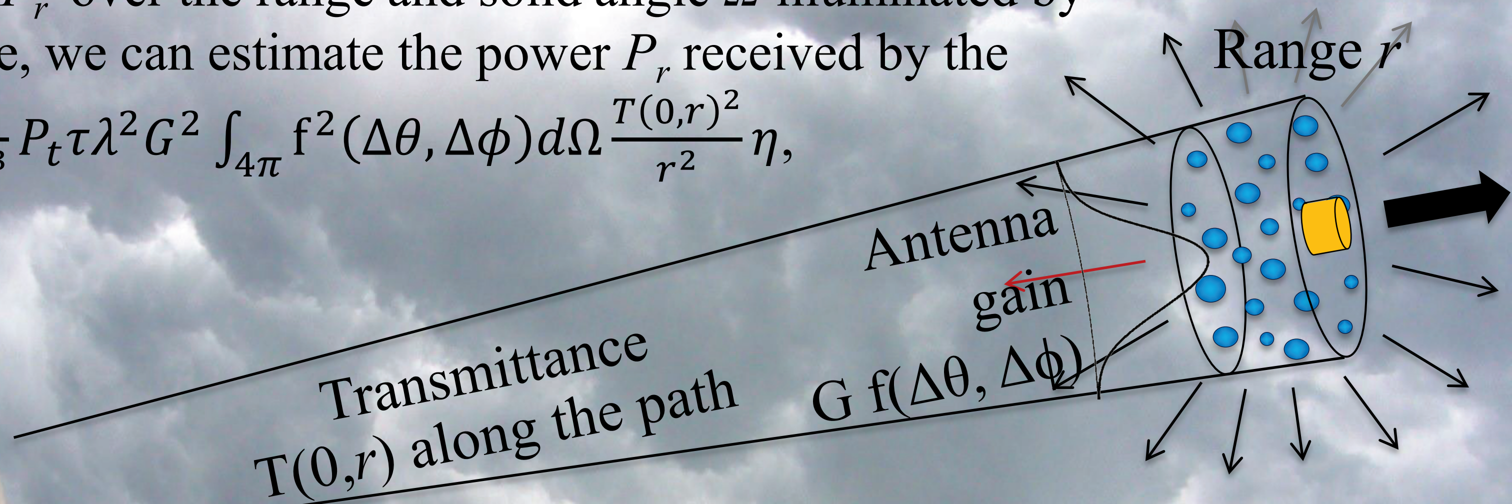
$$dP_r = [P_t G^2 \lambda^2 f^2(\Delta\theta, \Delta\phi)\, T(0,r)^2 / (64\pi^3 r^4)]\, \eta\, dA\, dr.$$

# Reflected power at range *r* captured by the antenna



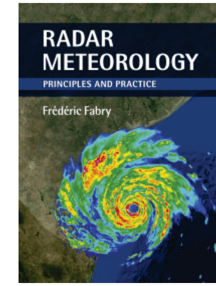

Recalling that the transmit pulse τ illuminates a range interval *cτ*/2, and integrating $dP_r$ over the range and solid angle *Ω* illuminated by the transmit pulse, we can estimate the power $P_r$ received by the radar: $P_r = \frac{c}{128\pi^3} P_t \tau \lambda^2 G^2 \int_{4\pi} \mathrm{f}^2(\Delta\theta, \Delta\phi) d\Omega \frac{T(0,r)^2}{r^2} \eta$, using $d\Omega = dA/r^2$.

