The Input-Output Analysis Computational Workbook:

Annotated Exercises to accompany

Input-Output Analysis: Foundations and Extensions, 3rd Edition

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Introduction

This collection of computational exercises accompanies the Third Edition *of Input-Output Analysis: Foundations and Extensions* which is a textbook and desk reference for students and scholars in the input–output research and applications community. The book includes extensively referenced and indexed coverage of most subtopics in the field. It is an ideal introduction to the subject for advanced undergraduate and graduate students in a wide variety of fields, including economics, regional science, regional economics, city, regional and urban planning, environmental planning, public policy analysis, and public management. This workbook is an expanded discussion of exercise problems aligned with the book chapters, illustrating major concepts and key analytical approaches as well as exploring applications using examples and selected real-world data.

The new edition of this book has been fully revised and updated to reflect important developments in the field since earlier editions. New topics covered include expanded coverage of non-survey estimation procedures, structural decomposition analysis (SDA), environmentally extended input-output models, social accounting matrices (SAMs), and international input–output models.

This edition is also supported by an accompanying website with supplemental appendices including further information for more advanced readers, exercise problems and solutions, and a sampling of real-world data sets (http://cambridge.com/millerandblair).

Overview of Input-Output Analysis

Professor Wassily Leontief's 1971 presidential address to the American Economic Association was entitled "Theoretical Assumptions and Non-observed Facts." The address took many in the economics profession to task for failing to underscore the necessity of empirically verifying economic theory. This was a longstanding concern of Leontief's about how much of the economics profession had evolved in the post-World War II period and one that he was particularly focused on in developing his own research on systematically analyzing the interdependence of industries in an economy.

Leontief characterized his work as expressing mathematically the efforts of 18th century French economist, Francois Quesnay, to produce a diagrammatic representation of how expenditures can be traced through an economy in a systematic way, known as the *Tableau Économique.* Leontief referred to the analytical framework he had been devising since the 1930s as *input-output analysis* (IOA), referring to the essence of his approach of capturing from observed economic data for a specific geographic region (e.g., a nation, state, or county) the activity of a group of industries that both produce goods and services (outputs) and consume goods and services from other industries (inputs) in the process of producing each industry's own output. In recognition of this work, Leontief received the 1973 Nobel Memorial Prize in Economic Sciences. Today, the basic concepts of IOA set forth by Leontief are key if not central components of many types of economic analysis and, indeed, IOA and its extensions over the last three-quarters of a century remain one of the most widely applied methods in economics.

The number of industries considered in an IOA model may vary from only a few, to hundreds or thousands. The observed data are the flows among or transactions of products between each of an economy's industries (as a producer/seller) and each of the industries (as a purchaser/buyer) over a standard time-period, usually a year. In more contemporary terms, depending upon the level of industry and geographic aggregation and accounting for the role of imports, IOA equations quantify essentially the complete and detailed supply chains for all products and services in the economy.

As noted at the beginning of this overview, one of Leontief's central concerns was the degree to which the transactions table presented an empirically accurate and stable picture of economic activity and what time-period was suitable for sufficiently and faithfully capturing the production characteristics of the economy. Leontief often referred to production functions of industries in his model as production *recipes*, found by normalizing each column of the transactions table by the value of total output of the corresponding industry in the economy to produce a matrix of technical coefficients.

Mathematically, in its simplest form, IOA is based on a matrix of interindustry transactions, **Z**, the rows of which correspond with producing industry sectors in the economy and the columns to those same industries as consumers of industrial products from across the economy, usually measured value terms such as dollars. The most common form of IOA is called an open model in which a schedule of *final* consumption is specified of industrial products in the economy, i.e., consumption outside the network of interindustry production, such as the total of personal consumption, government expenditures, capital expenditures, and exports. For this vector of total final demand, $\mathbf{f}$, the total industrial production for all sectors in the economy, including both deliveries to interindustry and final consumers, is specified as $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$ where $\mathbf{x}$ is the vector of total industrial outputs for all sectors in the economy. The production recipes or technical coefficients are defined by normalizing each column of **Z** by the value of total production for the industry designated by the column, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, i.e., elements of this matrix of technical coefficients or direct requirements designate the dollars' worth of input from each industry in the economy consumed directly to produce one dollar's worth of the output for the industry designated by the column. The total production accounting can then also be written as $\mathbf{x} = \mathbf{Ax} + \mathbf{f}$ or, rearranging terms, as $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ or $\mathbf{x} = \mathbf{Lf}$ where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. The matrix, **L**, is known as the Leontief inverse or matrix of total requirements.

Since, mathematically, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A}^2 + \mathbf{A}^3 + ... + \mathbf{A}^n$, we can interpret the terms of this power series expression as the "rounds" of industrial production necessary to ultimately supply the final consumption. That is, the production necessary to directly supply final consumption is **Af**. The production necessary to supply the inputs to that direct production (i.e., induced by the direct production) is $\mathbf{A}(\mathbf{Af})$ or $\mathbf{A}^2\mathbf{f}$ and subsequent "rounds" of induced production are $\mathbf{A}^3\mathbf{f}$, $\mathbf{A}^4\mathbf{f}$,..., $\mathbf{A}^n\mathbf{f}$ so that the value of total industry production in the economy, including the final consumption itself as well as the direct and all the induced production necessary to supply that final consumption, is $\mathbf{x} = \mathbf{f} + \mathbf{A}^2\mathbf{f} + \mathbf{A}^3\mathbf{f} + ... + \mathbf{A}^n\mathbf{f} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$. In a Leontief economy, for any newly projected increment of final consumption, $\Delta\mathbf{f}$, the additional total industrial production, $\Delta\mathbf{x}$ necessary to satisfy that new increment of final consumption is then found by $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f}$.

The matrix of technical coefficients, **A**, incorporates the central assumptions of the basic IOA model, i.e., that the interindustry flows from one industry to another for a given time-period depend entirely on the total output the consuming industry for that same time-period, i.e., industries exhibit a linear production function defined by fixed technical coefficients for that time-period. Thus, in a basic Leontief economy, an industry uses inputs in fixed proportions and ignores returns to scale.

Throughout Leontief's research career he spent much of his effort exploring the robustness of these assumptions and devising extensions and enhancements to the basic model to accommodate the situations when such assumptions were less suitable and to identifying characteristics of the approach and application that most influenced error and uncertainty in its use. Extensions to IOA became an important area of research in economics, many efforts of which focused on the practical challenges of implementing IOA, including managing the prodigious data and computational requirements, effects of industrial and geographic aggregation, and devising methods to characterize secondary industrial production, final consumer consumption, and the role of capital investment and use. Other extensions enabled IOA to be focused on analyzing structural change in the economy or on specific sectoral issues such as analyzing energy use, environment impacts, and labor utilization.

With growing confidence in the utility of IOA in many different types of economic analysis, much more attention was paid by governments to assembling local, regional, and national data suitable for IOA. A pivotal development in broad implementation of IOA was a widespread, if not essentially uniform adoption of a standardized System of National Accounts (SNA) for economic activity. Work spearheaded by British economist Richard Stone, for which he received the 1984 Nobel Memorial Prize in Economic Sciences, and subsequently promulgated by the United Nations, the SNA enabled systematic tracking of economic activities on a national and international scale. Since the late 1950s most developed nations and many developing ones routinely construct IOA tables along with governments or related agencies for many regions and localities and, increasingly, IOA efforts capture transactions between regions or nations in multiregional models, some at a global scale.

Perhaps the three most significant early limitations to widespread use of IOA were: (1) the lack of reliable data from which to construct the basic interindustry accounts, (2) the lack of uniform standards in the kinds and scale of data collected for IOA, and (3) the extraordinary computational requirements of IOA relative to computer capacity at the time. In the earliest days of IOA, the computational requirements were dominant constraints, limiting its application, even if the necessary data were available, to scores of industries rather than the hundreds or thousands today. Even the most basic of IOA applications involves a large system of linear equations. While conceptually straightforward, computational solution at the time was challenging for even the most powerful computers of the day.

The constraint on computational capacity at the time put IOA front and center in use of the earliest electronic computers, but with the exponential growth in computing capacity over the last half century, such limitations have all but evaporated today. With standardized and much more readily available data, supplemented with methods for utilizing alternative sources of data, IOA is experiencing a fresh resurgence of interest in its utility for many economic issues, especially for global issues requiring large multiregional models. As a result, long avoided because its data and computational burden was often considered a bridge too far, IOA has re-emerged as a central tool in economics and, increasingly, in other areas such as accounting for pollution emissions and mitigation (and related ecosystem models), social accounting models, and many others.

In the last decade IOA's integration with other modeling frameworks has blossomed as well, including links with econometrics, resource planning, demographic modeling, and many others. Leontief's original framework conceived of industry production functions as measured in physical units, such as specifying the technical coefficients in terms of tons of coal or bushels of wheat, as inputs, required per dollars' worth of an industry's output or per ton of steel output. However, the data collection requirements and other constraints rendered implementation of the framework measured in physical units too unwieldy, certainly at the time and even today to a lesser extent. But, while the basic methodology for IOA evolved, in both theory and application, largely through measuring all quantities in value terms with implicit fixed prices, its use expressed in physical units was always considered desirable, both to moderate the impact of prices in analysis and to allow IOA to relate more easily with other modeling frameworks.

The generalization of IOA techniques to a broader conceptual level, such as accounting for economic activity beyond its primary focus on interindustry production, also originated with simpler attempts to link IOA models and other national income accounting techniques. Such generalizations enabled extension of IOA to explore the roles of labor, households, and the social institutions of the economy. Extensions to IOA, such as social accounting matrices and other related constructs, capture many different socioeconomic characteristics of an economy associated with interindustry activity, and enable analysis, for example, of income from employment and its disposition, labor costs, and the demographics of the work force that comprise the market for the supply and demand of labor.

Even late in his own life, Leontief continued to explore ways in which his framework could be implemented more widely, e.g., using physical units rather than value terms to facilitate wider use. These techniques involved many measurable quantities associated with interindustry activity, such as employment, energy use, and environmental pollution. Integration with ecosystem models, for example, addresses the interface between the economy and ecosystems, enabling systematic analysis of such contemporary issues as consumption accounting of global carbon emissions, measuring the energy and environmental resource "footprint" of nations, or the environmental emissions embodied in international trade.

Today, IOA is a well-established and widely utilized tool for analyzing economic activity at any geographic scale, most recently at a global scale. Enabled by increasingly standardized data characteristics and availability of data as well as the formidable computational capacity available today, IOA will continue to grow in its use and utility for addressing many types of economic policy and planning issues. Our text captures most of the important features and extensions of IOA since its conception and its initial applications nearly a century ago. The computational exercise problems in this workbook illustrate many of these features.

Chapter 2, Foundations of Input–Output Analysis

Chapter 2 introduces Leontief's conceptual input–output framework and explains how to develop the fundamental mathematical relationships from the interindustry transactions table. The key assumptions associated with the basic Leontief model and implications of those assumptions are recounted and the economic interpretation of the basic framework is explored. The basic framework is illustrated with a highly aggregated model of the US economy. In addition, the "price model" formulation of the input–output framework is introduced to explore the role of prices in input–output models. Appendices to this chapter include a fundamental set of mathematical conditions for input–output models, known as the Hawkins–Simon conditions. The exercise problems for this chapter explore applications of the basic mathematical relationships of input-output analysis.

Problem 2.1

This problem explores the relationships of the fundamental input-output analysis identities developed in chapter 2: $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$ and $\mathbf{x} = \mathbf{Ax} + \mathbf{f}$ where $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$. Consider a two-sector economy (agriculture and manufacturing), the basic data for which are the matrix of interindustry transactions, **Z**, and vector of total outputs, **x**, expressed in dollar values, specified as:

$$\mathbf{Z} = \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1,000 \\ 800 \end{bmatrix}$$

Rearranging terms in the first input-output identity, $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$, to $\mathbf{f} = \mathbf{x} - \mathbf{Zi}$ makes it easy to calculate the vector of final demands, **f**, for this economy as

$$\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 1,000 \\ 800 \end{bmatrix} - \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1,000 \\ 800 \end{bmatrix} - \begin{bmatrix} 850 \\ 680 \end{bmatrix} = \begin{bmatrix} 150 \\ 120 \end{bmatrix}$$

To illustrate the process of impact analysis, i.e., computing the impact on industrial production in the economy resulting from a new final demands presented to the economy, we specify new final demands as f_1 increased by \$50 and f_2 decreased by \$20, so that the vector of new final demands is $\mathbf{f}^{new} = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$. To determine the production of total output for each sector in this economy necessary to support these new levels of final demand, we first invoke the basic Leontief model assumptions defining the matrix of technical coefficients or direct requirements:

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix} \begin{bmatrix} 1/1000 & 0 \\ 0 & 1/800 \end{bmatrix} = \begin{bmatrix} .5 & .4375 \\ .32 & .45 \end{bmatrix}$$

We can compute a "round-by-round" approximation of the impacts of the new final demands on this economy to intuitively illustrate the effect of these new final demands on total industrial production throughout the economy by computing, first, the direct requirements to satisfy the new final demand vector added to the final demands themselves, $\mathbf{f}^{new} + \mathbf{Af}^{new}$, then added the production necessary to supply that first "round" of direct requirements, $\mathbf{A}(\mathbf{Af}^{new})$, and so on. Mathematically, as discussed in chapter 2, this is expressed as the infinite power series $\mathbf{x}^{new} = \mathbf{f}^{new} + \mathbf{Af}^{new} + \mathbf{A}(\mathbf{Af}^{new}) + \cdots = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots)\mathbf{f}^{new}$. This infinite power series ultimately converges to the total amount of production, required directly and indirectly through the successive rounds of intermediate industrial production, to support the new final demands.

Terminating the power series is an approximation of the "exact" values found by rearranging $\mathbf{x} = \mathbf{Ax} + \mathbf{f}$ as $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ or $\mathbf{x} = \mathbf{Lf}$ where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^n$ with increasing precision of the approximation as n increases. The matrix of total requirements, **L**, is often referred to as the Leontief inverse.

For this economy, computing the "round by round" requirements for the first five terms yields only a rough approximation of the total outputs in the economy necessary to satisfy the new final demands: $\tilde{\mathbf{x}}^{new} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^4)\mathbf{f}^{new} = \begin{bmatrix} 650.81 \\ 453.98 \end{bmatrix}$, compared with the "exact" value, $\mathbf{x}^{new} = \mathbf{Lf}^{new} = \begin{bmatrix} 1{,}138.90 \\ 844.40 \end{bmatrix}$ where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 4.07 & 3.24 \\ 2.37 & 3.7 \end{bmatrix}$. It is a rough approximation because, in this particular case, the power series converges very slowly, e.g., for $n = 25$, the approximation is $\tilde{\mathbf{x}}^{new} = (\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{25})\mathbf{f}^{new} = \begin{bmatrix} 1{,}122.80 \\ 831.60 \end{bmatrix}$, compared again with the "exact" value, $\mathbf{x}^{new} = \begin{bmatrix} 1{,}138.90 \\ 844.40 \end{bmatrix}$, and it is not until $n = 57$ that $\tilde{\mathbf{x}}^{new} = \mathbf{x}^{new}$ for both elements within 0.1. This feature of slow convergence, however, is not always the case depending upon the characteristics of **A**. For example, if $\overline{\mathbf{A}} = .01 \times \mathbf{A}$, for the same vector of final demands, convergence, i.e., when $\tilde{\mathbf{x}}^{new} = \mathbf{x}^{new}$ for both elements within 0.1, occurs at $n = 6$. This result is analogous to the result in ordinary algebra, $1/(1-a) = 1 + a + a^2 + a^3 + \cdots + a^n$ for a scalar a where $|a| < 1$. For example, if $a = .427$, this series converges to within 0.001 at $n = 8$ while, for .0427, i.e. $.01a$, the series converges at $n = 2$.

Problem 2.2

This problem explores a more extensive example of basic input-output relationships. We specify interindustry sales and industry total outputs in a three-sector national economy for year *t*, given in the following table, where values are shown in thousands of dollars. (S_1, S_2, and S_3 designate the three industry sectors).

	Interindustry Sales			Total Output
	S_1	S_2	S_3	
S_1	350	0	0	1,000
S_2	50	250	150	500
S_3	200	150	550	1,000

From the table, the matrix of interindustry transactions, $\mathbf{Z}^t$, and the vector of total outputs, $\mathbf{x}^t$, are defined as $\mathbf{Z}^t = \begin{bmatrix} 350 & 0 & 0 \\ 50 & 250 & 150 \\ 200 & 150 & 550 \end{bmatrix}$, and $\mathbf{x}^t = \begin{bmatrix} 1{,}000 \\ 500 \\ 1{,}000 \end{bmatrix}$. The matrix of technical coefficients for year t, $\mathbf{A}^t$, and the corresponding matrix of total requirements, $\mathbf{L}^t$, are then found as

$$\mathbf{A}^t = \mathbf{Z}^t(\hat{\mathbf{x}}^t)^{-1} = \begin{bmatrix} .35 & 0 & 0 \\ .05 & .5 & .15 \\ .2 & .3 & .55 \end{bmatrix}, \text{ and } \mathbf{L}^t = (\mathbf{I}-\mathbf{A}^t)^{-1} = \begin{bmatrix} 1.538 & 0 & 0 \\ .449 & 2.5 & .833 \\ .983 & 1.667 & 2.778 \end{bmatrix}.$$

Suppose that government tax policy changes generate final demands for the products delivered by sectors 1, 2, and 3 projected for next year (year $t + 1$) to be 1,300, 100, and 200 for the three sectors, respectively (also measured in thousands of dollars). The corresponding total outputs that would be necessary from the three sectors to meet this projected new demand, assuming that there is no change in the technological structure of the economy (that is, assuming that the **A** matrix does not change from year t to year $t + 1$), would be $\mathbf{x}^{t+1} = \mathbf{L}^t\mathbf{f}^{t+1} = \begin{bmatrix} 2{,}000 \\ 1{,}000 \\ 2{,}000 \end{bmatrix}$ for

$\mathbf{f}^{t+1} = \begin{bmatrix} 1{,}300 \\ 100 \\ 200 \end{bmatrix}$. The original vector of final demands for year t is computed as

$\mathbf{f}^t = \mathbf{x}^t - \mathbf{Z}^t\mathbf{i} = \begin{bmatrix} 650 \\ 50 \\ 100 \end{bmatrix}$, from which we can observe that $\mathbf{f}^{t+1} = 2\mathbf{f}^t$, so it can be easily verified that

that $\mathbf{x}^{t+1} = 2\mathbf{x}^t$ since $\mathbf{x}^{t+1} = \mathbf{L}^t\mathbf{f}^{t+1} = 2\mathbf{L}^t\mathbf{f}^t = 2\mathbf{x}^t$, illustrating the linearity of the Leontief model.

Problem 2.3

This problem illustrates the distinctions between the open and closed Leontief models using the data of problem 2.1, where the interindustry transaction matrix and vector of total outputs, respectively, were defined as $\mathbf{Z} = \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1{,}000 \\ 800 \end{bmatrix}$.

Suppose that the part of the original final demands attributable to household (consumption) expenditures for this economy are \$90 from sector 1 and \$50 from sector 2 with the remaining parts of final demand reported as exports of products 1 and 2. Suppose, further, that (1) payments from sectors 1 and 2 for household labor services were \$100 and \$60, respectively; (2) that total household (labor) income in the economy was \$300; (3) that household purchases of labor services was \$40; and (4) that any new final demands presented to the economy are for exports.

This additional information allows us to expand **Z** and **x** of the basic data for two-sector model from problem 2.1, to be $\mathbf{Z}^c = \begin{bmatrix} 500 & 350 & 90 \\ 320 & 360 & 50 \\ 100 & 60 & 40 \end{bmatrix}$ and $\mathbf{x}^c = \begin{bmatrix} 1{,}000 \\ 800 \\ 300 \end{bmatrix}$. This illustrates the process known as closing the model to households. The result is a three-sector representation of the economy for which the matrices of direct and total requirements, respectively, are

$$\mathbf{A}^c = \mathbf{Z}^c(\hat{\mathbf{x}}^c)^{-1} = \begin{bmatrix} .5 & .438 & .3 \\ .32 & .45 & .167 \\ .1 & .075 & .133 \end{bmatrix} \text{ and } \mathbf{L}^c = (\mathbf{I}-\mathbf{A}^c)^{-1} = \begin{bmatrix} 5.820 & 5.036 & 2.983 \\ 3.686 & 5.057 & 2.248 \\ 0.990 & 1.019 & 1.693 \end{bmatrix}.$$

We can now find the impacts in terms of required new production for sectors 1 and 2 of the new final demands specified in problem 2.1, but this time using the Leontief inverse for the new, expanded matrix of technical coefficients of dimension 3×3. The vector of new final demands (now attributed solely to exports) is $\tilde{\mathbf{f}}^c = \begin{bmatrix} 200 \\ 100 \\ 0 \end{bmatrix}$, and we compute the resulting new vector of total outputs necessary to support those final demands as $\tilde{\mathbf{x}}^c = \mathbf{L}^c\tilde{\mathbf{f}}^c = \begin{bmatrix} 1{,}667.5 \\ 1{,}242.9 \\ 300 \end{bmatrix}$. Since in problem 2.1 we found $\mathbf{x}^o = \mathbf{L}^o\mathbf{f}^o = \begin{bmatrix} 1{,}138.9 \\ 844.4 \end{bmatrix}$ for $\mathbf{f}^0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$, the increases in outputs for both sectors 1 and 2 using the closed model reflect increased interindustry production resulting from the inclusion of households as an endogenous sector in the 3-sector model.

Problem 2.4

This problem explores the Hawkins-Simon conditions for the Leontief model developed in chapter 2. Consider an economy organized into three industries: (1) lumber, (2) machinery, and (3) paper characterized by the following:

- A consulting firm estimates that last year the lumber industry had an output valued at \$50 (assume all monetary values are in units of \$100,000), 5 percent of which the industry consumed itself; 70 percent of the lumber industry's output was consumed by final demand; 20 percent by the paper industry; and 5 percent by the machinery industry.
- The machinery industry consumed 15 percent of its own products, out of a total of \$l00; 25 percent went to final demand; 30 percent to the lumber industry; 30 percent to the paper industry.
- Finally, the paper industry produced \$50, of which it consumed 10 percent; 80 percent went to final demand; 5 percent went to the lumber industry; and 5 percent to the machinery industry.

Using this information the matrix of interindustry transactions and the vector of total outputs for this economy are $\mathbf{Z} = \begin{bmatrix} 2.5 & 10 & 2.5 \\ 2.5 & 5 & 2.5 \\ 30 & 30 & 15 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 35 \\ 40 \\ 25 \end{bmatrix}$, respectively, so the vector of total outputs, $\mathbf{x}$, and the matrix of technical coefficients, $\mathbf{A}$, are then $\mathbf{f} = \begin{bmatrix} 35 \\ 40 \\ 25 \end{bmatrix}$,

$\mathbf{x} = \mathbf{Zi} + \mathbf{f} = \begin{bmatrix} 50 \\ 50 \\ 100 \end{bmatrix}$ and $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .05 & .2 & .025 \\ .05 & .1 & .025 \\ .6 & .6 & .15 \end{bmatrix}$. The Hawkins-Simon conditions require positivity of all principal minors of $(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} .95 & -.2 & -.025 \\ -.05 & .9 & -.025 \\ -.6 & -.6 & .85 \end{bmatrix}$. Here the three first-order principal minors are the main diagonal elements, 0.95, 0.9 and 0.85; the three second-order principal minors are 0.845, 0.75 and 0.793, and the third-order principal minor is just the determinant $|\mathbf{I} - \mathbf{A}| = 0.687$, so all the principal minors are positive (see Appendix A of the text for discussion of minors in matrix operations).

The Leontief inverse for this economy is $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.092 & .269 & .04 \\ .084 & 1.154 & .036 \\ .830 & 1.005 & 1.23 \end{bmatrix}$. If we anticipate an economic recession reflected in decreased final demands for lumber, machinery, and paper of 25, 10, and 5 percent, respectively. The vector of new final demands is then $\mathbf{f}^{new} = \begin{bmatrix} (.75)f_1 \\ (.90)f_2 \\ (.95)f_3 \end{bmatrix} = \begin{bmatrix} 26.25 \\ 36.00 \\ 23.75 \end{bmatrix}$ and the corresponding vector of total outputs supporting this change in final demand is found by $\mathbf{x}^{new} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^{new} = \begin{bmatrix} 39.317 \\ 44.606 \\ 87.181 \end{bmatrix}$ for $\mathbf{f}^{new} = \begin{bmatrix} (.75)f_1 \\ (.90)f_2 \\ (.95)f_3 \end{bmatrix} = \begin{bmatrix} 26.25 \\ 36.00 \\ 23.75 \end{bmatrix}$. The new matrix of interindustry transactions is $\mathbf{Z}^{new} = \mathbf{A}(\hat{\mathbf{x}}^{new}) = \begin{bmatrix} 1.97 & 8.92 & 2.18 \\ 1.97 & 4.46 & 2.18 \\ 23.59 & 26.76 & 13.08 \end{bmatrix}$, so the vectors of value-added inputs and of intermediate outputs, respectively, are then computed as

$$\mathbf{v}^{new} = (\mathbf{x}^{new})' - \mathbf{i}'(\mathbf{Z}^{new}) = \begin{bmatrix} 11.795 & 4.461 & 69.744 \end{bmatrix} \text{ and } \mathbf{u}^{new} = (\mathbf{Z}^{new})\mathbf{i} = \begin{bmatrix} 13.067 \\ 8.606 \\ 63.431 \end{bmatrix}.$$

Problem 2.5

This problem assembles an input-output transactions table and explores the Hawkins-Simon conditions along with impact analysis for a new vector of final demands for the defined economy.

Consider a simple two-sector economy containing industries *A* and *B*. Industry *A* requires \$2 million worth of its own product and \$6 million worth of Industry *B*'s output in the process of supplying \$20 million worth of its own product to final consumers. Similarly, Industry *B* requires \$4 million worth of its own product and \$8 million worth of Industry *A*'s output in the process of supplying \$20 million worth of its own product to final consumers.

Using these data, we define the matrix of interindustry transactions and vector of final demands as $\mathbf{Z} = \begin{bmatrix} 2 & 8 \\ 6 & 4 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$, respectively, so the corresponding vector of total outputs is computed as $\mathbf{x} = \mathbf{f} + \mathbf{Zi} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$. Hence, the matrix of direct requirements is found by $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .067 & .267 \\ .2 & .133 \end{bmatrix}$ and the Hawkins-Simon conditions are satisfied as positive values for the determinant and the principal minors of the matrix $(\mathbf{I} - \mathbf{A})$, i.e., $|\mathbf{I} - \mathbf{A}| = 0.756$, $(1 - a_{11}) = 0.993$, and $(1 - a_{22}) = 0.867$. The matrix of total requirements is then found as $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.147 & .353 \\ .265 & 1.235 \end{bmatrix}$ and, for new vector of final demands, $\mathbf{f}^{new} = \begin{bmatrix} 15 \\ 18 \end{bmatrix}$, the corresponding vector of total outputs is computed as $\mathbf{x}^{new} = \mathbf{Lf}^{new} = \begin{bmatrix} 23.559 \\ 26.206 \end{bmatrix}$ and related interindustry activity (matrix of interindustry transactions) is $\mathbf{Z}^{new} = \mathbf{A}\hat{\mathbf{x}}^{new} = \begin{bmatrix} 1.571 & 6.988 \\ 4.712 & 3.494 \end{bmatrix}$.

Problem 2.6

While computer advances have considerably reduced the computational constraints for many applications of input-output analysis, it has also made possible the construction and use of much larger scale input-output models with thousands of sectors specified. This problem illustrates, on a small scale, practical considerations in working with very large input-output models for determining when using round-by-round calculations for impact analysis, $\tilde{\mathbf{x}} = \sum_{i=0}^{r} \mathbf{A}^i \mathbf{f}$, is a cost-effective substitute for using the direct computation of the Leontief inverse in impact analysis, $\mathbf{x} = \mathbf{Lf}$ where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$.

Consider the following transactions table, $\mathbf{Z}$, and total outputs vector, $\mathbf{x}$, for a two-sector economy:

$$\mathbf{Z} = \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$$

For this economy, the vectors of value-added inputs and final demands are computed as $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 10 & 11 \end{bmatrix}$ and $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$, respectively. With $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .3 & .133 \\ .2 & .133 \end{bmatrix}$ we show first that the Hawkins-Simon conditions are satisfied by positive values for the determinant and the principal minors of the matrix $(\mathbf{I} - \mathbf{A})$: $|\mathbf{I} - \mathbf{A}| = 0.58$, $(1 - a_{11}) = 0.7$, and $(1 - a_{22}) = 0.867$, so we consider the economy to be "well behaved." The *r*-order round-by-round approximation of $\mathbf{x} = \mathbf{Lf} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$ is found as: $\tilde{\mathbf{x}} = \sum_{i=0}^{r} \mathbf{A}^i \mathbf{f}$ (remember that $\mathbf{A}^0 = \mathbf{I}$), shown in the following table.

Round-by-round approximation of total outputs for $r = 1, 2, \cdots, 10$

r	1	2	3	4	5	**6**	7	8	9	10
$\tilde{x}_1$	16.800	18.720	19.488	19.795	19.918	**19.967**	19.987	19.995	19.998	19.999
$\tilde{x}_2$	12.600	14.040	14.616	14.846	14.939	**14.975**	14.990	14.996	14.998	14.999

We see from the table that $x_j - \tilde{x}_j < 0.05$ for both sectors ($j = 1, 2$) at $r = 6$.

The specified cost of performing impact analysis on the computer using the round-by-round method is then computed as $C_r = c_1 r + c_2(r - 1.5)$ where r is the order of the approximation (c_1 is the cost of an addition operation and c_2 is the cost of a multiplication operation). Also, we assume further that $c_1 = 0.5c$, that the cost of computing $(\mathbf{I}-\mathbf{A})^{-1}$ exactly rather than via successive approximation is given by $C_e = 20c_2$, and that the cost of using this inverse in impact analysis (multiplying it by a final-demand vector) is given by $C_f = c_2$.

If we want to determine whether to use the round-by-round method or to compute the exact inverse and then perform impact analysis, i.e., to determine the least-cost method for computing the solution, for one final demand vector, the equation defining the computation cost is $C_r = 0.5c_2 r + c_2 r - 1.5c_2 = (0.5r + r - 1.5)c_2 = (1.5r - 1.5)c_2 = 1.5(r-1)c_2$ and, from the table, for $x_j - \tilde{x}_j < 0.2$ then $r = 5$ so $C_r = 6c_2$.

Hence, for one final demand vector, the cost of the round-by-round approximation is $C_r = 6c_2$ which is less than the cost of using the exact inverse $C_e + C_f = 21c_2$, it is much more cost effective to use the round-by-round round method. For five final demand vectors, however, $C_r = 5(6c_2) = 30c_2 \;>\; C_e + 4C_f = 20c_2 + 4c_2 = 24c_2$, so it is more cost effective to use the exact method. For four final demand vectors, it turns out, the total cost of computation is $C_r = 4(6c_2) = 24c_2 \;=\; C_e + 4C_f = 20c_2 + 4c_2 = 24c_2$, i.e., the costs of both methods are identical so at least in terms of cost effectiveness we are indifferent as to which method to employ.

Problem 2.7

This problem explores computation of the Leontief inverse and impact analysis for an eight-sector economy (practical only with computer tools).

Consider the following matrix of interindustry transactions, **Z**, and vector of total outputs, **x**, for an eight-sector economy:

$$\mathbf{Z} = \begin{bmatrix} 8,565 & 8,069 & 8,843 & 3,045 & 1,124 & 276 & 230 & 3,464 \\ 1,505 & 6,996 & 6,895 & 3,530 & 3,383 & 365 & 219 & 2,946 \\ 98 & 39 & 5 & 429 & 5,694 & 7 & 376 & 327 \\ 999 & 1,048 & 120 & 9,143 & 4,460 & 228 & 210 & 2,226 \\ 4,373 & 4,488 & 8,325 & 2,729 & 2,9671 & 1,733 & 5,757 & 14,756 \\ 2,150 & 36 & 640 & 1,234 & 165 & 821 & 90 & 6,717 \\ 506 & 7 & 180 & 0 & 2,352 & 0 & 18,091 & 26,529 \\ 5,315 & 1,895 & 2,993 & 1,071 & 13,941 & 434 & 6,096 & 46,338 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 37,610 & 45,108 & 46,323 & 41,059 & 209,403 & 11,200 & 55,992 & 161,079 \end{bmatrix}$$

We compute the matrices of direct requirements, **A**, and the matrix of total requirements, **L**, as the following:

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .228 & .179 & .191 & .074 & .005 & .025 & .004 & .022 \\ .040 & .155 & .149 & .086 & .016 & .033 & .004 & .018 \\ .003 & .001 & .000 & .010 & .027 & .001 & .007 & .002 \\ .027 & .023 & .003 & .223 & .021 & .020 & .004 & .014 \\ .116 & .099 & .180 & .066 & .142 & .155 & .103 & .092 \\ .057 & .001 & .014 & .030 & .001 & .073 & .002 & .042 \\ .013 & 0 & .004 & 0 & .011 & 0 & .323 & .165 \\ .141 & .042 & .065 & .026 & .067 & .039 & .109 & .288 \end{bmatrix}$$

$$\mathbf{L} - (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.339 & .296 & .312 & .172 & .034 & .058 & .030 & .067 \\ .089 & 1.214 & .209 & .153 & .038 & .057 & .025 & .051 \\ .013 & .009 & 1.011 & .019 & .034 & .008 & .018 & .013 \\ .065 & .056 & .034 & 1.306 & .038 & .041 & .021 & .041 \\ .265 & .215 & .320 & .174 & 1.207 & .230 & .229 & .240 \\ .100 & .029 & .045 & .059 & .011 & 1.089 & .018 & .074 \\ .109 & .049 & .068 & .035 & .054 & .030 & 1.547 & .372 \\ .321 & .162 & .210 & .117 & .135 & .103 & .269 & 1.506 \end{bmatrix}$$

For a case where final demands in sectors 1 and 2 increase by 30 percent while in sector 5 they decrease by 20 percent with all other final demands unchanged, we first compute the base final demands as $\mathbf{f} = \mathbf{x} - \mathbf{A}\mathbf{x}$, then $\mathbf{x}^{new} = \mathbf{L}\mathbf{f}^{new}$, for $\mathbf{f} = \mathbf{x} - \mathbf{A}\mathbf{x} = \begin{bmatrix} 3{,}994 \\ 19{,}269 \\ 39{,}348 \\ 22{,}625 \\ 137{,}571 \\ \text{-}653 \\ 8{,}327 \\ 82{,}996 \end{bmatrix}$, and, applying the changes indicated, $\mathbf{f}^{new} = \begin{bmatrix} 5{,}192 \\ 25{,}050 \\ 39{,}348 \\ 22{,}625 \\ 110{,}057 \\ -653 \\ 8{,}327 \\ 82{,}996 \end{bmatrix}$ to yield $\mathbf{x}^{new} = \mathbf{L}\mathbf{f}^{new} = \begin{bmatrix} 39{,}998 \\ 51{,}181 \\ 45{,}455 \\ 40{,}404 \\ 177{,}756 \\ 11{,}182 \\ 54{,}929 \\ 158{,}687 \end{bmatrix}$.

