Appendix 12W.1
Hypothetical Extractions with Partitioned Matrices

Partitioned matrices (Appendix A) provide a useful and comprehensive framework in which to examine various kinds of possible hypothetical extraction linkage measures. Consider the standard representation of an *n*-sector **A** matrix, partitioned so that *k* sectors ($k < n$) are shown in the upper left (square) submatrix, identified as $\mathbf{A}_{11}$;

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$$

Then the Leontief inverse of this partitioned matrix can be expressed as[1]

$$\mathbf{L} = \left[\begin{array}{c|c} \mathbf{H} & \mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\left(\mathbf{I} + \mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22}\right) \end{array}\right] \quad \text{(A12W.1.1)}$$

where $\mathbf{H} = (\mathbf{I} - \mathbf{A}_{11} - \mathbf{A}_{12}\boldsymbol{\alpha}_{22}\mathbf{A}_{21})^{-1}$ and $\boldsymbol{\alpha}_{22} = (\mathbf{I} - \mathbf{A}_{22})^{-1}$. Final demands and gross outputs can be partitioned similarly, leading to

$$\mathbf{x} = \left[\begin{array}{c} \mathbf{x}_1 \\ \hline \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H} & \mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\left(\mathbf{I} + \mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22}\right) \end{array}\right]\left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{array}\right] \quad \text{(A12W.1.2)}$$

Assume that the sectors (or regions) to be extracted from the economic system occupy the first *k* rows and columns. For concreteness in what follows, we will generally assume that we are examining sectors (not regions) and that only one sector is being extracted ($k = 1$). This is consistent with much of the key sector literature.

There are essentially two issues in the literature on this kind of linkage measurement. First, there is the objective of providing a comprehensive *total linkage* (economic "importance") indicator for a sector—$\mathbf{i}'\mathbf{x} - \mathbf{i}'\overline{\mathbf{x}}$ (or variants) is one such measure.[2] (See Hewings, 1982, Harrigan and McGilvray, 1988, or Miller and Lahr, 2001, for reviews of much of the material on this topic.) Secondly, researchers have explored the question of how a total linkage measure might be disaggregated into (or built up from) *backward* and *forward linkage* components.

The partitioned form of the Leontief inverse in (A12W.1.1) and (A12W.1.2) suggests some fairly straightforward parallels to the early descriptions of backward and forward linkages as column and row sums from a Leontief inverse. Meller and Marfán (1981) were the first to measure forward linkages as a residual in the extraction approach. They identified total linkages through a kind of extraction procedure, backward linkages as column sums from the Leontief inverse (possibly weighted) and forward linkages as a

[1] We use $\boldsymbol{\alpha}_{ii}$ to denote the Leontief inverse of $\mathbf{A}_{ii}$. We do not use $\mathbf{L}_{ii}$ since that would identify the upper left partition of the Leontief inverse, and in general $\mathbf{L}_{ii} \neq (\mathbf{I} - \mathbf{A}_{ii})^{-1}$, as seen in (A12W.1.1). Alternative expressions for the partitioned inverse are possible (Appendix A).

[2] Hirschman (1958, Chap. 6) originally suggested the idea of measuring the "total linkage" of a sector. He cites two major works on linkage measurement, namely Chenery and Watanabe (1958) and Rasmussen (1957).

residual—the difference between total and backward linkage. Cella (1984) made this identification easier by formalizing the partitioned matrix approach. We examine some of the alternatives suggested by this partitioning format. [As noted, we assume for simplicity that only one industry is extracted ($k = 1$) so that, for example, $\mathbf{x}_1$ and $\mathbf{f}_1$ are scalars.]