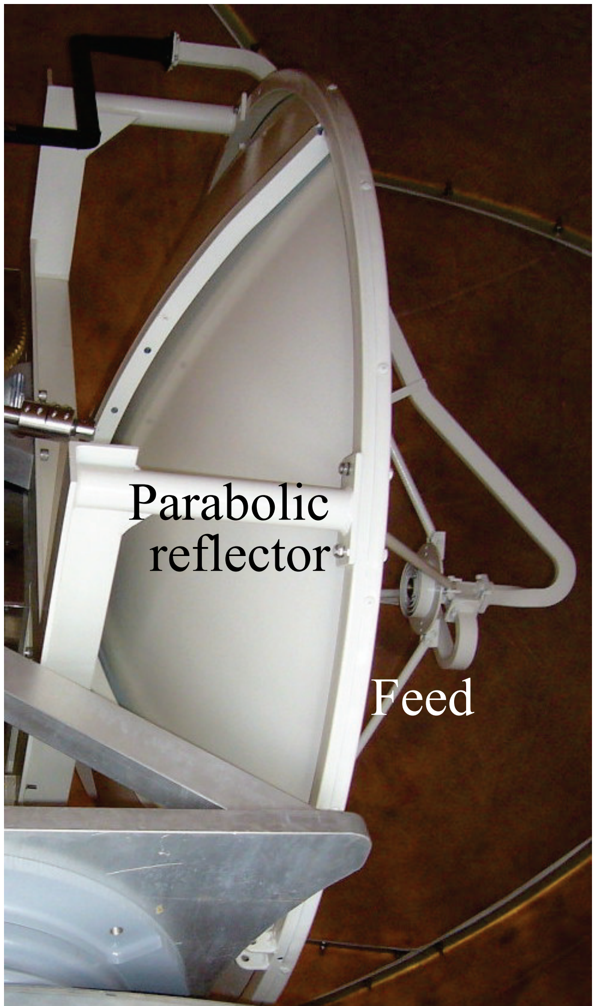


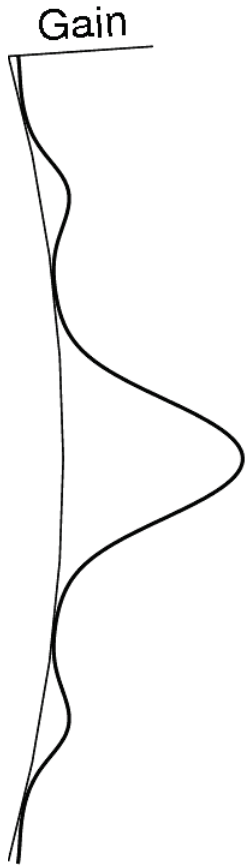




This constitutes the first (and most general) radar equation.
It is, however, very difficult to use.
We must hence make the radar equation more specific by better constraining both f(*Δθ*, *Δϕ*) and *η*.

# Antenna pattern function for parabolic reflector antenna



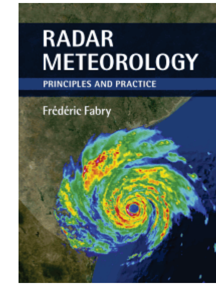



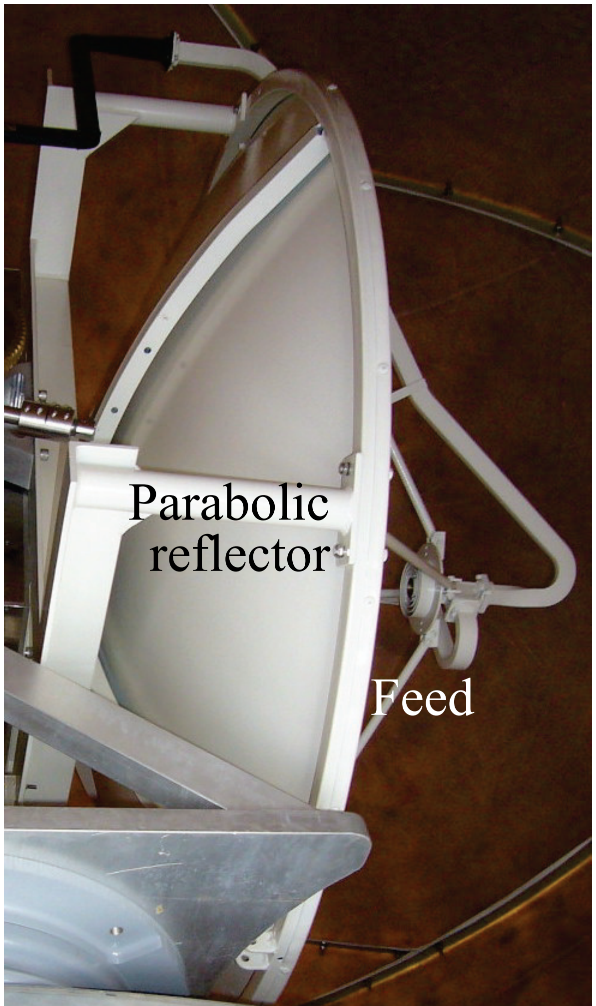

The exact shape of the antenna pattern function is determined by the diameter of the parabolic reflector as well the pattern function with which the feed illuminates the reflector itself.

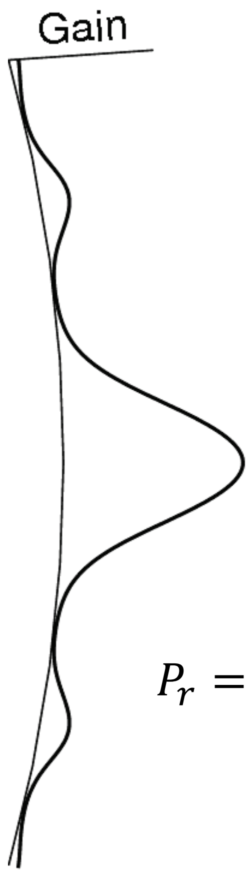
The antenna pattern that comes out of a parabolic reflector illuminated by a feed with a standard feed pattern function resembles a Gaussian function with some small sidelobes. Such antenna patterns are often determined by their half-power beamwidth $\theta_{beam}$ and $\phi_{beam}$ such that:

$$f^2(\Delta\theta, \Delta\phi) = \exp\left[-8 \log_e 2 \left(\frac{\Delta\theta^2}{\theta_{beam}^2} + \frac{\Delta\phi^2}{\phi_{beam}^2}\right)\right]$$