Problem 2.8
The problem explores changes in relative prices in an input-output formulation resulting from changes in value-added inputs. Consider a two-sector input-output table measured in millions of dollars:

	Manuf.	Services	Final Demand	Total Output
Manufacturing	10	40	50	100
Services	30	25	85	140
Value Added	60	75	135	
Total Output	100	140		

Using the table data, we define the matrix of interindustry transactions, $\mathbf{Z}^0 = \begin{bmatrix} 10 & 40 \\ 30 & 25 \end{bmatrix}$, the vector of total outputs, $\mathbf{x}^0 = \begin{bmatrix} 100 \\ 140 \end{bmatrix}$, and the vector of total value-added inputs $(\mathbf{w}^0)' = \begin{bmatrix} 60 & 75 \end{bmatrix}$, and we can then compute

$\mathbf{A}^0 = \mathbf{Z}^0(\hat{\mathbf{x}}^0)^{-1} = \begin{bmatrix} 10 & 40 \\ 30 & 25 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/140 \end{bmatrix} = \begin{bmatrix} .1 & .286 \\ .3 & .179 \end{bmatrix}$ and $\mathbf{L}^0 = (\mathbf{I} - \mathbf{A}^0)^{-1} = \begin{bmatrix} 1.257 & .437 \\ .459 & 1.377 \end{bmatrix}$. For this formulation we will need the matrix transposes of $\mathbf{A}^0$ and $\mathbf{L}^0$, i.e., $(\mathbf{A}^0)' = \begin{bmatrix} .1 & .3 \\ .286 & .179 \end{bmatrix}$ and $(\mathbf{L}^0)' = [\mathbf{I} - (\mathbf{A}^0)']^{-1} = \begin{bmatrix} 1.257 & .459 \\ .437 & 1.377 \end{bmatrix}$. The value-added coefficients are computed as $\mathbf{v}_c^0 = (\mathbf{w}^0)'(\hat{\mathbf{x}}^0)^{-1} = \begin{bmatrix} 60/100 & 75/140 \end{bmatrix} = \begin{bmatrix} .6 & .536 \end{bmatrix}$ for which the normalized prices are found, not surprisingly, as $\tilde{\mathbf{p}}^0 = (\mathbf{L}^0)'(\mathbf{v}_c^0)' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or, perhaps more intuitively, since the transpose of a product of matrices is the product of the transposes of the individual matrices in the reverse order (see Appendix A) and the row vector, $(\tilde{\mathbf{p}}^0)' = \mathbf{v}_c^0\mathbf{L}^0 = \begin{bmatrix} 1 & 1 \end{bmatrix}$. This is not surprising since, beginning with the basic accounting identity, $\mathbf{w} = \mathbf{x}' - \mathbf{i}'\mathbf{Z}$, which we can express as $\mathbf{w} = \mathbf{x}' - \mathbf{i}'\mathbf{A}\hat{\mathbf{x}}$, we first postmultiply through by $\hat{\mathbf{x}}^{-1}$ to obtain $\mathbf{v}_c = \mathbf{w}\hat{\mathbf{x}}^{-1} = \mathbf{x}'\hat{\mathbf{x}}^{-1} - \mathbf{i}'\mathbf{A}\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} = \mathbf{i}' - \mathbf{i}\mathbf{A} = \mathbf{i}'(\mathbf{I} - \mathbf{A})$. Then, postmultiplying through by $(\mathbf{I} - \mathbf{A})^{-1}$, the result is $\mathbf{v}_c(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{i}'(\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}$ which reduces to the general result, $\mathbf{v}_c\mathbf{L} = \mathbf{i}'$.

If labor costs in the services sector increase, causing a 25 percent increase in value added inputs required per unit of services and labor costs in manufacturing decrease by 25 percent, the new value-added coefficients, reflecting the changes are $\mathbf{v}_c^1 = \begin{bmatrix} 1.25 & 0 \\ 0 & .75 \end{bmatrix} \mathbf{v}_c^0 = \begin{bmatrix} .75 & .402 \end{bmatrix}$, so the prices for the new period 1 relative to the current period 0 are $(\tilde{\mathbf{p}}_0^1)' = \mathbf{v}_c^1\mathbf{L}^0 = \begin{bmatrix} 1.127 & .881 \end{bmatrix}$.

Problem 2.9

This problem explores changes in relative product prices resulting from a change in value-added inputs generated by a national corporate income tax. We use the 2003 U.S. direct requirements table given in Table 2.6. For the matrix of direct requirements, **A**, given in the table, the transpose of the Leontief inverse is

$$\mathbf{L}' = (\mathbf{I} - \mathbf{A}')^{-1} = \begin{bmatrix} 1.262 & 0.009 & 0.008 & 0.229 & 0.149 & 0.238 & 0.024 \\ 0.006 & 1.075 & 0.003 & 0.119 & 0.085 & 0.293 & 0.024 \\ 0.013 & 0.012 & 1.005 & 0.262 & 0.137 & 0.270 & 0.023 \\ 0.057 & 0.034 & 0.006 & 1.342 & 0.156 & 0.292 & 0.037 \\ 0.004 & 0.019 & 0.007 & 0.069 & 1.089 & 0.271 & 0.028 \\ 0.007 & 0.003 & 0.011 & 0.086 & 0.060 & 1.412 & 0.030 \\ 0.007 & 0.007 & 0.025 & 0.126 & 0.085 & 0.314 & 1.034 \end{bmatrix}.$$

Suppose that the new corporate income tax generates increases in the total value-added inputs of 10 percent for primary industries (agriculture and mining), of 15 percent for construction and manufacturing, and of 20 percent for all other sectors. The vector of value-added coefficients for the original input output economy is found as

$\mathbf{v}_c^0 = \mathbf{i} - \mathbf{i}'\mathbf{A} = \begin{bmatrix} .486 & .633 & .580 & .470 & .699 & .629 & .640 \end{bmatrix}'$, so that $\tilde{\mathbf{p}}^0 = \mathbf{L}'\mathbf{v}_c^0 = \mathbf{i}$. We define the vector of value-added growth factors, reflecting the value-added changes indicated, as $\mathbf{d} = [1.1 \quad 1.1 \quad 1.15 \quad 1.15 \quad 1.2 \quad 1.2 \quad 1.2]'$ so that we can find the new vector of

value-added coefficients by $\mathbf{v}_c^1 = \hat{\mathbf{d}}\mathbf{v}_c^0 = \begin{bmatrix} 1.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.2 \end{bmatrix} \begin{bmatrix} .486 \\ .633 \\ .58 \\ .47 \\ .699 \\ .629 \\ .64 \end{bmatrix} = \begin{bmatrix} .534 \\ .696 \\ .667 \\ .540 \\ .839 \\ .754 \\ .768 \end{bmatrix}$.

Hence the new prices are found as

$\tilde{\mathbf{p}}^1 = \mathbf{L}'\mathbf{v}_c^1 = \begin{bmatrix}1.133 & 1.129 & 1.163 & 1.163 & 1.197 & 1.197 & 1.195\end{bmatrix}'$.

Problem 2.10

This problem explores the process of opening a "U.S.-style" input-output economy (adopting the accounting conventions of national input-output tables assembled in the United States) to imports by "scrubbing" from the assembled interindustry transactions matrix the portion of interindustry transactions that represent competitive imports from outside the economy and reassigning them as value-added imports (noncompetitive imports are already treated as value added inputs).

Consider an input-output economy with three sectors: agriculture, services, and personal computers. The matrix of interindustry transactions and vector of total outputs are given, respectively, by $\mathbf{Z} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ so that the associated vector of final demands is $\mathbf{f} = \mathbf{x} - \mathbf{Z}\mathbf{i} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Notice, first, that this is a closed economy where all industry outputs become inputs. That is, with the given vector of total outputs, $\mathbf{x}$, the vector of total value-added inputs is found by $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and, of course, the gross domestic product is $\mathbf{v}'\mathbf{i} = \mathbf{i}'\mathbf{f} = 0$. For this economy, $\mathbf{A} = \begin{bmatrix} .4 & 1 & .5 \\ .2 & 0 & 0 \\ .4 & 0 & .5 \end{bmatrix}$, so we can compute $|\mathbf{I} - \mathbf{A}| = 0$. This means that $(\mathbf{I} - \mathbf{A})$ is a singular matrix and $\mathbf{L}$ does not exist.

Suppose that we determine *all* the inputs for the personal computers sector are imported. We can create a domestic transactions matrix by "opening" the economy to imports, i.e., transfer the value of all inputs to personal computers to final demand. For a "U.S. style" input-output table, competitive imports are included in the matrix of transactions and a corresponding negative entry for imports is included in final demand.

To "scrub" this transactions table of competitive imports we need to, first, subtract the value of imports from the first and third entries in the column for personal computers, then, add those amounts to the first and third entries of final demand. We define $\mathbf{D}$ as the matrix of domestic transactions where the values of competitive imports are subtracted to remove them from the matrix of interindustry transactions, $\mathbf{Z}$, and $\mathbf{g}$ as the new vector of final demands where the values of competitive imports are added to remove imports from final demand, $\mathbf{f}$, (recall that they were included originally in final demand as negative values).

Thus, $\mathbf{Z} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ becomes $\mathbf{D} = \begin{bmatrix} 2 & 2 & 1-1 \\ 1 & 0 & 0 \\ 2 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ and $\mathbf{f}$ becomes $\mathbf{g} = \begin{bmatrix} 0+1 \\ 1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The vector of total outputs, $\mathbf{x}$, is unchanged, but the new vector of total value added is $\overline{\mathbf{v}}' = \mathbf{x}' - \mathbf{iD} = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$ and, hence, gross domestic product is $\mathbf{v}'\mathbf{i} = \mathbf{i}'\mathbf{f} = 2$. We then can compute the matrix of direct requirements as $\overline{\mathbf{A}} = \begin{bmatrix} .4 & 1 & 0 \\ .2 & 0 & 0 \\ .4 & 0 & 0 \end{bmatrix}$, for which $|\mathbf{I} - \overline{\mathbf{A}}| = 0.4$ so the matrix $(\mathbf{I} - \overline{\mathbf{A}})$ is non-singular and the matrix of total requirements can be computed as

$$\overline{\mathbf{L}} = (\mathbf{I} - \overline{\mathbf{A}})^{-1} = \begin{bmatrix} 2.5 & 2.5 & 0 \\ .5 & 1.5 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Chapter 3, Input–Output Models at the Regional Level

Chapter 3 extends the basic input–output framework to analysis of regions and the relationships between regions. First, "single-region" models are presented, and the various assumptions employed in formulating regional models versus national models are explored. Next, the structure of an interregional input–output (IRIO) model, designed to expand the basic input–output framework to capture transactions between industrial sectors in regions, is presented. An important simplification of the IRIO model designed to deal with the most common of data limitations in constructing such models is known as the multiregional input–output (MRIO) model. This chapter introduces the basic MRIO formulation and explores the implications of its simplifying assumptions along with the features of the balanced regional model which captures the distinction between industrial production for regional versus national markets. Finally, the chapter summarizes the fast-growing range of applications of MRIO analysis to multinational and global economic models and issues. The exercise problems for this chapter explore various characteristics of regional, IRIO, and MRIO model configurations and their applications.

Problem 3.1

This problem explores the use of regional purchase coefficients to analyze regional interindustry activity. We begin with the data from problem 2.2, which describes a small national economy that contains firms producing in each of the three industry sectors.

Suppose that for a regional economy within this national economy, the technological structure of production of firms within the region is estimated to be the same as that reflected in the national data, but that there is need to import into the region (from producers elsewhere in the country) some of the inputs used in production in each of the regional sectors. In particular, the percentages of required inputs from sectors 1, 2, and 3 that come from within the region are 60, 90, and 75, respectively, which defines the vector of regional purchase coefficients as $\mathbf{p} = \begin{bmatrix} 0.60 \\ 0.90 \\ 0.75 \end{bmatrix}$. Using the matrix of technical coefficients, $\mathbf{A}$, from problem 2.2, we compute the regional direct requirements matrix as $\mathbf{A}^R = \hat{\mathbf{p}}\mathbf{A} = \begin{bmatrix} .210 & 0 & 0 \\ .045 & .450 & .135 \\ .150 & .225 & .413 \end{bmatrix}$ and $(\mathbf{I}-\mathbf{A}^R)^{-1} = \begin{bmatrix} 1.266 & 0 & 0 \\ .202 & 2.007 & .461 \\ .401 & .759 & 1.879 \end{bmatrix}$ is the regional total requirements matrix. If new final demands for the outputs of the regional producers are projected to be 1300, 100, and 200, respectively, or $\mathbf{f}^{new} = \begin{bmatrix} 1{,}300 \\ 100 \\ 200 \end{bmatrix}$, the total regional production necessary to support those final demands is computed as the vector of regional total outputs, $\mathbf{x}^{new} = (\mathbf{I}-\mathbf{A}^R)^{-1}\mathbf{f}^{new} = \begin{bmatrix} 1{,}645.57 \\ 555.346 \\ 973.257 \end{bmatrix}$.

Problem 3.2

This problem explores the basic structure of an interregional input-output (IRIO) model. The following table shows sales (in dollars) between and among two industry sectors in two regions, *r* and *s*.

		Region *r*		Region *s*	
		Industry 1	Industry 2	Industry 1	Industry 2
Region *r*	Industry 1	40	50	30	45
	Industry 2	60	10	70	45
Region *s*	Industry 1	50	60	50	80
	Industry 2	70	70	50	50

In addition, sales to final demand purchasers for each region are designated, respectively for regions *r* and *s*, are $\mathbf{f}^r = \begin{bmatrix} 200 \\ 200 \end{bmatrix}$ and $\mathbf{f}^s = \begin{bmatrix} 300 \\ 400 \end{bmatrix}$.

These data are sufficient to create a two-region IRIO model connecting regions *r* and *s*. Using the data from the table, $\mathbf{Z}$ is defined as the matrix of IRIO transactions, the corresponding vector of final demands is found as $\mathbf{f} = \begin{bmatrix} \mathbf{f}^r \\ \hdashline \mathbf{f}^s \end{bmatrix}$, and the vector of total outputs is found as

$\mathbf{x} = \mathbf{f} + \mathbf{Zi} = \begin{bmatrix} 365 \\ 385 \\ \hdashline 540 \\ 640 \end{bmatrix}$. Consequently, the matrix of technical coefficients is found as $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$

which can be partitioned into $\mathbf{A} = \left[\begin{array}{c:c} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \hdashline \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{array}\right] = \left[\begin{array}{cc:cc} 0.110 & 0.130 & 0.056 & 0.070 \\ 0.164 & 0.026 & 0.130 & 0.070 \\ \hdashline 0.137 & 0.156 & 0.093 & 0.125 \\ 0.192 & 0.182 & 0.093 & 0.078 \end{array}\right]$. If, because of a stimulated region *r* economy, household demand increased by \$280 for the output of sector 1 in region *r* and by \$360 for the output of sector 2 in region *r*, the vector of *changes* in final demand is $\Delta\mathbf{f} = \begin{bmatrix} 280 \\ 360 \\ \hdashline 0 \\ 0 \end{bmatrix}$. Computing, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \left[\begin{array}{cc:cc} 1.205 & 0.202 & 0.115 & 0.123 \\ 0.263 & 1.116 & 0.189 & 0.131 \\ \hdashline 0.273 & 0.262 & 1.177 & 0.200 \\ 0.330 & 0.289 & 0.179 & 1.156 \end{array}\right]$, the matrix of total requirements, then we can compute $\Delta\mathbf{x} = \begin{bmatrix} \Delta\mathbf{x}^r \\ \hdashline \Delta\mathbf{x}^s \end{bmatrix} = \mathbf{L}\Delta\mathbf{f} = \begin{bmatrix} 409.98 \\ 475.67 \\ \hdashline 170.62 \\ 196.24 \end{bmatrix}$, defining the new necessary gross outputs from each of the sectors in each of the two regions to satisfy this new final demand. Note that the increased outputs in region s for sector 1 of 170.62 and 196.24 for sector 2 are attributable solely to the interregional feedback effects associated with the new final demands in region r.

Problem 3.3

This problem explores the basic structure of the multiregional input-output (MRIO) model.

Suppose that you have assembled the following information on (1) the dollar values of purchases of each of two goods in each of two regions and (2) on the shipments of each of the two goods between regions:

Purchases in Region r		Purchases in Region s	
$z_{11}^r = 40$	$z_{12}^r = 50$	$z_{11}^s = 30$	$z_{12}^s = 45$
$z_{21}^r = 60$	$z_{22}^r = 10$	$z_{21}^s = 70$	$z_{22}^s = 45$
Shipments of Good 1		Shipments of Good 2	
$z_1^{rr} = 50$	$z_1^{rs} = 60$	$z_2^{rr} = 50$	$z_2^{rs} = 80$
$z_1^{sr} = 70$	$z_1^{ss} = 70$	$z_2^{sr} = 50$	$z_2^{ss} = 50$

These data are sufficient to generate the necessary matrices for a two-region MRIO model involving regions r and s. There will be six necessary matrices—$\mathbf{A}^r$, $\mathbf{A}^s$, $\hat{\mathbf{c}}^{rr}$, $\hat{\mathbf{c}}^{rs}$, $\hat{\mathbf{c}}^{sr}$, and $\hat{\mathbf{c}}^{ss}$. All of these will be 2×2 matrices, configured from the transactions and trade shipments for each region. First, from the table we can construct the matrix of total transactions for each region as $\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^r & 0 \\ 0 & \mathbf{Z}^s \end{bmatrix} = \left[\begin{array}{cc|cc} 40 & 50 & 0 & 0 \\ 60 & 10 & 0 & 0 \\ \hline 0 & 0 & 30 & 45 \\ 0 & 0 & 70 & 45 \end{array}\right]$. These transactions for each region, $\mathbf{Z}^r$ or $\mathbf{Z}^s$, include the inputs from all regions to support production in that region. We can configure the shipments of goods 1 and 2 in a matrix defined as $\mathbf{Q} = \left[\begin{array}{cc|cc} z_1^{rr} & 0 & z_1^{rs} & 0 \\ 0 & z_2^{rr} & 0 & z_2^{rs} \\ \hline z_1^{sr} & 0 & z_1^{ss} & 0 \\ 0 & z_2^{sr} & 0 & z_2^{ss} \end{array}\right] = \left[\begin{array}{cc|cc} 50 & 0 & 60 & 0 \\ 0 & 50 & 0 & 80 \\ \hline 70 & 0 & 70 & 0 \\ 0 & 50 & 0 & 50 \end{array}\right]$ so the vector of row sums of $\mathbf{Q}$ is $\mathbf{x} = \mathbf{Qi} = \left[\begin{array}{c} 110 \\ 130 \\ \hline 140 \\ 100 \end{array}\right]$, which is the vector of total deliveries of commodities of each type for each region to all regions, and the vector of the column sums of $\mathbf{Q}$ is $\mathbf{q} = \mathbf{i}'\mathbf{Q} = \begin{bmatrix} 120 & 100 & 130 & 130 \end{bmatrix}$, which is the vector of total availability from all regions of each commodity in each region. Hence, we can define matrix of technical coefficients as

$$\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}=\left[\begin{array}{c|c}\mathbf{A}^r & 0\\ \hline 0 & \mathbf{A}^s\end{array}\right]=\left[\begin{array}{cc|cc}.364 & .385 & 0 & 0\\ .545 & .077 & 0 & 0\\ \hline 0 & 0 & .214 & .45\\ 0 & 0 & .5 & .45\end{array}\right]$$ and the matrix of trade coefficients as

$$\mathbf{C}=\mathbf{Q}\hat{\mathbf{q}}^{-1}=\left[\begin{array}{c|c}\hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs}\\ \hline \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss}\end{array}\right]=\left[\begin{array}{cc|cc}.417 & 0 & .462 & 0\\ 0 & .5 & 0 & .615\\ \hline .583 & 0 & .538 & 0\\ 0 & .5 & 0 & .385\end{array}\right].$$ Now we compute the matrix of multiregional total requirements as $$(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}=\left[\begin{array}{cc|cc}0.971 & 0.556 & 1.024 & 0.524\\ 0.882 & 1.197 & 0.889 & 1.251\\ \hline 1.297 & 0.714 & 1.264 & 0.677\\ 0.663 & 1.010 & 0.673 & 0.854\end{array}\right].$$ If the projected demands for the coming period are $\mathbf{f}^{r(new)}=\begin{bmatrix}50\\50\end{bmatrix}$ and $\mathbf{f}^{s(new)}=\begin{bmatrix}40\\60\end{bmatrix}$, then $\mathbf{f}^{new}=\left[\begin{array}{c}\mathbf{f}^{r(new)}\\ \hline \mathbf{f}^{s(new)}\end{array}\right]=\left[\begin{array}{c}50\\50\\ \hline 40\\60\end{array}\right]$. The corresponding vector of new total outputs for each sector in each region; $\mathbf{x}^r$ and $\mathbf{x}^s$, necessary to satisfy this new vector of final demands is found as the vector of new total outputs,

$$\mathbf{x}^{new}=\left[\begin{array}{c}\mathbf{x}^{r(new)}\\ \hline \mathbf{x}^{s(new)}\end{array}\right]=(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}\mathbf{f}^{new}=\left[\begin{array}{c}148.778\\214.539\\ \hline 191.718\\161.772\end{array}\right].$$

Problem 3.4

This problem illustrates several important features of regional input-output data. Suppose that a federal government agency for a three-region country has collected the following data on input purchases for two sectors, (1) manufacturing and (2) agriculture, measured in dollars for previous year. These flows are not specific with respect to region of origin; that is, they can be described as of the $z_{ij}^{\bullet s}$ sort rather than of the z_{ij}^{rs} sort. The three regions are denoted by *A*, *B,* and *C*.

	Region *A*		Region *B*		Region *C*	
	1	2	1	2	1	2
1	200	100	700	400	100	0
2	100	100	100	200	50	0

Also, gross (total) outputs for each of the two sectors in each of the three regions are known and specified by the vectors:

$$\mathbf{x}^A = \begin{bmatrix} 600 \\ 300 \end{bmatrix},\ \mathbf{x}^B = \begin{bmatrix} 1,200 \\ 700 \end{bmatrix} \text{ and } \mathbf{x}^C = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

From the table we can define total regional interindustry transactions for each region as: $\mathbf{Z}^A = \begin{bmatrix} 200 & 100 \\ 100 & 100 \end{bmatrix}$, $\mathbf{Z}^B = \begin{bmatrix} 700 & 400 \\ 100 & 200 \end{bmatrix}$ and $\mathbf{Z}^C = \begin{bmatrix} 100 & 0 \\ 50 & 0 \end{bmatrix}$. We can then construct matrices of regional technical coefficients as $\mathbf{A}^r = \mathbf{Z}^r(\hat{\mathbf{x}}^r)^{-1}$ for regions $r = A$, B and C as $\mathbf{A}^A = \begin{bmatrix} 0.333 & 0.333 \\ 0.167 & 0.333 \end{bmatrix}$, $\mathbf{A}^B = \begin{bmatrix} 0.583 & 0.571 \\ 0.083 & 0.286 \end{bmatrix}$, and $\mathbf{A}^C = \begin{bmatrix} 0.500 & 0 \\ 0.250 & 0 \end{bmatrix}$. It is also straightforward to assemble the matrix of national transactions as the sum of all the regional transactions matrices, $\mathbf{Z}^N = \mathbf{Z}^A + \mathbf{Z}^B + \mathbf{Z}^C = \begin{bmatrix} 1,000 & 500 \\ 250 & 300 \end{bmatrix}$, and the vector of national total outputs as the sum of regional total output vectors, $\mathbf{x}^N = \mathbf{x}^A + \mathbf{x}^B + \mathbf{x}^C = \begin{bmatrix} 2,000 \\ 1,000 \end{bmatrix}$. Hence, the national technical coefficients matrix is found by $\mathbf{A}^N = \mathbf{Z}^N(\hat{\mathbf{x}}^N)^{-1} = \begin{bmatrix} .500 & .500 \\ .125 & .300 \end{bmatrix}$.

Since origin-destination data on shipments of each good have not been specified it is not yet possible to construct these data as an IRIO or MRIO model, but using the data specified, if the federal government is considering spending \$5,000 on manufactured goods and \$4,500 on agricultural products next year, we can define the vector of changes in final demand as $\mathbf{f}^{new} = \begin{bmatrix} 5,000 \\ 4,500 \end{bmatrix}$. Using $\mathbf{A}^N$ and $\mathbf{f}^{new}$, we compute $(\mathbf{I} - \mathbf{A}^N)^{-1} = \begin{bmatrix} 2.435 & 1.739 \\ 0.435 & 1.739 \end{bmatrix}$ and we find that $\mathbf{x}^{new} = (\mathbf{I} - \mathbf{A}^N)^{-1}\mathbf{f}^{new} = \begin{bmatrix} 20,000 \\ 10,000 \end{bmatrix}$. Note that the original national total outputs vector was $\mathbf{x}^N = \begin{bmatrix} 2,000 \\ 1,000 \end{bmatrix}$ and the corresponding national final demand vector is $\mathbf{f}^N = \begin{bmatrix} 500 \\ 450 \end{bmatrix}$, found as $\mathbf{f}^N = \mathbf{x}^N - \mathbf{Z}^N\mathbf{i}$ or as $\mathbf{f}^N = \mathbf{f}^A + \mathbf{f}^B + \mathbf{f}^C$ where $\mathbf{f}^r = \mathbf{x}^r - \mathbf{Z}^r\mathbf{i}$ for regions $r = A$, B and C. This simply illustrates the linearity of the input-output model, since $\mathbf{x}^{new} = 10\mathbf{x}^N$ follows directly from $\mathbf{f}^{new} = 10\mathbf{f}^N$. (See also the solution to problem 2.2.)

Problem 3.5

This problem illustrates the key features of an interregional input-output (IRIO) model configuration, especially the role of interregional linkages. Consider the following two-region, three-sector interregional input-output transactions table:

		North			South			
		Agric.	Mining	Const.& Manuf.	Agric.	Mining	Const.& Manuf.	Total Output
North	Agriculture	277,757	3,654	1,710,816	8,293	26	179,483	3,633,382
	Mining	319	2,412	598,591	15	112	30,921	743,965
	Construction & Manufacturing	342,956	39,593	6,762,703	45,770	3,499	1,550,298	10,931,024
South	Agriculture	7,085	39	98,386	255,023	3,821	1,669,107	3,697,202
	Mining	177	92	15,966	365	3,766	669,710	766,751
	Construction & Manufacturing	71,798	7,957	2,017,905	316,256	36,789	8,386,751	14,449,941

Using the table's data to define the IRIO transactions matrix, $\mathbf{Z} = \left[\begin{array}{c|c} \mathbf{Z}^{NN} & \mathbf{Z}^{NS} \\ \hline \mathbf{Z}^{SN} & \mathbf{Z}^{SS} \end{array}\right]$, the total regional final demand vector is found as $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \left[\begin{array}{c} \mathbf{f}^{N} \\ \hline \mathbf{f}^{S} \end{array}\right] = \left[\begin{array}{r} 1{,}453{,}353 \\ 111{,}595 \\ 2{,}186{,}205 \\ \hline 1{,}663{,}741 \\ 76{,}675 \\ 3{,}612{,}485 \end{array}\right]$. Hence the matrices of regional technical coefficients for the North and South regions, respectively are

$$\mathbf{A}^{NN} = \mathbf{Z}^{NN}(\hat{\mathbf{x}}^{N})^{-1} = \begin{bmatrix} 0.076 & 0.005 & 0.157 \\ 0.000 & 0.003 & 0.055 \\ 0.094 & 0.053 & 0.619 \end{bmatrix} \text{ and } \mathbf{A}^{SS} = \mathbf{Z}^{SS}(\hat{\mathbf{x}}^{S})^{-1} = \begin{bmatrix} 0.069 & 0.005 & 0.116 \\ 0.000 & 0.005 & 0.046 \\ 0.086 & 0.048 & 0.580 \end{bmatrix};$$

the matrices of interregional trade coefficients between the two regions are found as

$$\mathbf{A}^{SN} = \mathbf{Z}^{SN}(\hat{\mathbf{x}}^{N})^{-1} = \begin{bmatrix} 0.002 & 0.000 & 0.009 \\ 0.000 & 0.000 & 0.001 \\ 0.020 & 0.011 & 0.185 \end{bmatrix} \text{ and } \mathbf{A}^{NS} = \mathbf{Z}^{NS}(\hat{\mathbf{x}}^{S})^{-1} = \begin{bmatrix} 0.002 & 0.000 & 0.012 \\ 0.000 & 0.000 & 0.002 \\ 0.012 & 0.005 & 0.107 \end{bmatrix}.$$

If we assume that a constrained availability of imported oil (upon which the economy is totally dependent) has forced the construction and manufacturing industry (sector 3) to reduce total output by 10 percent in the South and 5 percent in the North and, further, that interindustry relationships remain the same, i.e., the technical coefficients matrix remains unchanged, the corresponding amounts of output available for final demand in the economy are found by first assembling the IRIO coefficients matrix as $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}^{NN} & \mathbf{A}^{NS} \\ \hline \mathbf{A}^{SN} & \mathbf{A}^{SS} \end{array}\right]$. The new constrained total outputs vector can be computed as $\mathbf{x}^{new} = \hat{\mathbf{r}}\mathbf{x} = \left[\begin{array}{r} 3{,}633{,}382 \\ 743{,}965 \\ 10{,}384{,}473 \\ \hline 3{,}697{,}202 \\ 766{,}751 \\ 13{,}004{,}947 \end{array}\right]$ where the vector $\mathbf{r}$ is defined as

$\mathbf{r}=\begin{bmatrix}1 & 1 & .95 & \vdots & 1 & 1 & .9\end{bmatrix}$, reflecting the specified reduced total outputs for construction and manufacturing in the two regions. The corresponding new vector of final demands is found by

$$\mathbf{f}^{new}=\mathbf{x}^{new}-\mathbf{A}\mathbf{x}^{new}=\begin{bmatrix}1{,}556{,}842\\144{,}617\\2{,}132{,}819\\\hline 1{,}835{,}571\\144{,}444\\3{,}107{,}061\end{bmatrix}.$$

If we assume that tough import quotas imposed in Western Europe and the US on this country's goods have reduced the final demand for output from the country's construction and manufacturing industries by 15 percent in the North, the impact on the output vector for the North region (as an example) is computed by first expressing the modified final demand vector as $\mathbf{f}^{new}=\hat{\mathbf{r}}\mathbf{f}=\begin{bmatrix}1{,}453{,}353\\111{,}595\\1{,}858{,}274\\\hline 1{,}663{,}741\\76{,}675\\3{,}612{,}485\end{bmatrix}$ where $\mathbf{r}=\begin{bmatrix}1 & 1 & .85 & \vdots & 1 & 1 & 1\end{bmatrix}$, which reflects the specified reduction in final demand for construction and from the North region. The corresponding impact on total outputs is found as

$$\mathbf{x}^{new}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}^{new}=\left[\begin{array}{ccc|ccc}1.145 & 0.038 & 0.567 & 0.028 & 0.012 & 0.188\\0.020 & 1.014 & 0.180 & 0.007 & 0.004 & 0.054\\0.348 & 0.183 & 3.218 & 0.124 & 0.058 & 0.875\\\hline 0.033 & 0.016 & 0.219 & 1.111 & 0.024 & 0.365\\0.011 & 0.006 & 0.075 & 0.014 & 1.012 & 0.135\\0.215 & 0.112 & 1.500 & 0.284 & 0.147 & 2.868\end{array}\right]\begin{bmatrix}1{,}453{,}353\\111{,}595\\1{,}858{,}274\\\hline 1{,}663{,}741\\76{,}675\\3{,}612{,}485\end{bmatrix}=\begin{bmatrix}3{,}447{,}445\\684{,}913\\9{,}875{,}755\\\hline 3{,}625{,}440\\742{,}265\\13{,}957{,}983\end{bmatrix}$$

and for the North region, in particular, $\mathbf{x}^{N,new}=\begin{bmatrix}3{,}447{,}445\\684{,}913\\9{,}875{,}755\end{bmatrix}$.

If, for comparison, we ignore interregional linkages by using the Leontief inverse for the North region only, i.e., using only $\mathbf{A}^{NN}$, we find $(\mathbf{I}-\mathbf{A}^{NN})^{-1}=\begin{bmatrix}1.131 & 0.031 & 0.468\\0.016 & 1.011 & 0.152\\0.282 & 0.149 & 2.760\end{bmatrix}$. In conjunction with $\mathbf{f}^{N,new}=\begin{bmatrix}1{,}453{,}353\\111{,}595\\1{,}858{,}274\end{bmatrix}$, we find $\mathbf{x}^{N,new}=(\mathbf{I}-\mathbf{A}^{NN})^{-1}\mathbf{f}^{N,new}=\begin{bmatrix}2{,}517{,}159\\417{,}336\\5{,}554{,}462\end{bmatrix}$.

Compared with the IRIO results we can conclude that interregional linkages are important in this economy since the outputs found for the three industries using the North region alone are 27, 39, and 44 percent below their corresponding values for each industry respectively using the full two-region IRIO model.