A12W.1.1 Case 1. Complete Extraction of Sector 1

In this case, set $\mathbf{A}_{11} = \mathbf{A}_{12} = \mathbf{A}_{21} = \mathbf{0}$, and from the inverse form in (A12W.1.1)[3]

$$\overline{\mathbf{A}}^1 = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right] \text{ and } \overline{\mathbf{L}}^1 = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\alpha}_{22} \end{array}\right] \qquad \text{(A12W.1.3)}$$

This is the method of extraction originally conceived by Paelinck, de Caevel, and Degueldre (1965), and later employed by Strassert (1968), Schultz (1976, 1977), Meller and Marfán (1981), Milana (1985), Heimler (1991) and others.[4] The pre-extraction total output vector is given in (A12W.1.2). From $\overline{\mathbf{L}}^1$ in (A12W.1.3)

$$\overline{\mathbf{x}}^1 = \left[\begin{array}{c} \overline{\mathbf{x}}_1^1 \\ \hline \overline{\mathbf{x}}_2^1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\alpha}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{array}\right] \qquad \text{(A12W.1.4)}$$

and from (A12W.1.2) and (A12W.1.4),

$$\Delta\mathbf{x}^1 = (\mathbf{L} - \overline{\mathbf{L}}^1)\mathbf{f} = \Delta\mathbf{L}^1\mathbf{f}$$

Here this is

$$\Delta\mathbf{x}^1 = \left[\begin{array}{c} \mathbf{x}_1 - \overline{\mathbf{x}}_1^1 \\ \hline \mathbf{x}_2 - \overline{\mathbf{x}}_2^1 \end{array}\right] = \left[\begin{array}{c} \Delta\mathbf{x}_1^1 \\ \hline \Delta\mathbf{x}_2^1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H} - \mathbf{I} & \mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{\Delta}_{11}^{1L} & \boldsymbol{\Delta}_{12}^{1L} \\ \hline \boldsymbol{\Delta}_{21}^{1L} & \boldsymbol{\Delta}_{22}^{1L} \end{array}\right] \left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{array}\right] \qquad \text{(A12W.1.5)}$$

where "*L*" denotes extractions from the Leontief model. (Later we use "*G*" for extractions from the Ghosh model.)[5]

This is one comprehensive measure of sector 1's importance to the economy; it reflects removal of all connections—forward, backward and internal. Since sector 1 ceases to exist ($\mathbf{A}_{11} = \mathbf{A}_{12} = \mathbf{A}_{21} = \mathbf{0}$), $\overline{\mathbf{x}}_1^1 = \mathbf{0}$ and the amount of its output that goes to satisfy final demand for sector 1 goods is also zero; then the (original) amount of f_1 would have to be satisfied by imports. The importance of sector 1 in the total economy from which it is "completely extracted" in this manner could be measured by $\mathbf{i}'\boldsymbol{\Delta}\mathbf{x}^1 = \mathbf{i}'\Delta\mathbf{x}_1^1 + \mathbf{i}'\boldsymbol{\Delta}\mathbf{x}_2^1$. To examine the importance of the excluded sector to just those sectors that remain, it is the vector $\boldsymbol{\Delta}\mathbf{x}_2^1 = \mathbf{x}_2 - \overline{\mathbf{x}}_2^1$ that is of interest, so the appropriate

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[3] We use the overbar to indicate a model with extraction, and a "1" to indicate that this is the first of several possible extraction scenarios.

[4] Groenewold, Hagger, and Madden (1993) call this scenario "shut-down of [the] industry."

[5] In this and subsequent expressions for $\Delta\mathbf{x}$, the reader should bear in mind that the $\mathbf{A}_{ij}$, $\boldsymbol{\alpha}_{ii}$ and $\mathbf{H}$ are from the original $\mathbf{x}$ in (A12W.1.2), before the "zeroing out."

measure is $\mathbf{i}'\Delta\mathbf{x}_2^1 = \mathbf{i}'\boldsymbol{\Delta}_{21}^{1L}\mathbf{f}_1 + \mathbf{i}'\boldsymbol{\Delta}_{22}^{1L}\mathbf{f}_2$. This is used, for example, in Schultz (1977). (Note that final demands act as "weights" in these expressions. Other choices are possible.)[6]

