Nonlinear and parametric resonator

This section describes the most common nonlinearity that occurs in mechanical resonators. Typically nonlinearity enters the equation of motion as the nonlinear stiffness (nonlinear spring). While absolute displacement in a MEMS device are small compared to the macro world, their size can be relatively large compared to the size of the structure itself. Geometric nonlinearities, such as nonlinearity of the pendulum for large displacement, rarely occur in MEMS devices and will not be discussed.

1. Duffing equation (nonlinear spring)

The restoring force of a linear spring is proportional to its stiffness and the constitutive equation of the linear spring is $F_k = -kx$. A spring force always acts against the external force. If we expand the force of a general nonlinear spring in Taylor series, the first nonlinear term that is retained is the cubic term $\sim x^3$. The quadratic term does not have the restoring property of the spring, i.e., it does not always counteract the external force. If the constitutive equation of the nonlinear spring is $F_k = -kx - \alpha kx^3$ is inserted into Newton's law, the equation of motion becomes

$$m\ddot{x} + b\dot{x} + kx + \alpha kx^3 = F_e \qquad \text{(B1)}$$

or in normalized form

$$\ddot{x} + \frac{\omega_o}{Q}\dot{x} + \omega_o^2 x + \alpha\omega_o^2 x^3 = \frac{F_e}{m} \qquad \text{(B2)}$$

When the force is harmonic, $F_e = F_o\cos(\omega t + \phi)$, this equation of motion called Duffing equation, after the scientist who first analyzed it in 1918 [1]. For $\alpha > 0$ the nonlinearity increases overall stiffness (spring hardening effect). For $\alpha < 0$, the nonlinearity decreases overall stiffness (spring softening effect). Figure 1 shows the simulated amplitude and phase response. We see that for spring hardening the amplitude characteristics bends to the right and the peak is lower than the peak of the associated linear system.

In the case of spring softening the amplitude characteristics bends to the left and the peak is higher than the peak of the associated linear system.

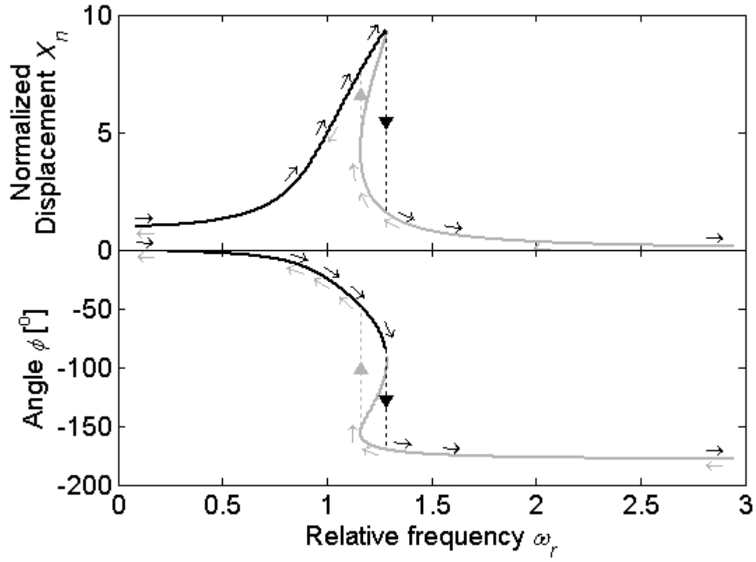

Figure 1: Amplitude and phase characteristic of a Duffing oscillator with a hardening spring.

More detailed analysis reveals a hysteresis in amplitude and phase characteristics. This hysteresis can be problematic when the mechanical system is driven in resonance using a phase locked loop (PLL) that locks on quadrature, i.e., 90 ^{o} phase difference between the excitation and the response. The hysteresis in the phase characteristic gives rise to different amplitudes when the PLL locks from above the resonance than when it locks from below the resonance.

2. Analytical derivation of the amplitude and phase response

The harmonic analysis using successive approximation starts by assuming the response in the form $x^{(1)} = X\cos(\omega t)$[*]. Then the first approximation is substituted in the equation of motion to obtain

$$\begin{aligned}\ddot{x}^{(2)} &= \frac{F_e}{m}\cos(\omega t+\phi)+\frac{\omega_o\omega}{Q}X_o\sin(\omega t)\\ &-\omega_o^2 X_o\cos(\omega t)-\frac{1}{4}\alpha\omega_o^2 X_o^3\left(\cos(3\omega t)+3\cos(\omega t)\right)\end{aligned} \tag{B3}$$

where we used $\cos^3(\alpha) = ¼\cos(3\alpha) + ¾\cos(\alpha)$ [†]. After direct integration, we get the second iteration

[*] The derivation is similar to that of Jorge and Saletan [2].