Problem 3.6

This problem illustrates some key features of the interregional linkage using data from a highly aggregated version of the 2000 China MRIO table. Consider the following three-region, three-sector interregional transactions table:

China 2000		North Nat. Res.	North Manuf. & Const.	North Services	South Nat. Res.	South Manuf. & Const.	South Services	Rest of China Nat. Res.	Rest of China Manuf. & Const.	Rest of China Services
North	Natural Resources	1,724	6,312	406	188	1,206	86	14	49	4
	Manuf. & Const.	2,381	18,458	2,987	301	3,331	460	39	234	57
	Services	709	3,883	1,811	64	432	138	5	23	5
South	Natural Resources	149	656	42	3,564	8,828	806	103	178	15
	Manuf. & Const.	463	3,834	571	3,757	34,931	5,186	202	1,140	268
	Services	49	297	99	1,099	6,613	2,969	31	163	62
ROC	Natural Resources	9	51	3	33	254	18	1,581	3,154	293
	Manuf. & Const.	32	272	41	123	1,062	170	1,225	6,704	1,733
	Services	4	25	7	25	168	47	425	2,145	1,000
Total Output		16,651	49,563	15,011	27,866	81,253	23,667	11,661	21,107	8,910

If we denote the interindustry tractions matrix in this table by $\mathbf{Z}$ and the vector of total outputs by $\mathbf{x}$, the corresponding direct and total requirements matrices, are found by $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$, respectively. Suppose, however, for this economy all the inputs to the North region from the South region were replaced with corresponding industry production from the Rest of China (ROC) region. We would reflect this change in the transactions table by removing the transactions from the 3×3 matrix partition showing transactions from South to North (i.e., each element in that partition becomes zero) and add those transactions, element-by-element, to the partition showing transactions from ROC to the North (lower left 3×3 partition) with the rest of the table unchanged. This change corresponds to the situation where all inputs to the North from the South came instead from the Rest of China, and the resulting transactions table would be:

China 2000		North Nat. Res.	North Manuf. & Const.	North Services	South Nat. Res.	South Manuf. & Const.	South Services	Rest of China Nat. Res.	Rest of China Manuf. & Const.	Rest of China Services
North	Natural Resources	1,724	6,312	406	188	1,206	86	14	49	4
	Manuf. & Const.	2,381	18,458	2,987	301	3,331	460	39	234	57
	Services	709	3,883	1,811	64	432	138	5	23	5
South	Natural Resources	0	0	0	3,564	8,828	806	103	178	15
	Manuf. & Const.	0	0	0	3,757	34,931	5,186	202	1,140	268
	Services	0	0	0	1,099	6,613	2,969	31	163	62
ROC	Natural Resources	158	707	46	33	254	18	1,581	3,154	293
	Manuf. & Const.	494	4,106	613	123	1,062	170	1,225	6,704	1,733
	Services	53	321	105	25	168	47	425	2,145	1,000
Total Output		16,651	49,563	15,011	27,866	81,253	23,667	11,661	21,107	8,910

Note that we have not changed the vector of total outputs, **x**. We can denote the revised transactions matrix as $\overline{\mathbf{Z}}$ and the revised direct and total requirements table then become $\overline{\mathbf{A}} = \overline{\mathbf{Z}}\hat{\mathbf{x}}^{-1}$ and $\overline{\mathbf{L}} = (\mathbf{I} - \overline{\mathbf{A}})^{-1}$ are the following:

Revised direct requirements:

China 2000		North Nat. Res.	North Manuf. & Const.	North Services	South Nat. Res.	South Manuf. & Const.	South Services	Rest of China Nat. Res.	Rest of China Manuf. & Const.	Rest of China Services
North	Natural Resources	0.1035	0.1273	0.0270	0.0067	0.0148	0.0036	0.0012	0.0023	0.0005
	Manuf. & Const.	0.1430	0.3724	0.1990	0.0108	0.0410	0.0194	0.0034	0.0111	0.0064
	Services	0.0426	0.0783	0.1206	0.0023	0.0053	0.0058	0.0004	0.0011	0.0006
South	Natural Resources	0.0000	0.0000	0.0000	0.1279	0.1087	0.0340	0.0089	0.0084	0.0017
	Manuf. & Const.	0.0000	0.0000	0.0000	0.1348	0.4299	0.2191	0.0173	0.0540	0.0301
	Services	0.0000	0.0000	0.0000	0.0394	0.0814	0.1255	0.0026	0.0077	0.0070
ROC	Natural Resources	0.0095	0.0143	0.0030	0.0012	0.0031	0.0008	0.1356	0.1494	0.0329
	Manuf. & Const.	0.0297	0.0828	0.0408	0.0044	0.0131	0.0072	0.1050	0.3176	0.1945
	Services	0.0032	0.0065	0.0070	0.0009	0.0021	0.0020	0.0364	0.1016	0.1122

Revised total requirements:

China 2003		North Nat. Res.	North Manuf. & Const.	North Services	South Nat. Res.	South Manuf. & Const.	South Services	Rest of China Nat. Res.	Rest of China Manuf. & Const.	Rest of China Services
North	Natural Resources	1.1603	0.2494	0.0929	0.0225	0.0575	0.0264	0.0064	0.0159	0.0084
	Manuf. & Const.	0.2938	1.7104	0.3988	0.0530	0.1579	0.0840	0.0189	0.0523	0.0311
	Services	0.0826	0.1651	1.1775	0.0114	0.0303	0.0200	0.0034	0.0092	0.0054
South	Natural Resources	0.0032	0.0077	0.0041	1.1897	0.2441	0.1081	0.0237	0.0438	0.0220
	Manuf. & Const.	0.0137	0.0332	0.0180	0.3165	1.8924	0.4892	0.0710	0.1923	0.1136
	Services	0.0026	0.0063	0.0034	0.0834	0.1879	1.1943	0.0137	0.0362	0.0244
ROC	Natural Resources	0.0365	0.0775	0.0367	0.0087	0.0236	0.0120	1.1966	0.2816	0.1075
	Manuf. & Const.	0.1028	0.2545	0.1374	0.0258	0.0714	0.0399	0.2096	1.5757	0.3577
	Services	0.0202	0.0471	0.0298	0.0060	0.0158	0.0098	0.0735	0.1930	1.1724

To illustrate the impact on total outputs of all regions and sectors for a final demand of ¥100,000 on export demand for manufactured goods produced in the North, we first specify the change in final demand as $(\Delta\mathbf{f}^N)' = \begin{bmatrix} 0 & 100 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 & 0 & 0 \end{bmatrix}$. The corresponding vector of total outputs is then $(\Delta\mathbf{x})' = (\overline{\mathbf{L}}\Delta\mathbf{f}^N)' = \begin{bmatrix} 24.9 & 171.0 & 16.5 & \vdots & 0.8 & 3.3 & 0.6 & \vdots & 7.7 & 25.4 & 4.7 \end{bmatrix}$. If we recast $\Delta\mathbf{x}$ in the format of Table 3.10 we have the following table which shows the changes in production by region and sector generated by the shift in the location of inputs to production from the South to the Rest of China:

	Produced in the North (original)			**Produced in the North (revised)**		
Sector	North	South	ROC	North	South	ROC
Nat. Res.	25.6	6.8	0.8	24.9	0.8	7.7
Mfg. &Const.	172.8	29.4	2.5	171	3.3	25.4
Services	16.9	4.5	0.5	16.5	0.6	4.7
Total	215.3	40.7	3.8	212.4	4.7	37.8

Problem 3.7

This exercise problem illustrates application of a multiregional input-output (MRIO) model, using a three-region, five-sector version of the U.S. multiregional input-output economy (shown below and in Table A4.1-3 of Appendix S4.1 in the text).

Five-Sector, Three-Region Multiregional Input–Output Tables for the United States (1963)

Regional Transactions (millions of dollars)

	Agric	Mining	Const & Manuf	Services	Transport & Utilities
East					
Agriculture	2,013	0	7,863	44	0
Mining	35	335	3,432	44	843
Const & Manuf	2,029	400	78,164	11,561	2,333
Services	1,289	294	19,699	26,574	2,301
Transport & Util	225	384	7,232	4,026	3,534
Central					
Agriculture	10,303	0	13,218	97	0
Mining	82	472	8,686	15	1,271
Const & Manuf	4,422	1,132	93,816	10,155	2,401
Services	4,952	2,378	21,974	22,358	2,473
Transport & Util	667	406	9,296	3,468	4,513
West					
Agriculture	2,915	0	3,452	65	0
Mining	4	292	2,503	0	353
Const & Manuf	1,214	466	27,681	4,925	1,015
Services	1,307	721	8,336	10,809	991
Transport & Util	338	160	2,936	1,659	1,576

Commodity Trade Flows and Total Outputs (millions of dollars)

	East	West	Central
Agriculture			
East	6,007	2,124	208
West	3,845	28,885	2,521
Central	403	2,922	7,028
Mining			
East	2,904	415	53
West	1,108	10,942	271
Central	71	772	3,996
Const & Manuf			
East	158,679	42,150	8,368
West	44,589	201,025	11,778
Central	4,702	6,726	61,385
Services			
East	146,336	16,116	2,955
West	9,328	121,079	3,185
Central	1,939	3,643	58,663
Transp & Util			
East	21,434	4,974	263
West	4,396	23,811	1,948
Central	1,009	1,334	9,635
Total Output			
Agriculture	10,259	33,939	9,753
Mining	4,084	12,129	4,319
Const & Manuf	207,948	249,840	81,512
Services	157,468	140,850	64,803
Transport & Util	26,847	30,130	11,841

Suppose that a new government military project is initiated in the western United States, stimulating new final demand in that region of (in millions of dollars) which we can express as $\Delta\mathbf{f}^W = \begin{bmatrix} 0 & 0 & 100 & 50 & 25 \end{bmatrix}'$. The impact on total production of all sectors in all three regions of the United States economy stimulated by this new final demand in the West, can be found by first defining the new final demand for the entire economy as

$$\left[\Delta\mathbf{f}\right]' = \left[(\Delta\mathbf{f}^E)' \mid (\Delta\mathbf{f}^C)' \mid (\Delta\mathbf{f}^W)'\right] = \left[0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \mid 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \mid 0 \;\; 0 \;\; 100 \;\; 50 \;\; 25\right].$$

Finally, we find the vector of changes in total output as $\Delta\mathbf{x} = \bar{\mathbf{L}}\Delta\mathbf{f}$:

$$\left[\Delta\mathbf{x}\right]' = \left[0.75 \; 1.125 \; 23.3 \; 13.2 \; 8.225 \mid 3.525 \; 5.375 \; 38.175 \; 20.475 \; 13.4 \mid 4.825 \; 4.65 \; 96.45 \; 68.125 \; 25.6\right].$$

Problem 3.8

This problem explores the use of an interregional input-output (IRIO) model for impact analysis using the three-region, five-sector version of an interregional input-output economy of Japan for 1965 given in Table A4.1-1 of Appendix S4.1. For the final demand vector

$$\left[\Delta\mathbf{f}\right]' = \left[(\Delta\mathbf{f}^{C})' \;\vdots\; (\Delta\mathbf{f}^{N})' \;\vdots\; (\Delta\mathbf{f}^{S})'\right] = \left[0\,0\,0\,0\,0 \;\vdots\; 0\,0\,0\,0\,0 \;\vdots\; 0\,0\;\; 100\;\; 50\;\; 25\right]$$

This is the same vector as that used in problem 3.6, but it is used in this case for the Japanese IRIO economy where the regions are Central, North, and South. Using the vector, $\Delta\mathbf{f}$, we find the corresponding vector of total outputs, $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f}$, for this very interconnected interregional economy:

$$[\Delta\mathbf{x}]' = [.386\;\; .024\;\; 13.061\;\; 0.892\;\; 3.024 \;\vdots\; .145\;\; .021\;\; 3.669\;\; 1.376\;\; .339 \;\vdots\; 3.634\;\; .475\;\; 181.630\;\; 56.029\;\; 42.904]$$

Problem 3.9

This problem explores IRIO analysis using the 4 region, three sector IRIO model for China, Japan, the United States, and an aggregation of other Asian nations including Indonesia, Malaysia, the Philippines, Singapore, and Thailand for the year 2000. The interindustry transactions and total outputs are specified in the following table.

2000		United States			Japan			China			Rest of Asia		
		Nat Res	Manuf & Const	Services	Nat Res	Manuf & Const	Services	Nat Res	Manuf & Const	Services	Nat Res	Manuf & Const	Services
US	Nat Res	75,382	296,016	17,829	351	4,764	473	174	403	17	103	2,740	83
	Manuf & Const	68,424	1,667,042	960,671	160	21,902	3,775	587	8,863	1,710	383	45,066	4,391
	Services	95,115	1,148,999	3,094,357	118	6,695	807	160	1,466	296	197	7,393	953
Japan	Nat Res	7	52	53	8,721	78,936	11,206	13	66	2	14	180	27
	Manuf & Const	859	41,484	11,337	28,088	1,414,078	484,802	764	20,145	2,809	462	72,258	4,108
	Services	97	4,390	1,424	24,901	662,488	1,001,832	107	2,763	335	270	7,816	1,189
China	Nat Res	72	343	147	50	2,316	229	49,496	183,509	15,138	102	2,430	99
	Manuf & Const	331	15,657	6,442	93	10,199	1,989	89,384	892,227	181,932	157	15,093	1,237
	Services	38	2,218	1,099	17	1,780	280	25,391	210,469	136,961	23	2,078	132
ROA	Nat Res	322	1,068	203	64	11,906	266	64	1,475	14	12,153	92,647	6,402
	Manuf & Const	503	56,287	18,129	278	35,418	3,562	1,141	41,496	4,685	23,022	566,274	144,417
	Services	152	4,578	1,921	41	3,982	447	138	3,669	422	15,163	213,470	239,053
Total Output		468,403	5,866,935	11,609,307	140,622	3,883,455	4,658,191	408,153	2,000,741	702,248	173,080	1,727,367	1,225,460

The table of direct requirements, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, is the following:

2000		United States			Japan			China			Rest of Asia		
		Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
U.S.	Nat. Res.	0.1609	0.0505	0.0015	0.0025	0.0012	0.0001	0.0004	0.0002	0.0000	0.0006	0.0016	0.0001
	Manuf. & Const.	0.1461	0.2841	0.0828	0.0011	0.0056	0.0008	0.0014	0.0044	0.0024	0.0022	0.0261	0.0036
	Services	0.2031	0.1958	0.2665	0.0008	0.0017	0.0002	0.0004	0.0007	0.0004	0.0011	0.0043	0.0008
Japan	Nat. Res.	0.0000	0.0000	0.0000	0.0620	0.0203	0.0024	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000
	Manuf. & Const.	0.0018	0.0071	0.0010	0.1997	0.3641	0.1041	0.0019	0.0101	0.0040	0.0027	0.0418	0.0034
	Services	0.0002	0.0007	0.0001	0.1771	0.1706	0.2151	0.0003	0.0014	0.0005	0.0016	0.0045	0.0010
China	Nat. Res.	0.0002	0.0001	0.0000	0.0004	0.0006	0.0000	0.1213	0.0917	0.0216	0.0006	0.0014	0.0001
	Manuf. & Const.	0.0007	0.0027	0.0006	0.0007	0.0026	0.0004	0.2190	0.4459	0.2591	0.0009	0.0087	0.0010
	Services	0.0001	0.0004	0.0001	0.0001	0.0005	0.0001	0.0622	0.1052	0.1950	0.0001	0.0012	0.0001
ROA	Nat. Res.	0.0007	0.0002	0.0000	0.0005	0.0031	0.0001	0.0002	0.0007	0.0000	0.0702	0.0536	0.0052
	Manuf. & Const.	0.0011	0.0096	0.0016	0.0020	0.0091	0.0008	0.0028	0.0207	0.0067	0.1330	0.3278	0.1178
	Services	0.0003	0.0008	0.0002	0.0003	0.0010	0.0001	0.0003	0.0018	0.0006	0.0876	0.1236	0.1951

The table of total requirements, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$, is:

2000		United States			Japan			China			Rest of Asia		
		Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
U.S.	Nat. Res.	1.2103	0.0889	0.0126	0.0043	0.0036	0.0008	0.0013	0.0019	0.0010	0.0022	0.0071	0.0016
	Manuf. & Const.	0.2953	1.4643	0.1660	0.0071	0.0164	0.0039	0.0082	0.0183	0.0113	0.0147	0.0641	0.0163
	Services	0.4140	0.4158	1.4113	0.0054	0.0096	0.0021	0.0040	0.0083	0.0051	0.0079	0.0289	0.0076
Japan	Nat. Res.	0.0002	0.0005	0.0001	1.0755	0.0366	0.0082	0.0004	0.0010	0.0006	0.0007	0.0027	0.0006
	Manuf. & Const.	0.0085	0.0201	0.0048	0.3924	1.6463	0.2196	0.0161	0.0420	0.0233	0.0235	0.1117	0.0238
	Services	0.0026	0.0061	0.0015	0.3280	0.3663	1.3236	0.0052	0.0136	0.0074	0.0090	0.0345	0.0083
China	Nat. Res.	0.0007	0.0012	0.0004	0.0013	0.0024	0.0005	1.1997	0.2184	0.1025	0.0020	0.0061	0.0013
	Manuf. & Const.	0.0043	0.0096	0.0028	0.0046	0.0108	0.0027	0.5518	2.0243	0.6666	0.0077	0.0317	0.0075
	Services	0.0009	0.0021	0.0007	0.0011	0.0026	0.0006	0.1649	0.2816	1.3374	0.0018	0.0071	0.0016
ROA	Nat. Res.	0.0015	0.0018	0.0005	0.0024	0.0070	0.0011	0.0022	0.0060	0.0028	1.0906	0.0914	0.0205
	Manuf. & Const.	0.0081	0.0239	0.0062	0.0101	0.0261	0.0051	0.0255	0.0711	0.0368	0.2440	1.5530	0.2293
	Services	0.0023	0.0055	0.0015	0.0028	0.0070	0.0014	0.0061	0.0166	0.0086	0.1562	0.2487	1.2798

If we assume that annual final demand growth in China is 8 percent, growth in the U.S. and Japan is 4 percent, and that of other Asian nations is 3 percent, we can compute the original and projected final demand vectors as the following. The original vector of final demands is computed by $\mathbf{f} = \mathbf{x} - \mathbf{Zi}$, so

$$\mathbf{f}' = \left[70{,}067 \;\; 3{,}083{,}962 \;\; 7{,}252{,}751 \;\vdots\; 41{,}344 \;\; 1{,}802{,}261 \;\; 2{,}950{,}579 \;\vdots\; 154{,}222 \;\; 786{,}002 \;\; 321{,}762 \;\vdots\; 46{,}495 \;\; 832{,}154 \;\; 742{,}423\right]$$

For a level of growth in China at 8 percent, in the U.S. and Japan at 4 percent, and in the rest of Asia at 3 percent, the final demand in the next year is found by multiplying the first three elements of f (U.S. final demand) by 1.04, the next three (Japanese final demand) by 1.04, the next three (Chinese final demand) by 1.08, and the last three (final demand for the other nations in Asia) by 1.03, to yield

$$(\mathbf{f}^{new})' = \left[72{,}869 \;\; 3{,}207{,}321 \;\; 7{,}542{,}861 \;\vdots\; 42{,}998 \;\; 1{,}874{,}352 \;\; 3{,}068{,}602 \;\vdots\; 166{,}560 \;\; 848{,}883 \;\; 347{,}503 \;\vdots\; 47{,}890 \;\; 857{,}119 \;\; 764{,}696\right]$$

The corresponding vector of total outputs is then found as $\mathbf{x}^{new} = \mathbf{L}\mathbf{f}^{new}$:

$$(\mathbf{x}^{new})' = \left[487{,}149 \;\; 6{,}101{,}723 \;\; 12{,}073{,}729 \;\vdots\; 146{,}262 \;\; 4{,}039{,}397 \;\; 4{,}844{,}723 \;\vdots\; 440{,}002 \;\; 2{,}156{,}077 \;\; 757{,}348 \;\vdots\; 178{,}822 \;\; 1{,}784{,}590 \;\; 1{,}263{,}502\right]$$

Finally, the vector of the percentage growth in total output for each and all regions and sectors is then found as

$$100 \times \frac{(\mathbf{x}^{new} - \mathbf{x})}{\mathbf{x}} = \left[4.002 \;\; 4.002 \;\; 4.000 \;\vdots\; 4.011 \;\; 4.016 \;\; 4.004 \;\vdots\; 7.803 \;\; 7.764 \;\; 7.846 \;\vdots\; 3.317 \;\; 3.313 \;\; 3.104\right]$$

Problem 3.10

This problem illustrates recursive use expressing the Leontief inverse of a matrix in terms of partitions of the original matrix, sometimes necessary for very large matrices (thousands of sectors). Assume that a limited computer that can directly determine the inverse of matrices no larger than of dimension 2×2 (in practice this might be more like $5,000\times5,000$). For

$$\mathbf{A}=\begin{bmatrix} 0 & 0.1 & 0.3 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0 & 0 \\ 0.2 & 0 & 0.1 & 0.3 & 0.1 \\ 0.3 & 0 & 0 & 0.1 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix}$$

, we first partition the matrix $(\mathbf{I}-\mathbf{A})$ as

$$(\mathbf{I}-\mathbf{A})=\left[\begin{array}{cc:ccc} 1.0 & -0.1 & -0.3 & -0.2 & -0.2 \\ -0.1 & 0.9 & -0.1 & 0.0 & 0.0 \\ \hdashline -0.2 & 0.0 & 0.9 & -0.3 & -0.1 \\ -0.3 & 0.0 & 0.0 & 0.9 & -0.3 \\ -0.3 & -0.2 & -0.1 & -0.1 & 0.8 \end{array}\right]=\left[\begin{array}{c:c} \mathbf{E} & \mathbf{F} \\ \hdashline \mathbf{G} & \mathbf{H} \end{array}\right]$$

and then further partition the matrix **H** by

$$\mathbf{H}=\left[\begin{array}{cc:c} 0.9 & -0.3 & -0.1 \\ 0 & 0.9 & -0.3 \\ \hdashline -0.1 & -0.1 & 0.8 \end{array}\right]=\left[\begin{array}{c:c} \mathbf{H}_1 & \mathbf{H}_2 \\ \hdashline \mathbf{H}_3 & \mathbf{H}_4 \end{array}\right]$$

. We then can define $(\mathbf{I}-\mathbf{A})^{-1}=\left[\begin{array}{c:c} \mathbf{S} & \mathbf{T} \\ \hdashline \mathbf{U} & \mathbf{V} \end{array}\right]$ where partitions in similar positions in $(\mathbf{I}-\mathbf{A})$ and $(\mathbf{I}-\mathbf{A})^{-1}$ have the same dimensions. From the results on the inverse of a partitioned inverse (Appendix A), we find that we need $\mathbf{E}^{-1}$ and $\mathbf{H}^{-1}$, the inverses of a 2×2 and a 3×3 matrix. Therefore, to find $\mathbf{H}^{-1}$ we again use the results on the inverse of a partitioned matrix, where **H** is partitioned as above. This requires that $\mathbf{H}^{-1}$ and $\mathbf{H}_4^{-1}$ be found; since these are 2×2 and 1×1 matrices, respectively, this is easily accomplished. Hence, we have

$$\mathbf{H}^{-1}=\left[\begin{array}{cc:c} 1.144 & 0.415 & 0.299 \\ 0.050 & 1.177 & 0.448 \\ \hdashline 0.149 & 0.199 & 1.343 \end{array}\right]$$

. This in conjunction with $\mathbf{E}^{-1}$, **F** and **G** allows us to find **S**, **T**, **U** and **V** which comprise $(\mathbf{I}-\mathbf{A})^{-1}$: $\mathbf{S}=\begin{bmatrix} 1.566 & 0.332 \\ 0.253 & 1.172 \end{bmatrix}$, $\mathbf{T}=\begin{bmatrix} 0.638 & 0.640 & 0.711 \\ 0.231 & 0.150 & 0.148 \end{bmatrix}$,

$\mathbf{U}=\begin{bmatrix} 0.708 & 0.217 \\ 0.802 & 0.270 \\ 0.839 & 0.478 \end{bmatrix}$, and $\mathbf{V}=\begin{bmatrix} 1.441 & 0.707 & 0.622 \\ 0.388 & 1.509 & 0.815 \\ 0.525 & 0.554 & 1.733 \end{bmatrix}$. Therefore

$$(\mathbf{I}-\mathbf{A})^{-1}=\left[\begin{array}{cc:ccc} 1.566 & 0.332 & 0.638 & 0.640 & 0.711 \\ 0.253 & 1.172 & 0.231 & 0.150 & 0.148 \\ \hdashline 0.708 & 0.217 & 1.441 & 0.707 & 0.622 \\ 0.802 & 0.270 & 0.388 & 1.509 & 0.815 \\ 0.839 & 0.478 & 0.525 & 0.554 & 1.733 \end{array}\right]$$

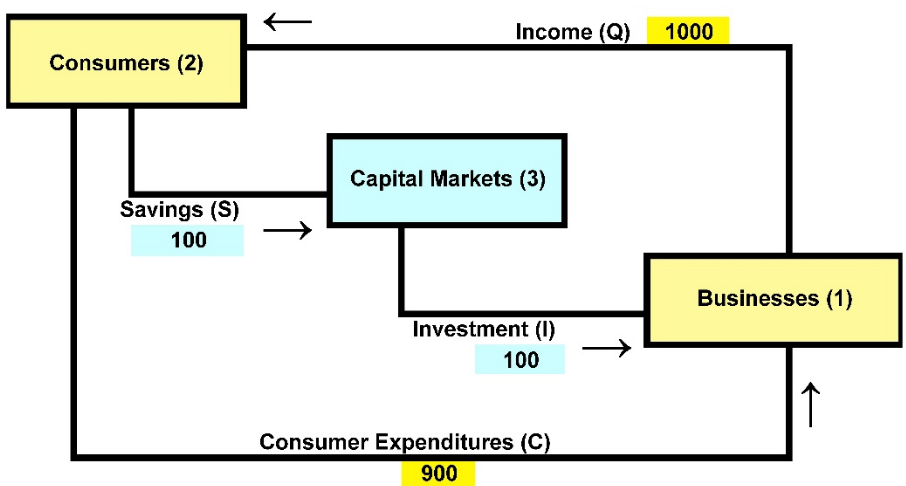
. In this problem we used the method of partitioning repeatedly (sometimes called recursive application of the method) on sub-partitions of the original four partitions of $(\mathbf{I}-\mathbf{A})$. We can in theory invert an infinitely large matrix by recursively partitioning it into smaller and smaller submatrices.

Chapter 4, Organization of Basic Data for Input–Output Models

Chapter 4 deals with the construction of input–output tables from standardized conventions of national economic accounts, such as the widely used System of National Accounts (SNA) promoted by the United Nations, including a basic introduction to the so-called commodity-by-industry or supply-use input–output framework developed in additional detail in Chapter 5. A simplified SNA is derived from fundamental economic concepts of the circular flow of income and expenditure, that, as additional sectoral details are defined for businesses, households, government, foreign trade, and capital formation, ultimately result in the basic commodity-by-industry formulation of input–output accounts. The process is illustrated with the US input–output model and some of the key traditional conventions widely applied for such considerations as secondary production (multiple products or commodities produced by a business), competitive imports (commodities that are also produced domestically) versus non-competitive imports (commodities not produced domestically), trade and transportation margins on interindustry transactions, or the treatment of scrap and secondhand goods. The exercise problems for this chapter illustrate the key features of the SNA and relationships with input-output accounts and models.

Problem 4.1

This problem illustrates the basic concepts of the circular flow of income and expenditure in a simple macroeconomy and corresponding set of national accounts. Consider a macroeconomy show below where transactions are measured in millions of dollars.



The balance equations, found by equating the sum of all flows into an account with the sum of all flows leaving the account, are the following:

$$
\begin{aligned}
&Q = 1,000 = C + I = 900 + 100 = 1,000 \\
&C + S = 900 + 90 = Q + D = 1000 - 10 = 990 \\
&I + D = 100 - 10 = S = 90
\end{aligned}
$$

The corresponding "T" account tables are the following:

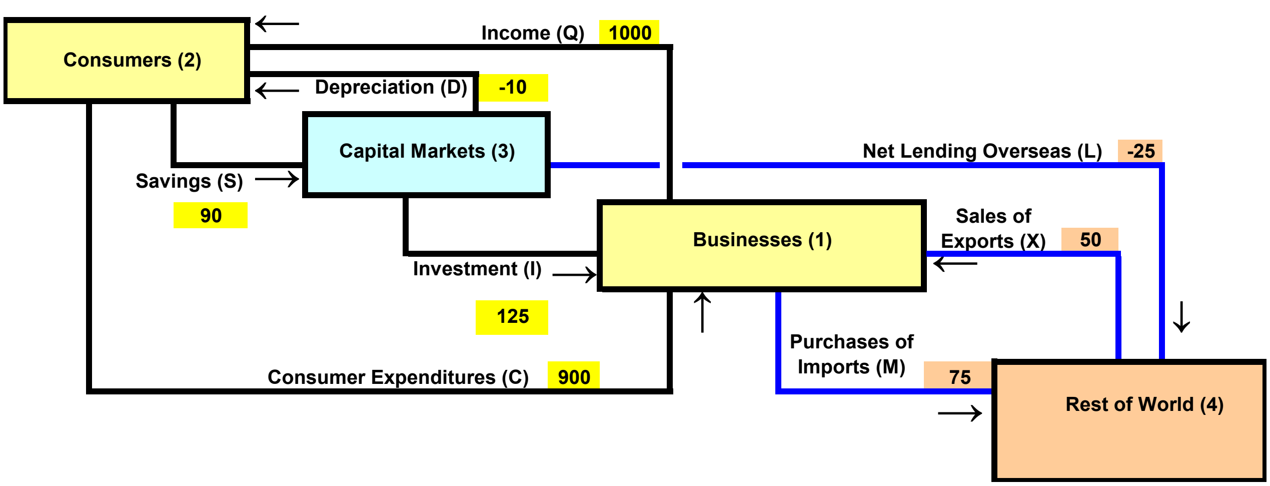
Production (Domestic Product Account)			
Debits		**Credits**	
Income (Q)	1000	Sales of consumption goods (C)	900
		Sales of capital goods (I)	100
Total	1000	Total	1000

Consumption (Income and Outlay Account)			
Debits		**Credits**	
Purchases of consumption goods (C)	900	Income (Q)	1000
Savings (S)	90	Depreciation (D)	-10
Total	990	Total	990

Accumulation (Capital Transactions Account)			
Debits		**Credits**	
Purchase of capital goods (I)	100	Savings (S)	90
Depreciation (D)	-10		
Total	90	Total	90

Problem 4.2

This problem illustrates adding depreciation and rest-of-world accounts to the macroeconomy from problem 4.1. We presume new transactions added are a capital consumption allowance to account for depreciation of capital investments of 10 percent of total investment (I), international trade allowances with a. "rest of world" account to accommodate purchases of imports of $75 million (M), sales of exports of $50 million (X), and savings made available to capital markets from overseas lenders of $25 million (L), resulting in a new total amount of capital available for businesses of $125 million.



The modified balance equations for the businesses, consumers, capital, and trade accounts are:

$$Q + M = 1{,}000 + 75 = C + I + X = 900 + 125 + 50 = 1075$$
$$C + S = 900 + 90 = Q + D = 1{,}000 - 10 = 990$$
$$I + D + L = 125 - 10 - 25 = S = 90$$
$$X = W - L = 75 - 25 = 50$$

and the corresponding set of "T" accounts are the following:

Production (Domestic Product Account)			
Debits		**Credits**	
Income (Q)	1000	Sales of consumption goods (C)	900
Imports (W)	75	Sales of capital goods (I)	125
		Exports (X)	50
Total	1075	Total	1075

Consumption (Income and Outlay Account)			
Debits		**Credits**	
Purchases of consumption goods (C)	900	Income (Q)	1000
Savings (S)	90	Depreciation (D)	-10
Total	990	Total	990

Accumulation (Capital Transactions Account)			
Debits		**Credits**	
Purchase of capital goods (I)	125	Savings (S)	90
Depreciation (D)	-10		
Net Lending Overseas (L)	-25		
Total	90	Total	90

Rest of World (Balance of Payments Account)			
Debits		**Credits**	
Sales of exports (X)	50	Purchases of Imports (W)	75
		Net Overseas Lending (L)	-25
Total	50	Total	50

Problem 4.3

This problem illustrates expressing a national economic balance sheet for an economy as a collection of balance equations and as a matrix representation of the consolidated national accounts. Consider a national economic balance sheet for an economy is given by the following:

Debits					Economic Transaction	Credits				
		Capital		Rest of				Capital		Rest of
Prod.	Cons.	Accum.	Govt	World		Prod.	Cons.	Accum.	Govt	World
	475				Consumption Goods (C)	475				
		54			Capital Goods (I)	54				
				46	Exports (X)	46				
46					Imports (M)					46
554					Income (Q)		554			
		-29			Depreciation (D)		-29			
	30				Savings (S)			30		
			25		Govt. Expenditures (G)	25				
	20				Taxes (T)				20	
		5			Govt Deficit Spending (B)				5	
600	525	30	25	46	Totals	600	525	30	25	46

The corresponding balance equations are:

Domestic Product Account: $Q + M = C + I + X + G$

Income and Outlay Account: $C + S + T = Q + D$

Capital Transactions Account: $I + D + B = S$

Balance of Payments Account: $X = M$

Government Account: $G = T + B$

The corresponding matrix representation of the consolidated national accounts is the following:

	Prod.	Cons.	Cap.	ROW	Govt.	Total
Production		475	54	46	25	600
Consumption	554		-29			525
Capital Accum.		30				30
Rest of World	46					46
Govt.		20	5			25
Total	600	525	30	46	25	

Problem 4.4

This problem illustrates the application of double deflation to adjusting interindustry transactions according the changes in relative prices. Consider the following 4-sector input-output transactions table for the year 2015 along with industry prices for 2015 and 2020.

	Industry Transactions				Total	Price	Price
	1	2	3	4	Output	Year 2000	Year 2005
1	24	86	56	64	398	2	5
2	32	15	78	78	314	3	6
3	104	49	62	94	469	5	9
4	14	16	63	78	454	7	12

To compute the matrices of interindustry transactions and technical coefficients as well as the vector of total outputs deflated to year 2015 value terms, first, the vector of price indices is

$\mathbf{p} = [2/5 \quad 3/6 \quad 5/9 \quad 7/12] = [0.400 \;\; 0.500 \;\; 0.556 \;\; 0.583]$. This vector is comprised of the ratios of the year 2000 prices to the year 2005 prices. Hence, $\mathbf{Z}^{2000}$, $\mathbf{A}^{2000}$ and $\mathbf{x}^{2000}$ can be computed as

$$\mathbf{Z}^{2000} = \hat{\mathbf{p}}\mathbf{Z}^{2005} = \begin{bmatrix} .4 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ 0 & 0 & .556 & 0 \\ 0 & 0 & 0 & .583 \end{bmatrix} \begin{bmatrix} 24 & 86 & 56 & 64 \\ 32 & 15 & 78 & 78 \\ 104 & 49 & 62 & 94 \\ 14 & 16 & 63 & 78 \end{bmatrix} = \begin{bmatrix} 9.6 & 34.4 & 22.4 & 25.6 \\ 16 & 7.5 & 39 & 39 \\ 57.78 & 27.22 & 34.44 & 52.22 \\ 8.17 & 9.33 & 36.75 & 45.5 \end{bmatrix},$$

$$\mathbf{A}^{2000} = \mathbf{Z}^{2000}(\hat{\mathbf{x}}^{2000})^{-1} = \hat{\mathbf{p}}\mathbf{Z}^{2005}(\hat{\mathbf{p}}\hat{\mathbf{x}}^{2005}) = \hat{\mathbf{p}}\mathbf{Z}^{2005}(\hat{\mathbf{x}}^{2005})^{-1}\hat{\mathbf{p}}^{-1} = \hat{\mathbf{p}}\mathbf{A}^{2005}\hat{\mathbf{p}}^{-1} =$$

$$\begin{bmatrix} .0603 & .2191 & .0860 & .0967 \\ .1005 & .0478 & .1497 & .1473 \\ .3629 & .1734 & .1322 & .1972 \\ .0513 & .0594 & .1410 & .1718 \end{bmatrix}, \text{ and } \mathbf{x}^{2000} = \hat{\mathbf{p}}\mathbf{x}^{2005} = \begin{bmatrix} 159.1 \\ 157 \\ 260.56 \\ 264.83 \end{bmatrix}.$$

Problem 4.5

This problem illustrates the impact of sector aggregation on the accounting for production of total outputs in an input-output economy using the transactions data given in problem 2.7:

$$\mathbf{Z} = \begin{bmatrix} 8,565 & 8,069 & 8,843 & 3,045 & 1,124 & 276 & 230 & 3,464 \\ 1,505 & 6,996 & 6,895 & 3,530 & 3,383 & 365 & 219 & 2,946 \\ 98 & 39 & 5 & 429 & 5,694 & 7 & 376 & 327 \\ 999 & 1,048 & 120 & 9,143 & 4,460 & 228 & 210 & 2,226 \\ 4,373 & 4,488 & 8,325 & 2,729 & 2,9671 & 1,733 & 5,757 & 14,756 \\ 2,150 & 36 & 640 & 1,234 & 165 & 821 & 90 & 6,717 \\ 506 & 7 & 180 & 0 & 2,352 & 0 & 18,091 & 26,529 \\ 5,315 & 1,895 & 2,993 & 1,071 & 13,941 & 434 & 6,096 & 46,338 \end{bmatrix}$$

One way of illustrating the effects of aggregation is as follows. Using a final-demand vector of all 1's, determine the effect on total of total outputs throughout the entire economy (i.e., summed over all the sectors) by successively aggregating transactions from 8 to 7 to 6 sectors and so on (also aggregating the corresponding final-demand vector) and evaluating the relative impact on vectors of total outputs and the total of total outputs. Consider the following sequence of aggregations:

- Case 1 (8×8) No sectoral aggregation
- Case 2 (7×7) Combine sector 6 with sector 2
- Case 3 (6×6) Also combine sector 5 with sector 1
- Case 4 (5×5) Also combine sector 8 with sector 3
- Case 5 (4×4) Also combine sector 7 with previously combined 6 and 2
- Case 6 (3×3) Also combine sector 4 with previously combined 5 and 1

The impact of the sum of total outputs is indicated in the following table at each level of aggregation:

Aggregation Level	x'i	Aggregated Sector Total Output							
		1	2	3	4	5	6	7	8
8	16.26	2.31	1.84	1.12	1.6	2.88	1.43	2.26	2.82
7	16.56	2.48	3.33	1.13	1.61	2.87	2.28	2.87	
6	15.64	4.85	3.14	1.15	1.58	2.2	2.62		
5	15.62	4.78	3.11	3.86	1.58	2.29			
4	15.72	4.73	5.51	3.91	1.57				
3	15.53	6.15	5.44	3.94					

Problem 4.6

This problem illustrates the computation of first order and total aggregation bias using the seven-sector input-output table of technical coefficients for the U.S. economy (1972) given in Appendix SD1 (located on the supplemental resources website). Consider the following vector of final demands: $\Delta\mathbf{f} = \begin{bmatrix}100 & 100 & 100 & 100 & 100 & 100 & 100\end{bmatrix}'$. To compute the first order and total aggregation bias associated with, as an example, combining agriculture with mining, construction with manufacturing, and transportation-utilities with services and other sectors to yield a new three-sector model we first compute the interindustry transactions,

$$\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}} = \begin{bmatrix} 26{,}370 & 9 & 465 & 41{,}257 & 377 & 2{,}768 & 193 \\ 160 & 1{,}647 & 1{,}511 & 22{,}531 & 6{,}038 & 104 & 322 \\ 579 & 857 & 50 & 3{,}273 & 5{,}887 & 13{,}734 & 2{,}676 \\ 12{,}056 & 2{,}865 & 58{,}464 & 287{,}046 & 15{,}360 & 46{,}582 & 1{,}257 \\ 5{,}172 & 1{,}462 & 17{,}314 & 59{,}830 & 36{,}984 & 23{,}082 & 3{,}256 \\ 7{,}262 & 4{,}470 & 11{,}387 & 44{,}987 & 43{,}664 & 84{,}651 & 1{,}693 \\ 193 & 191 & 697 & 8{,}906 & 4{,}453 & 5{,}013 & 532 \end{bmatrix}.$$

The aggregation matrix for the specified sectoral aggregation is $\mathbf{S} = \begin{bmatrix} 1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,1 \end{bmatrix}$ so we compute the aggregated transactions and total outputs as

$$\mathbf{Z}^* = \mathbf{SZS}' = \begin{bmatrix} 28{,}186 & 65{,}763 & 9{,}804 \\ 16{,}358 & 348{,}834 & 85{,}496 \\ 18{,}750 & 143{,}121 & 203{,}328 \end{bmatrix},\ \mathbf{x}^* = \mathbf{Sx} = \begin{bmatrix} 114{,}341 \\ 927{,}192 \\ 1{,}060{,}811 \end{bmatrix} \text{ and}$$

$$\mathbf{A}^* = \mathbf{Z}^*(\mathbf{x}^*)^{-1} = \begin{bmatrix} .247 & .071 & .009 \\ .143 & .376 & .081 \\ .164 & .154 & .192 \end{bmatrix},$$ respectively. We subsequently compute

$$\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.365 & 0.163 & 0.032 \\ 0.358 & 1.686 & 0.172 \\ 0.345 & 0.355 & 1.276 \end{bmatrix}; \ \tilde{\mathbf{x}}^* = \begin{bmatrix} 315.2 \\ 460.5 \\ 523.0 \end{bmatrix} \text{ and } \tilde{\mathbf{x}} = (\mathbf{I} - \mathbf{A})^{-1}\Delta\mathbf{f} = \begin{bmatrix} 174.5 \\ 120.9 \\ 116.5 \\ 329.7 \\ 184.6 \\ 218.3 \\ 110.1 \end{bmatrix}$$

. The vector of first order bias for individual sectors is found by $\boldsymbol{\varphi} = (\mathbf{A}^*\mathbf{S} - \mathbf{S}\mathbf{A})\Delta\mathbf{f} = \begin{bmatrix} 17.069 \\ 7.726 \\ 5.790 \end{bmatrix}$ and the total first order bias is $\mathbf{i}'\boldsymbol{\varphi} = 30.585$. The vector of the total aggregation bias for individual sectors is found by $\boldsymbol{\tau} = \tilde{\mathbf{x}}^* - \mathbf{S}\tilde{\mathbf{x}} = \begin{bmatrix} 19.751 \\ 14.326 \\ 10.025 \end{bmatrix}$ and the overall total aggregation bias is $\mathbf{i}'\boldsymbol{\tau} = 44.102$.