A12W.1.2 Case 2. Extraction of Sector 1's Intersectoral Relations

Here all of sector 1's linkages to the rest of the economy are eliminated but internal linkages remain; it differs from Case 1 only by the retention of $\mathbf{A}_{11}$ (*intra*sectoral linkage for sector 1). In this case, let $\mathbf{A}_{12} = \mathbf{A}_{21} = \mathbf{0}$. Then

$$\bar{\mathbf{A}}^2 = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right] \text{ and } \bar{\mathbf{L}}^2 = \left[\begin{array}{c|c} \boldsymbol{\alpha}_{11} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\alpha}_{22} \end{array}\right] \qquad \text{(A12W.1.6)}$$

Gross output differences without and with sector 1 extracted in this manner are

$$\Delta\mathbf{x}^2 = \left[\begin{array}{c} \Delta\mathbf{x}_1^2 \\ \hline \Delta\mathbf{x}_2^2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H} - \boldsymbol{\alpha}_{11} & \mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{array}\right] \qquad \text{(A12W.1.7)}$$

The total linkage measure presented by Cella (1984) is $\mathbf{i}'(\Delta\mathbf{x}^2)$. He argues that this particular extraction, namely $\mathbf{A}_{12} = \mathbf{0}$ and $\mathbf{A}_{21} = \mathbf{0}$, sets up the appropriate measure of "…the quantities of *n* goods directly and indirectly stimulated by the intermediate functions (both as purchaser and as supplier)" of sector 1 (p. 74). Miller (1966, 1969), Miller and Blair (1983), and Dietzenbacher, van der Linden, and Steenge (1993)—among others—applied this structure in a spatial (interregional) setting to measure interregional feedback effects (interregional linkages).

Cella developed this approach partly in response to Schultz (1976, 1977) and to Meller and Marfán (1981), because he believed that they had accounted for too little and too much linkage, respectively, in using (A12W.1.3).[7] He suggested this modification because it removes the extracted sector's internal linkage to itself [in the upper left submatrix in (A12W.1.7)], and one might argue that an industry's self-supply can be considered both a forward and backward link.[8]

Furthermore, Cella proposed a decomposition of this total linkage indicator into forward and backward linkage components, suggesting that the two submatrices in the left half of the partitioned inverse serve to capture backward linkages,

$$BL_1 = \mathbf{i}'(\mathbf{H} - \boldsymbol{\alpha}_{11})\mathbf{f}_1 + \mathbf{i}'(\boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H})\mathbf{f}_1$$

and that forward linkages are measured in the two submatrices in the right half of that inverse,

$$FL_1 = \mathbf{i}'(\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22})\mathbf{f}_2 + \mathbf{i}'(\boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22})\mathbf{f}_2$$

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[6] Meller and Marfán (1981), for example, "normalized" all final demands to 1 and then premultiplied all inverses by labor input coefficients to convert results to employment terms.

[7] Cella (1984, p.79) suggests that he is "sharpening up" the approach of Schultz.

[8] The magnitude of this internal linkage effect depends in part on the level of aggregation in the input-output model. If sector 1 is "manufacturing," this effect will be large; if sector 1 is "brass bolts," it is likely to be very small.

This reflects the logical conditions that sector 1's backward linkage is zero if and only if $\mathbf{A}_{21} = \mathbf{0}$ (making $\mathbf{H} = \boldsymbol{\alpha}_{11}$) and its forward linkage is zero if and only if $\mathbf{A}_{12} = \mathbf{0}$.