[†] To derive the trigonometric identity one can start with $\exp(j3\alpha) = \cos(3\alpha) + j\sin(3\alpha) = (\cos(\alpha) + j\sin(\alpha))^3$, equate the real parts, and get $\cos(3\alpha) = \cos^3(\alpha) - 3\cos(\alpha)\sin^2(\alpha) = \cos^3(\alpha) - 3\cos(\alpha) - 3\cos^3(\alpha)$, from which $\cos^3(\alpha) = ¼\cos(3\alpha) + ¾\cos(\alpha)$ follows.

$$x^{(2)}=-\frac{F_e}{m\omega^2}\cos(\omega t+\phi)-\frac{\omega_o}{\omega Q}X_o\sin(\omega t)$$
$$+\left(\frac{\omega_o}{\omega}\right)^2 X_o\cos(\omega t)+\frac{1}{4}\alpha\left(\frac{\omega_o}{\omega}\right)^2 X_o^3\left(\cos(3\omega t)+3\cos(\omega t)\right) \tag{B4}$$

$x^{(2)}$ is then equated to $x^{(1)}$. Ignoring the harmonic term (at 3ω), and introducing relative frequency $\omega_r=\omega/\omega_0$, we have the following identities

$$\frac{F}{k}\sin\phi=\frac{\omega_r}{Q}X_o \qquad \text{equating terms multiplying } \sin(\omega t)$$
$$\frac{F}{k}\cos\phi=(1-\omega_r^2)X_o+\frac{3}{4}\alpha X_o^3 \qquad \text{equating terms multiplying } \cos(\omega t) \tag{B5}$$

If the two identities (B-73) are squared and summed, after some manipulation, we get a biquadratic equation

$$\omega_n^4+\left(\frac{1}{Q^2}-2-\frac{3}{2}\alpha\left(\frac{F}{k}\right)^2X_n^2\right)\omega_n^2+1-\frac{1}{X_n^2}+\frac{3}{2}\alpha\left(\frac{F}{k}\right)^2X_n^2+\frac{9}{16}\alpha^2\left(\frac{F}{k}\right)^4X_n^4=0 \tag{B6}$$

where $X_n = X_o/(F/k)$. The solution of the biquadratic equation is given by

$$\omega_n=\sqrt{1-\frac{1}{Q^2}+\frac{3}{4}\alpha\left(\frac{F}{k}\right)^2X_n^2\pm\sqrt{\frac{1}{Q^4}-\frac{2}{Q^2}-\frac{3}{2Q^2}\alpha\left(\frac{F}{k}\right)^2X_n^2}} \tag{B7}$$

The phase is simply the ratio of the two identities above

$$\phi=\arctan\left(\frac{\omega_r/Q}{1-\omega_r^2+3/4\alpha X_o^2}\right) \tag{B8}$$

After completing the derivation it is instructive to look at Figure 1 again. The darker trace correspond to "-" solution for ω_n, while the lighter traces correspond to the "+" solution of ω_n. The darker arrows follow the frequency sweep from low to high frequencies, while the lighter arrows follow the frequency sweep from high to low frequencies. The paths of the two sweeps are different and we say that the resonator exhibits hysteretic behavior*. This behavior can be readily observed experimentally: a jump in the response amplitude during a frequency sweep and the dependence of the peak location on the direction of the frequency sweep are telltale signs of nonlinearity.

* The bistable response of Duffing oscillator is studied by Batista et al. [3].

Mathieu Equation

The general linear parametric oscillator is described by Hill's equation

$$\ddot{x} + G(t)x = 0\,, \tag{B9}$$

where $G(t)$ is a general periodic function. The parametric term is due to electromechanical coupling. When the parametric term of the oscillator is in the form of cosine,

$$\ddot{x} + \frac{\omega_o}{Q}\dot{x} + \omega_o^2\left(1 + \frac{\Delta k}{k}\cos(\omega_p t)\right)x = F(t)/m \tag{B10}$$

the equation is known as Mathieu equation. An example of the application is quality factor enhancements (Grasser et al.[4]). Numerical simulations showed that amplitude of the response increases tenfold when $\omega_p = 2\omega_o$ is used to modulate stiffness.

Often real devices have both nonlinear and parametric terms due to coupling. Nonlinear, parametrically-excited MEMS oscillators can exhibit abrupt changes in their behavior. They have found application in filtering, mass sensing, and scanning probe microscopy. For detail analysis of a nonlinear, parametrically excited oscillator see Rhoads et al [5].

3. To probe further

Nonlinearities can be quite important in MEMS, especially for devices that feature resonant operation and large deflections, as some gyroscopes do. There are several excellent introductory texts on nonlinear dynamics including [6-8], but our favorite is Strogatz [9].

References

[1] G. Duffing, *Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung*: F. Vieweg & sohn, 1918.

[2] J. V. José and E. J. Saletan, *Classical dynamics : a contemporary approach*. Cambridge England ; New York: Cambridge University Press, 1998.

[3] A. A. Batista, F. A. Oliveira, and H. N. Nazareno, "Duffing oscillators: Control and memory effects," *Physical Review E,* vol. 77, p. 066216, 2008.

[4] L. Grasser, H. Mathias, F. Parrain, X. L. Roux, and J. P. Gilles, "Mems Q-Factor Enhancement Using Parametric Amplification: Theoretical Study and Design of a Parametric Device," *arXiv preprint arXiv:0802.3073,* 2008.

[5] J. F. Rhoads, S. W. Shaw, K. L. Turner, J. Moehlis, B. E. DeMartini, and W. Zhang, "Generalized parametric resonance in electrostatically actuated microelectromechanical oscillators," *Journal of Sound and Vibration,* vol. 296, pp. 797-829, 2006.

[6] E. A. Jackson, *Perspectives of nonlinear dynamics*. Cambridge Cambridgeshire ;: New York : Cambridge University Press, 1989.

[7] V. I. Arnol′d, *Ordinary differential equations*. Cambridge,: MIT Press, 1973.

[8] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential equations, dynamical systems & an introduction to chaos* vol. 60: Academic Press, 2004.

[9] S. H. Strogatz, *Nonlinear dynamics and Chaos : with applications to physics, biology, chemistry, and engineering*. Reading, Mass.: Addison-Wesley Pub., 1994.