Problem 4.7

This problem illustrates construction of a table of consolidated national accounts in matrix form from a set of national accounting equations. Consider the following national accounting equations:

(1) $Q + M = C + I + X + G$
(2) $C + S + T = Q + D$
(3) $L + I + D + B = S$
(4) $X = M + L$
(5) $G = T + B$

where $Q =$ total consumer income payments; $M =$ purchases of imports; $C =$ total sales of consumption goods; $S =$ total consumer savings; $T =$ total taxes paid to government; $I =$ total purchases of capital goods; $D =$ total capital consumption allowance (depreciation); $L =$ net lending from overseas; $B =$ total government deficit spending; $X =$ total sales of exports; $G =$ total government purchases and the following are known: $Q = -500$, $M = 75$, $S = 60$, $T = 20$, $D = 10$, $L = -20$, and $B = 10$.

First, note that there are missing quantities C, I, X and G that are necessary to complete the table, which can be found with equations (2), (3), (4), and (5), respectively as: $C = Q + D - S - T = 410$; $I = -D + S - L - B = 80$; $X = L + M = 55$; and $G = T + B = 30$. The resulting consolidated table of national accounts represented in matrix form is the following:

	Prod.	Cons.	Cap.	ROW	Govt.	Total
Prod.		**C=410**	**I=80**	**X=55**	**G=30**	575
Cons.	Q=500		D=-10			490
Cap.		S=60				60
ROW	M=75		L=-20			55
Govt.		T=20	B=10			30
Total	575	490	60	55	30	

Problem 4.8

This problem illustrates conversion of table of national accounts to a consolidated table of supply and use input-output accounts. Consider the following table of national accounts (generated in problem 4.7).

	Prod.	Cons.	Cap.	ROW	Govt.	Total
Prod.		410	80	55	30	575
Cons.	500		-10			490
Cap.		60				60
ROW	75		-20			55
Govt.		20	10			30
Total	575	490	60	55	30	

Suppose the following tables become available providing the interindustry supply and use detail for this economy.

Use of commodities by industries:

		Industry			Total Intermed. Output
		Nat. Res.	Manuf.	Serv.	
Commodity	Agriculture	20	12	18	50
	Mining	5	30	12	47
	Manufacturing	10	13	11	34
	Services	12	17	40	69

Final uses of commodity production:

	Households	Government	Investment	Exports
Agriculture	30	6	16	5
Mining	60	9	16	17
Manufacturing	50	3	40	22
Services	70	12	8	11
Totals	210	30	80	55

Supply of commodities by industries:

		Commodity				Total Industry Output
		Agric.	Mining	Manuf.	Services	
Industry	Natural Resources	99			10	109
	Manufacturing	8	143	137	10	298
	Services		6	12	150	168
	Total Commodity Output	107	149	149	170	575

The corresponding consolidated set of supply and use accounts including the sector detail for interindustry transactions becomes the following:

		Commodities				Industries			Final Demand	Total Output
		Agric.	Mining	Manuf.	Serv.	Nat. Res.	Manuf.	Serv.		
Comm.	Agriculture					20	12	18	57	107
	Mining					5	30	12	102	149
	Manufacturing					10	13	11	115	149
	Services					12	17	40	101	170
Ind.	Natural Resources	99			10					109
	Manufacturing	8	143	137	10					298
	Services		6	12	150					168
	Value Added					62	226	87	375	
	Total Output	107	149	149	170	109	298	168		575

Problem 4.9

This problem illustrates a process of "scrubbing" a U.S. style input-output transactions table of competitive imports to yield a domestic transactions table. We define an input-output economy with $\mathbf{Z} = \begin{bmatrix} 500 & 0 & 0 \\ 50 & 300 & 150 \\ 200 & 150 & 550 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1{,}000 \\ 750 \\ 1{,}000 \end{bmatrix}$. We also know the vector of the total value of competitive imports, $\mathbf{m} = \begin{bmatrix} 150 \\ 105 \\ 210 \end{bmatrix}$. Knowing $\mathbf{m}$, we can define $\mathbf{g} = \mathbf{f} + \mathbf{m} = \begin{bmatrix} 650 & 355 & 310 \end{bmatrix}'$, which is the vector of total final demands, including imports. In some cases, the accounting is such that $\mathbf{m}$ is recorded as a negative final demand so that the vector of total outputs, $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$, reflects total domestic output, which is the convention used for the US input-output tables. Using the assumption of import similarity, we can compute the domestic transactions matrix where competitive imports are removed from interindustry transactions by the following steps. First, we compute the vectors of total final demand and intermediate outputs, $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 500 \\ 250 \\ 100 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 500 \\ 500 \\ 900 \end{bmatrix}$, respectively. The import similarity scaling factors are found for each commodity as the ratio of the value of total interindustry imports of that commodity divided by the total output (including imports), $r_i = \dfrac{m_i}{u_i + f_i}$, or $\mathbf{r} = \begin{bmatrix} .15 & .24 & .21 \end{bmatrix}'$.

We can then compute the scaled quantities $\overline{\mathbf{D}} = \mathbf{Z} - \hat{\mathbf{r}}\mathbf{Z} = \begin{bmatrix} 425 & 0 & 0 \\ 43 & 258 & 129 \\ 158 & 118.5 & 434.5 \end{bmatrix}$,

$\overline{\mathbf{M}} = \hat{\mathbf{r}}\mathbf{Z} = \begin{bmatrix} 75 & 0 & 0 \\ 7 & 42 & 21 \\ 42 & 31.5 & 115.5 \end{bmatrix}$, $\mathbf{h} = \hat{\mathbf{r}}\mathbf{f} = \begin{bmatrix} 75 \\ 35 \\ 21 \end{bmatrix}$, and $\overline{\mathbf{g}} = \mathbf{g} - \hat{\mathbf{r}}\mathbf{f} = \begin{bmatrix} 575 \\ 320 \\ 289 \end{bmatrix}$. Note that the identity $\mathbf{x} = \overline{\mathbf{D}}\mathbf{i} + \overline{\mathbf{g}}$ (analogous to $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$) still holds, but this balance equation now accounts for only domestic transactions with interindustry imports reassigned to total value added. The new total value-added vector is $\overline{\mathbf{v}}' = \mathbf{x}' - \mathbf{i}'\overline{\mathbf{D}} = \begin{bmatrix} 374 & 373.5 & 436.5 \end{bmatrix}$, which inflates the original vector of total valued added, $\mathbf{v}' = \begin{bmatrix} 250 & 300 & 300 \end{bmatrix}$ by interindustry imports to each industry, i.e., $\overline{\mathbf{m}}' = \begin{bmatrix} 124 & 73.5 & 136.5 \end{bmatrix}$, excluding the value of imports consumed directly in final demand.

If we subsequently learn that $\mathbf{M} = \begin{bmatrix} 150 & 0 & 0 \\ 25 & 50 & 30 \\ 35 & 75 & 100 \end{bmatrix}$, i.e., we know precisely the competitive imports associated with interindustry transactions, we can compute the domestic transactions matrix (rather than approximate it with import similarity scaling factors) by $\mathbf{D} = \mathbf{Z} - \mathbf{M} = \begin{bmatrix} 350 & 0 & 0 \\ 25 & 250 & 120 \\ 165 & 75 & 450 \end{bmatrix}$ and compute $\mathbf{m} = \mathbf{Mi} = \begin{bmatrix} 150 \\ 105 \\ 210 \end{bmatrix}$, and $\mathbf{g} = \mathbf{f} + \mathbf{m} = \begin{bmatrix} 650 \\ 355 \\ 310 \end{bmatrix}$ where $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 500 \\ 250 \\ 100 \end{bmatrix}$. Note that the balance equation $\mathbf{x} = \mathbf{Di} + \mathbf{g}$ holds here as well. Then the new total value added vector, $\tilde{\mathbf{v}}' = \mathbf{x}' - \mathbf{i}'\mathbf{D} = \begin{bmatrix} 460 & 425 & 430 \end{bmatrix}$, inflates the original vector of total valued added, $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 250 & 300 & 300 \end{bmatrix}$, by the total value of all imports to each industry, $\tilde{\mathbf{m}}' = \begin{bmatrix} 210 & 125 & 130 \end{bmatrix}$, i.e. $\tilde{\mathbf{m}}' = \mathbf{i}'\mathbf{M} = \tilde{\mathbf{v}}' - \mathbf{v}'$.

We compute the Leontief inverse for the first case as

$$\mathbf{L}^{I} = (\mathbf{I} - \mathbf{A}^{I})^{-1} = \begin{bmatrix} 1.739 & 0 & 0 \\ .222 & 1.613 & .368 \\ .548 & .451 & 1.871 \end{bmatrix} \text{ for } \mathbf{A}^{I} = \overline{\mathbf{D}}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .425 & 0 & 0 \\ .043 & .344 & .129 \\ .158 & .158 & .435 \end{bmatrix} \text{ and for the}$$

second case as $\mathbf{L}^{II} = (\mathbf{I} - \mathbf{A}^{II})^{-1} = \begin{bmatrix} 1.538 & 0 & 0 \\ .146 & 1.551 & .338 \\ .488 & .282 & 1.88 \end{bmatrix}$ for $\mathbf{A}^{II} = \mathbf{D}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .35 & 0 & 0 \\ .025 & .333 & .12 \\ .165 & .1 & .45 \end{bmatrix}$.

The mean absolute deviation (*mad*) between $\mathbf{A}^{I}$ and $\mathbf{A}^{II}$ is

$mad(\mathbf{A}) = (1/9)\sum_{i=1}^{3}\sum_{j=1}^{3}\left|a_{ij}^{I} - a_{ij}^{II}\right| = .0215$ and the *mad* between $\mathbf{L}^{I}$ and $\mathbf{L}^{II}$ is found to be

$mad(\mathbf{L}) = (1/9)\sum_{i=1}^{3}\sum_{j=1}^{3}\left|l_{ij}^{I} - l_{ij}^{II}\right| = .0673$.

Problem 4.10

This problem illustrates the calculation of spatial aggregation bias using the three-region, three-sector Chinese interregional model (for the year 2000) specified in problem 3.6. Using that table as a point of departure, we aggregate regions 1 and 2 and leave region 3 unaggregated to yield a two-region model.

To aggregate the North and South regions and leave the Rest of China region unaggregated, we construct the aggregation matrix $\mathbf{S} = \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$ and compute the aggregated matrix of interindustry transactions, $\mathbf{Z}^{(a)} = \mathbf{SZS}'$, and the aggregated vector of total outputs, $\mathbf{x}^{(a)} = \mathbf{Sx}$, which are shown in the following table:

China 2000		North and South			Rest of China		
		Nat. Res.	Manuf & Const.	Services	Nat. Res.	Manuf & Const.	Services
North & South	Nat. Resources	5,625	17,001	1,339	117	227	19
	Manuf. & Const.	6,902	60,554	9,203	241	1,374	325
	Services	1,920	11,225	5,017	35	186	68
Rest of China	Nat. Resources	43	305	22	1,581	3,154	293
	Manuf. & Const.	155	1,334	212	1,225	6,704	1,733
	Services	29	193	53	425	2,145	1,000
Total Chinese Output		44,517	130,816	38,678	11,661	21,107	8,910

The corresponding technical coefficients matrix and Leontief inverse, respectively, are

$$\mathbf{A}^{(a)} = \mathbf{Z}^{(a)}(\hat{\mathbf{x}}^{(a)})^{-1} = \left[\begin{array}{ccc|ccc} .126 & .13 & .035 & .01 & .011 & .002 \\ .155 & .463 & .238 & .021 & .065 & .037 \\ .043 & .086 & .13 & .003 & .009 & .008 \\ \hline .001 & .002 & .001 & .136 & .149 & .033 \\ .003 & .01 & .005 & .105 & .318 & .194 \\ .001 & .001 & .001 & .036 & .102 & .112 \end{array}\right] \text{ and}$$

$$\mathbf{L}^{(a)} = (\mathbf{I} - \mathbf{A}^{(a)})^{-1} = \left[\begin{array}{ccc|ccc} 1.207 & .315 & .135 & .031 & .062 & .032 \\ .395 & 2.055 & .580 & .093 & .252 & .149 \\ .099 & .219 & 1.213 & .018 & .046 & .03 \\ \hline .005 & .013 & .006 & 1.196 & .279 & .106 \\ .015 & .039 & .022 & .207 & 1.567 & .353 \\ .004 & .009 & .006 & .073 & .191 & 1.171 \end{array}\right].$$

To calculate the aggregation bias measured as a percent of gross outputs with a reference vector of final demands given by $\tilde{\mathbf{f}} = \begin{bmatrix} 100 & 100 & \cdots & 100 \end{bmatrix}'$ for the unaggregated model, we can specify $\tilde{\mathbf{f}} = \left[\begin{array}{ccc|ccc|ccc} 100 & 100 & 100 & 100 & 100 & 100 & 100 & 100 & 100 \end{array}\right]'$ for the unaggregated case and we can write $\tilde{\mathbf{f}}^{(a)} = \mathbf{S}\tilde{\mathbf{f}} = \left[\begin{array}{ccc|ccc} 200 & 200 & 200 & 100 & 100 & 100 \end{array}\right]'$ for the aggregated case. We can now compute $\tilde{\mathbf{x}} = \mathbf{L}\tilde{\mathbf{f}} = \left[\begin{array}{ccc|ccc|ccc} 165 & 284 & 151 & 178 & 371 & 164 & 163 & 227 & 147 \end{array}\right]'$ and $\tilde{\mathbf{x}}^{(a)} = \mathbf{L}^{(a)}\tilde{\mathbf{f}}^{(a)} = \left[\begin{array}{ccc|ccc} 344 & 655 & 316 & 163 & 228 & 147 \end{array}\right]'$ from which we can compute the aggregation bias as $100 \times \dfrac{\left|\mathbf{S}\tilde{\mathbf{x}} - \tilde{\mathbf{x}}^{(a)}\right|\mathbf{i}}{\mathbf{S}\tilde{\mathbf{x}}\mathbf{i}} = 100 \times (2.115/1{,}850.718) = 0.114$ percent.

Problem 4.11

This exercise applies the same procedure for removing competitive imports from the interindustry transactions table utilized in problem 4.9, but this time applied to real-world data for a six-sector input-output table for Nepal for the year 2013, defined by the matrix of interindustry transactions, **Z**, and vector of total outputs, **x**, in the following:

Interindustry Transactions	Agric.	Mining	Manuf.	Const.	Utilities	Services	Total Output
Agriculture	774	0	1,149	45	0	719	9,766
Mining	0	0	87	0	119	1	252
Manufacturing	1,037	22	2,029	166	1,654	1,743	13,015
Construction	47	5	109	47	79	376	834
Utilities	11	1	13	6	2	394	3,963
Services	780	22	781	201	377	3,443	20,446

This table includes both domestic transactions, **D**, and competitive imports, **M**, such that $\mathbf{Z} = \mathbf{D} + \mathbf{M}$. However, for the present, we presume we do not know the detailed transactions reported as **M** and, instead, know only $\mathbf{m} = \mathbf{Mi} = \begin{bmatrix} 68 & 48 & 3{,}227 & 65 & 1 & 457 \end{bmatrix}'$, the value of all imports of each commodity to the economy. To estimate the interindustry import transactions we assume import similarity, i.e., the imports as a fraction of interindustry activity are the same as that of the entire economy. To do this, first, we compute the vectors of intermediate outputs and total final demand, respectively, as $\mathbf{u} = \mathbf{Zi} = \begin{bmatrix} 2{,}686 & 206 & 6{,}651 & 663 & 427 & 5{,}603 \end{bmatrix}'$ and $\mathbf{f} = \mathbf{x} - \mathbf{u} = \begin{bmatrix} 7{,}080 & 45 & 6{,}363 & 171 & 3{,}536 & 14{,}843 \end{bmatrix}'$.

Knowing **m**, we can define $\mathbf{g} = \mathbf{f} + \mathbf{m} = \begin{bmatrix} 7{,}147 & 94 & 9{,}591 & 236 & 3{,}537 & 15{,}300 \end{bmatrix}'$, which is the vector of total final demands, including imports. Note that in some cases the accounting is such that **m** is recorded as a negative final demand so that $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$ reflects total domestic output, which is the convention used for the US input-output tables. Using the assumption of import similarity, we can estimate the domestic transactions matrix where competitive imports are removed from interindustry transactions by the following steps.

We can develop import similarity scaling factors for each commodity as the ratio of the value of total interindustry imports of that commodity divided by the total output (including imports), $r_i = \dfrac{m_i}{u_i + f_i}$, or $\mathbf{r} = \begin{bmatrix} 0.007 & 0.193 & 0.248 & 0.078 & 0.000 & 0.022 \end{bmatrix}'$. We can then compute the scaled quantities for imports, domestic transactions, and final demands as

$$\bar{\mathbf{M}} = \hat{\mathbf{r}}\mathbf{Z} = \begin{bmatrix} 5 & 0 & 8 & 0 & 0 & 5 \\ 0 & 0 & 17 & 0 & 23 & 0 \\ 257 & 6 & 503 & 41 & 410 & 432 \\ 4 & 0 & 8 & 4 & 6 & 29 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 17 & 0 & 17 & 4 & 8 & 77 \end{bmatrix},$$

$$\bar{\mathbf{D}} = \mathbf{Z} - \bar{\mathbf{M}} = \begin{bmatrix} 769 & 0 & 1{,}141 & 45 & 0 & 714 \\ 0 & 0 & 70 & 0 & 96 & 0 \\ 780 & 17 & 1{,}526 & 125 & 1{,}244 & 1{,}311 \\ 43 & 5 & 100 & 43 & 73 & 347 \\ 11 & 1 & 13 & 6 & 2 & 394 \\ 762 & 22 & 763 & 196 & 368 & 3{,}366 \end{bmatrix},$$

$\mathbf{h} = \hat{\mathbf{r}}\mathbf{f} = \begin{bmatrix} 49 & 9 & 1{,}578 & 13 & 1 & 332 \end{bmatrix}'$, and $\bar{\mathbf{g}} = \mathbf{g} - \mathbf{h} = \begin{bmatrix} 7{,}098 & 85 & 8{,}013 & 223 & 3{,}536 & 14{,}968 \end{bmatrix}'$.

Note that with these scaled quantities the identity $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{g}}$ (analogous to $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$) still holds., but this balance equation now accounts for only domestic transactions with interindustry imports reassigned to total value added. The new total value-added vector is $\bar{\mathbf{v}}' = \mathbf{x}' - \mathbf{i}'\bar{\mathbf{D}} = \begin{bmatrix} 7{,}401 & 208 & 9{,}402 & 419 & 2{,}180 & 14{,}314 \end{bmatrix}$, which inflates the original vector of total valued added, $\mathbf{v}' = \begin{bmatrix} 7{,}118 & 201 & 8{,}848 & 369 & 1{,}732 & 13{,}770 \end{bmatrix}$ by interindustry imports to each industry, i.e., $\bar{\mathbf{m}}' = \mathbf{i}'\bar{\mathbf{M}} = \begin{bmatrix} 283 & 6 & 554 & 50 & 448 & 544 \end{bmatrix}$, excluding the value of imports consumed directly in final demand.

If we subsequently learn that $\mathbf{M} = \begin{bmatrix} 18 & 0 & 30 & 1 & 0 & 18 \\ 0 & 0 & 20 & 0 & 28 & 0 \\ 406 & 16 & 793 & 124 & 672 & 1,216 \\ 5 & 0 & 12 & 5 & 10 & 33 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 19 & 2 & 54 & 10 & 22 & 350 \end{bmatrix}$, i.e., we know precisely the competitive imports associated with interindustry transactions, we can compute the domestic transactions matrix by $\mathbf{D} = \mathbf{Z} - \mathbf{M} = \begin{bmatrix} 756 & 0 & 1,118 & 44 & 0 & 701 \\ 0 & 0 & 67 & 0 & 91 & 0 \\ 631 & 7 & 1,236 & 42 & 982 & 527 \\ 42 & 5 & 97 & 42 & 69 & 344 \\ 11 & 1 & 13 & 6 & 2 & 393 \\ 760 & 20 & 727 & 190 & 355 & 3,093 \end{bmatrix}$ (rather than approximate it with import similarity scaling factors) and compute $\mathbf{m} = \mathbf{Mi} = \begin{bmatrix} 68 & 48 & 3,227 & 65 & 1 & 457 \end{bmatrix}'$ and $\mathbf{g} = \mathbf{f} - \mathbf{m} = \begin{bmatrix} 7,012 & -3 & 3,136 & 106 & 3,535 & 14,386 \end{bmatrix}'$ where $\mathbf{f} = \mathbf{x} - \mathbf{u} = \begin{bmatrix} 7,080 & 45 & 6,363 & 171 & 3,536 & 14,843 \end{bmatrix}'$, as earlier. Note that the balance equation $\mathbf{x} = \mathbf{Di} + \mathbf{g}$ holds here as well. Then the new total value-added vector, $\tilde{\mathbf{v}}' = \mathbf{x}' - \mathbf{i}'\mathbf{D} = \begin{bmatrix} 7,565 & 219 & 9,757 & 510 & 2,465 & 15,388 \end{bmatrix}$, inflates the original vector of total valued added, $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 7,118 & 201 & 8,848 & 369 & 1,732 & 13,770 \end{bmatrix}$, by the total value of all imports to each industry, $\tilde{\mathbf{m}}' = \begin{bmatrix} 447 & 18 & 909 & 141 & 733 & 1,618 \end{bmatrix}$, i.e., $\tilde{\mathbf{m}}' = \mathbf{i}'\mathbf{M} = \tilde{\mathbf{v}}' - \mathbf{v}'$.

We compute the Leontief inverse for the first case as

$$\mathbf{L}^{I} = (\mathbf{I} - \mathbf{A}^{I})^{-1} = \begin{bmatrix} 1.1007 & 0.0145 & 0.1141 & 0.0947 & 0.0433 & 0.0577 \\ 0.0007 & 1.0006 & 0.0063 & 0.0015 & 0.0264 & 0.0012 \\ 0.1105 & 0.0909 & 1.1537 & 0.2180 & 0.3784 & 0.1063 \\ 0.0081 & 0.0234 & 0.0118 & 1.0632 & 0.0260 & 0.0234 \\ 0.0036 & 0.0047 & 0.0033 & 0.0146 & 1.0041 & 0.0238 \\ 0.1134 & 0.1175 & 0.0960 & 0.3253 & 0.1524 & 1.2193 \end{bmatrix}$$ for

$$\mathbf{A}^{I} = \bar{\mathbf{D}}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.0787 & 0.0001 & 0.0876 & 0.0534 & 0.0000 & 0.0349 \\ 0.0000 & 0.0000 & 0.0054 & 0.0000 & 0.0243 & 0.0000 \\ 0.0798 & 0.0665 & 0.1172 & 0.1499 & 0.3139 & 0.0641 \\ 0.0044 & 0.0193 & 0.0077 & 0.0520 & 0.0184 & 0.0170 \\ 0.0011 & 0.0021 & 0.0010 & 0.0075 & 0.0005 & 0.0193 \\ 0.0780 & 0.0857 & 0.0587 & 0.2352 & 0.0930 & 0.1646 \end{bmatrix}$$ and for the second case

as $$\mathbf{L}^{II} = (\mathbf{I} - \mathbf{A}^{II}) = \begin{bmatrix} 1.0961 & 0.0084 & 0.1079 & 0.0786 & 0.0328 & 0.0498 \\ 0.0005 & 1.0003 & 0.0058 & 0.0007 & 0.0245 & 0.0008 \\ 0.0828 & 0.0346 & 1.1167 & 0.0771 & 0.2829 & 0.0452 \\ 0.0076 & 0.0228 & 0.0110 & 1.0590 & 0.0236 & 0.0221 \\ 0.0035 & 0.0042 & 0.0030 & 0.0137 & 1.0036 & 0.0233 \\ 0.1085 & 0.1051 & 0.0872 & 0.2984 & 0.1362 & 1.1943 \end{bmatrix}$$ for

$$\mathbf{A}^{II} = \mathbf{D}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.0774 & 0.0000 & 0.0859 & 0.0526 & 0.0000 & 0.0343 \\ 0.0000 & 0.0000 & 0.0051 & 0.0000 & 0.0230 & 0.0000 \\ 0.0646 & 0.0259 & 0.0949 & 0.0506 & 0.2477 & 0.0258 \\ 0.0043 & 0.0195 & 0.0075 & 0.0499 & 0.0174 & 0.0168 \\ 0.0011 & 0.0020 & 0.0010 & 0.0074 & 0.0004 & 0.0192 \\ 0.0779 & 0.0810 & 0.0559 & 0.2281 & 0.0896 & 0.1513 \end{bmatrix}.$$

The mean absolute deviations between $\mathbf{L}^{I}$ and $\mathbf{L}^{II}$ and between $\mathbf{A}^{I}$ and $\mathbf{A}^{II}$ are found to be $mad(\mathbf{L}) = (1/36)\sum_{i=1}^{6}\sum_{j=1}^{6}\left|l_{ij}^{I} - l_{ij}^{II}\right| = .016$ and $mad(\mathbf{A}) = (1/36)\sum_{i=1}^{6}\sum_{j=1}^{6}\left|a_{ij}^{I} - a_{ij}^{II}\right| = .009$, respectively.

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Chapter 5, The Commodity-by-Industry Approach in Input–Output Models

Chapter 5 explores variations to the commodity-by-industry input–output framework introduced in Chapter 4, expanding the basic input–output framework to include distinguishing between commodities and industries, i.e., the supply of specific commodities in the economy and the use of those commodities by collections of businesses defined as industries. The chapter introduces the fundamental commodity-by-industry accounting relationships and how they relate to the basic input–output framework. Alternative assumptions are defined for handling the common accounting issue of secondary production, and economic interpretations of those alternative assumptions are presented. The formulations of commodity-driven and industry-driven models are also presented along with illustrations of variants on combining alternative assumptions for secondary production. Finally, the chapter illustrates the problems encountered with commodity-by-industry models, such as nonsquare commodity–industry systems, mixed technology options or the interpretation of negative elements. The exercise problems for this chapter illustrate key features of commodity-by-industry accounts and their applications.

Problem 5.1

This problem illustrates the basic configuration a commodity by industry model using make and use matrices. For a system of commodity-by-industry accounts, suppose we have defined three commodities and two industries.

The use matrix, **U**, and the make matrix, **V**, and are the following:

$$\mathbf{U} = \begin{bmatrix} 3 & 5 \\ 2 & 7 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 15 & 5 & 10 \\ 5 & 25 & 0 \end{bmatrix}$$

From these matrices we can compute the vector of commodity final demands, **e**, the vector of industry value added inputs, $\mathbf{v}'$, the vector of total commodity outputs, **q**, and the vector of total industry outputs, **x**, as the following:

$$\mathbf{x} = \mathbf{Vi} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}, \quad \mathbf{q} = (\mathbf{i}'\mathbf{V})' = \begin{bmatrix} 20 \\ 30 \\ 10 \end{bmatrix}, \quad \mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{U} = \begin{bmatrix} 23 & 15 \end{bmatrix}, \text{ and } \mathbf{e} = \mathbf{x} - \mathbf{Ui} = \begin{bmatrix} 12 \\ 21 \\ 5 \end{bmatrix}$$

The matrix of commodity-by-industry direct requirements is $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .1 & .167 \\ .067 & .233 \\ .067 & .1 \end{bmatrix}$ and the matrix of commodity output proportions is $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} .75 & 0.167 & 1 \\ .25 & 0.833 & 0 \end{bmatrix}$.

Among the various configurations for total requirements matrices, as an example, if we assume a fixed commodity sales structure, the industry-by-industry total requirements matrix is found by $(\mathbf{I} - \mathbf{DB})^{-1}\mathbf{D} = \begin{bmatrix} 1.021 & .555 & 1.22 \\ .435 & 1.149 & 0.129 \end{bmatrix}$.

Problem 5.2

This problem illustrates commodity-by-industry total requirements matrices under alternative assumptions of industry-based and commodity-based technology. Consider the following system of commodity and industry accounts for a region:

		Commodities		Industries		Final Demand	Total Output
		1	2	1	2		
Commodities	1			1	2	7	10
	2			3	4	3	10
Industries	1	10	2				12
	2	0	8				8
Value Added				8	2	10	
Total Inputs		10	10	12	8		

From this table, the use matrix is $\mathbf{U} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; the make matrix is $\mathbf{V} = \begin{bmatrix} 10 & 2 \\ 0 & 8 \end{bmatrix}$; the vector of commodity final demands is $\mathbf{e} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$; the vector of total commodity output is $\mathbf{q} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$; the vector of total industry output is $\mathbf{x} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$; and the vector of industry value added is $\mathbf{v}' = \begin{bmatrix} 8 & 2 \end{bmatrix}$. We these definitions we can compute the commodity-by-industry matrix of direct requirements as $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .083 & .25 \\ .25 & .5 \end{bmatrix}$; the commodity output proportions matrix as $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} 1.0 & .2 \\ 0 & .8 \end{bmatrix}$; and the industry output proportions matrix as $\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .833 & 0 \\ .167 & 1 \end{bmatrix}$.

As one common variant, the industry-based technology, industry-demand-driven, commodity-by-industry total requirements matrix is $(\mathbf{I} - \mathbf{DB})^{-1}\mathbf{D} = \begin{bmatrix} 1.333 & 0.889 \\ .444 & 1.63 \end{bmatrix}$. As another variant, the commodity-based technology, industry-demand drive, commodity-by-industry total requirements matrix is $(\mathbf{I} - \mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{C}^{-1} = \begin{bmatrix} 1.412 & .706 \\ .235 & 2.118 \end{bmatrix}$ under the fixed industry sale structure assumption, $\Delta\mathbf{x} = \begin{bmatrix} 14.11 \\ 10.95 \end{bmatrix}$, and under a commodity-based technology assumption, $\Delta\mathbf{x} = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. These are different because the accounting of secondary production is different in the two assumptions.

Problem 5.3

This problem illustrates the adoption of mixed technology assumptions in construction commodity by industry models. Consider again the system of accounts given in problem 5.1:

$\mathbf{U} = \begin{bmatrix} 3 & 5 \\ 2 & 7 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} 15 & 5 & 10 \\ 5 & 25 & 0 \end{bmatrix}$ so that $\mathbf{x} = \mathbf{Vi} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$ v and we can compute the industry input requirements matrix as $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1/30 & 0 \\ 0 & 1/30 \end{bmatrix} = \begin{bmatrix} .1 & .167 \\ .067 & .233 \\ .067 & .1 \end{bmatrix}$.

Suppose we can split the make matrix, $\mathbf{V} = \begin{bmatrix} 15 & 5 & 10 \\ 5 & 25 & 0 \end{bmatrix}$ into two components, $\mathbf{V}_1 = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 0 \end{bmatrix}$ and $\mathbf{V}_2 = \begin{bmatrix} 10 & 0 & 5 \\ 0 & 20 & 0 \end{bmatrix}$ such that $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$. This means that $\mathbf{x}_1 = \mathbf{V}_1\mathbf{i} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$ and $\mathbf{q}_1 = \mathbf{iV} = \begin{bmatrix} 10 \\ 10 \\ 5 \end{bmatrix}$.

We might be interested in comparing the two "mixed technology" assumptions that were covered in sections 5.7.1 and 5.7.2 in computing the industry-by-commodity total requirements matrix for this system of accounts. However, since **V** is nonsquare, the matrix of industry output proportions, **C**, will be nonsquare and hence no unique $\mathbf{C}^{-1}$ exists.

Since no unique $\mathbf{C}^{-1}$ exists we cannot use the mixed-technology assumption requiring computation of $\mathbf{C}^{-1}$; that is, we cannot determine either $(\mathbf{I} - \mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{C}^{-1}$ or $\mathbf{R}(\mathbf{I} - \mathbf{BR})^{-1}$ where $\mathbf{R} = [\mathbf{C}_1^{-1}(\mathbf{I} - \langle \mathbf{D}_2' \rangle) + \mathbf{D}_2]$. Nonetheless, we can use the industry-based technology assumption with $\mathbf{T} = [(\mathbf{I} + \mathbf{D}_1\mathbf{C}_2 - \langle \mathbf{i}'\mathbf{C}_2 \rangle)^{-1}\mathbf{D}_1]$ where $\mathbf{D}_1 = \mathbf{V}_1\hat{\mathbf{q}}_1^{-1} = \begin{bmatrix} .5 & .5 & 1 \\ .5 & .5 & 0 \end{bmatrix}$ and $\mathbf{C}_2 = \mathbf{V}_2'\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .333 & 0 \\ 0 & .667 \\ .137 & 0 \end{bmatrix}$, which in this case is $\mathbf{T} = \begin{bmatrix} .333 & .333 & 1.333 \\ .667 & .667 & -.333 \end{bmatrix}$ and then we compute the matrix of total requirements as $\mathbf{T}(\mathbf{I} - \mathbf{BT})^{-1} = \begin{bmatrix} .685 & .685 & 1.476 \\ .949 & .949 & -.264 \end{bmatrix}$. Note the negative element in this matrix of total requirements, the implications of which are discussed in Section 5.5 of the text.