These definitions have been criticized. For example, Clements (1990) argues that $\mathbf{i}'(\boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22})\mathbf{f}_2$ belongs as a third term in BL_1, leaving only $\mathbf{i}'(\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22})\mathbf{f}_2$ as FL_1. A more fundamental disagreement appears initially to have been raised by Guccione (1986), namely that the two terms in Cella's FL_1 are in fact more appropriately viewed as the *backward* linkage of sector(s) 2—the rest of the economy—on 1 (see also Cella, 1986, 1988b, for some reactions to this and other criticisms). Dietzenbacher, van der Linden and Steenge (1993) have reiterated this point of view, insisting that only backward linkages are to be found from the Leontief model and (harking back to Beyers, 1976, and Jones, 1976) that forward linkage measures must come from elements of the Ghosh model.[9]

A12W.1.3 Case 3. Extraction of Sector 1's Intermediate Purchases

Here $\mathbf{A}_{11} = \mathbf{A}_{21} = \mathbf{0}$, so

$$\bar{\mathbf{A}}^3 = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right] \text{ and } \bar{\mathbf{L}}^3 = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \mathbf{0} & \boldsymbol{\alpha}_{22} \end{array}\right] \qquad \text{(A12W.1.8)}$$

The difference between gross outputs in the economy without and with sector 1 extracted in this manner is

$$\Delta\mathbf{x}^3 = \left[\begin{array}{c} \Delta\mathbf{x}_1^3 \\ \hline \Delta\mathbf{x}_2^3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{H}-\mathbf{I} & (\mathbf{H}-\mathbf{I})\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{array}\right] \qquad \text{(A12W.1.9)}$$

This can be viewed as another measure of the strength of sector 1's backward linkage, since all intermediate inputs into the sector are removed. This measure appears in Szyrmer and Walker (1983) and is also used by Dietzenbacher and van der Linden (1997) to generate their preferred (spatial) backward linkage measure.

A12W.1.4 Case 4. Extraction of Sector 1's Intermediate Sales

In this case, $\mathbf{A}_{11} = \mathbf{A}_{12} = \mathbf{0}$, meaning

$$\bar{\mathbf{A}}^4 = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] \text{ and } \bar{\mathbf{L}}^4 = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21} & \boldsymbol{\alpha}_{22} \end{array}\right] \qquad \text{(A12W.1.10)}$$

with the following total change in output

[9] Cella (1984, 1988a) seems to have been the first to argue that indices from Leontief and Ghosh models cannot be combined, basically because of inconsistent stability assumptions about the coefficient matrices that underpin the two models. This is the "joint stability" problem (section 12.1.4, above).

$$\Delta \mathbf{x}^4 = \begin{bmatrix} \Delta \mathbf{x}_1^4 \\ \hline \Delta \mathbf{x}_2^4 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{H} - \mathbf{I} & \mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}(\mathbf{H} - \mathbf{I}) & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \end{array} \right] \begin{bmatrix} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{bmatrix} \qquad \text{(A12W.1.11)}$$

Parallel to the argument in Case 3, this can be viewed as another measure of the strength of sector 1's forward linkage, since all intermediate shipments from the sector are removed. Groenewold, Hagger, and Madden (1987) discuss this measure as a partial improvement over that given in Case 1, which they criticize for "overcounting" the individual effects associated with a sector's extraction.

A12W.1.5 Case 5. Extraction of Sector 1's Intersectoral Intermediate Purchases

This has been suggested as another measure of sector 1's backward linkage (as in Case 3) but with emphasis on the linkages external to sector 1. Here, then,

$$\overline{\mathbf{A}}^5 = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array} \right] \text{ and } \overline{\mathbf{L}}^5 = \left[\begin{array}{c|c} \boldsymbol{\alpha}_{11} & \boldsymbol{\alpha}_{11}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \mathbf{0} & \boldsymbol{\alpha}_{22} \end{array} \right] \qquad \text{(A12W.1.12)}$$

and it is easily seen that

$$\Delta \mathbf{x}^5 = \begin{bmatrix} \Delta \mathbf{x}_1^5 \\ \hline \Delta \mathbf{x}_2^5 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{H} - \boldsymbol{\alpha}_{11} & (\mathbf{H} - \boldsymbol{\alpha}_{11})\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \end{array} \right] \begin{bmatrix} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{bmatrix} \qquad \text{(A12W.1.13)}$$