# Antenna pattern function for parabolic reflector antenna




$$\mathrm{f}^2(\Delta\theta,\Delta\phi) = \exp\left[-8\log_e 2\left(\frac{\Delta\theta^2}{\theta_{beam}^2}+\frac{\Delta\phi^2}{\phi_{beam}^2}\right)\right]$$

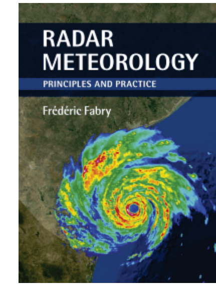
Integrated over all steradians, if the beam widths are small enough,

$$\int_{4\pi}\mathrm{f}^2(\Delta\theta,\Delta\phi)d\Omega \approx \frac{\pi\theta_{beam}\phi_{beam}}{8\log_e 2}$$

This leads to:

$$P_r = \frac{c}{1024\pi^2\log_e 2}P_t\tau\lambda^2 G^2\theta_{beam}\phi_{beam}\frac{\mathrm{T}(0,r)^2}{r^2}\eta$$

# A radar equation for spherical Rayleigh targets




From (2.6) and (3.3), we know that for spherical Rayleigh targets,

$$\eta = \frac{\pi^5}{\lambda^4}\left\|\frac{n^2(\lambda)-1}{n^2(\lambda)+2}\right\|^2 \int_0^\infty N(D)D^6 \equiv \frac{\pi^5}{\lambda^4}\|K\|^2 10^{-18} Z,$$

when $Z$ is expressed in mm$^6$ m$^{-3}$ and everything else in SI units.

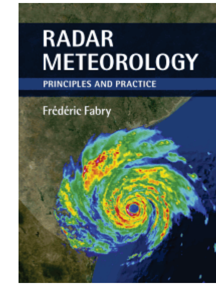
Once the above is integrated in the radar equation, we obtain:

$$P_r = \frac{10^{-18}\pi^3 c}{1024 \log_e 2}\,\frac{P_t \tau G^2 \theta_{beam}\phi_{beam}}{\lambda^2}\,\frac{\mathrm{T}(0,r)^2}{r^2}\|K\|^2 Z.$$

This is the standard radar equation as derived by Probert-Jones (1962) with minor modifications to accommodate $Z$ in its usual units and to make transmittance appear explicitly as a path property.

Probert-Jones, J.R., 1962: The radar equation in meteorology. *Quarterly Journal of the Royal Meteorological Society,* **88**, 485–495.

# Modifying the Probert-Jones equation for Rayleigh targets




$$P_r = \frac{10^{-18}\pi^3 c}{1024 \log_e 2} \frac{P_t \tau G^2 \theta_{beam} \phi_{beam}}{\lambda^2} \frac{\mathrm{T}(0,r)^2}{r^2} \|K\|^2 Z.$$

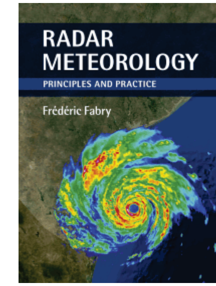
The Probert-Jones equation is great to use when one knows the peak gain *G* and the beam widths of one's antenna, information generally available in the engineering specifications of the radar.

For trying to understand what radar characteristics shape the power returned to the radar, it is not as useful because:

- Some quantities, such as peak gain and beam widths, are linked; yet,
- It is not immediately obvious how to relate peak antenna gain with other radar characteristics; and,
- The diameter of the antenna, one of the easiest radar parameter to grasp, does not appear explicitly in the equation.

We hence use a few more approximations to obtain other versions of the equation.

# Modifying the Probert-Jones equation for Rayleigh targets




$$P_r = \frac{10^{-18}\pi^3 c}{1024 \log_e 2} \frac{P_t \tau G^2 \theta_{beam} \phi_{beam}}{\lambda^2} \frac{\mathrm{T}(0,r)^2}{r^2} \|K\|^2 Z.$$

For standard radar antennas with parabolic reflectors of diameter $D_a$, we have:

$\theta_{beam} \approx \phi_{beam} \approx 1.22 \frac{\lambda}{D_a}$ and $G = \frac{\pi^2 D_a^2}{\lambda^2} e_A$,

where $e_A$ is the aperture efficiency of the antenna and can be approximated by 0.55 for radars with low sidelobes. With these approximations, we can get:

$$P_r = \frac{1.22^2 0.55^2 10^{-18} \pi^7 c}{1024 \log_e 2} \frac{P_t \tau D_a^2}{\lambda^4} \frac{\mathrm{T}(0,r)^2}{r^2} \|K\|^2 Z, \text{ and} \qquad (3.2)$$

$$P_r = \frac{1.22^4 0.55^2 10^{-18} \pi^7 c}{1024 \log_e 2} \frac{P_t \tau}{\lambda^2 \theta_{beam}^2} \frac{\mathrm{T}(0,r)^2}{r^2} \|K\|^2 Z.$$