Problem 5.4

This problem explores further the use mixed technology assumptions in deriving industry-by-commodity total requirements matrices, this time for the system of accounts given in problem 5.2. In this case the make matrix, **V**, is a square matrix so it is possible to compute the inverse of matrix of industry output proportions, **C**. First, we split $\mathbf{V} = \begin{bmatrix} 10 & 2 \\ 0 & 8 \end{bmatrix}$ into two components,

$\mathbf{V}_1 = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{V}_2 = \begin{bmatrix} 0 & 2 \\ 0 & 6 \end{bmatrix}$ such that $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$. The vectors of industry and commodity outputs are then found by $\mathbf{x}_1 = \mathbf{V}_1\mathbf{i} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, $\mathbf{q}_1 = \mathbf{i}'\mathbf{V}_1 = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$, respectively.

With this configuration, we can compute $\mathbf{C}_1 = \mathbf{V}_1'(\hat{\mathbf{x}}_1)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{D}_2 = \mathbf{V}_2(\hat{\mathbf{q}})^{-1} = \begin{bmatrix} 0 & .2 \\ 0 & .6 \end{bmatrix}$, and, subsequently, $\mathbf{C}_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. From **V**, we can compute $\mathbf{q} = (\mathbf{i}'\mathbf{V})' = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ and $\mathbf{x} = \mathbf{V}\mathbf{i} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$, and, subsequently, one variant of direct requirements using mixed technology assumptions as $\mathbf{R} = \mathbf{C}_1^{-1}(\mathbf{I} - \langle \mathbf{D}_2'\mathbf{i} \rangle) + \mathbf{D}_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 0.8 \end{bmatrix}$ and $\mathbf{R}(\mathbf{I} - \mathbf{B}\mathbf{R})^{-1} = \begin{bmatrix} 1.333 & .889 \\ .444 & 1.630 \end{bmatrix}$. As another variant of direct requirements using mixed technology assumptions, **T**, we first compute $\mathbf{D}_1 = \mathbf{V}_1(\hat{\mathbf{q}}_1)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{C}_2 = \mathbf{V}_2'(\hat{\mathbf{x}})^{-1} = \begin{bmatrix} 0 & 0 \\ 0.167 & 0.750 \end{bmatrix}$, so we can compute $\mathbf{T} = (\mathbf{I} + \mathbf{D}_1\mathbf{C}_2 - \langle \mathbf{C}_2'\mathbf{i} \rangle)^{-1}\mathbf{D}_1 = \begin{bmatrix} 1.2 & 0 \\ -0.2 & 1 \end{bmatrix}$ and $\mathbf{T}(\mathbf{I} - \mathbf{B}\mathbf{T})^{-1} = \begin{bmatrix} 1.412 & .706 \\ .235 & 2.118 \end{bmatrix}$.

Of course, there are alternative partitions of the **V** matrix into its $\mathbf{V}_1$ and $\mathbf{V}_2$ components with the requirement that $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, depending upon the suitable assumptions for industries and commodities in the economy.

Problem 5.5

In this problem we explore further the characteristics technology assumptions of commodity-by-industry models. In a system of commodity-by-industry accounts, suppose we have defined four commodities and three industries. The make matrix, **V**, and the use matrix, **U**, are given as $\mathbf{U} = \begin{bmatrix} 20 & 12 & 18 \\ 5 & 30 & 12 \\ 10 & 13 & 11 \\ 12 & 17 & 40 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} 99 & 0 & 0 & 10 \\ 8 & 143 & 137 & 10 \\ 0 & 6 & 12 & 150 \end{bmatrix}$. We can compute vectors of total commodity outputs and total industry outputs, respectively, as $\mathbf{q} = \mathbf{V}'\mathbf{i} = \begin{bmatrix} 107 \\ 149 \\ 149 \\ 170 \end{bmatrix}$ and $\mathbf{x} = \mathbf{V}\mathbf{i} = \begin{bmatrix} 109 \\ 298 \\ 168 \end{bmatrix}$.

Recall that the commodity-by-industry total requirements matrix with the assumption of industry-based technology is $\mathbf{D}^{-1}(\mathbf{I} - \mathbf{B}\mathbf{D})^{-1}$. In this case since there are more commodities than industries, the matrix **D** is non-square, hence, $\mathbf{D}^{-1}$ does not exist so it is impossible to compute $\mathbf{D}^{-1}(\mathbf{I} - \mathbf{B}\mathbf{D})^{-1}$.

For industry-by-commodity total requirements using the assumption of industry-based technology for commodity-driven final demand, we can compute:

$$\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1} = \begin{bmatrix} 1.164 & .077 & .082 & .25 \\ .321 & 1.159 & 1.122 & .321 \\ .182 & .148 & .197 & .1.187 \end{bmatrix}.$$

To illustrate mixed technology assumptions, we aggregate the first two commodities to one in the make and use matrices. Hence, we have $\mathbf{U} = \begin{bmatrix} 25 & 42 & 30 \\ 10 & 13 & 11 \\ 12 & 17 & 40 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} 99 & 0 & 10 \\ 151 & 137 & 10 \\ 6 & 12 & 150 \end{bmatrix}$. We can compute vectors of total commodity outputs and total industry outputs, respectively, as $\mathbf{q} = \mathbf{V}'\mathbf{i} = \begin{bmatrix} 256 \\ 149 \\ 170 \end{bmatrix}$ and $\mathbf{x} = \mathbf{Vi} = \begin{bmatrix} 109 \\ 298 \\ 168 \end{bmatrix}$. We assume that $\mathbf{V}$ can be decomposed into $\mathbf{V}_1$ and $\mathbf{V}_2$ where $\mathbf{V}_1 = \begin{bmatrix} 99 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 30 \end{bmatrix}$, so we can compute $\mathbf{V}_2 = \mathbf{V} - \mathbf{V}_1 = \begin{bmatrix} 0 & 0 & 10 \\ 151 & 127 & 10 \\ 6 & 12 & 120 \end{bmatrix}$. With these definitions we can compute the commodity-by-industry matrix of direct requirements as $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .229 & .141 & .179 \\ .092 & .044 & .066 \\ .110 & .057 & .238 \end{bmatrix}$ and the commodity output proportions matrix as $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} .387 & 0 & .059 \\ .590 & .919 & .059 \\ .023 & .081 & .882 \end{bmatrix}$. If we assume a commodity-based technology for $\mathbf{V}_1$ and an industry-based technology for $\mathbf{V}_2$, the four total requirements matrices (i.e., commodity-by-commodity, industry-by-commodity, commodity-by-industry and industry-by-industry) to be used with commodity-driven demand calculations are found by first computing $\mathbf{x}_1 = \mathbf{V}_1\mathbf{i} = \begin{bmatrix} 99 \\ 10 \\ 30 \end{bmatrix}$. Also, in this case, since $\mathbf{V}_1$ is diagonal, we can easily find

$$\mathbf{x}_1 = \mathbf{V}_1\mathbf{i} = \mathbf{q}_1 = \mathbf{V}_1'\mathbf{i} = \begin{bmatrix} 99 \\ 10 \\ 30 \end{bmatrix}.$$

From these quantities we can compute $\mathbf{C}_1 = \mathbf{V}_1'\hat{\mathbf{x}}_1^{-1} = \mathbf{C}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and, subsequently,

$\mathbf{D}_2 = \mathbf{V}_2(\hat{\mathbf{q}})^{-1} = \begin{bmatrix} 0 & 0 & .059 \\ .59 & .852 & .059 \\ .023 & .081 & .706 \end{bmatrix}$ so that $\mathbf{R} = \mathbf{C}_1^{-1}\left[\mathbf{I} - \langle \mathbf{D}_2'\mathbf{i} \rangle\right] + \mathbf{D}_2 = \begin{bmatrix} .387 & 0 & .059 \\ .59 & .925 & .059 \\ .023 & .081 & .882 \end{bmatrix}$ and

$\mathbf{R}^{-1} = \begin{bmatrix} 2.573 & .015 & -.173 \\ -1.656 & 1.084 & .038 \\ .083 & -.099 & 1.134 \end{bmatrix}$. (Note the negative elements in $\mathbf{R}^{-1}$). We can now compute the family of total requirements matrices as

$$(\mathbf{I} - \mathbf{BR})^{-1} = \begin{bmatrix} 1.260 & .213 & .308 \\ .093 & 1.070 & .111 \\ .141 & .121 & 1.324 \end{bmatrix}, \ (\mathbf{I} - \mathbf{BR})^{-1}\mathbf{R}^{-1} = \begin{bmatrix} 2.916 & .22 & .14 \\ -1.524 & 1.15 & .15 \\ .272 & .001 & 1.483 \end{bmatrix},$$

$$\mathbf{R}(\mathbf{I} - \mathbf{BR})^{-1} = \begin{bmatrix} .496 & .09 & .197 \\ .837 & 1.117 & .362 \\ .161 & .198 & 1.185 \end{bmatrix} \text{ and } (\mathbf{I} - \mathbf{RB})^{-1} = \begin{bmatrix} 1.144 & .085 & .141 \\ .334 & 1.187 & .309 \\ .186 & .099 & 1.324 \end{bmatrix}.$$

Problem 5.6

This problem explores further the properties of mixed technology assumptions for commodity-by-industry total requirements matrices. Recall first from the numerical results in section 5.7.3 for the examples provided that the column sums of both the mixed technology direct requirements matrices, **R** and **T**, are one, i.e., $\mathbf{i}'\mathbf{R} = \mathbf{i}'\mathbf{T}$. We can show that such is generally the case for **C**, **D**, **R**, and **T**.

We start with the matrix of industry output proportions, $\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1}$. The column sums of **C** are found by premultiplying **C** by $\mathbf{i}'$, so $\mathbf{i}'\mathbf{C} = \mathbf{i}'\mathbf{V}'\hat{\mathbf{x}}^{-1}$. Since for any pair of matrices, **A** and **B**, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$, we rewrite this as $\mathbf{i}'\mathbf{C} = \mathbf{i}'\mathbf{V}'\hat{\mathbf{x}}^{-1} = (\mathbf{Vi})'\hat{\mathbf{x}}^{-1}$, and substituting $\mathbf{x} = \mathbf{Vi}$ yields $\mathbf{i}'\mathbf{C} = \mathbf{x}'\hat{\mathbf{x}}^{-1} = \mathbf{i}'$ which proves the case generally for **C**.

Similarly, if we start with the matrix of commodity output proportions, $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$, the column sums of **D** are found with $\mathbf{i}'\mathbf{D} = \mathbf{i}'\mathbf{V}'\hat{\mathbf{q}}^{-1}$, using the same property of the transpose of a product of matrices as in the previous case, yields $\mathbf{i'D} = (\mathbf{Vi})'\hat{\mathbf{q}}^{-1}$ and substituting $\mathbf{q} = \mathbf{Vi}$ yields $\mathbf{i'D} = \mathbf{q}'\hat{\mathbf{q}}^{-1} = \mathbf{i}'$ which proves the case generally for **D**.

One variant of mixed technology began with the identity, $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{C}_1^{-1}\mathbf{q}_1 + \mathbf{D}_2\mathbf{q}$ developed in section 5.7.1 but expressing $\mathbf{q}_1$ as a function of **q** to yield $\mathbf{x} = \mathbf{Rq} = [\mathbf{C}_1^{-1}(\mathbf{I} - \langle \mathbf{D}_2'\mathbf{i}\rangle) + \mathbf{D}_2]\mathbf{q}$. Another variant, derived in section 5.7.2, again expressing **x** as a function of **q** was $\mathbf{x} = \mathbf{Tq} = [(\mathbf{I} + \mathbf{D}_1\mathbf{C}_2 - \langle \mathbf{i'C}_2\rangle)^{-1}\mathbf{D}_1]\mathbf{q}$. Applying property of the transpose of a product of matrices once again, this time on the term, $\langle \mathbf{D}_2'\mathbf{i}\rangle$, yields $\mathbf{i'R} = \mathbf{i'C}_1^{-1} - \mathbf{i'C}_1^{-1}\mathbf{i'D}_2 + \mathbf{i'D}_2$. Finally, substituting $\mathbf{C}_1 = (\mathbf{V}_1')(\hat{\mathbf{x}}_1)^{-1}$ or $\mathbf{C}_1^{-1} = \hat{\mathbf{x}}_1(\mathbf{V}_1')^{-1}$ and $\mathbf{D}_2 = \mathbf{V}_2\hat{\mathbf{q}}^{-1}$ yields $\mathbf{i'R} = \mathbf{i}'\hat{\mathbf{x}}_1(\mathbf{V}_1')^{-1} - \mathbf{i}'\hat{\mathbf{x}}_1(\mathbf{V}_1')^{-1}\mathbf{i'V}_2\hat{\mathbf{q}}_2^{-1} + \mathbf{i'V}_2\hat{\mathbf{q}}_2^{-1}$ and it follows directly that $\mathbf{i'C}_1 = \mathbf{i}'$, $\mathbf{i'R} = \mathbf{i}'$, $\mathbf{i'D}_1 = \mathbf{i}'$, and $\mathbf{i'T} = \mathbf{i}'$.

Problem 5.7

This problem explores the use of total industry-by-commodity requirements under an assumption of industry-based technology for impact analysis. Consider the following make and use matrices.

$$\mathbf{U} = \begin{bmatrix} 20 & 12 & 18 \\ 5 & 30 & 12 \\ 10 & 13 & 11 \\ 12 & 17 & 40 \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} 99 & 0 & 0 & 10 \\ 8 & 143 & 137 & 10 \\ 0 & 6 & 12 & 150 \end{bmatrix}. \text{ We compute } \mathbf{q} = \mathbf{V'i} = \begin{bmatrix} 107 \\ 149 \\ 149 \\ 170 \end{bmatrix} \text{ and}$$

$$\mathbf{x} = \mathbf{Vi} = \begin{bmatrix} 109 \\ 298 \\ 168 \end{bmatrix} \text{ so } \mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .183 & .040 & .107 \\ .046 & .101 & .071 \\ .092 & .044 & .065 \\ .110 & .057 & .238 \end{bmatrix} \text{ and } \mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} .925 & 0 & 0 & .059 \\ .075 & .960 & .919 & .059 \\ 0 & .040 & .081 & .882 \end{bmatrix}.$$

We assume that the three industries are: Agriculture, Oil Production, and Manufacturing and the four commodities are Agricultural Products, Crude Oil, Natural Gas, and Manufactured Products. We can interpret this as meaning in this case that natural gas is considered a secondary product of the oil industry.

To compute the levels of oil and natural gas industry production necessary to support a final demand of 100 manufactured products, first generate the total industry-by-commodity requirements using an industry-based technology: $\mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1} = \begin{bmatrix} 1.164 & .078 & .082 & .25 \\ .321 & 1.160 & 1.122 & .321 \\ .182 & .148 & .197 & 1.187 \end{bmatrix}$.

For the final demand of 100 for manufactured products, $\Delta\mathbf{f} = \begin{bmatrix} 0 & 0 & 0 & 100 \end{bmatrix}'$, we have

$$\Delta\mathbf{x} = \mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1}\Delta\mathbf{f} = \begin{bmatrix} 25.02 \\ 32.08 \\ 118.65 \end{bmatrix}.$$

Problem 5.8

This problem explores the issues of using commodity-by-commodity models with commodity-based technology. Consider the following make and use matrices: $\mathbf{U} = \begin{bmatrix} 20 & 15 & 18 \\ 5 & 30 & 12 \\ 10 & 16 & 11 \end{bmatrix}$ and

$\mathbf{V} = \begin{bmatrix} 30 & 0 & 0 \\ 10 & 50 & 35 \\ 0 & 25 & 150 \end{bmatrix}$. First, we compute $\mathbf{q} = \mathbf{i}'\mathbf{V} = \begin{bmatrix} 40 \\ 75 \\ 185 \end{bmatrix}$ and then

$\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} .75 & 0 & 0 \\ .25 & .667 & .189 \\ 0 & .333 & .811 \end{bmatrix}$.

A standard calculation for producing the commodity-by-commodity transactions matrix with commodity-base technology begins with the matrix of technical requirements, $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{BC}^{-1} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\mathbf{V}'\hat{\mathbf{x}}^{-1}]^{-1} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\hat{\mathbf{x}}(\mathbf{V}')^{-1}] = \mathbf{U}(\mathbf{V}')^{-1}$. To express in terms of inter-commodity transactions, we postmultiply through by $\hat{\mathbf{q}}$ to obtain $\mathbf{Z}_C = \mathbf{A}_C\hat{\mathbf{q}} = \mathbf{U}[(\mathbf{V}')^{-1}\hat{\mathbf{q}}]$.

Recall that the definition of $\mathbf{D}$ is $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$. Since the transpose of a product of matrices is the reverse product of the transposes of each matrix and that the transpose of a diagonal matrix is the original matrix itself, we can rewrite this definition as $\hat{\mathbf{q}}^{-1}\mathbf{V}' = \mathbf{D}'$. Further, since the inverse of a product of matrices is the reverse product of the inverses of each matrix, this equation becomes $(\mathbf{V}')^{-1}\hat{\mathbf{q}} = (\mathbf{D}')^{-1}$ which we can substitute in the equation defining $\mathbf{Z}_C$ above so that $\mathbf{Z}_C = \mathbf{A}_C\hat{\mathbf{q}} = \mathbf{U}[(\mathbf{V}')^{-1}\hat{\mathbf{q}}] = \mathbf{U}(\mathbf{D}')^{-1}$.

For this case, the commodity-by-commodity matrix of interindustry transactions is $\mathbf{Z}_C = \mathbf{U}(\mathbf{D}')^{-1} = \begin{bmatrix} 26.667 & 7.019 & 19.315 \\ 6.667 & 43.359 & -3.025 \\ 13.333 & 17.151 & 6.516 \end{bmatrix}$. Note that there is a negative element. So, we can apply the Almon purifying algorithm (Appendix 5.2 in the text) which iteratively distributes negative elements across positive elements to remove them while preserving the essential accounting identities. The result is a "purified" non-negative transactions matrix:

$\tilde{\mathbf{Z}}_C = \mathbf{U}(\mathbf{D}')^{-1} = \begin{bmatrix} 26.667 & 7.019 & 19.315 \\ 6.667 & 40.334 & 0 \\ 13.333 & 17.151 & 6.516 \end{bmatrix}$.

Problem 5.9

This problem explores application of an industry-based technology commodity-by-industry model using the use and make matrices for highly aggregated U.S. input-output tables for 2003. The following are the use and make tables:

US Use Table for 2003	1	2	3	4	5	6	7
1. Agriculture	61,946	1	1,270	147,559	231	18,453	2,093
2. Mining	441	33,299	6,927	174,235	89,246	1,058	11,507
3. Construction	942	47	1,278	8,128	10,047	65,053	48,460
4. Manufacturing	47,511	22,931	265,115	1,249,629	132,673	516,730	226,689
5. Trade, Transport & Utils	24,325	13,211	100,510	382,630	190,185	297,537	123,523
6. Services	25,765	42,276	147,876	509,084	490,982	2,587,543	442,674
7. Other	239	1,349	2,039	48,835	35,110	83,322	36,277

US Make Table for 2003	1	2	3	4	5	6	7
1. Agriculture	273,244	-	-	67	-	1,748	-
2. Mining	-	232,387	-	10,843	-	-	-
3. Construction	-	-	1,063,285	-	-	-	-
4. Manufacturing	-	-	-	3,856,583	-	30,555	3,278
5. Trade, Transport & Utils	-	570	-	-	2,855,126	41	957
6. Services	-	475	-	-	133	9,136,001	3,278
7. Other	3,359	896	-	3,936	104,957	323,996	1,827,119

Below is a table providing the detail of the components of total commodity final demand. Note that the total final demand entry for mining is negative due to a negative trade balance, i.e., the value of net exports (exports minus imports) is negative and is sufficiently large to offset other components of final demand to render total final demand negative.

Commodity Final Demands for U.S. 2003 Input-Output Tables

Commodity\Final Demand	Personal consumption expenditures	Private fixed investment	Change in private inventories	Exports of goods and services	Imports of goods and services	Government consumption expenditures and gross investment	Total Final Demand
Agriculture	47,922	-	175	24,859	(26,769)	(1,136)	45,050
Mining	72	35,698	1,912	4,739	(125,508)	702	(82,384)
Construction	-	704,792	-	71	-	224,468	929,331
Manufacturing	1,301,616	573,197	8,983	506,780	(1,075,128)	94,705	1,410,152
Trade, Transportation & Utili	1,549,792	125,271	2,994	131,884	8,065	10,289	1,828,294
Services	4,780,516	303,426	461	175,546	(44,060)	30,256	5,246,145
Other	80,963	(75,404)	(15,748)	98,989	(177,578)	1,716,238	1,627,459
Total	7,760,881	1,666,980	(1,224)	942,868	(1,440,979)	2,075,522	11,004,047

Suppose that the value for total imports of manufactured goods is projected to increase by \$1 trillion from its 2003 value with, for simplicity, all other elements of total final demand remaining identical to those for 2003. To compute the impact on gross national product and on total output of all sectors of the economy, we first observe that if net exports are reduced by a rise in imports of \$1 trillion, then final demand is reduced by the same amount and, all other values remaining constant, so GDP is also reduced by the same amount.

To estimate the vector total outputs, we must first determine the commodity-by-industry input matrix, **B**, and the commodity output proportions matrix, **D**, to specify the industry-based technology, commodity-by-industry total requirements matrix, $\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}$:

$$\mathbf{B} = \begin{bmatrix} 0.225 & 0.000 & 0.001 & 0.038 & 0.000 & 0.002 & 0.001 \\ 0.002 & 0.137 & 0.007 & 0.045 & 0.031 & 0.000 & 0.005 \\ 0.003 & 0.000 & 0.001 & 0.002 & 0.004 & 0.007 & 0.021 \\ 0.173 & 0.094 & 0.249 & 0.321 & 0.046 & 0.057 & 0.100 \\ 0.088 & 0.054 & 0.095 & 0.098 & 0.067 & 0.033 & 0.055 \\ 0.094 & 0.174 & 0.139 & 0.131 & 0.172 & 0.283 & 0.196 \\ 0.001 & 0.006 & 0.002 & 0.013 & 0.012 & 0.009 & 0.016 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0.988 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.992 & 0.000 & 0.003 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.996 & 0.000 & 0.003 & 0.002 \\ 0.000 & 0.002 & 0.000 & 0.000 & 0.964 & 0.000 & 0.001 \\ 0.000 & 0.002 & 0.000 & 0.000 & 0.000 & 0.962 & 0.002 \\ 0.012 & 0.004 & 0.000 & 0.001 & 0.035 & 0.034 & 0.996 \end{bmatrix}$$

$$\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1} = \begin{bmatrix} 1.290 & 0.011 & 0.023 & 0.076 & 0.007 & 0.011 & 0.012 \\ 0.029 & 1.163 & 0.036 & 0.092 & 0.045 & 0.011 & 0.021 \\ 0.009 & 0.004 & 1.006 & 0.008 & 0.008 & 0.012 & 0.025 \\ 0.377 & 0.207 & 0.421 & 1.551 & 0.118 & 0.145 & 0.206 \\ 0.170 & 0.104 & 0.157 & 0.185 & 1.060 & 0.068 & 0.096 \\ 0.284 & 0.341 & 0.313 & 0.355 & 0.292 & 1.387 & 0.339 \\ 0.044 & 0.035 & 0.030 & 0.048 & 0.068 & 0.068 & 1.035 \end{bmatrix}$$

The revised vector of total final demands is specified by simply reducing the value for total imports of manufactured goods by \$1 trillion, so computing the corresponding change in total outputs is found by

$$\Delta\mathbf{x} = \mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}\Delta\mathbf{f} = -\begin{bmatrix} 75{,}840 & 91{,}880 & 7{,}742 & 1{,}550{,}525 & 184{,}877 & 355{,}257 & 48{,}461 \end{bmatrix}'$$

Chapter 6, Multipliers in the Input–Output Model

Chapter 6 examines key summary analytical measures known as multipliers that can be derived from input–output models to estimate the effects of exogenous changes on (1) new outputs of economic sectors, (2) income earned by households resulting from new outputs, and (3) employment generated from new outputs or (4) value-added generated by production or (5) energy and environmental effects.

The chapter develops the general structure of multiplier analysis and special considerations associated with regional, IRIO, and MRIO models. Extensions to capture the effects of income generation for various household groups are then explored, as well as additional multiplier variants. Chapter appendices expand on mathematical formulations of household and income multipliers.

The exercise problems for this chapter illustrate various types of input-output multipliers and their applications.

Problem 6.1

This problem explores the use of total output multipliers as an indicator of relative importance to the economy using the input-output tables utilized in the exercise problems 2.1 through 2.10. The output multipliers, the column sums of the Leontief inverse in each case (with the largest multiplier in each case highlighted in boldface), are the following:

Problem	Output Multipliers							
2.1	6.444	**6.944**						
2.2	2.970	**4.167**	3.611					
2.3	6.444	**6.944**						
2.4	2.006	**2.428**	1.307					
2.5	1.412	**1.588**						
2.6	**1.839**	1.437						
2.7	2.301	2.031	2.209	2.035	1.551	1.616	2.156	**2.364**
2.8	1.716	**1.814**						
2.9	1.919	1.605	1.722	**1.925**	1.487	1.608	1.599	
2.10	4.000	**5.000**	1.000					

Problem 6.2

This problem explores the use of output multipliers to derive the total value of output (across all sectors) associated with the new final demands, again using the exercise problems in Chapter 2. We have already calculated the multipliers in problem 6.1, so in conjunction with the new final demands in the problems in Chapter 2, we can derive the total value of output (across all sectors) associated with the new final demands.

Using problem 2.2 as an example, the row vector of output multipliers is

$\mathbf{m}(o) = [2.970 \quad 4.167 \quad 3.611]$. In conjunction with the final-demand vector used in that problem, namely $\mathbf{f}^{t+1} = \begin{bmatrix} 1{,}300 \\ 100 \\ 200 \end{bmatrix}$, we find $\mathbf{m}(o)^{t+1} = 5{,}000$. In the solution to problem 2.2, we found that

$\mathbf{x}^{t+1} = \begin{bmatrix} 2,000 \\ 1,000 \\ 2,000 \end{bmatrix}$, and the sum of these elements is 5,000; that is, $\mathbf{i}'\mathbf{x}^{t+1} = 5,000$. In matrix notation, this is comparing $\mathbf{m}(o)\Delta\mathbf{f}$ with $\mathbf{i}'\Delta\mathbf{x} = \mathbf{i}'\mathbf{L}\Delta\mathbf{f}$; we know that they must be equal, since output multipliers are the column sums of the Leontief inverse—$\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$.

Problem 6.3

This problem explores type I and type II income multipliers in addition to total output multipliers, using the data in problem 2.3 of a model closed to households, which included the matrix of interindustry transactions, $\mathbf{Z}^c = \begin{bmatrix} 500 & 350 & 90 \\ 320 & 360 & 50 \\ 100 & 60 & 40 \end{bmatrix}$, and vector of total outputs, $\mathbf{x}^c = \begin{bmatrix} 1,000 \\ 800 \\ 300 \end{bmatrix}$, from which we could compute the matrices of direct and total requirements, respectively, as $\mathbf{A}^c = \mathbf{Z}^c(\hat{\mathbf{x}}^c)^{-1} = \begin{bmatrix} .5 & .438 & .3 \\ .32 & .45 & .167 \\ .1 & .075 & .133 \end{bmatrix}$ and $\mathbf{L}^c = (\mathbf{I} - \mathbf{A}^c)^{-1} = \begin{bmatrix} 5.820 & 5.036 & 2.983 \\ 3.686 & 5.057 & 2.248 \\ 0.990 & 1.019 & 1.693 \end{bmatrix}$

The output multipliers for the three-sector model, closed with respect to households, are $\mathbf{m}(o) = [10.496 \quad 11.112 \quad 6.924]$. The type I income multipliers require that we have the labor-input coefficients, which are $a_{31} = 0.100$ and $a_{32} = 0.075$, along with the Leontief inverse of the model that is open with respect to households (from problem 2.3), $(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 4.074 & 3.241 \\ 2.370 & 3.704 \end{bmatrix}$.

Then $m(h)_1 =$ (0.1)(4.074)+(0.075)(2.370) = 0.5852 and $m(h)_2 =$ (0.1)(3.241) + (0.075)(3.704) = 0.6019; $m(h)_1^I = 0.5852/0.1 = 5.852$ and $m(h)_2^I = 0.6019/0.075 = 8.025$.

The total household income multipliers can be found as the first two elements in the bottom row of the Leontief inverse of the model closed with respect to households, $\mathbf{L}^c = (\mathbf{I} - \mathbf{A}^c)^{-1}$, which are $m(h)_1 =$ 0.990 and $m(h)_2 =$ 1.019, so the type II income multipliers are therefore $m(h)_1^{II} = 0.990 / 0.1 = 9.90$ and $m(h)_2^{II} = 1.019 / 0.075 = 13.59$. Note that, for both sectors, the ratio of the type II to the type I income multiplier is 1.69.

Problem 6.4

This problem configures a typical policy question that can be addressed with input-output multipliers. Suppose we assemble the following facts about the two sectors that make up the economy of a small country under study where the available data pertain to the most recent quarter. Total interindustry inputs were \$50 and \$100, respectively, for Sectors 1 and 2. Sector 1's sales to final demand were \$60 and Sector 1's total output was \$100. Sector 2's sales to Sector 1 were \$30 and this represented 10 percent of Sector 2's total output. If we define the matrix of interindustry transactions as $\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$ and the vectors of final demands, interindustry inputs, and total outputs, respectively, as $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we can summarize the known values as $z_{21} = 30$, $f_1 = 10$, $v_1 = 50$, $v_2 = 100$, and $x_1 = 100$. The value of x_2 is easily found from $x_2 = z_{21} / 0.1 = 120$ and the unavailable data for z_{11}, z_{12}, z_{22}, and f_2 can be computed from the basic accounting identities, $\mathbf{Zi} + \mathbf{f} = \mathbf{x}$ and $\mathbf{i'Z} + \mathbf{v} = \mathbf{x}$ so that **Z**, **v**, **f**, and **x** are, respectively, $\mathbf{Z} = \begin{bmatrix} 20 & 20 \\ 30 & 80 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 50 & 100 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 60 \\ 10 \end{bmatrix}$. and $\mathbf{x} = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$, from which we can then calculate the direct requirements and total requirements matrices, respectively, as $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .2 & .067 \\ .3 & .267 \end{bmatrix}$ and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.294 & .118 \\ .529 & 1.412 \end{bmatrix}$.

If we project that, after national elections are held, it may turn out that different government policy will be forthcoming during the first quarter of the coming year. For example, if there is an increase of \$100 in government purchases of sector 1's output, we specify the projected change in total final demand as $\Delta\mathbf{f}^1 = \begin{bmatrix} 160 \\ 190 \end{bmatrix}$, while if the same increase is of sector 2's output, we specify the change in final demand as $\Delta\mathbf{f}^2 = \begin{bmatrix} 60 \\ 290 \end{bmatrix}$.

We can compare the stimulative effect of the two scenarios by calculating the sum of total outputs for each that would be necessary to support the changed final demands, i.e., $\Delta\mathbf{x}^1 = \mathbf{i'L}\Delta\mathbf{f}^1 = 582.35$ and $\Delta\mathbf{x}^2 = \mathbf{i'L}\Delta\mathbf{f}^2 = 552.94$. The first option generates the larger stimulative effect by $\Delta\mathbf{x}^1 - \Delta\mathbf{x}^2 = 582.35 - 552.94 = 29.41$.

Problem 6.5

This problem explores a typical economic planning question. Consider an input output economy defined by $\mathbf{Z} = \begin{bmatrix} 140 & 350 \\ 800 & 50 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1,000 \\ 1,000 \end{bmatrix}$. Suppose economic planners are asked to design an advertising campaign to stimulate export sales of one of the goods produced in the country and need to determine which of the two sectors on which to concentrate their efforts or perhaps if some combination would be more effective. The answer rests on the relative size of the output multipliers, which will indicate the relative stimulative effect of focusing on one sector or the other (or a combination).

The output multipliers are found by first computing the technical coefficients matrix, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .14 & .35 \\ .8 & .05 \end{bmatrix}$, and the total requirements matrix, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.769 & .652 \\ 1.490 & 1.601 \end{bmatrix}$, so the vector of output multipliers is $\mathbf{m}(o) = \mathbf{i'L} = \begin{bmatrix} 3.259 & 2.253 \end{bmatrix}$. So, in terms of relative stimulative effect, it is more effective to concentrate on stimulating export demand for the product of sector 1; since it has a considerably larger output multiplier.

If we determine labor income coefficients for the two sectors in the region to be $a_{31} = 0.1$ and $a_{32} = 0.18$, and the focus is on job creation, it is possible the priorities may change. In this case, knowing $a_{31} = 0.1$ and $a_{32} = 0.18$, we can find $H_1 = 0.4451$ and $H_2 = 0.3534$ by $\mathbf{H} = \mathbf{lL} = \begin{bmatrix} .4451 & .3534 \end{bmatrix}$ where the vector of labor coefficients is $\mathbf{l} = \begin{bmatrix} .1 & .18 \end{bmatrix}$. Thus, converting output effects to income earned per dollar of new final demand for each of the sectors does not change the ranking, so, in this case, stimulation of export demand for the output of sector 1 is still more beneficial.

Problem 6.6

This problem explores interregional input-output multipliers using the elements in the full two-region interregional Leontief inverse from problem 3.2. First, recall the interregional Leontief inverse from that problem: $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 1.205 & .202 & .115 & .123 \\ .263 & 1.116 & .189 & .131 \\ \hline .273 & .262 & 1.177 & .2 \\ .33 & .289 & .179 & 1.156 \end{array}\right]$.

We can first calculate the vectors of simple intraregional output multipliers for sectors 1 and 2 as $\mathbf{m}(o)^{rr} = \mathbf{i}'\mathbf{L}_{11} = \begin{bmatrix} 1.468 & 1.318 \end{bmatrix}$ and $\mathbf{m}(o)^{ss} = \mathbf{i}'[\mathbf{L}_{22}] = [1.356 \quad 1.356]$. The vectors of simple national (total) output multipliers for sectors 1 and 2 are $\mathbf{m}(o)^{r} = \mathbf{i}'\begin{bmatrix} \mathbf{L}_{11} \\ \mathbf{L}_{21} \end{bmatrix} = \begin{bmatrix} 2.070 & 1.869 \end{bmatrix}$ and $\mathbf{m}(o)^{s} = \mathbf{i}'\begin{bmatrix} \mathbf{L}_{12} \\ \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 1.660 & 1.610 \end{bmatrix}$.

Finally, the sector-specific simple national output multipliers for sectors 1 and 2 in regions *r* and *s*. show the impact on sector *i* throughout the entire country, because of a dollar's worth of final demand for sector *j* in either region. In this case it means finding the four multipliers for each region as:

$\mathbf{m}(o)^{\bullet r} = \begin{bmatrix} m(o)^{\bullet r}_{11} & m(o)^{\bullet r}_{21} & m(o)^{\bullet r}_{12} & m(o)^{\bullet r}_{22} \end{bmatrix} = \begin{bmatrix} 1.478 & 0.593 & 0.464 & 1.405 \end{bmatrix}$ and

$\mathbf{m}(o)^{\bullet s} = \begin{bmatrix} m(o)^{\bullet s}_{11} & m(o)^{\bullet s}_{21} & m(o)^{\bullet s}_{12} & m(o)^{\bullet s}_{22} \end{bmatrix} = \begin{bmatrix} 1.292 & 0.368 & 0.323 & 1.287 \end{bmatrix}$.