A12W.1.6 Case 6. Extraction of Sector 1's Intersectoral Intermediate Sales

Using the same logic as that behind Case 5, this is another measure of sector 1's forward linkage (as in Case 4) but with emphasis on the linkages external to sector 1. In this case,

$$\overline{\mathbf{A}}^6 = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{0} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array} \right] \text{ and } \overline{\mathbf{L}}^6 = \left[\begin{array}{c|c} \boldsymbol{\alpha}_{11} & \mathbf{0} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\boldsymbol{\alpha}_{11} & \boldsymbol{\alpha}_{22} \end{array} \right] \qquad \text{(A12W.1.13)}$$

and

$$\Delta \mathbf{x}^6 = \begin{bmatrix} \Delta \mathbf{x}_1^6 \\ \hline \Delta \mathbf{x}_2^6 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{H} - \boldsymbol{\alpha}_{11} & \mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \hline \boldsymbol{\alpha}_{22}\mathbf{A}_{21}(\mathbf{H} - \boldsymbol{\alpha}_{11}) & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H}\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \end{array} \right] \begin{bmatrix} \mathbf{f}_1 \\ \hline \mathbf{f}_2 \end{bmatrix} \qquad \text{(A12W.1.14)}$$

A12W.1.7 The Ghosh Model and Some Comparisons

There are the same possibilities for extractions from a partitioned version of the Ghosh model. It has been argued (Dietzenbacher, van der Linden, and Steenge, 1993; Dietzenbacher and van der Linden, 1997) that one of these provides an alternative and superior measure of total forward linkage. Here, for the partitioned case,

$$\mathbf{B} = \left[\begin{array}{c|c} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \hline \mathbf{B}_{21} & \mathbf{B}_{22} \end{array} \right]$$

Recall that **A** and **B** are similar matrices (section 12.1.2). Let $\hat{\mathbf{x}} = \left[\begin{array}{c|c} \hat{\mathbf{x}}_1 & \mathbf{0} \\ \hline \mathbf{0} & \hat{\mathbf{x}}_2 \end{array}\right]$, so

$\hat{\mathbf{x}}^{-1} = \left[\begin{array}{c|c} (\hat{\mathbf{x}}_1)^{-1} & \mathbf{0} \\ \hline \mathbf{0} & (\hat{\mathbf{x}}_2)^{-1} \end{array}\right]$, and from similarity we have

$$\mathbf{B} = \left[\begin{array}{c|c} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \hline \mathbf{B}_{21} & \mathbf{B}_{22} \end{array}\right] = \hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}} = \left[\begin{array}{c|c} (\hat{\mathbf{x}}_1)^{-1}\mathbf{A}_{11}\hat{\mathbf{x}}_1 & (\hat{\mathbf{x}}_1)^{-1}\mathbf{A}_{12}\hat{\mathbf{x}}_2 \\ \hline (\hat{\mathbf{x}}_2)^{-1}\mathbf{A}_{21}\hat{\mathbf{x}}_1 & (\hat{\mathbf{x}}_2)^{-1}\mathbf{A}_{22}\hat{\mathbf{x}}_2 \end{array}\right]$$

