Equation of motion via energy methods

SDF case

An alternative method to derive the equation of motion is based energy consideration. The energy methods are very powerful especially for analyzing conservative (lossless) systems. We derive here a lossless form of the equation of motion. Although all real mechanical resonators dissipate energy, most analyses of many degrees of freedom and continuous system start the lossless idealization and thus we analyze here a lossless version of the resonator (no damper!). For a conservative system we have two types of energy: kinetic energy T (energy of motion), and potential energy V (energy that is a function of the position, or the configuration of the system).

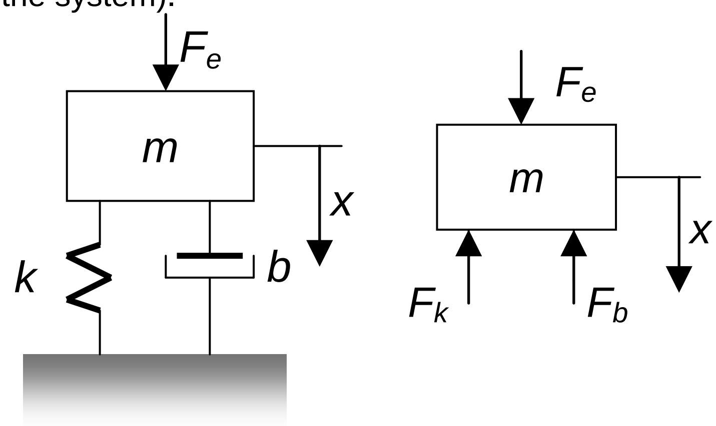

Figure 1: A spring-mass system (a) a symbolic diagram, (b) free-body diagram.

2-DOF model.

In our system, the kinetic energy is the energy of a lumped mass constrained to move along a line and is given by $T = m\dot{x}^2/2$. The potential energy is the elastic energy of the spring is given by $V = ½\, kx^2$. We sketch the derivation of Lagrange equations and then apply them to the SDF. The complete derivation can be found in texts on dynamics e.g. [1-5] Lagrangian is a quantity defined by the difference between kinetic and potential energy $L = T - V$. The time integral of Lagrangian is called action

$$\int \underbrace{(T - V)}_{L} dt .$$

Setting the variation of action to zero with the fixed start and the fixed end point gives Hamilton' principle

$$\delta\left(\int_{t_1}^{t_2} L(t, x, \dot{x}) dt \right) = 0 ,$$

which, in turn, gives rise to Lagrange equations as follows

$$\delta\left(\int_{t_1}^{t_2} L(t,x,\dot{x},t)dt\right) = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}\delta x + \underbrace{\frac{\partial L}{\partial \dot{x}}\delta\dot{x}}_{\text{perform partial integration on this term}}\right)dt = \underbrace{\left.\frac{\partial L}{\partial \dot{x}}\delta x\right|_{t1}^{t2}}_{\text{=0 because there is no variation at the end points}} + \int_{t_1}^{t_2}\underbrace{\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)}_{\text{Independent of varion, must be zero}}\delta x dt$$

Thus, the Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\,.$$

If we substitute the kinetic and potential energy in the Lagrange equation, we have

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) = 0$$
$$m\ddot{x} + kx = 0$$

The non-conservative forces, including dissipative and external forces are added to the right hand side of the Lagrange equation. Hamilton principle and Lagrange equations are very useful tools for analyzing more complex, systems. It is important to note that Lagrange equations are much easier to use than to derive.

MDF equation of motion via energy methods

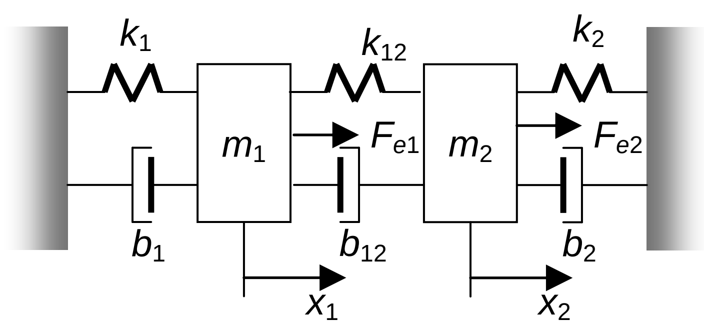

Figure 2: 2-DOF model.

As is the case with SDF mode, the conservative version of the equations can be derived from the energy consideration. Generally, as the number of degrees increase, obtaining equations of motion via energy methods becomes easier than obtaining from the force balance (Newton's law). The kinetic energy of this 2 DOF model is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

and the potential energy is

$$V = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 + \frac{1}{2}k_{12}\left(x_1 - x_2\right)^2$$

The Lagrange equations give us the conservative part of the left-hand side of (A2-16)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 + k_1 x_1 - k_{12}x_2$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = m_2\ddot{x}_2 + k_2 x_2 - k_{12}x_1$$

Note that the kinetic energy and the potential energy can be written compactly in terms of matrices:

$$T = 1/2\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}}$$
$$V = 1/2\mathbf{x}^T\mathbf{K}\mathbf{x}$$

These relations are general and, because T and V are positive quantities, the relations gives us some insight in properties of **M** and **K** – they are both positive definite matrices[1].

A SDF system has one resonant frequency $\omega_0 = (k/m)^{1/2}$, but n DOF system has n resonances. The resonances are obtained using eigen analysis.

References

[1] L. Meirovitch, *Methods of analytical dynamics*. Mineola, N.Y.: Dover, 2003.

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[3] V. I. Arnol*d, *Mathematical methods of classical mechanics*, 2nd ed. New York: Springer, 1997.

[4] J. V. José and E. J. Saletan, *Classical dynamics : a contemporary approach*. Cambridge England ; New York: Cambridge University Press, 1998.

[5] C. Lanczos, *The variational principles of mechanics*, 3d ed. Toronto,: University of Toronto Press, 1966.

[1] This is not generally true. The system exhibiting rigid body modes have $V = 0$ and positive semi-definite **K**. However, all present MEMS systems are constrained, with $V > 0$.