Problem 6.7

This problem further explores the characteristics of interregional input-output multipliers using the results of problem 6.6.

To determine which sector's output increases the most for an arbitrary new final demand in the two regions, we simply compare the intraregional multipliers for each sector in each region, $\mathbf{m}(o)^{rr} = \mathbf{i}'[\mathbf{L}_{11}] = [1.468 \quad 1.318]$ and $\mathbf{m}(o)^{ss} = \mathbf{i}'[\mathbf{L}_{22}] = [1.356 \quad 1.356]$. In region *r* sector 1's multiplier is larger than sector 2's (1.468>1.318) and in region *s* the multipliers are equal for the two sectors.

To determine which sector in which region produces the largest national (two-region) impact for an arbitrary increase in final demand we compare the sector-specific simple national output multipliers:

$\mathbf{m}(o)^{\bullet r} = \begin{bmatrix} m(o)^{\bullet r}_{11} & m(o)^{\bullet r}_{21} & m(o)^{\bullet r}_{12} & m(o)^{\bullet r}_{22} \end{bmatrix} = \begin{bmatrix} 1.478 & 0.593 & 0.464 & 1.405 \end{bmatrix}$ and

$\mathbf{m}(o)^{\bullet s} = \begin{bmatrix} m(o)_{11}^{\bullet s} & m(o)_{21}^{\bullet s} & m(o)_{12}^{\bullet s} & m(o)_{22}^{\bullet s} \end{bmatrix} = \begin{bmatrix} 1.292 & 0.368 & 0.323 & 1.287 \end{bmatrix}$, the largest of which is 1.478 for sector 1 in region *r*.

To determine whether it would be better to institute policies that would increase household demand in region *r* or in region *s*, increasing the output of sector 1 nationally (i.e., in both regions), we compare the total interregional output multipliers $\mathbf{m}(o)^{r} = \mathbf{i}'\begin{bmatrix} \mathbf{L}_{11} \\ \mathbf{L}_{21} \end{bmatrix} = \begin{bmatrix} 2.070 & 1.869 \end{bmatrix}$ and $\mathbf{m}(o)^{s} = \mathbf{i}'\begin{bmatrix} \mathbf{L}_{12} \\ \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 1.660 & 1.610 \end{bmatrix}$. The multiplier for sector 1 in region *r* is larger than the corresponding multiplier in region *s* (2.07>1.66) so increasing household demand in region *r* is more beneficial. The same is true for sector 2 (1.869>1.610).

Problem 6.8

This problem explores the same characteristics of multipliers as problems 6.6 and 6.7 but in the multiregional rather than interregional case using the elements in $(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}$ from problem 3.3:

$$(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C} = \left[\begin{array}{cc|cc} 0.971 & 0.556 & 1.024 & 0.524 \\ 0.882 & 1.197 & 0.889 & 1.251 \\ \hline 1.297 & 0.714 & 1.264 & 0.677 \\ 0.663 & 1.010 & 0.673 & 0.854 \end{array}\right]$$

To determine which sector's output increases the most for an arbitrary new final demand in the two regions, we simply compare the intraregional multipliers for each sector in each region, $\mathbf{m}(o)^{rr} = \mathbf{i}'[\mathbf{L}_{11}] = [1.853 \quad 1.753]$ and $\mathbf{m}(o)^{ss} = \mathbf{i}'[\mathbf{L}_{22}] = [1.937 \quad 1.530]$. In region *r* sector 1's multiplier is greater than sector 2's (1.853>1.753) and in region s the same is true (1.937>1.530).

To determine which sector in which region produces the largest national (two-region) impact for an arbitrary increase in final demand we compare the sector-specific simple national output multipliers:

$\mathbf{m}(o)^{\bullet r} = \begin{bmatrix} m(o)_{11}^{\bullet r} & m(o)_{21}^{\bullet r} & m(o)_{12}^{\bullet r} & m(o)_{22}^{\bullet r} \end{bmatrix} = \begin{bmatrix} 2.269 & 1.545 & 1.270 & 2.207 \end{bmatrix}$ and

$\mathbf{m}(o)^{\bullet s} = \begin{bmatrix} m(o)_{11}^{\bullet s} & m(o)_{21}^{\bullet s} & m(o)_{12}^{\bullet s} & m(o)_{22}^{\bullet s} \end{bmatrix} = \begin{bmatrix} 2.288 & 1.562 & 1.201 & 2.105 \end{bmatrix}$, the largest of which is 2.288 for sector 1 in region *s*.

To determine whether it would be better to institute policies that would increase household demand in region *r* or in region *s* so as to increase the output of sector 1 nationally (i.e., in both regions), we compare the total interregional output multipliers $\mathbf{m}(o)^{r} = \mathbf{i}'\begin{bmatrix} \mathbf{L}_{11} \\ \mathbf{L}_{21} \end{bmatrix} = \begin{bmatrix} 3.813 & 3.477 \end{bmatrix}$ and $\mathbf{m}(o)^{s} = \mathbf{i}'\begin{bmatrix} \mathbf{L}_{12} \\ \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 3.849 & 3.306 \end{bmatrix}$. The multiplier for sector 1 in region *s* is larger than the corresponding multiplier in region *r* (3.849>3.813) so increasing household demand in region *s* is more beneficial. The opposite is true for sector 2, i.e., The multiplier for sector 1 in region *r* is larger than the corresponding multiplier in region *s* (3.477>3.306) so increasing household demand in region *r* is more beneficial.

Problem 6.9

This problem explores the use of regional output multipliers in analysis of a typical policy problem using the basic data introduced in problem 3.4. Suppose the government is interested in starting an overseas advertising and promotion campaign aimed at increasing export sales of the products of the country. There is specialization of production in the regions of the country; in particular, the products are shown in the table below:

	Region A	Region B	Region C
Manufacturing	Scissors	Cloth	Pottery
Agriculture	Oranges	Walnuts	None

To determine the product for which an increase in export sales would produce the greatest stimulation of the national economy, we calculate the total regional output multipliers for each region:

$$\mathbf{m}(o)^A = \mathbf{i}'(\mathbf{I}-\mathbf{A}^A)^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.714 & 0.857 \\ 0.429 & 1.714 \end{bmatrix} = \begin{bmatrix} 2.143 & 2.571 \end{bmatrix}$$

$$\mathbf{m}(o)^B = \mathbf{i}'(\mathbf{I}-\mathbf{A}^B)^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2.857 & 2.286 \\ 0.333 & 1.667 \end{bmatrix} = \begin{bmatrix} 3.190 & 3.952 \end{bmatrix}$$

$$\mathbf{m}(o)^C = \mathbf{i}'(\mathbf{I}-\mathbf{A}^C)^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2.0 & 0 \\ 0.5 & 1.0 \end{bmatrix} = \begin{bmatrix} 2.5 & 1.0 \end{bmatrix}$$

The largest total output multiplier is associated with sector 2 in region B (3.952); that is, with walnuts (this of course ignores and interregional multiplier effects that might be found with an IRIO or MRIO model).

Problem 6.10

This problem explores the relationships between Type I and Type II income multipliers. We use the example provided in section 6.2.1 (revisited from section 2.5), which began with the matrix of interindustry transactions, $\mathbf{Z} = \begin{bmatrix} 150 & 500 \\ 200 & 100 \end{bmatrix}$ and the vector of total outputs, $\mathbf{x} = \begin{bmatrix} 1{,}000 \\ 2{,}000 \end{bmatrix}$ from which we can derive the matrix of technical coefficients, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ and corresponding matrix of total requirements, $\mathbf{L} = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix}$.

We developed this model closed to households as $\overline{\mathbf{A}} = \left[\begin{array}{cc|c} .15 & .25 & .05 \\ .20 & .05 & .40 \\ \hline .30 & .25 & .05 \end{array}\right]$ with the corresponding matrix of total requirements computed as $\overline{\mathbf{L}} = (\mathbf{I}-\overline{\mathbf{A}})^{-1} = \left[\begin{array}{cc|c} 1.365 & .425 & .251 \\ .527 & 1.348 & .595 \\ \hline .570 & .489 & 1.289 \end{array}\right]$.

Here, $|(\mathbf{I}-\mathbf{A})| = 0.7575$ and $|(\mathbf{I}-\overline{\mathbf{A}})| = 0.587875$, giving $|(\mathbf{I}-\mathbf{A})| / |(\mathbf{I}-\overline{\mathbf{A}})| = 1.289$, which is the same as $\overline{l}_{33}$ in $\overline{\mathbf{L}}$.

In (A6.2.2) from Appendix 6.2 we showed in general for $\overline{\mathbf{L}}$, partitioned as $\overline{\mathbf{L}} = (\mathbf{I}-\overline{\mathbf{A}})^{-1} = \begin{bmatrix} \overline{\mathbf{L}}_{11} & \overline{\mathbf{L}}_{12} \\ \overline{\mathbf{L}}_{21} & \overline{\mathbf{L}}_{22} \end{bmatrix}$, that $\overline{\mathbf{L}}_{21} = -\overline{\mathbf{L}}_{21}(\mathbf{GE}^{-1}) = -\overline{\mathbf{L}}_{22}(\mathbf{GL})$ and $\mathbf{h}'_c\mathbf{L} = -\mathbf{GL}$ where **E**, **F**, and **G** are defined by $(\mathbf{I}-\overline{\mathbf{A}}) = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$ and $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1}$. Finally, $\mathbf{h}'_c = \mathbf{h}'\hat{\mathbf{x}}^{-1}$, where **h** is household employment in units such as person-years, **h**. Then it was shown that the row vector of the ratios of the Type II to Type I income multipliers, $R_j = m(h)^{II}_j / m(h)^{I}_j$, is $\mathbf{R} = -\underset{(1\times 1)}{\overline{\mathbf{L}}_{22}}\underset{(1\times n)}{(\mathbf{GL})}[\underset{(n\times n)}{\langle -\mathbf{GL}\rangle}]^{-1} = \underset{(1\times 1)}{\overline{\mathbf{L}}_{22}}\underset{(1\times n)}{[1,\cdots,1]} = \overline{\mathbf{L}}_{22}\mathbf{i}'$; that is, the ratios are all the same and are equal to the element in the lower-right of the closed model inverse. Here $\overline{\mathbf{L}}_{22} = \overline{l}_{33} = 1.289$.

Chapter 7, Supply-Side Models, Linkages, and Important Coefficients

Chapter 7 presents the supply side input–output model. It is discussed both as a quantity model (the early interpretation) and as a price model (the more modern interpretation). Relationships to the standard Leontief quantity and price models are also explored. In addition, the fast-growing literature on quantification of economic linkages and analysis of the overall structure of economies using input–output data is examined. Finally, approaches for identifying key or important coefficients in input–output models and alternative measures of coefficient importance are presented.

The exercise problems for this chapter illustrate the configuration of supply side input-models and measures of forward and backward economic linkages in both demand and supply models.

Problem 7.1

This problem explores the properties of the output inverse in a supply side input output model. Consider the centrally planned economy of Czaria, which is involved in its planning for the next fiscal year. The matrix of technical coefficients, **A**, and vector of total industry outputs, **x**, for Czaria are given as the following:

	1	2	3	4	Total Output
1. Agriculture	0.168	0.155	0.213	0.212	12,000
2. Mining	0.194	0.193	0.168	0.115	15,000
3. Military Manufacturing	0.105	0.025	0.126	0.124	12,000
4. Civilian Manufacturing	0.178	0.101	0.219	0.186	16,000

From the table we define $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z} = \begin{bmatrix} .168 & .194 & .213 & .283 \\ .155 & .193 & .134 & .123 \\ .105 & .031 & .126 & .165 \\ .134 & .095 & .164 & .186 \end{bmatrix}$ for $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$. The output inverse for this economy is then $\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = \begin{bmatrix} 1.468 & .455 & .558 & .692 \\ .376 & 1.393 & .384 & .418 \\ .253 & .155 & 1.300 & .375 \\ .336 & .268 & .399 & 1.466 \end{bmatrix}$. The next year's value-added inputs for agriculture, mining, military manufacturing products and civilian manufacturing in Czaria are projected to be \$4,558 million, \$5,665 million, \$2,050 million and \$5,079 million, respectively. The nation's projected *GDP*, since it is the sum of either all final demands or value added, i.e., $GDP = \mathbf{i}'\mathbf{f} = \mathbf{v}'\mathbf{i}$, can be computed very simply for the projected new final demands, $(\mathbf{v}^{new})' = \begin{bmatrix} 4,558 & 5,665 & 2,050 & 5,079 \end{bmatrix}$, as $GDP = (\mathbf{v}^{new})'\mathbf{i} = 17,352$, the sum of all new value-added inputs. The corresponding vector of new total gross production is $\mathbf{x}^{new} = \begin{bmatrix} 13,928.5 & 12,518.4 & 6,606.6 & 11,313.2 \end{bmatrix}'$, found by $\mathbf{x}^{new} = (\mathbf{I} - \mathbf{B}')^{-1}\mathbf{v}^{new}$, the supply side model. Note that this is the "old view" of the Ghosh model as described in section 7.1.1.

Problem 7.2

This problem illustrates the use of mean absolute percentage difference (MAPD) as a measure for comparing output coefficients in supply-side input-output models. Consider a case where $\mathbf{Z} = \begin{bmatrix} 13 & 75 & 45 \\ 53 & 21 & 48 \\ 67 & 68 & 93 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 130 \\ 150 \\ 220 \end{bmatrix}$ for a base year. If final demands for the next year are projected to be $\mathbf{f}^{new} = \begin{bmatrix} 200 \\ 300 \\ 500 \end{bmatrix}$ and the change in interindustry transactions is expected to be $\Delta\mathbf{Z} = \begin{bmatrix} 0 & 5 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ the MAPD between the direct output coefficients for the base year and next year is found by first computing the output coefficients for the two years. First, $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z} = \begin{bmatrix} .049 & .285 & .171 \\ .195 & .077 & .176 \\ .150 & .152 & .208 \end{bmatrix}$, and, since $\mathbf{Z}^{new} = \mathbf{Z} + \Delta\mathbf{Z} = \begin{bmatrix} 13 & 80 & 45 \\ 63 & 21 & 48 \\ 67 & 68 & 108 \end{bmatrix}$ and $\mathbf{x}^{new} = \mathbf{f}^{new} + \mathbf{Z}^{new}\mathbf{i} = \begin{bmatrix} 338 \\ 432 \\ 743 \end{bmatrix}$, $\mathbf{B}^{new}$ is found as $\mathbf{B}^{new} = (\hat{\mathbf{x}}^{new})^{-1}\mathbf{Z}^{new} = \begin{bmatrix} .038 & .123 & .133 \\ .146 & .049 & .111 \\ .090 & .092 & .145 \end{bmatrix}$.

The MAPD between $\mathbf{B}$ and $\mathbf{B}^{new}$ is found by $MAPD = (1/n^2)\sum_{i=1}^{n}\sum_{j=1}^{n}[|b_{ij} - b_{ij}^{new}|/b_{ij}]\times 100 = 29.1$. For the total output coefficients or output inverses, $\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = \begin{bmatrix} 1.195 & .427 & .353 \\ .307 & 1.235 & .341 \\ .284 & .317 & 1.394 \end{bmatrix}$ and $\mathbf{G}^{new} = (\mathbf{I} - \mathbf{B}^{new})^{-1} = \begin{bmatrix} 1.104 & .295 & .210 \\ .185 & 1.114 & .174 \\ .136 & .150 & 1.211 \end{bmatrix}$, the MAPD between $\mathbf{G}$ and $\mathbf{G}^{new}$ is 32.806.

Problem 7.3

This problem explores the calculation of relative price changes using the Ghosh price model. For an input-output transactions matrix of $\mathbf{Z} = \begin{bmatrix} 384 & 520 & 831 \\ 35 & 54 & 530 \\ 672 & 8 & 380 \end{bmatrix}$ and total outputs of $\mathbf{x} = \begin{bmatrix} 2,500 \\ 1,200 \\ 3,000 \end{bmatrix}$ given for a base year, if additional growth in value added for the next year is projected to result in $\mathbf{v}^{new} = \begin{bmatrix} 2,000 \\ 1,000 \\ 1,500 \end{bmatrix}$, the corresponding price changes of output for the three industries for the new year relative to the base year are found by first calculating $\mathbf{B} = (\hat{\mathbf{x}})^{-1}\mathbf{Z} = \begin{bmatrix} .154 & .208 & .332 \\ .029 & .045 & .442 \\ .224 & .003 & .127 \end{bmatrix}$ and

$\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = \begin{bmatrix} 1.37 & .3 & .673 \\ .205 & 1.093 & .631 \\ .352 & .080 & 1.320 \end{bmatrix}$. Using the Ghosh price model, the new total outputs are

found as $\mathbf{x}^{new} = \mathbf{G}'\mathbf{v}^{new} = \begin{bmatrix} 3,472.7 \\ 1,814.5 \\ 3,956.9 \end{bmatrix}$ and the relative price changes between the two years are

$\boldsymbol{\pi} = \hat{\mathbf{x}}^{-1}\mathbf{x}^{new} = \begin{bmatrix} 1.389 & 1.512 & 1.319 \end{bmatrix}$.

Problem 7.4

This problem explores the calculation of relative price changes using the Leontief price model and demonstrates that the Leontief price model produces the same relative price changes of industrial output for the new year relative to the base year as found in problem 7.3, thus showing that the Ghosh and Leontief price models produce the same result. Using the basic data in Problem 7.3, first recall that $\mathbf{Z} = \begin{bmatrix} 384 & 520 & 831 \\ 35 & 54 & 530 \\ 672 & 8 & 380 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 2,500 \\ 1,200 \\ 3,000 \end{bmatrix}$ produces

$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .154 & .433 & .277 \\ .014 & .045 & .177 \\ .269 & .007 & .127 \end{bmatrix}$ and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.37 & .626 & .561 \\ .098 & 1.093 & .252 \\ .422 & .201 & 1.32 \end{bmatrix}$. Also, with the new

year's value added defined in problem 7.3 as $\mathbf{v}^{new} = \begin{bmatrix} 2,000 \\ 1,000 \\ 1,500 \end{bmatrix}$, we calculate the vector of the new

value added as a fraction of the base year total outputs, $\mathbf{v}_c^{new} = \hat{\mathbf{x}}^{-1}\mathbf{v}^{new} = \begin{bmatrix} .8 & .833 & .5 \end{bmatrix}'$.

The vector of relative price changes using the Leontief price model as $\tilde{\mathbf{p}} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{v}_c^{new} = \mathbf{L}'\mathbf{v}_c^{new} = \begin{bmatrix} 1.389 & 1.512 & 1.319 \end{bmatrix}'$ which are identical to the Ghosh model price changes, i.e., $\tilde{\mathbf{p}} = \boldsymbol{\pi}$ from problem 7.3.

Problem 7.5

This problem explores the basic concepts of forward and backward linkages in input-output models. Consider the case of a matrix of transactions, $\mathbf{Z} = \begin{bmatrix} 418 & 687 & 589 & 931 \\ 847 & 527 & 92 & 654 \\ 416 & 702 & 911 & 763 \\ 263 & 48 & 737 & 329 \end{bmatrix}$, and vector of total final demands, $\mathbf{f}=\begin{bmatrix}2,000\\3,000\\2,500\\1,500\end{bmatrix}$. With $\mathbf{x}=\mathbf{f}+\mathbf{Zi}=\begin{bmatrix}4,625\\5,120\\5,292\\2,877\end{bmatrix}$ we compute the matrix of direct

requirements, $\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}=\begin{bmatrix}.090 & .134 & .111 & .324\\.183 & .103 & .017 & .227\\.090 & .137 & .172 & .265\\.057 & .009 & .139 & .114\end{bmatrix}$, and

$\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}1.207 & .227 & .264 & .579\\.280 & 1.182 & .138 & .447\\.214 & .241 & 1.332 & .539\\.114 & .065 & .228 & 1.256\end{bmatrix}$, for the demand-driven model. We next compute

$\mathbf{B}=\hat{\mathbf{x}}^{-1}\mathbf{Z}=\begin{bmatrix}.090 & .149 & .127 & .201\\.165 & .103 & .018 & .128\\.079 & .133 & .172 & .144\\.091 & .017 & .256 & .114\end{bmatrix}$, the matrix of direct requirements, and

$\mathbf{G}=(\mathbf{I}-\mathbf{B})^{-1}=\begin{bmatrix}1.207 & .251 & .303 & .360\\.253 & 1.182 & .142 & .251\\.187 & .233 & 1.332 & .293\\.183 & .116 & .419 & 1.256\end{bmatrix}$, the matrix of total requirements for supply driven input-output models.

The vectors of direct and total backward linkages are found as $\mathbf{i}'\mathbf{A}=\begin{bmatrix}.420 & .384 & .440 & .930\end{bmatrix}$ and $\mathbf{i}'\mathbf{L}=\begin{bmatrix}1.815 & 1.716 & 1.962 & 2.820\end{bmatrix}$, respectively, from the demand-driven model. The vectors of direct and total forward linkages are found as $\mathbf{Bi}=\begin{bmatrix}0.568 & 0.414 & 0.528 & 0.479\end{bmatrix}'$ and $\mathbf{Gi}=\begin{bmatrix}2.121 & 1.828 & 2.046 & 1.974\end{bmatrix}'$, respectively, from the supply-driven model.

Problem 7.6

This problem explores spatial forward and backward linkages in an interregional input-output (IRIO) model using the three region IRIO table for Japan given in problem in Table A4.1.1 of Appendix S4.1.

First define $B(d)^{rr}=(1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$ for regions, $r=1$, 2, and 3 designating the average direct spatial linkage of a region to itself as the average of the intraregional technical coefficients. Also designate $B(d)^{sr}=(1/n)\mathbf{i}'\mathbf{A}^{sr}\mathbf{i}$ for regions $r=1$, 2, and 3 and $s=1$, 2, and 3 but for $r\neq s$ to designate the average direct interregional spatial linkage of a region to other regions as the average of the interregional technical coefficients relating the regions. Similarly,

$B(t)^{rr} = (1/n)\mathbf{i}'\mathbf{L}^{rr}\mathbf{i}$ and $B(t)^{sr} = (1/n)\mathbf{i}'\mathbf{L}^{sr}\mathbf{i}$ designate the total spatial linkage of a region to itself and the total spatial linkage between regions, respectively.

For the Japanese IRIO model provided, $n = 5$ industry sectors, and $r = c,\ n,$ and s for the Japanese central, north, and south regions, the specific direct backward linkage measures are $B(d)^c = B(d)^{cc} + B(d)^{cn} + B(d)^{cs}$, $B(d)^n = B(d)^{nn} + B(d)^{nc} + B(d)^{ns}$ and $B(d)^s = B(d)^{ss} + B(d)^{sn} + B(d)^{sc}$. These are the direct backward linkages for the central, north, and south regions, respectively. Analogous notation applies for the total backward linkages and the direct and total forward linkages which use **B** and **G** instead of **A** and **L**.

The results of these calculations are the following:

	$r =$ Central	$r =$ North	$r =$ South
$b(d)^r$	.865	.741	.939
$b(t)^r$	3.177	2.731	3.434
$f(d)^r$	.579	.453	.597
$f(t)^r$	2.615	2.483	2.595

From the table of results, we can observe that the North region is both the least backward-linked and forward-linked among the 3 regions.

Problem 7.7

This problem explores the concept of hypothetically extracting an industry sector from the economy and calculating the decrease in total output of the economy resulting from the hypothetical extraction using a highly aggregated version of the 2005 U.S. input-output table:

US Technical Coefficients 2005	1	2	3	4	5	6	7	Tot. Output
1 Agriculture	0.2258	0.0000	0.0015	0.0384	0.0001	0.0017	0.0007	312,754
2 Mining	0.0027	0.1432	0.0075	0.0675	0.0367	0.0004	0.0070	396,563
3 Construction	0.0051	0.0002	0.0010	0.0018	0.0037	0.0071	0.0215	1,302,388
4 Manufacturing	0.1955	0.0877	0.2591	0.3222	0.0547	0.0566	0.1010	4,485,529
5 Trade, Transport & Utilities	0.0819	0.0422	0.1011	0.0994	0.0704	0.0334	0.0487	3,355,944
6 Services	0.0843	0.1276	0.1225	0.1172	0.1760	0.2783	0.2026	10,477,640
7 Other	0.0099	0.0095	0.0093	0.0219	0.0215	0.0188	0.0240	2,526,325

To hypothetically extract the agriculture sector (sector 1), we set the first row and first column of the matrix **A** to zero, the result of which we define as $\mathbf{A}^{(1)}$ and set the first element of the vector **f** to zero which we define as $\mathbf{f}^{(1)}$. Then we compute $\mathbf{L}^{(1)} = (\mathbf{I} - \mathbf{A}^{(1)})^{-1}$ and subsequently $t_1 = \mathbf{i}'\mathbf{x} - \mathbf{i}'\mathbf{L}^{(1)}\mathbf{f}^{(1)} = 54{,}744{,}946$, which would be the reduction in total output of the economy if the agriculture sector were extracted.

If we now define $p_i = 100 \times (\mathbf{i}'\mathbf{x} - \mathbf{i}'\mathbf{L}^{(i)}\mathbf{f}^{(i)}) / \mathbf{i}'\mathbf{x}$ as the percentage reduction in total output by extracting industry *i*, we can compute the vector of all the seven p_i's for this economy as $\mathbf{p} = \begin{bmatrix} 2.4 & 2.6 & 11.5 & 29.8 & 22.0 & 54.8 & 18.8 \end{bmatrix}'$, which indicates that the services sector (sector 6) would yield the highest reduction in output from a hypothetical extraction with a 54.8 percent reduction in total output.

Problem 7.8

This problem explores the concept of "inverse important" coefficients in a Leontief model.

Consider an economy with $\mathbf{Z} = \begin{bmatrix} 8 & 64 & 89 \\ 28 & 44 & 77 \\ 48 & 24 & 28 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 300 \\ 250 \\ 200 \end{bmatrix}$. Using the element a_{13} as an example, we define the criteria for a sector designated as "inverse important" as the following.

We define parameters $\alpha = 30$ and $\beta = 5$ as specifying that a 30 percent change in a_{13} generates a 5 percent change in one or more elements in the associated Leontief inverse. We can explore the sensitivity of the results to the values of α and β as a relative indication of inverse importance. First, compute the matrix of technical coefficients,

$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.0267 & 0.2560 & 0.4450 \\ 0.0933 & 0.1760 & 0.3850 \\ 0.1600 & 0.0960 & 0.1400 \end{bmatrix}$, and the matrix of total requirements,

$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.2107 & 0.4738 & 0.8386 \\ 0.2557 & 1.3805 & 0.7503 \\ 0.2538 & 0.2423 & 1.4026 \end{bmatrix}$.

If a_{13} is increased to 0.5785 $(\alpha = 30)$ we find $\mathbf{L}^*_{(13)} = \begin{bmatrix} 1.2531 & 0.5144 & 1.0732 \\ 0.2647 & 1.3890 & 0.7999 \\ 0.2627 & 0.2507 & 1.4517 \end{bmatrix}$ and consequently $\mathbf{P}_{(13)} = 100\{[\mathbf{L}^*_{(13)} - \mathbf{L}] \oslash \mathbf{L}\} = \begin{bmatrix} 3.5069 & 8.5531 & 27.9806 \\ 3.5069 & 0.6201 & 6.6051 \\ 3.5069 & 3.5069 & 3.5069 \end{bmatrix}$, which is the element by element normalized percentage difference between elements in $\mathbf{L}^*_{(13)}$ and $\mathbf{L}$. In this case, a_{13} is identified as inverse important because a 30 percent change in its value causes a greater than 5 percent change in three inverse elements—l_{12}, l_{13}, and l_{23}. Notice that, as expected, the largest impact of a change in a_{13} is on the corresponding element in $\mathbf{L}$, namely l_{13}.

If we change the parameters to $\alpha = 20$ and $\beta = 10$ then,

$\mathbf{L}^*_{(13)} = \begin{bmatrix} 1.2387 & 0.5005 & 0.9932 \\ 0.2616 & 1.3861 & 0.7830 \\ 0.2597 & 0.2478 & 1.4350 \end{bmatrix}$ and $\mathbf{P}_{(13)} = \begin{bmatrix} 2.3109 & 5.6362 & 18.4382 \\ 2.3109 & 0.4086 & 4.3525 \\ 2.3109 & 2.3109 & 2.3109 \end{bmatrix}$, so a_{13} would still be classified as inverse important, since there is (now only) one element, l_{13}, that is changed by more than $\beta = 10$ percent. Finally, as another illustration, with $\alpha = 10$ and $\beta = 10$, we find

$\mathbf{L}^*_{(13)} = \begin{bmatrix} 1.2245 & 0.4870 & 0.9150 \\ 0.2586 & 1.3832 & 0.7664 \\ 0.2567 & 0.2450 & 1.4186 \end{bmatrix}$ and $\mathbf{P}_{(13)} = \begin{bmatrix} 1.1423 & 2.7859 & 9.1138 \\ 1.1423 & 0.2020 & 2.1514 \\ 1.1423 & 1.1423 & 1.1423 \end{bmatrix}$. In this case, a_{13}

would not be labeled inverse important since the largest percentage change in an element of the Leontief inverse is less than the threshold of $\beta = 10$ percent.

Problem 7.9

This problem explores the use of a supply-driven model to determine the sensitivity of an economy to an interruption in availability of a scarce-factor input—for example, a strike—in one of the sectors using the U.S. economy for 2005 (using the data presented in problem 7.7).

For this economy, from problem 7.7, specified are **A** and **x** from which we can compute the interindustry transaction matrix,

$$\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}} = \begin{bmatrix} 70,629 & 10 & 1,973 & 172,428 & 435 & 18,296 & 1,739 \\ 832 & 56,798 & 9,707 & 302,783 & 123,117 & 4,273 & 17,745 \\ 1,597 & 74 & 1,329 & 7,886 & 12,449 & 74,678 & 54,282 \\ 61,158 & 34,779 & 3,37,412 & 1,445,451 & 183,602 & 593,372 & 255,282 \\ 25,620 & 16,748 & 131,675 & 445,685 & 236,309 & 350,316 & 123,084 \\ 26352 & 50611 & 159600 & 525827 & 590537 & 2915594 & 511919 \\ 3091 & 3773 & 12087 & 98416 & 72256 & 197062 & 60628 \end{bmatrix}.$$

With **Z** specified we can now find the direct and total supply coefficients, $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$ and $\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1}$, respectively as:

$$\mathbf{B} = \begin{bmatrix} 0.2258 & 0.0000 & 0.0063 & 0.5513 & 0.0014 & 0.0585 & 0.0056 \\ 0.0021 & 0.1432 & 0.0245 & 0.7635 & 0.3105 & 0.0108 & 0.0447 \\ 0.0012 & 0.0001 & 0.0010 & 0.0061 & 0.0096 & 0.0573 & 0.0417 \\ 0.0136 & 0.0078 & 0.0752 & 0.3222 & 0.0409 & 0.1323 & 0.0569 \\ 0.0076 & 0.0050 & 0.0392 & 0.1328 & 0.0704 & 0.1044 & 0.0367 \\ 0.0025 & 0.0048 & 0.0152 & 0.0502 & 0.0564 & 0.2783 & 0.0489 \\ 0.0012 & 0.0015 & 0.0048 & 0.0390 & 0.0286 & 0.0780 & 0.0240 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 1.3139 & 0.0129 & 0.1027 & 1.1313 & 0.0811 & 0.3446 & 0.0987 \\ 0.0365 & 1.1863 & 0.1691 & 1.5050 & 0.4944 & 0.4018 & 0.1883 \\ 0.0026 & 0.0010 & 1.0054 & 0.0257 & 0.0190 & 0.0930 & 0.0499 \\ 0.0301 & 0.0169 & 0.1284 & 1.5707 & 0.0997 & 0.3280 & 0.1182 \\ 0.0165 & 0.0102 & 0.0674 & 0.2631 & 1.1072 & 0.2229 & 0.0716 \\ 0.0085 & 0.0102 & 0.0379 & 0.1502 & 0.1005 & 1.4409 & 0.0868 \\ 0.0041 & 0.0036 & 0.0154 & 0.0863 & 0.0454 & 0.1363 & 1.0390 \end{bmatrix}$$

We can also determine the corresponding vector of total value added as:

$$\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = [123,475 \;\; 233,770 \;\; 648,605 \;\; 1,487,054 \;\; 2,137,239 \;\; 6,324,050 \;\; 1,501,645]$$

As an example, a 10 percent reduction in construction primary inputs (sector 3) would reduce v_3 to 583,745 from 648,605.

If we define the new **v** incorporating the reduced manufacturing labor input as $\bar{\mathbf{v}}$, then we can compute

$\bar{\mathbf{x}}' = \bar{\mathbf{v}}'\mathbf{G} = [312,584 \;\; 396,496 \;\; 1,237,177 \;\; 4,483,860 \;\; 3,354,712 \;\; 10,471,611 \;\; 2,523,091]$,

which represents a 0.33 percent reduction in total output ($\mathbf{i}'\bar{\mathbf{x}}$ compared with $\mathbf{i}'\mathbf{x}$). For comparison, a 10 percent reduction in the services sector (sector 6) generates a 5.08 percent reduction in total output.

Problem 7.10

This problem explores the direct and the direct and indirect forward and backward linkages for the sectors in the U.S. economy and examine how these linkages have changed over time, using the seven-sector input-output data for the United States presented in the Supplemental Resources to this text (Appendix SD1, described in Appendix B).

The following shows, for the seven industry sectors for the years 1919-2018, the backward direct linkages, $\mathbf{b}(d) = \mathbf{i}'\mathbf{A}$ [denoted as B(d) in the table, found as the column sums of **A** for each year along with the average across the 7 sectors, b(d)], the backward total linkages, $\mathbf{b}(t) = \mathbf{i}'\mathbf{L}$ [denoted as B(t) in the table found by the column sums of **L** along with the average across the 7 sectors; b(t)];the forward direct linkages $\mathbf{f}(d) = \mathbf{Bi}$ [denoted by F(d) in the table, found as the row sums of **B** along with the average across the 7 sectors, f(d)]; and the forward total linkages, $\mathbf{f}(t) = \mathbf{Gi}$ [denoted by F(t) in the table, found by the row sums of **G**, along with the average across the 7 sectors, f(t)].