The associated partitioned inverse is

$$\mathbf{G} = \hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}} = \left[\begin{array}{c|c} \mathbf{K} & \mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22} \\ \hline \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}\left(\mathbf{I} + \mathbf{B}_{21}\mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22}\right) \end{array}\right] \qquad \text{(A12W.1.15)}$$

where $\mathbf{K} = (\mathbf{I} - \mathbf{B}_{11} - \mathbf{B}_{12}\boldsymbol{\beta}_{22}\mathbf{B}_{21})^{-1} = (\hat{\mathbf{x}}_1)^{-1}\mathbf{H}\hat{\mathbf{x}}_1$ and $\boldsymbol{\beta}_{22} = (\mathbf{I} - \mathbf{B}_{22})^{-1} = (\hat{\mathbf{x}}_2)^{-1}\boldsymbol{\alpha}_{22}\hat{\mathbf{x}}_2$ .Value added (a row vector) can also be partitioned, as $\mathbf{v}' = \left[\mathbf{v}_1' \mid \mathbf{v}_2'\right]$, so that

$$\mathbf{x}' = \left[\mathbf{x}_1' \mid \mathbf{x}_2'\right] = \left[\mathbf{v}_1' \mid \mathbf{v}_2'\right]\left[\begin{array}{c|c} \mathbf{K} & \mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22} \\ \hline \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}(\mathbf{I} + \mathbf{B}_{21}\mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22}) \end{array}\right] \qquad \text{(A12W.1.16)}$$

We examine only one of the possibilities for the Ghosh case; others have obvious parallels to the Leontief model. Removing all of sector 1's interindustry sales in order to quantify that sector's *total forward linkage*, we have

$$\bar{\mathbf{B}}^4 = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{B}_{21} & \mathbf{B}_{22} \end{array}\right] \text{ and } \bar{\mathbf{G}}^4 = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \boldsymbol{\beta}_{22}\mathbf{B}_{21} & \boldsymbol{\beta}_{22} \end{array}\right] \qquad \text{(A12W.1.17)}$$

Then $(\Delta\mathbf{x}^4)' = \mathbf{v}'(\Delta\mathbf{G}^4)$; here this is

$$(\Delta\mathbf{x}^4)' = \left[(\Delta\mathbf{x}_1^4)' \mid (\Delta\mathbf{x}_2^4)'\right] = \left[\mathbf{v}_1' \mid \mathbf{v}_2'\right]\left[\begin{array}{c|c} \mathbf{K} - \mathbf{I} & \mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22} \\ \hline \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22} \end{array}\right] \qquad \text{(A12W.1.18)}$$

In particular,

$$(\Delta\mathbf{x}_2^4)'\mathbf{i} = \mathbf{v}_1'\mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22}\mathbf{i} + \mathbf{v}_2'\boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K}\mathbf{B}_{12}\boldsymbol{\beta}_{22}\mathbf{i} \qquad \text{(A12W.1.19)}$$

This is the forward linkage measure advocated by Dietzenbacher and van der Linden (1997).

Table A12W.1-1 summarizes the $\Delta\mathbf{x}$ results in terms of the $\Delta\mathbf{L}$ or $\Delta\mathbf{G}$ matrix for the various cases.[10] In each case, outcomes on the unextracted (remaining) sectors are

[10] In all of the submatrices for any of the cases in the Ghosh column in Table 1 it is also easily shown that $\Delta_{ij}^{kG} = (\hat{\mathbf{x}}_i)^{-1}\Delta_{ij}^{kL}\hat{\mathbf{x}}_j$. When a single sector is excluded, $\mathbf{H} = \mathbf{K} = s$ (a scalar) and so $\Delta_{11}^{kG} = \Delta_{ij}^{kL}$ for $k = 1,\ldots, 6$.

found by summing over the elements in $\mathbf{\Delta x}_2$.[11] For Case k in the Leontief model, this means the sum of elements from the bottom *row* of the partitioned difference matrix, weighted by final demands—$\mathbf{i'\Delta x}_2 = \mathbf{i'\Delta}_{21}^{kL}\mathbf{f}_1 + \mathbf{i'\Delta}_{22}^{kL}\mathbf{f}_2$. For the Case k Ghosh model, as in (A12W.1.18), it is the sum of elements from the right-hand *column* of the partitioned difference matrix, weighted by value added—$\mathbf{\Delta x}_2\mathbf{i} = \mathbf{v}_1'\mathbf{\Delta}_{12}^{kG}\mathbf{i} + \mathbf{v}_2'\mathbf{\Delta}_{22}^{kG}\mathbf{i}$.