B(d)	1	2	3	4	5	6	7	b(d)
1919	0.556	0.748	0.729	0.722	0.57	0.546	0.524	0.628
1929	0.57	0.653	0.59	0.706	0.53	0.638	0.444	0.59
1938	0.624	0.724	0.517	0.807	0.639	0.449	0.626	0.627
1947	0.38	0.467	0.58	0.657	0.348	0.358	0.161	0.422
1958	0.462	0.472	0.609	0.633	0.35	0.333	0.266	0.446
1963	0.528	0.459	0.586	0.62	0.346	0.326	0.238	0.443
1967	0.548	0.492	0.553	0.611	0.334	0.335	0.265	0.448
1972	0.541	0.488	0.596	0.619	0.302	0.335	0.239	0.446
1977	0.572	0.479	0.559	0.656	0.357	0.332	0.263	0.46
1982	0.581	0.447	0.557	0.665	0.379	0.325	0.302	0.465
1987	0.547	0.454	0.573	0.636	0.347	0.351	0.308	0.459
1992	0.546	0.494	0.544	0.629	0.344	0.347	0.294	0.457
1997	0.579	0.463	0.521	0.645	0.351	0.372	0.312	0.463
2002	0.604	0.425	0.491	0.63	0.36	0.374	0.334	0.46
2007	0.585	0.34	0.466	0.659	0.395	0.393	0.357	0.456
2012	0.603	0.415	0.485	0.666	0.412	0.374	0.363	0.474
2018	0.627	0.444	0.478	0.627	0.415	0.389	0.358	0.477

B(t)	1	2	3	4	5	6	7	b(t)
1919	2.366	2.922	3.01	2.879	2.547	2.354	2.47	2.65
1929	2.359	2.5	2.475	2.701	2.292	2.415	2.141	2.412
1938	2.915	3.245	2.702	3.525	2.988	2.401	2.978	2.965
1947	1.742	1.916	2.219	2.359	1.648	1.703	1.296	1.84
1958	1.899	1.913	2.262	2.304	1.636	1.634	1.528	1.882
1963	2.066	1.843	2.219	2.291	1.623	1.6	1.467	1.873
1967	2.1	1.92	2.133	2.259	1.596	1.613	1.509	1.876
1972	2.071	1.886	2.2	2.263	1.524	1.602	1.454	1.857
1977	2.218	1.944	2.215	2.42	1.662	1.622	1.522	1.943
1982	2.23	1.862	2.179	2.41	1.707	1.604	1.595	1.941
1987	2.088	1.838	2.142	2.302	1.616	1.63	1.59	1.887
1992	2.078	1.892	2.071	2.266	1.602	1.607	1.543	1.865
1997	2.178	1.872	2.068	2.374	1.617	1.658	1.599	1.909
2002	2.211	1.772	1.974	2.294	1.622	1.649	1.62	1.877
2007	2.214	1.634	1.967	2.38	1.716	1.707	1.692	1.901
2012	2.271	1.797	2.011	2.415	1.752	1.667	1.704	1.945
2018	2.299	1.837	1.964	2.297	1.739	1.687	1.684	1.929

F(d)	1	2	3	4	5	6	7	f(d)
1919	0.821	0.806	0.671	0.531	0.679	0.307	0.631	0.635
1929	0.743	0.835	0.624	0.554	0.691	0.412	0.52	0.625
1938	0.717	0.871	0.732	0.571	0.959	0.18	0.879	0.701
1947	0.866	0.812	0.161	0.555	0.406	0.41	0.123	0.476
1958	0.808	0.917	0.151	0.598	0.448	0.39	0.137	0.493
1963	0.847	0.947	0.114	0.607	0.412	0.373	0.138	0.491
1967	0.844	0.915	0.139	0.6	0.416	0.386	0.128	0.49
1972	0.846	0.976	0.162	0.601	0.42	0.383	0.12	0.501
1977	0.793	0.252	0.133	0.619	0.442	0.378	0.128	0.535
1982	0.786	0.957	0.122	0.62	0.448	0.384	0.144	0.494
1987	0.888	0.096	0.146	0.619	0.417	0.404	0.151	0.532
1992	0.8	0.115	0.173	0.6	0.412	0.39	0.163	0.522
1997	0.812	0.14	0.127	0.627	0.4	0.418	0.171	0.528
2002	0.812	0.244	0.134	0.628	0.391	0.425	0.16	0.542
2007	0.811	0.294	0.155	0.657	0.403	0.43	0.159	0.558
2012	0.848	0.197	0.218	0.643	0.404	0.416	0.159	0.555
2018	0.84	0.879	0.179	0.659	0.403	0.43	0.164	0.508

F(t)	1	2	3	4	5	6	7	f(t)
1919	3.357	3.081	2.725	2.36	2.902	1.8	2.571	2.685
1929	2.941	3.038	2.444	2.316	2.75	1.947	2.269	2.529
1938	3.153	4.006	3.379	2.789	3.981	1.579	3.702	3.227
1947	2.885	2.744	1.271	2.122	1.813	1.782	1.208	1.975
1958	2.802	3.016	1.251	2.194	1.895	1.747	1.237	2.02
1963	2.97	3.041	1.185	2.22	1.81	1.693	1.252	2.024
1967	2.989	2.949	1.229	2.197	1.809	1.712	1.226	2.016
1972	2.974	3.089	1.258	2.184	1.812	1.691	1.216	2.032
1977	2.872	3.858	1.222	2.293	1.902	1.715	1.244	2.158
1982	2.851	3.125	1.199	2.266	1.905	1.72	1.271	2.048
1987	3.105	3.388	1.231	2.224	1.812	1.725	1.276	2.109
1992	2.819	3.355	1.27	2.168	1.785	1.686	1.29	2.053
1997	2.952	3.511	1.22	2.283	1.766	1.752	1.31	2.113
2002	2.917	3.675	1.228	2.234	1.724	1.748	1.285	2.116
2007	2.976	3.966	1.268	2.334	1.77	1.772	1.29	2.197
2012	3.073	3.784	1.369	2.309	1.783	1.74	1.29	2.193
2018	3.04	2.96	1.298	2.306	1.761	1.763	1.297	2.061

Chapter 8, Decomposition Approaches

Chapter 8 introduces and illustrates the basic concepts of structural decomposition analysis (SDA) within an input–output framework, in related additive and multiplicative formulations. The application of SDA to MRIO is developed to introduce a spatial context. Appendices to this chapter develop extended presentations of additional decomposition results as well as an overview of early applied studies and some further mathematical results. The exercise problems for this chapter illustrate various analytical features of SDA.

Problem 8.1

This problem explores the basic principles of SDA. Consider an input-output economy specified at two points in time, t^0 and t^1 by matrices of interindustry transactions and final demand vectors: $\mathbf{Z}^0 = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 2 & 25 \\ 20 & 40 & 60 \end{bmatrix}$, $\mathbf{f}^0 = \begin{bmatrix} 60 \\ 40 \\ 55 \end{bmatrix}$, $\mathbf{Z}^1 = \begin{bmatrix} 15 & 25 & 40 \\ 12 & 7.5 & 30 \\ 10 & 30 & 40 \end{bmatrix}$, and $\mathbf{f}^1 = \begin{bmatrix} 75 \\ 55 \\ 40 \end{bmatrix}$. To measure how the economy has changed in structure over the period, we can compute for each sector the change in total output between the two years that was attributable to changing final demand or to changing technology by the following.

First, we compute $\mathbf{A}^0 = \mathbf{Z}^0(\hat{\mathbf{x}}^0)^{-1} = \begin{bmatrix} .083 & .260 & .176 \\ .042 & .026 & .147 \\ .167 & .519 & .353 \end{bmatrix}$ and

$\mathbf{A}^1 = \mathbf{Z}^1(\hat{\mathbf{x}}^1)^{-1} = \begin{bmatrix} .097 & .239 & .333 \\ .077 & .072 & .250 \\ .065 & .287 & .333 \end{bmatrix}$. Next, $\mathbf{L}^0 = (\mathbf{I} - \mathbf{A}^0)^{-1} = \begin{bmatrix} 1.199 & .562 & .455 \\ .111 & 1.221 & .308 \\ .398 & 1.125 & 1.910 \end{bmatrix}$ and

$\mathbf{L}^1 = (\mathbf{I} - \mathbf{A}^1)^{-1} = \begin{bmatrix} 1.214 & .566 & .82 \\ .15 & 1.289 & .558 \\ .182 & 1.61 & 1.82 \end{bmatrix}$. Then $\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0 = \begin{bmatrix} 15 \\ 10 \\ -10 \end{bmatrix}$ and

$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \begin{bmatrix} .015 & .004 & .365 \\ .039 & .068 & .251 \\ -.216 & -.515 & -.09 \end{bmatrix}$.

We can find $\mathbf{x}^0 = \mathbf{L}^0\mathbf{f}^0 = \begin{bmatrix} 120 \\ 77 \\ 170 \end{bmatrix}$ and $\mathbf{x}^1 = \mathbf{L}^1\mathbf{f}^1 = \begin{bmatrix} 155 \\ 104.5 \\ 120 \end{bmatrix}$ so $\Delta\mathbf{x} = \begin{bmatrix} 35 \\ 27.5 \\ -50 \end{bmatrix}$. Then, using the basic structural decomposition relationship, $\Delta\mathbf{x} = \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change effect}} + \underbrace{(1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change effect}}$, we have the results in the following table (figures in parentheses are percentages of the total output change in each row).

	Output Change	Technology Change Contribution	Final-Demand Change Contribution
Sector 1	35	17.65 (50)	17.35 (50)
Sector 2	27.5	17.3 (63)	10.2 (37)
Sector 3	-50	-44.4 (89)	-5.6 (11)
Economy-wide Total	12.5	-9.45 (-75)	21.95 (175)

Problem 8.2

This problem illustrates the basic principles of SDA using input-output data for the U.S. economy for the years 1972 and 2002 aggregated to 7 industry sectors.

A and **x** for US, 2002	1	2	3	4	5	6	7	Tot. Output
1 Agriculture	0.2637	0.0020	0.0028	0.0374	0.0007	0.0008	0.0008	270,514
2 Mining	0.0032	0.0467	0.0097	0.0377	0.0226	0.0005	0.0040	184,516
3 Construction	0.0040	0.0336	0.0007	0.0030	0.0053	0.0078	0.0186	967,568
4 Manufacturing	0.1502	0.0942	0.2399	0.3464	0.0645	0.0464	0.0939	3,850,417
5 Trade, Transport & Utils	0.0868	0.0676	0.0960	0.0920	0.0816	0.0302	0.0475	2,811,865
6 Services	0.1310	0.2416	0.1436	0.1349	0.1813	0.2640	0.1954	8,948,582
7 Other	0.0098	0.0159	0.0083	0.0160	0.0276	0.0179	0.0203	2,146,282

A and **x** for US, 1972	1	2	3	4	5	6	7	Tot. Output
1 Agriculture	0.3141	0.0003	0.0028	0.0542	0.0010	0.0053	0.0012	83,955
2 Mining	0.0019	0.0542	0.0091	0.0296	0.0160	0.0002	0.0020	30,386
3 Construction	0.0069	0.0282	0.0003	0.0043	0.0156	0.0263	0.0166	165,998
4 Manufacturing	0.1436	0.0943	0.3522	0.3771	0.0407	0.0892	0.0078	761,194
5 Trade, Transport & Utils	0.0616	0.0481	0.1043	0.0786	0.0980	0.0442	0.0202	377,389
6 Services	0.0865	0.1471	0.0686	0.0591	0.1157	0.1621	0.0105	522,215
7 Other	0.0023	0.0063	0.0042	0.0117	0.0118	0.0096	0.0033	161,207

To compute the changes in total output between 1972 and 2002 for all sectors attributed to changes in final demand and to changes in technology, we employ the basic SDA relationship, $\Delta \mathbf{x} = \underbrace{(1/2)(\Delta \mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change effect}} + \underbrace{(1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta \mathbf{f})}_{\text{Final demand change effect}}$, to yield:

Sector	$\Delta \mathbf{x}$	Technology	Final Demand
1	186,559	-107,931	294,490
2	154,131	27,372	126,758
3	801,570	-73,551	875,121
4	3,089,223	-209,242	3,298,465
5	2,434,476	-3,750	2,438,225
6	8,426,367	1,224,317	7,202,050
7	1,985,075	109,282	1,875,793

Problem 8.3

This problem illustrates a special case of SDA. Consider an input-output economy specified by transactions matrices and final demand vectors for periods 0 and 1.

$$\mathbf{Z}^0 = \begin{bmatrix} 10 & 20 & 25 \\ 15 & 5 & 30 \\ 30 & 40 & 5 \end{bmatrix}, \ \mathbf{f}^0 = \begin{bmatrix} 45 \\ 30 \\ 25 \end{bmatrix}, \ \mathbf{Z}^1 = \begin{bmatrix} 15 & 30 & 37.5 \\ 22.5 & 7.5 & 45 \\ 45 & 60 & 7.5 \end{bmatrix}, \text{ and } \mathbf{f}^1 = \begin{bmatrix} 67.5 \\ 45 \\ 37.5 \end{bmatrix}$$

If we apply the basic SDA formulation in this case, we find that the changes represent uniform growth, i.e., both transactions and final demand grow uniformly by 50 percent between periods 0 and 1 as will the resulting total outputs (perhaps obvious in retrospect).

Problem 8.4

This problem explores a more complex form of SDA involving changes in technology and final demand as well as interactions between technology and final demand. Again, consider two observations on an input-output economy specified by matrices of interindustry transactions and vectors of total outputs for two years, designate 0 and 1:

$$\mathbf{Z}^0 = \begin{bmatrix} 12 & 15 & 35 \\ 24 & 11 & 30 \\ 36 & 50 & 8 \end{bmatrix}, \ \mathbf{f}^0 = \begin{bmatrix} 50 \\ 35 \\ 26 \end{bmatrix}, \ \mathbf{Z}^1 = \begin{bmatrix} 20 & 30 & 45 \\ 35 & 23 & 50 \\ 50 & 65 & 24 \end{bmatrix}, \ \mathbf{f}^1 = \begin{bmatrix} 55 \\ 50 \\ 60 \end{bmatrix}.$$

From these basic data we can compute the vectors of total outputs and the matrices of technical requirements and total requirements for both years: $\mathbf{x}^0 = \mathbf{f}^0 + \mathbf{Z}^0\mathbf{i} = \begin{bmatrix} 112 \\ 100 \\ 120 \end{bmatrix}$,

$$\mathbf{x}^1 = \mathbf{f}^1 + \mathbf{Z}^1\mathbf{i} = \begin{bmatrix} 150 \\ 158 \\ 199 \end{bmatrix}, \ \mathbf{A}^0 = \mathbf{Z}^0(\hat{\mathbf{x}}^0)^{-1} = \begin{bmatrix} .107 & .150 & .292 \\ .214 & .110 & .25 \\ .321 & .5 & .067 \end{bmatrix}, \ \mathbf{A}^1 = \mathbf{Z}^1(\hat{\mathbf{x}}^1)^{-1} = \begin{bmatrix} .133 & .19 & .226 \\ .233 & .146 & .251 \\ .333 & .411 & .121 \end{bmatrix},$$

$$\mathbf{L}^0 = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.491 & .604 & .628 \\ .592 & 1.563 & .604 \\ .831 & 1.045 & 1.611 \end{bmatrix}, \text{ and } \mathbf{L}^1 = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.541 & .618 & .573 \\ .687 & 1.633 & .643 \\ .905 & .998 & 1.655 \end{bmatrix}.$$

The changes in total outputs, final demands, and elements of the total requirements matrices are the $\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0 = \begin{bmatrix} 38 \\ 58 \\ 79 \end{bmatrix}$, $\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0 = \begin{bmatrix} 5 \\ 15 \\ 34 \end{bmatrix}$, and

$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \begin{bmatrix} .05 & .014 & -.055 \\ .095 & .07 & .039 \\ .075 & -.047 & .044 \end{bmatrix}$, respectively. Now we can compute a variety of alternative structural decompositions accounting for the interaction term as summarized in equations (8.3) through (8.7):

(8.3) $\Delta\mathbf{x} = \mathbf{L}^1(\mathbf{f}^0 + \Delta\mathbf{f}) - (\mathbf{L}^1 + \Delta\mathbf{L})\mathbf{f}^0 = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f})$

(8.4) $\Delta\mathbf{x} = (\mathbf{L}^0 + \Delta\mathbf{L})\,\mathbf{f}^1 - \mathbf{L}^0(\mathbf{f}^1 - \Delta\mathbf{f}) = (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f})$

(8.5) $\Delta\mathbf{x} = (\mathbf{L}^0 + \Delta\mathbf{L})(\mathbf{f}^0 + \Delta\mathbf{f}) - \mathbf{L}^0\mathbf{f}^0 = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^0(\Delta\mathbf{f}) + (\Delta\mathbf{L})(\Delta\mathbf{f})$

(8.6) $\Delta\mathbf{x} = \mathbf{L}^1\mathbf{f}^1 - (\mathbf{L}^1 - \Delta\mathbf{L})(\mathbf{f}^1 - \Delta\mathbf{f}) = (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{f}) - (\Delta\mathbf{L})(\Delta\mathbf{f})$

(8.7) $\Delta\mathbf{x} = (1/2)\underbrace{(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change}} + (1/2)\underbrace{(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change}}$

The following are the results from applying these equations:

		Total Output Change		Technology Change Contribution		Final-Demand Change Contribution		Interaction Term	
		Output	Percent	Output	Percent	Output	Percent	Output	Percent
Equation (8.3)	Sector 1	38	100	1.55	4.09	36.45	95.91	0.00	0.00
	Sector 2	58	100	8.21	14.15	49.79	85.85	0.00	0.00
	Sector 3	79	100	3.23	4.09	75.77	95.91	0.00	0.00
	Total	175	100	12.99	7.42	162.01	92.58	0.00	0.00
Equation (8.4)	Sector 1	38	100	0.15	0.39	37.85	99.61	0.00	0.00
	Sector 2	58	100	11.08	19.10	46.92	80.90	0.00	0.00
	Sector 3	79	100	4.40	5.57	74.60	94.43	0.00	0.00
	Total	175	100	15.62	8.93	159.38	91.07	0.00	0.00
Equation (8.5)	Sector 1	38	100	1.55	4.09	37.85	99.61	-1.41	-3.70
	Sector 2	58	100	8.21	14.15	46.92	80.90	2.87	4.94
	Sector 3	79	100	3.23	4.09	74.60	94.43	1.17	1.48
	Total	175	100	12.99	7.42	159.38	91.07	2.63	1.50
Equation (8.6)	Sector 1	38	100	0.15	0.39	36.45	95.91	1.41	3.70
	Sector 2	58	100	11.08	19.10	49.79	85.85	-2.87	-4.94
	Sector 3	79	100	4.40	5.57	75.77	95.91	-1.17	-1.48
	Total	175	100	15.62	8.93	162.01	92.58	-2.63	-1.50
Equation (8.7)	Sector 1	38	100	0.85	2.24	37.15	97.76	0.00	0.00
	Sector 2	58	100	9.64	16.63	48.36	83.37	0.00	0.00
	Sector 3	79	100	3.81	4.83	75.19	95.17	0.00	0.00
	Total	175	100	14.31	8.17	160.69	91.83	0.00	0.00

Problem 8.5

This problem explores further sector-specific and economy-wide structural decomposition with additional details for sectoral technology and final-demand decomposition of level, mix, and distribution using the input-output economy specified in problem 8.4. First, we assume that the final demand vectors can be specified with two components: $\mathbf{F}^0 = \begin{bmatrix} \mathbf{f}_1^0 & \mathbf{f}_2^0 \end{bmatrix} = \begin{bmatrix} 20 & 30 \\ 15 & 20 \\ 12 & 14 \end{bmatrix}$ and

$\mathbf{F}^1 = \begin{bmatrix} \mathbf{f}_1^1 & \mathbf{f}_2^1 \end{bmatrix} = \begin{bmatrix} 25 & 30 \\ 30 & 20 \\ 35 & 25 \end{bmatrix}$ (in both cases $\mathbf{f} = \mathbf{Fi}$).

The quantities $\mathbf{x}^0$, $\mathbf{x}^1$, $\mathbf{A}^0, \mathbf{A}^1$, $\mathbf{L}^0$, $\mathbf{L}^1$, $\Delta\mathbf{f}$, $\Delta\mathbf{x}$ and $\Delta\mathbf{L}$ were computed in problem 8.4. From **F** we can now compute distribution across final demand categories [from (8.13)]:

$$\mathbf{d}^0 = \begin{bmatrix} 47/111 \\ 64/111 \end{bmatrix} = \begin{bmatrix} 0.4234 \\ 0.5766 \end{bmatrix} \quad \text{and} \quad \mathbf{d}^1 = \begin{bmatrix} 58/111 \\ 53/111 \end{bmatrix} = \begin{bmatrix} 0.5455 \\ 0.4545 \end{bmatrix}$$

The bridge matrices [from (8.14)] are found as

$$\mathbf{B}^0 = \begin{bmatrix} 20 & 30 \\ 15 & 20 \\ 12 & 14 \end{bmatrix} \begin{bmatrix} 1/47 & 0 \\ 0 & 1/64 \end{bmatrix} = \begin{bmatrix} 0.4255 & 0.4545 \\ 0.3191 & 0.3636 \\ 0.2553 & 0.1818 \end{bmatrix} \text{ and}$$

$$\mathbf{B}^1 = \begin{bmatrix} 25 & 30 \\ 30 & 20 \\ 35 & 25 \end{bmatrix} \begin{bmatrix} 1/90 & 0 \\ 0 & 1/75 \end{bmatrix} = \begin{bmatrix} 0.2778 & 0.4000 \\ 0.3333 & 0.2667 \\ 0.3889 & 0.3333 \end{bmatrix}$$

And changes are computed as

$$\Delta\mathbf{d} = \begin{bmatrix} .1220 \\ -.1220 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} -.1478 & -.0687 \\ .0142 & -.0458 \\ .1336 & .1146 \end{bmatrix} \text{ and } \quad \Delta f = 54$$

Equation (8.31) defines, for $\Delta\mathbf{x}$, both the final-demand decomposition (including distribution across final-demand categories) and the technology change decomposition in the same expression, now including all six of the change components, i.e., the three-sector specific technology change components, the final demand level component, the final demand mix component, and the final demand distribution component:

$$\begin{aligned}
\Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\
&= \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(1)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 1}} + \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(2)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 2}} \\
&+ \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(3)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 3}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta f)(\mathbf{B}^0\mathbf{d}^0 + \mathbf{B}^1\mathbf{d}^1)}_{\text{Effect of change in final-demand level}} \\
&+ \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)[f^0(\Delta\mathbf{B})\mathbf{d}^1 + f^1(\Delta\mathbf{P})\mathbf{d}^0]}_{\text{Effect of change in final-demand mix}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(f^0\mathbf{B}^0 + f^1\mathbf{B}^1)(\Delta\mathbf{d})}_{\text{Effect of change in final-demand distribution}}
\end{aligned}$$

Application of equation (8.31) yields the various change contributions given in the table below.

	Output Change		Technology Change Contribution				Final Demand Change Contribution			
			Sector 1	Sector 2	Sector 3	Total	Level	Mix	Dist.	Total
Sector 1	38	Output	7.72	4.03	-10.90	0.85	51.97	-13.43	-1.39	37.15
		Percentage	20	11	-29	2	137	-35	-4	98
Sector 2	58	Output	7.43	3.52	-1.30	9.64	50.27	-2.59	0.67	48.36
		Percentage	13	6	-2	17	87	-4	1	83
Sector 3	79	Output	8.17	-9.27	4.91	3.81	61.79	12.67	0.73	75.19
		Percentage	10	-12	6	5	78	16	1	95
Total	175	Output	23.32	-1.72	-7.30	14.31	164.02	-3.34	0.01	160.69
		Percentage	13	-1	-4	8	94	-2	0	92

Chapter 9, Nonsurvey and Partial-Survey Methods: Fundamentals

Chapter 9 introduces approaches designed to deal with a major challenge in input–output analysis that the kinds of information-gathering surveys needed to collect input–output data for an economy can be expensive and very time consuming, resulting in tables of input–output coefficients that are outdated before they are produced. These techniques, known as partial survey and nonsurvey approaches to input–output table construction, are central to modern applications of input–output analysis.

The chapter begins by reviewing the basic factors contributing to the stability of input–output data over time, such as changing technology, prices, and the scale and scope of business enterprises. Several techniques for updating input–output data are developed and the economic implications of each described. The bulk of the chapter is concerned with the widely utilized biproportional scaling (or RAS) technique and some related "hybrid model" variants. The exercise problems for this chapter explore various nonsurvey approaches to assembling input-output tables and measures for the accuracy of such tables.

Problem 9.1

This exercise explores the adjustment of input-output tables to express input-output relationships in constant value terms in prices of another point in time using highly aggregated U.S. input-output tables for 1997[1], 2003 and 2005. The following are the make and use tables for these years all expressed in current year dollars.

US Use Matrix 1997	1	2	3	4	5	6	7	Imports
1 Agriculture	74,938	15	1,121	150,341	2,752	13,400	11	(23,123)
2 Mining	370	19,461	4,281	112,513	53,778	5,189	30	(64,216)
3 Construction	1,122	29	832	7,499	11,758	50,631	27	-
4 Manufacturing	49,806	19,275	178,903	1,362,660	169,915	418,412	1,914	(765,454)
5 Trade, Transport & Utilities	21,650	11,125	76,056	380,272	199,004	224,271	612	6,337
6 Services	32,941	45,234	107,723	483,686	545,779	1,592,426	3,801	(16,942)
7 Other	63	781	422	33,905	19,771	26,730	-	(126,350)
US Make Matrix 1997	1	2	3	4	5	6	7	Ind. Output
1 Agriculture	284,511	-	65	356	455	1,152	-	286,539
2 Mining	-	158,239	109	9,752	295	258	-	168,653
3 Construction	-	-	670,210	-	-	-	-	670,210
4 Manufacturing	-	727	1,258	3,703,275	39,720	36,034	3,669	3,784,683
5 Trade, Transport & Utilities	556	381	21,393	15,239	2,201,532	141,674	-	2,380,776
6 Services	-	410	54,850	1,306	109,292	6,444,098	1,821	6,611,778
7 Other	-	-	6,206	-	-	7,010	947,023	960,238
Total Commodity Output	285,067	159,757	754,091	3,729,928	2,351,295	6,630,226	952,513	14,862,876

[1] These tables differ from those provided in the supplemental resources for this text (described in Appendix B in the text) in that they reflect data assembled "before redefinitions" as discussed in Chapter 4.

US Use Matrix 2003	1	2	3	4	5	6	7	Imports
1 Agriculture	61,946	1	1,270	147,559	231	18,453	2,093	(26,769)
2 Mining	441	33,299	6,927	174,235	89,246	1,058	11,507	(125,508)
3 Construction	942	47	1,278	8,128	10,047	65,053	48,460	-
4 Manufacturing	47,511	22,931	265,115	1,249,629	132,673	516,730	226,689	(1,075,128)
5 Trade, Transport & Utilities	24,325	13,211	100,510	382,630	190,185	297,537	123,523	8,065
6 Services	25,765	42,276	147,876	509,084	490,982	2,587,543	442,674	(44,060)
7 Other	239	1,349	2,039	48,835	35,110	83,322	36,277	(177,578)

US Make Matrix 2003	1	2	3	4	5	6	7	Ind. Output
1 Agriculture	273,244	-	-	67	-	1,748	-	275,058
2 Mining	-	232,387	-	10,843	-	-	-	243,231
3 Construction	-	-	1,063,285	-	-	-	-	1,063,285
4 Manufacturing	-	-	-	3,856,583	-	30,555	3,278	3,890,416
5 Trade, Transport & Utilities	-	570	-	-	2,855,126	41	957	2,856,693
6 Services	-	475	-	-	133	9,136,001	3,278	9,139,886
7 Other	3,359	896	-	3,936	104,957	323,996	1,827,119	2,264,263
Total Commodity Output	276,602	234,328	1,063,285	3,871,429	2,960,216	9,492,341	1,834,631	19,732,832

US Use Matrix 2005	1	2	3	4	5	6	7	Imports
1 Agriculture	71,682	1	1,969	174,897	335	18,047	1,671	(31,248)
2 Mining	524	57,042	8,045	297,601	123,095	1,290	16,570	(226,059)
3 Construction	1,597	74	1,329	7,886	12,449	74,678	54,282	-
4 Manufacturing	61,461	34,860	339,047	1,452,738	183,135	589,452	255,456	(1,372,424)
5 Trade, Transport & Utilities	26,501	17,197	136,193	460,348	244,153	362,324	127,266	6,790
6 Services	27,274	52,297	165,179	543,690	610,978	3,017,728	529,779	(50,588)
7 Other	240	1,323	2,021	61,316	44,561	90,071	39,656	(208,971)

US Make Matrix 2005	1	2	3	4	5	6	7	Ind. Output
1 Agriculture	310,868	-	-	65	-	1,821	-	312,754
2 Mining	-	373,811	-	22,752	-	-	-	396,563
3 Construction	-	-	1,302,388	-	-	-	-	1,302,388
4 Manufacturing	-	-	-	4,454,957	-	26,106	4,467	4,485,529
5 Trade, Transport & Utilities	-	808	-	-	3,354,043	47	1,046	3,355,944
6 Services	-	556	-	-	152	10,473,161	3,771	10,477,640
7 Other	4,657	1,410	-	4,111	115,428	339,582	2,061,136	2,526,325
Total Commodity Output	315,525	376,586	1,302,388	4,481,885	3,469,622	10,840,717	2,070,419	22,857,143

First, as one variant, we produce industry-by-industry transactions tables using the assumption of industry-based technology for these three years. That is, for each year from the corresponding table if the make and use matrices are $\mathbf{V}$ and $\mathbf{U}$, respectively, and the total industry and commodity outputs are $\mathbf{x}$ and $\mathbf{g}$, respectively, we construct transactions tables in current dollar terms by computing $\mathbf{Z} = \mathbf{DB}\hat{\mathbf{x}}$ where $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$ and $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1}$:

Z(1997)	1	2	3	4	5	6	7
1	74,807	27	1,170	150,337	2,897	13,738	12
2	501	19,330	4,722	115,074	53,759	6,331	35
3	997	26	739	6,665	10,450	44,999	24
4	49,998	19,663	179,517	1,362,631	175,369	428,076	1,932
5	21,358	11,509	74,280	372,728	199,152	247,197	663
6	33,123	44,541	108,369	489,159	540,798	1,562,040	3,725
7	106	825	540	34,282	20,330	28,677	4

Z(2003)	1	2	3	4	5	6	7
1	61,199	9	1,286	145,882	321	18,714	2,153
2	570	33,088	7,612	176,292	88,879	2,496	12,046
3	942	47	1,278	8,128	10,047	65,053	48,460
4	47,412	22,981	264,578	1,246,562	133,807	523,227	227,310
5	23,463	12,824	96,960	369,498	183,671	287,032	119,187
6	24,800	40,759	142,346	490,430	472,802	2,490,571	426,150
7	2,782	3,406	10,953	83,306	58,947	182,603	55,918

Z(2005)	1	2	3	4	5	6	7
1	70,629	10	1,973	172,428	435	18,296	1,739
2	832	56,798	9,707	302,783	123,117	4,273	17,745
3	1,597	74	1,329	7,886	12,449	74,678	54,282
4	61,158	34,779	337,412	1,445,451	183,602	593,372	255,282
5	25,620	16,748	131,675	445,685	236,309	350,316	123,084
6	26,352	50,611	159,600	525,827	590,537	2,915,594	511,919
7	3,091	3,773	12,087	98,416	72,256	197,062	60,628

Suppose historical price indices for these tables are given in the following table (price indices in percent relative to some arbitrary earlier year):

	1997	2003	2005
Agriculture	100	113.5	122.7
Mining	96.6	131.3	201
Construction	181.6	188.9	209.9
Manufactuirng	133.7	150.8	156.9
Trade, Transport & Utilities	200.4	205.7	217.1
Services	129.3	151.6	219.8
Other	140	144.7	161.4

To generate price indices relative to the year 2005, the elements in each row of the historical price indices are divided by the last element in that row to yield the following table of relative price indices:

	1997	2003	2005
1 Agriculture	0.815	0.925	1
2 Mining	0.481	0.653	1
3 Construction	0.865	0.900	1
4 Manufacturing	0.852	0.961	1
5 Trade, Transport & Utilities	0.923	0.947	1
6 Services	0.588	0.690	1
7 Other	0.867	0.897	1

The constant price transactions tables expressed relative to 2005 dollars are then found as $\mathbf{Z}(1997)^{(2005)} = \hat{\mathbf{p}}_{1997}^{(2005)}\mathbf{Z}(1997)$ where $\hat{\mathbf{p}}_{1997}^{(2005)}$ is a matrix with the first column of the relative price table placed along the diagonal and zeros elsewhere. The matrix $\mathbf{Z}(2003)^{(2003)}$ is computed in the same manner, i.e., $\mathbf{Z}(2003)^{(2005)} = \hat{\mathbf{p}}_{2003}^{(2005)}\mathbf{Z}(2003)$ where $\hat{\mathbf{p}}_{2003}^{(2005)}$ is matrix of price indices converting 2003 to 2005 year prices, but $\mathbf{Z}(2005)^{(2005)} = \hat{\mathbf{p}}_{2005}^{(2005)}\mathbf{Z}(2005)$ is, of course, identical to the $\mathbf{Z}(2005)$ since 2005 is the base year of the price indices, i.e., $\hat{\mathbf{p}}_{2005}^{(2005)} = \mathbf{I}$:

$\mathbf{Z}(1997)^{2005}$	1	2	3	4	5	6	7
1	60,967	22	953	122,524	2,361	11,197	10
2	241	9,290	2,269	55,304	25,837	3,043	17
3	863	22	640	5,766	9,041	38,932	21
4	42,605	16,755	152,973	1,161,145	149,438	364,778	1,646
5	19,715	10,624	68,566	344,057	183,832	228,182	612
6	19,485	26,202	63,749	287,754	318,131	918,889	2,191
7	92	715	469	29,737	17,635	24,874	4

$\mathbf{Z}(2003)^{2005}$	1	2	3	4	5	6	7
1	56,611	8	1,190	134,944	297	17,311	1,991
2	372	21,614	4,973	115,160	58,059	1,631	7,869
3	847	42	1,150	7,315	9,042	58,544	43,612
4	45,568	22,088	254,292	1,198,098	128,605	502,885	218,472
5	22,231	12,150	91,869	350,096	174,027	271,960	112,928
6	17,105	28,112	98,179	338,258	326,100	1,717,792	293,923
7	2,494	3,053	9,820	74,687	52,848	163,709	50,132

Problem 9.2

This exercise explores measurement of year-to-year changes in technical coefficients of an input-output model as the average of the absolute value of differences between the column sums of **A** for the same industry sectors in two different years using the series of transactions tables developed in exercise problem 9.1.

We arbitrarily pick the years 1997 and 2005 with 2005 assumed to be the base year. First, we must compute the technical coefficients matrices for 2005 and 1997 expressed in current year prices, $\mathbf{A}(2005) = \mathbf{Z}(2005)\hat{\mathbf{x}}(2005)^{-1}$ and $\mathbf{A}(1997) = \mathbf{Z}(1997)\hat{\mathbf{x}}(1997)^{-1}$, as well as the technical

coefficient matrix for 1997 expressed in 2005 (the base year) prices:
$\mathbf{A}(1997)^{(2005)} = \mathbf{Z}(1997)^{(2005)}\left[\hat{\mathbf{x}}(1997)^{(2005)}\right]^{-1}$:

A (1997)	1	2	3	4	5	6	7
1	0.2611	0.0002	0.0017	0.0397	0.0012	0.0021	0.0000
2	0.0017	0.1146	0.0070	0.0304	0.0226	0.0010	0.0000
3	0.0035	0.0002	0.0011	0.0018	0.0044	0.0068	0.0000
4	0.1745	0.1166	0.2679	0.3600	0.0737	0.0647	0.0020
5	0.0745	0.0682	0.1108	0.0985	0.0836	0.0374	0.0007
6	0.1156	0.2641	0.1617	0.1292	0.2272	0.2363	0.0039
7	0.0004	0.0049	0.0008	0.0091	0.0085	0.0043	0.0000

A(2005)	1	2	3	4	5	6	7
1	0.2258	0.0000	0.0015	0.0384	0.0001	0.0017	0.0007
2	0.0027	0.1432	0.0075	0.0675	0.0367	0.0004	0.0070
3	0.0051	0.0002	0.0010	0.0018	0.0037	0.0071	0.0215
4	0.1955	0.0877	0.2591	0.3222	0.0547	0.0566	0.1010
5	0.0819	0.0422	0.1011	0.0994	0.0704	0.0334	0.0487
6	0.0843	0.1276	0.1225	0.1172	0.1760	0.2783	0.2026
7	0.0099	0.0095	0.0093	0.0219	0.0215	0.0188	0.0240

$A(1997)^{(2005)}$	1	2	3	4	5	6	7
1	0.2611	0.0003	0.0016	0.0380	0.0011	0.0029	0.0000
2	0.0010	0.1146	0.0039	0.0171	0.0118	0.0008	0.0000
3	0.0037	0.0003	0.0011	0.0018	0.0041	0.0100	0.0000
4	0.1824	0.2067	0.2638	0.3600	0.0680	0.0938	0.0020
5	0.0844	0.1311	0.1182	0.1067	0.0836	0.0587	0.0007
6	0.0834	0.3233	0.1099	0.0892	0.1448	0.2363	0.0026
7	0.0004	0.0088	0.0008	0.0092	0.0080	0.0064	0.0000

Then, using the constant price technical coefficient tables, i.e., $\mathbf{A}(2005)$ and $\mathbf{A}(1997)^{(2005)}$, we compute the average of the absolute value of differences between the column sums of $\mathbf{A}$ for each industry:

$$\frac{1}{7}[\mathbf{i}'\left|\mathbf{A}(2005)-\mathbf{A}(1997)^{(2005)}\right|]=\begin{bmatrix}.009 & .062 & .007 & .02 & .014 & .017 & .057\end{bmatrix}.$$

The most changed sectors in decreasing order are 2, 7 and 4. If we, instead, compare the current price tables, $\mathbf{A}(2005)$ and $\mathbf{A}(1997)$, these values are:

$$\frac{1}{7}[\mathbf{i}'\left|\mathbf{A}(2005)-\mathbf{A}(1997)\right|]=\begin{bmatrix}.015 & .032 & .01 & .015 & .016 & .01 & .057\end{bmatrix}.$$

In this case, the most changed sectors in decreasing order are 7, 2 and 5. The differences in rates of inflation explain the difference between the comparisons in constant and current year prices.