Table A12W.1-1 Partitioned Difference Matrices for Cases 1 - 6

$$\Delta\mathbf{x}^k = \begin{bmatrix} \mathbf{\Delta}_{11}^{kL} & \mathbf{\Delta}_{12}^{kL} \\ \mathbf{\Delta}_{21}^{kL} & \mathbf{\Delta}_{22}^{kL} \end{bmatrix}\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \quad \text{and} \quad (\Delta\mathbf{x}^k)' = \begin{bmatrix} \mathbf{v}_1' & \mathbf{v}_2' \end{bmatrix}\begin{bmatrix} \mathbf{\Delta}_{11}^{kG} & \mathbf{\Delta}_{12}^{kG} \\ \mathbf{\Delta}_{21}^{kG} & \mathbf{\Delta}_{22}^{kG} \end{bmatrix}$$

k	Structure of $\bar{\mathbf{A}}$ or $\bar{\mathbf{B}}$	Leontief Model $\begin{bmatrix} \mathbf{\Delta}_{11}^{kL} & \mathbf{\Delta}_{12}^{kL} \\ \mathbf{\Delta}_{21}^{kL} & \mathbf{\Delta}_{22}^{kL} \end{bmatrix}$	Ghosh Model $\begin{bmatrix} \mathbf{\Delta}_{11}^{kG} & \mathbf{\Delta}_{12}^{kG} \\ \mathbf{\Delta}_{21}^{kG} & \mathbf{\Delta}_{22}^{kG} \end{bmatrix}$
1	$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \blacksquare \end{bmatrix}$	$\begin{bmatrix} \mathbf{H}-\mathbf{I} & \mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \\ \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \end{bmatrix}$	$\begin{bmatrix} \mathbf{K}-\mathbf{I} & \mathbf{KB}_{12}\boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{KB}_{12}\boldsymbol{\beta}_{22} \end{bmatrix}$
2	$\begin{bmatrix} \blacksquare & \mathbf{0} \\ \mathbf{0} & \blacksquare \end{bmatrix}$	$\begin{bmatrix} \mathbf{H}-\boldsymbol{\alpha}_{11} & \mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \\ \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \end{bmatrix}$	$\begin{bmatrix} \mathbf{K}-\boldsymbol{\beta}_{11} & \mathbf{KB}_{12}\boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{KB}_{12}\boldsymbol{\beta}_{22} \end{bmatrix}$
3	$\begin{bmatrix} \mathbf{0} & \blacksquare \\ \mathbf{0} & \blacksquare \end{bmatrix}$	$\begin{bmatrix} \mathbf{H}-\mathbf{I} & (\mathbf{H}-\mathbf{I})\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \end{bmatrix}$	$\begin{bmatrix} \mathbf{K}-\mathbf{I} & (\mathbf{K}-\mathbf{I})\mathbf{B}_{12}\boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{KB}_{12}\boldsymbol{\beta}_{22} \end{bmatrix}$
4	$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \blacksquare & \blacksquare \end{bmatrix}$	$\begin{bmatrix} \mathbf{H}-\mathbf{I} & \mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \\ \boldsymbol{\alpha}_{22}\mathbf{A}_{21}(\mathbf{H}-\mathbf{I}) & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \end{bmatrix}$	$\begin{bmatrix} \mathbf{K}-\mathbf{I} & \mathbf{KB}_{12}\boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22}\mathbf{B}_{21}(\mathbf{K}-\mathbf{I}) & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{KB}_{12}\boldsymbol{\beta}_{22} \end{bmatrix}$
5	$\begin{bmatrix} \blacksquare & \blacksquare \\ \mathbf{0} & \blacksquare \end{bmatrix}$	$\begin{bmatrix} \mathbf{H}-\boldsymbol{\alpha}_{11} & (\mathbf{H}-\boldsymbol{\alpha}_{11})\mathbf{A}_{12}\boldsymbol{\alpha}_{22} \\ \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{H} & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \end{bmatrix}$	$\begin{bmatrix} \mathbf{K}-\boldsymbol{\beta}_{11} & (\mathbf{K}-\boldsymbol{\beta}_{11})\mathbf{B}_{12}\boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{K} & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{KB}_{12}\boldsymbol{\beta}_{22} \end{bmatrix}$
6	$\begin{bmatrix} \blacksquare & \mathbf{0} \\ \blacksquare & \blacksquare \end{bmatrix}$	$\begin{bmatrix} \mathbf{H}-\boldsymbol{\alpha}_{11} & \mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \\ \boldsymbol{\alpha}_{22}\mathbf{A}_{21}(\mathbf{H}-\boldsymbol{\alpha}_{11}) & \boldsymbol{\alpha}_{22}\mathbf{A}_{21}\mathbf{HA}_{12}\boldsymbol{\alpha}_{22} \end{bmatrix}$	$\begin{bmatrix} \mathbf{K}-\boldsymbol{\beta}_{11} & \mathbf{KB}_{12}\boldsymbol{\beta}_{22} \\ \boldsymbol{\beta}_{22}\mathbf{B}_{21}(\mathbf{K}-\boldsymbol{\beta}_{11}) & \boldsymbol{\beta}_{22}\mathbf{B}_{21}\mathbf{KB}_{12}\boldsymbol{\beta}_{22} \end{bmatrix}$

where $\boldsymbol{\alpha}_{ii} = (\mathbf{I}-\mathbf{A}_{ii})^{-1}$, $\boldsymbol{\beta}_{ii} = (\mathbf{I}-\mathbf{B}_{ii})^{-1} = (\hat{\mathbf{x}}_i)^{-1}\boldsymbol{\alpha}_{ii}\hat{\mathbf{x}}_i$,

$\mathbf{H} = (\mathbf{I}-\mathbf{A}_{11}-\mathbf{A}_{12}\boldsymbol{\alpha}_{22}\mathbf{A}_{21})^{-1}$, $\mathbf{K} = (\mathbf{I}-\mathbf{B}_{11}-\mathbf{B}_{12}\boldsymbol{\beta}_{22}\mathbf{B}_{21})^{-1} = (\hat{\mathbf{x}}_1)^{-1}\mathbf{H}\hat{\mathbf{x}}_1$.

[11] Summing only over the elements of $\mathbf{\Delta x}_2$ often may be appropriate in an interindustry setting, where the usual story for the extraction is that one is measuring an industry's relative stimulative importance to the economy. This may not be the case in an interregional (spatial) setting where one may be less interested in analyzing the stimulative importance of a region but strictly the magnitude of its linkages to the rest of the economy. Because of this the $\mathbf{\Delta x}$ results, rather than $\mathbf{\Delta x}_2$, are often used in this context (unless one is applying Case 1). In an interindustry setting if one is strictly interested in interindustrial linkages as opposed to total linkages, the same observations apply.

It is clear from Table A12W.1-1 that Leontief Cases 1, 2, 3 and 5 generate identical results for $\mathbf{i}'\Delta\mathbf{x}_2$ (the first element in the bottom row of $\Delta\mathbf{L}^k$ is the same in all four cases, as is the second) and also that Ghosh Cases 1, 2, 4 and 6 produce identical results for $\Delta\mathbf{x}_2\mathbf{i}$ (first and second elements in the right-hand column of $\Delta\mathbf{G}^k$ are equal, respectively, across these four cases). This suggests that for certain questions one has several extraction scenarios that are equally appropriate.

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