Problem 9.3

This exercise problem illustrates the computation of marginal technical coefficients, using the current price transactions tables developed in problem 9.1 between the years 1997 and 2005. The matrix of marginal input coefficients, computed as $\left[\mathbf{Z}(2005)-\mathbf{Z}(1997)\right]\left[\langle\mathbf{x}(2005)-\mathbf{x}(1997)\rangle\right]^{-1}$, is:

	1	2	3	4	5	6	7
1	-0.1594	-0.0001	0.0013	0.0315	-0.0025	0.0012	0.0011
2	0.0126	0.1644	0.0079	0.2678	0.0711	-0.0005	0.0113
3	0.0229	0.0002	0.0009	0.0017	0.0020	0.0077	0.0346
4	0.4257	0.0663	0.2498	0.1182	0.0084	0.0428	0.1618
5	0.1626	0.0230	0.0908	0.1041	0.0381	0.0267	0.0782
6	-0.2583	0.0266	0.0810	0.0523	0.0510	0.3501	0.3245
7	0.1139	0.0129	0.0183	0.0915	0.0532	0.0436	0.0387

Note that these marginal coefficients deal with changes, so negative entries can appear and do in this case in industries 1, 2, 5, and 6.

Problem 9.4

This exercise problem illustrates use of the so-called RAS technique of biproportional scaling to generate an estimate of a future technical coefficients table for an economy based on a previous year's table and future estimates of the vectors for total final demand, total value-added, and total output.

Consider the following interindustry transactions and total outputs two-sector input-output economy for the year 2020:

2020	A	B	Total Output
A	1	2	10
B	3	4	10

Estimates for the year 2030 for the vectors of total final demand, total value-added, and total output are the following:

2030	Final Demand	Value Added	Total Output
A	12	10	25
B	6	8	20

To use the 2020 table as a base and the 2030 projections for final demand, value-added and total output in computing an estimate of the 2030 technical coefficients table first, from the matrix of transactions, $\mathbf{Z}(0)=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, and vector of total outputs, $\mathbf{x}(0)=\begin{bmatrix}10\\ 10\end{bmatrix}$, we can compute $\mathbf{A}(0)=\mathbf{Z}(0)\hat{\mathbf{x}}(0)^{-1}=\begin{bmatrix}.1 & .2\\ .3 & .4\end{bmatrix}$. We can compute $\mathbf{u}(1)=\mathbf{x}(1)-\mathbf{f}(1)=\begin{bmatrix}25\\ 20\end{bmatrix}-\begin{bmatrix}12\\ 6\end{bmatrix}=\begin{bmatrix}13\\ 14\end{bmatrix}$ and $\mathbf{v}(1)=\mathbf{x}(1)-\mathbf{va}(1)=\begin{bmatrix}25\\ 20\end{bmatrix}-\begin{bmatrix}10\\ 8\end{bmatrix}=\begin{bmatrix}15\\ 12\end{bmatrix}$. [Here we use **va**(1) for the value added vector in 2030 to differentiate it from **v**(1), the total intermediate inputs vector in 2030.] Performing the RAS procedure using **A**(0), **u**(1), **v**(1) and **x**(1) converges to the 2030 matrix in 6 iterations. That is, the result is $\tilde{\mathbf{A}}(1)=\begin{bmatrix}.262 & .323\\ .338 & .277\end{bmatrix}$, such that $\mathbf{u}(1)=\tilde{\mathbf{A}}(1)\mathbf{x}(1)$ and $\mathbf{v}(1)=\mathbf{i}'\tilde{\mathbf{A}}(1)\hat{\mathbf{x}}(1)$ within .0001 for each element of the intermediate inputs and outputs vectors, **u**(1) and **v**(1), respectively.

Problem 9.5

This exercise explores measurement of error between an RAS-estimated table of technical coefficients and a "real" table with the mean absolute percentage error (MAPE) of the element-by-element comparison of the two tables as the error metric. In this problem we use the 1997 input-output table expressed in 1997 dollars constructed in problem 9.1 and the vectors of intermediate inputs, intermediate outputs, and total outputs from the corresponding input-output table for 2005.

First, the 1997 input-output table, $\mathbf{A}(1997)=\mathbf{Z}(1997)\mathbf{x}(1997)^{-1}$ was computed in exercise problem 9.2. We can retrieve the year 2005 total outputs, **x**(2005), from exercise problem 9.1 and compute the year 2005 intermediate outputs, $\mathbf{u}(2005)=\mathbf{Z}(2005)\mathbf{i}$, and intermediate inputs, $\mathbf{v}(2005)=\mathbf{i}'\mathbf{Z}(2005)$, all given in the following table:

	1	2	3	4	5	6	7
u(2005)'	265,510	515,254	152,295	2,911,056	1,329,436	4,780,440	447,314
v(2005)'	189,279	162,793	653,783	2,998,476	1,218,705	4,153,590	1,024,680
x(2005)	312,754	396,563	1,302,388	4,485,529	3,355,944	10,477,640	2,526,325

Performing the RAS procedure using **A**(1997), **u**(2005), **v**(2005) and **x**(2005), yields the RAS-estimate of **A**(2005), which we designate as $\tilde{\mathbf{A}}(2005)$, given in the following table:

$\tilde{\mathbf{A}}(2005)$	1	2	3	4	5	6	7
1	0.2448	0.0001	0.0015	0.0357	0.0009	0.0021	0.0007
2	0.0037	0.1423	0.0140	0.0622	0.0373	0.0022	0.0048
3	0.0052	0.0001	0.0015	0.0025	0.0050	0.0109	0.0023
4	0.1592	0.0618	0.2274	0.3148	0.0519	0.0638	0.1129
5	0.0737	0.0392	0.1020	0.0933	0.0639	0.0400	0.0420
6	0.1172	0.1557	0.1525	0.1256	0.1780	0.2588	0.2419
7	0.0015	0.0112	0.0030	0.0343	0.0261	0.0185	0.0011

For the 2005 "real" input-output table, $\mathbf{A}(2005) = \mathbf{Z}(2005)\mathbf{x}(2005)^{-1}$ (also derived in exercise problem 9.2), since there are a total of $7 \times 7 = 49$ elements to compare, the MAPE is computed as $(\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7} 100 \times \left[\frac{\left|\tilde{a}_{ij}(2005) - a_{ij}(2005)\right|}{a_{ij}(2005)}\right] = 49.028$ [for $a_{ij}(2005) \neq 0$ and 0 otherwise]. Note that in this case the RAS estimate is very weak since the average error is nearly 50 percent.

Problem 9.6

This exercise demonstrates an example of the equivalence of performing an RAS-estimate using either interindustry transactions or technical coefficients. Suppose we have a baseline transactions matrix defined as $\mathbf{Z}(0) = \begin{bmatrix} 100 & 55 & 5 \\ 50 & 75 & 45 \\ 25 & 10 & 110 \end{bmatrix}$. We are provided with estimates of intermediate inputs and outputs, $\mathbf{v}(1) = \begin{bmatrix} 265 \\ 225 \\ 325 \end{bmatrix}$ and $\mathbf{u}(1) = \begin{bmatrix} 325 \\ 235 \\ 255 \end{bmatrix}$, respectively.

To compute an estimate of the transactions table for the next year, $\tilde{\mathbf{Z}}^Z(1)$, if $\mathbf{Z}(0)$, $\mathbf{v}(1)$ and $\mathbf{u}(1)$ are known, we use the RAS technique to biproportionately scale $\mathbf{Z}(0)$ iteratively to convergence of $\mathbf{u}(1) = \tilde{\mathbf{Z}}^Z(1)\mathbf{i}$ and $\mathbf{v}(1) = \mathbf{i}'\tilde{\mathbf{Z}}^Z(1)$ within .0001 for each element of $\mathbf{u}(1)$ and $\mathbf{v}(1)$ to yield: $\tilde{\mathbf{Z}}^Z(1) = \begin{bmatrix} 167.5 & 104.5 & 53 \\ 61.2 & 104.1 & 69.7 \\ 36.3 & 16.5 & 202.2 \end{bmatrix}$.

Alternatively, suppose we know the vector of total outputs, $\mathbf{x}(0) = \begin{bmatrix} 750 \\ 500 \\ 1{,}000 \end{bmatrix}$, corresponding to $\mathbf{Z}(0)$, and we also have an estimate of total outputs for next year, $\mathbf{x}(1) = \begin{bmatrix} 1{,}000 \\ 750 \\ 1{,}500 \end{bmatrix}$. Compute $\mathbf{A}(0) = \mathbf{Z}(0)\hat{\mathbf{x}}(0) = \begin{bmatrix} .133 & .11 & .025 \\ .067 & .15 & .045 \\ .033 & .02 & .11 \end{bmatrix}$ and use it [rather than $\mathbf{Z}(0)$] along with $\mathbf{v}(1)$ and $\mathbf{u}(1)$ to generate an estimate of the technical coefficients matrix for next year using the RAS technique, we find $\tilde{\mathbf{A}}^A(1) = \begin{bmatrix} .168 & .139 & .035 \\ .061 & .139 & .047 \\ .036 & .022 & .135 \end{bmatrix}$. If we also compute the matrix of technical coefficients matrix from the $\tilde{\mathbf{Z}}^Z(1)$ and $\hat{\mathbf{x}}(1)^{-1}$, we find that $\tilde{\mathbf{A}}^Z(1) = \tilde{\mathbf{Z}}^Z(1)\hat{\mathbf{x}}(1)^{-1} = \begin{bmatrix} .168 & .139 & .035 \\ .061 & .139 & .047 \\ .036 & .022 & .135 \end{bmatrix}$, which is identical to $\tilde{\mathbf{A}}^Z(1)$. The explanation for why this is true generally is discussed in section 9.4.3 of the text.

Problem 9.7

This exercise problem explores the prospects using of partial information about a target technical coefficients matrix to improve an RAS-estimated technical coefficients table compared with estimation absent such information. For the economy in problem 9.6, suppose we acquire a survey-based table of technical coefficients for next year of $\mathbf{A}(1) = \begin{bmatrix} .2 & .1 & .033 \\ .035 & .167 & .05 \\ .03 & .033 & .133 \end{bmatrix}$, which we consider to be the "real" target technical coefficients matrix since it is based on more comprehensive information.

At the beginning of the survey, however, suppose we know only $a(1)_{32} = .033$ of the nine survey-based coefficients and we use that value along with $\mathbf{A}(0), \mathbf{v}(1)$ and $\mathbf{u}(1)$ to generate an intermediate estimate of the entire matrix of coefficients, $\underset{\sim}{\tilde{\mathbf{A}}}(1)$. To so this we first define the matrix of known coefficients for the target table as $\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & .033 & 0 \end{bmatrix}$ and the reference table, $\bar{\mathbf{A}}(0) = \begin{bmatrix} .133 & .11 & .025 \\ .067 & .15 & .047 \\ .033 & 0 & .110 \end{bmatrix}$, where $a(0)_{32}$ (the location of the known coefficient) is set to 0. We must also revise $\mathbf{u}(1)$ and $\mathbf{v}(1)$ to reflect removal of the interindustry transaction associated with the know information, which we can compute as

$$\bar{\mathbf{u}}(1) = \mathbf{u}(1) - \mathbf{K}\mathbf{x}(1) = \begin{bmatrix} 325 \\ 235 \\ 255 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 24.75 \end{bmatrix} = \begin{bmatrix} 325 \\ 235 \\ 230.25 \end{bmatrix} \text{ and}$$

$$\bar{\mathbf{v}}(1) = \bar{\mathbf{v}}(1) - \mathbf{i}\mathbf{K}\hat{\mathbf{x}}(1) = \begin{bmatrix} 265 \\ 225 \\ 325 \end{bmatrix} - \begin{bmatrix} 0 \\ 24.75 \\ 0 \end{bmatrix} = \begin{bmatrix} 265 \\ 200.25 \\ 325 \end{bmatrix}.$$

The "intermediate estimate," $\underset{\sim}{\tilde{\mathbf{A}}}(1)$, for this case, is then found by adding $\mathbf{K}$ to the result of applying the RAS procedure using $\bar{\mathbf{A}}(1), \bar{\mathbf{u}}(1), \bar{\mathbf{v}}(1)$, and $\mathbf{x}(1)$, to yield $\underset{\sim}{\tilde{\mathbf{A}}}(1) = \begin{bmatrix} .169 & .134 & .037 \\ .062 & .133 & .049 \\ .034 & .033 & .131 \end{bmatrix}$, including the known value for $a(1)_{32}$. The MAPE for the RAS estimate, $\tilde{\mathbf{A}}(1)$, which excludes the additional information about $a(1)_{32}$, compared with the known $\mathbf{A}(1)$ is 24.05. The MAPE for the modified RAS estimate, $\tilde{\mathbf{A}}(1)$ (including the known coefficient), is 19.5, which we record for this case as $\underset{\sim}{\tilde{\mathbf{A}}}(1)^{(case1)} = 19.5$. The MAPE value is lower so the estimate with the additional information is better.

For a second case, we assume instead that we know only $a(1)_{33} = 0.133$, i.e., instead of $a(1)_{32} = 0.033$. If we apply the same procedure to determine $\tilde{\mathbf{A}}(1)$, we find that the MAPE for the modified RAS estimate, $\underset{\sim}{\tilde{\mathbf{A}}}(1)$, including the alternative known coefficient, is $\underset{\sim}{\tilde{\mathbf{A}}}(1)^{(case2)} = 24.18$. Recall that the MAPE of the estimate without additional information is 24.05, which is lower than that of the modified estimate in this case, so the estimate without additional information is better in this case. In general, introduction of more accurate exogenous information in applying RAS improves the resulting estimates, but it is not always the case as discussed in section 9.4.6 of the text.

Problem 9.8

This problem illustrates two degenerate cases that occur in applying RAS. First, consider the transactions matrix $\mathbf{Z}(0) = \begin{bmatrix} 100 & 55 & 25 \\ 0 & 75 & 25 \\ 25 & 10 & 110 \end{bmatrix}$ and projected vectors of intermediate inputs and outputs, $\mathbf{v}(1) = \begin{bmatrix} 125 \\ 140 \\ 160 \end{bmatrix}$ and $\mathbf{u}(1) = \begin{bmatrix} 180 \\ 100 \\ 145 \end{bmatrix}$, respectively. In this case $\mathbf{u}(1)$ and $\mathbf{v}(1)$ are identical to $\mathbf{u}(0)$ and $\mathbf{v}(0)$, respectively, so an RAS procedure attempting to produce $\tilde{\mathbf{Z}}(1)$ will converge immediately and is, of course, unnecessary.

If we project $v_1(1) = 100$ instead of 125, By reducing $v_1(1)$ to substantially below the existing value, without any other changes, then $\mathbf{i}'\mathbf{u}(1) \neq \mathbf{i}'\mathbf{v}(1)$. Successive RAS adjustments in this case fail to converge since both row and column constraints in the RAS procedure cannot be satisfied simultaneously.

Problem 9.9

The exercise explores the degree to which the accuracy of RAS estimates of technical coefficients relate to that of the total requirements matrices. We use the U.S. input-output tables for 1997 and 2005 (from problem 9.1, expressed in current dollars rather than constant dollars).

The matrices $\mathbf{A}$(1997), $\mathbf{A}$(2005) and $\tilde{\mathbf{A}}(2005)$ [produced by using RAS with $\mathbf{A}$(1997)], $\mathbf{u}$(2005), $\mathbf{v}$(2005) and $\mathbf{x}$(2005), were all computed in problems 7.1 and 7.5. The MAPE for $\tilde{\mathbf{A}}(2005)$ compared with $\mathbf{A}$(2005) is 49.03. The MAPE for $\tilde{\mathbf{L}}(2005) = [\mathbf{I} - \tilde{\mathbf{A}}(2005)]^{-1}$ compared with $\mathbf{L}$(2005) is 12.33, where the matrices $\mathbf{L}$(2005) and $\tilde{\mathbf{L}}(2005)$ are computed as:

L(2005)	1	2	3	4	5	6	7
1	1.3139	0.0102	0.0247	0.0789	0.0076	0.0103	0.0122
2	0.0462	1.1863	0.0515	0.1331	0.0584	0.0152	0.0296
3	0.0109	0.0034	1.0054	0.0075	0.0074	0.0116	0.0257
4	0.4324	0.1907	0.4421	1.5707	0.1332	0.1404	0.2098
5	0.1773	0.0865	0.1737	0.1969	1.1072	0.0714	0.0950
6	0.2861	0.2701	0.3053	0.3508	0.3136	1.4409	0.3600
7	0.0330	0.0231	0.0300	0.0486	0.0342	0.0329	1.0390

$\tilde{\mathbf{L}}$(2005)	1	2	3	4	5	6	7
1	1.3426	0.0081	0.0218	0.0746	0.0084	0.0114	0.0126
2	0.0407	1.1812	0.0534	0.1223	0.0585	0.0188	0.0266
3	0.0124	0.0044	1.0073	0.0094	0.0095	0.0164	0.0078
4	0.3670	0.1485	0.3931	1.5503	0.1297	0.1533	0.2197
5	0.1607	0.0795	0.1686	0.1853	1.1006	0.0807	0.0876
6	0.3324	0.3030	0.3381	0.3682	0.3148	1.4147	0.3999
7	0.0254	0.0261	0.0278	0.0665	0.0398	0.0339	1.0187

Chapter 10, Nonsurvey and Partial-Survey Methods: Extensions

Chapter 10 surveys a range of partial survey and nonsurvey estimation approaches for creating input–output tables at the regional level. Variants of the commonly used class of estimating procedures using location quotients are reviewed; these presume a regional estimate of input–output data can be derived using some information about a target region. Cross-hauling is discussed and approaches to address it are presented.

The RAS technique developed in Chapter 9 is applied using a base national table or a table for another region and some available data for the target region. Techniques for partial survey estimation of commodity flows between regions are also presented along with discussions of several real-world multinational applications, including the China–Japan Transnational Interregional Model and Leontief's World Model. The exercise problems for this chapter explore application of nonsurvey techniques for generating regional input-output models.

Problem 10.1

The exercise explores the use of the RAS technique to develop and use input-output tables for target economies with similar basic structural characteristics. Consider three different nations. The first, the economy of the Land of Lilliput, is described by the following input-output table:

	Interindustry Transactions		Total Outputs
	A	*B*	
A	1	6	20
B	4	2	15

The Land of Brobdingnag is described by another input-output table:

	Interindustry Transactions		Total Outputs
	A	*B*	
A	7	4	35
B	1	5	15

And finally, the economy of the distant land of the Houyhnhnms is described by yet another input-output table:

	Interindustry Transactions		Total Outputs
	A	*B*	
A	20	30.67	100
B	2.86	38.3	115

First, we compute the vectors of value-added, intermediate inputs, final-demand, and intermediate outputs for each economy, shown in the following table:

	Lilliput (L)	Brobdingnag (B)	Houyhnhnm (H)
Value Added	$[15 \quad 7]$	$[27 \quad 6]$	$[77.14 \quad 46.03]$
Intermediate Inputs ($\mathbf{v} = \mathbf{i}'\mathbf{Z}$)	$[5 \quad 8]$	$[8 \quad 9]$	$[22.86 \quad 68.97]$
Final Demands	$\begin{bmatrix} 13 \\ 9 \end{bmatrix}$	$\begin{bmatrix} 24 \\ 9 \end{bmatrix}$	$\begin{bmatrix} 49.33 \\ 73.84 \end{bmatrix}$
Intermediate Outputs ($\mathbf{u} = \mathbf{Zi}$)	$\begin{bmatrix} 7 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 50.67 \\ 41.16 \end{bmatrix}$

A Lilliputian economist is interested in examining the structure of the Brobdingnagian economy. Likewise, a Brobdingnagian economist is interested in examining the structure of the Lilliputian economy. However, each economist only has available to him the value-added, final-demand, and total-output vectors for the foreign economy. Each economist knows the RAS modification procedure and uses it with the technical coefficients matrix of her own economy serving as the base **A** matrix. To determine which of the two economists calculates a better estimate of the foreign economy's technical coefficients matrix in terms of mean absolute deviation (all elements of **A**), first we compute the true technical coefficients matrices for each economy: $\mathbf{A}^L = \begin{bmatrix} .050 & .400 \\ .200 & .133 \end{bmatrix}$ and $\mathbf{A}^B = \begin{bmatrix} .200 & .267 \\ .029 & .333 \end{bmatrix}$. We denote the L estimate of the $\mathbf{A}^B$ matrix as ${}^L\mathbf{A}^B = \begin{bmatrix} .088 & .529 \\ .141 & .071 \end{bmatrix}$; we use the metric of mean absolute deviation (MAD) to measure the relative accuracy of between ${}^L\mathbf{A}^B$ as an estimate of $\mathbf{A}^B$, which is 0.187. The B estimate of $\mathbf{A}^L$ is ${}^B\mathbf{A}^L = \begin{bmatrix} .207 & .190 \\ .043 & .343 \end{bmatrix}$, with a MAD comparing of ${}^B\mathbf{A}^L$ as an estimate of $\mathbf{A}^L$ found as 0.183. Therefore, the Brobdingnagian economist does slightly better.

Suppose now that an economist in the distant land of the Houyhnhnms learned of the two other economies from a world traveler. She becomes interested in the economic structures of these foreign lands but is only able to obtain the final-demand, value-added, and total-output vectors for each economy from the world traveler. The economist uses RAS with her own country's **A** matrix as a base to estimate the interindustry structure of the two distant lands. The two Houyhnhnm estimates are ${}^H\mathbf{A}^L = \begin{bmatrix} .207 & .190 \\ .043 & .343 \end{bmatrix}$ and ${}^H\mathbf{A}^B = \begin{bmatrix} .200 & .267 \\ .029 & .333 \end{bmatrix}$, respectively (note that $\mathbf{A}^H = \mathbf{A}^B = {}^H\mathbf{A}^B = \begin{bmatrix} .200 & .267 \\ .029 & .333 \end{bmatrix}$). The error, as measured by MAD, is 0.183 in the first case and, of course, zero in the second case since $\mathbf{A}^H = \mathbf{A}^B$, i.e., the Houyhnhnm and Brobdingnagian economies are identical.

Suppose now that the Land of Lilliput plans to build a new power plant which will require the following value of output (in millions of dollars) from each of the economy's industries (directly, so it can be thought of as a final demand presented to the Lilliputian economy) of $\mathbf{f} = [100 \quad 150]'$. To measure the accuracy of the Houyhnhnms' estimate of the total industrial activity (output) in the Lilliputian economy required to construct this power plant, measured as an average mean absolute deviation, we first compute the true impact as

$\Delta\mathbf{x}^L = \mathbf{L}^L \Delta\mathbf{f} = \begin{bmatrix} 197.3 \\ 218.6 \end{bmatrix}$ for $\Delta\mathbf{f} = \begin{bmatrix} 100 \\ 150 \end{bmatrix}$ and $\mathbf{L}^L = (\mathbf{I} - \mathbf{A}^L)^{-1} = \begin{bmatrix} 1.166 & .538 \\ .269 & 1.278 \end{bmatrix}$. Using the same final demand vector with $(\mathbf{I} - {}^H\mathbf{A}^L)^{-1} = \begin{bmatrix} 1.281 & .371 \\ .083 & 1.546 \end{bmatrix}$ yields $\Delta[{}^H\mathbf{x}^L] = \begin{bmatrix} 183.8 \\ 240.3 \end{bmatrix}$. The mean absolute deviation between these two vectors is 17.6.

Problem 10.2

This exercise expands the economies given in problem 10.1 to three economic sectors in each economy. The Land of Lilliput is described by the following input-output table:

	Interindustry Transactions			Total Outputs
	A	*B*	*C*	
A	1	6	6	20
B	4	2	1	15
C	4	1	1	12

The economy of the neighboring land of Brobdingnag is described by another input-output table:

	Interindustry Transactions			Total Outputs
	A	*B*	*C*	
A	7	4	8	35
B	1	5	1	15
C	6	2	7	30

The economy of the distant land of Houyhnhnms is described by yet another input-output table:

	Interindustry Transactions			Total Outputs
	A	*B*	*C*	
A	5.5	33	33	110
B	22	11	5.5	82.5
C	22	5.5	5.5	66

First, we find $\mathbf{A}^L = \begin{bmatrix} .050 & .400 & .500 \\ .200 & .133 & .083 \\ .200 & .067 & .083 \end{bmatrix}$ and $\mathbf{A}^B = \begin{bmatrix} .200 & .267 & .267 \\ .029 & .333 & .033 \\ .171 & .133 & .233 \end{bmatrix}$. The RAS estimates are ${}^B\mathbf{A}^L = \begin{bmatrix} .264 & .199 & .395 \\ .052 & .343 & .068 \\ .134 & .059 & .204 \end{bmatrix}$ and ${}^L\mathbf{A}^B = \begin{bmatrix} .033 & .460 & .365 \\ .106 & .123 & .049 \\ .261 & .151 & .120 \end{bmatrix}$. The table of value added, intermediate inputs, final demands, and intermediate outputs of the economies are given in the following table:

	Lilliput (L)	Brobdingnag (B)	Houyhnhnm (H)
Value Added ($\mathbf{v}'$)	[11 6 4]	[21 4 14]	[60.5 33.0 22.0]
Intermediate Inputs ($\mathbf{i}'\mathbf{Z}$)	[9 9 8]	[14 11 16]	[49.5 49.5 44.0]
Final Demands ($\mathbf{f}$)	$\begin{bmatrix} 7 \\ 8 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 16 \\ 8 \\ 15 \end{bmatrix}$	$\begin{bmatrix} 38.5 \\ 44.0 \\ 33.0 \end{bmatrix}$
Intermediate Outputs ($\mathbf{u}$)	$\begin{bmatrix} 13 \\ 7 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 19 \\ 7 \\ 15 \end{bmatrix}$	$\begin{bmatrix} 71.5 \\ 38.5 \\ 33.0 \end{bmatrix}$

The mean absolute deviation (MAD) for the L estimate of B is 0.109, while the MAD for the B estimate of L is 0.121.

The Houyhnhnm estimates of L and B, respectively are ${}^{H}\mathbf{A}^{L} = \begin{bmatrix} .050 & .400 & .500 \\ .200 & .133 & .083 \\ .200 & .067 & .083 \end{bmatrix}$ and

${}^{H}\mathbf{A}^{B} = \begin{bmatrix} .033 & .460 & .365 \\ .106 & .123 & .049 \\ .261 & .151 & .120 \end{bmatrix}$. Note that in this case that $\mathbf{A}^{H} = \mathbf{A}^{L}$, i.e., it is the Houyhnhnm and Lilliputian economies that are identical, so the error of the Houyhnhnm estimate of the Lilliputian economy, ${}^{H}\mathbf{A}^{L}$, compared with the true Lilliputian economy, $\mathbf{A}^{L}$, as measured by the MAD, is 0.0. The MAD for the Houyhnhnm estimate of the Brobdingnagian economy, ${}^{H}\mathbf{A}^{B}$, compared with the true Brobdingnagian economy, $\mathbf{A}^{B}$, is 0.109.

Problem 10.3

This exercise illustrates the considerations of analysis costs in estimation and impact analysis. Consider the following input-output transactions and total outputs table for Region 1:

	A	*B*	Total Output
A	1	2	10
B	3	4	10

We are interested in determining the impact of a particular final demand in another region (Region 2). Suppose we have the following data concerning Region 2.

	Value Added	Final Demand	Total Outputs
A	10	11	15
B	13	12	20

The cost of computing an RAS estimate of Region 2's input-output table using Region 1's **A** matrix as a base table is given by nc_1, where n is the number of RAS iterations, where for purposes here one iteration is defined by one row and one column adjustment, that is, $\mathbf{A}^k = \hat{\mathbf{r}}\mathbf{A}^{k-1}\hat{\mathbf{s}}$ (a row adjustment alone as the last iteration would also be counted as an iteration).

We ultimately wish to compute the impact of a new final demand in Region 2. This impact (the total outputs required to support the new final demand) can be computed exactly or by using the round-by-round approximation of the inverse. We know that: (1) The cost of computing the inverse exactly on a computer is c_1 and the cost of using this inverse in impact analysis is c_2 (let us assume that $c_2 = 10c_1$, i.e., the cost of computing the inverse is ten times the cost of using it in impact analysis). (2) The cost of a round-by-round approximation of impact analysis is mc_1, where m is the order of the round-by-round approximation, that is, $\mathbf{f} + \mathbf{Af} + \mathbf{A}^2\mathbf{f} + \cdots + \mathbf{A}^m\mathbf{f}$.

If we assume that a fourth-order round-by-round approximation is sufficiently accurate ($m = 4$), to determine whether the first or second method of impact analysis would minimize cost, we observe that the cost of using the first method, i.e., computing the exact inverse, is $c_1 + c_2$; with $c_2 = 10c_1$, and the total cost is $11c_1$. With $m = 4$, the cost of using the second method, i.e., round-by-round approximation in impact analysis, $4c_1$, so it is the least cost method in this case.

To determine the total cost of performing impact analysis, including the cost of the RAS approximation (tolerance of 0.01) and of the impact analysis scheme, we first note that, since the RAS procedure converges to within a tolerance of 0.01 in 2 iterations, the cost of the RAS estimate of region 2's coefficients matrix is $5c_1$. Then utilizing the result in a round-by-round application, with $m = 4$, gives a total cost of $6c_1$.

Finally, if we presume the budget for the entire impact-analysis calculation is $7c_1$, the level of tolerance that is affordable, among the options of 0.01, 0.001, 0.0001, 0.00001, or 0.000001, is found by in the following table of cost calculations:

RAS Tolerance	Number of Iterations	RAS Cost	Impact Analysis Cost	Total Cost
.01	3	$3c_1$	$4c_1$	$7c_1$
.001	4	$4c_1$	$4c_1$	$8c_1$
.0001	5	$5c_1$	$4c_1$	$9c_1$
.00001	6	$6c_1$	$4c_1$	$10c_1$
.000001	7	$7c_1$	$4c_1$	$11c_1$

Therefore, the maximum affordable tolerance is .01.

Problem 10.4

This exercise explores the behavior of the adjustment term that converts location-quotient Flagg Location quotient approach (*FLQ)* to an "augmented" *FLQ*, designated *AFLQ*, which adjusts for a measure of regional size. First, recall that the *FLQ* is defined as an adjustment to the cross-industry quotient, *CIQ*, defined by $FLQ_{ij}^r = (\lambda)CIQ_{ij}^r$ where $\lambda = \left\{\log_2[1+(x^r/x^n)]\right\}^\delta$, $0 \le \delta < 1$, and the modified technical coefficients are defined by $a_{ij}^{rr} = \begin{Bmatrix} (FLQ_{ij}^r)a_{ij}^n & \text{if } FLQ_{ij}^r < 1 \\ a_{ij}^n & \text{if } FLQ_{ij}^r \ge 1 \end{Bmatrix}$. The

$AFLQ$ is defined by $AFLQ_{ij}^{r} = \begin{Bmatrix} \left[\log_2(1+LQ_j^r)\right] FLQ_{ij}^r & \text{if } LQ_j^r > 1 \\ FLQ_{ij}^r & \text{if } LQ_j^r \le 1 \end{Bmatrix}$ and the modified technical coefficients by $a_{ij}^{rr} = \begin{Bmatrix} (AFLQ_{ij}^r)a_{ij}^n & \text{if } LQ_j^r > 1 \\ (FLQ_{ij}^r)a_{ij}^n & \text{if } LQ_j^r \le 1 \end{Bmatrix}$. The adjustment term for *AFLQ*, $\lambda = \log_2(1+LQ_j^r) = \{\log_2[1+(x^r/x^n)]\}^{\delta}$, varies with the degree of specialization in a region, i.e., when $LQ_j^r > 1$, then $\left[\log_2(1+LQ_j^r) > 1\right]$, as discussed in section 10.2.5.

The following table shows $\lambda = \log_2(1+LQ_j^r) = \{\log_2[1+(x^r/x^n)]\}^{\delta}$ for values of x^r/x^n = .01, .1, .25, .5, .75 and 1 cross tabulated with values of $\delta =$ 0, .1, .3, .5 and 1.

x^r/x^n	0.01	0.1	0.25	0.5	0.75	1.0
$\log_2[1+(x^r/x^n)]$	0.0144	0.1375	0.3219	0.5850	0.8074	1
$\{\log_2[1+(x^r/x^n)]\}^0$	1	1	1	1	1	1
$\{\log_2[1+(x^r/x^n)]\}^{0.1}$	0.6542	0.8200	0.8928	0.9478	0.9788	1
$\{\log_2[1+(x^r/x^n)]\}^{0.3}$	0.2800	0.5514	0.7118	0.8514	0.9378	1
$\{\log_2[1+(x^r/x^n)]\}^{0.5}$	0.1198	0.3708	0.5647	0.7648	0.8985	1
$\{\log_2[1+(x^r/x^n)]\}^{1}$	0.0144	0.1375	0.32	0.5850	0.8074	1

Problem 10.5

This exercise illustrates the use of simple location quotients (SLQ) to estimate the matrix of regional technical coefficients. First, we define the matrix of technical coefficients for a national economy, $\mathbf{A}^N$, and the vector of total outputs, $\mathbf{x}^N$, as

$$\mathbf{A}^N = \begin{bmatrix} .1830 & .0668 & .0087 \\ .1377 & .3070 & .0707 \\ .2084 & .2409 & .2999 \end{bmatrix} \quad \mathbf{x}^N = \begin{bmatrix} 518{,}288.6 \\ 4{,}953{,}700.6 \\ 14{,}260{,}843.0 \end{bmatrix}$$

as well as the corresponding values for a target region, $\mathbf{A}^R$ and $\mathbf{x}^R$, as

$$\mathbf{A}^R = \begin{bmatrix} .1092 & .0324 & .0036 \\ .0899 & .0849 & .0412 \\ .1603 & .1170 & .2349 \end{bmatrix} \quad \mathbf{x}^R = \begin{bmatrix} 8{,}262.7 \\ 95{,}450.8 \\ 170{,}690.3 \end{bmatrix}.$$

We calculate the simple location quotients by $LQ_i^r = \left(\dfrac{x_i^r / x^r}{x_i^n / x^n} \right)$, but set equal to 1 when the calculation of LQ_i^r exceeds 1. In this case, the matrix of simple location quotients is

$\mathbf{SLQ} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ .8607 & .8607 & .8607 \end{bmatrix}$. The corresponding estimate of the matrix of regional technical coefficients, found by element-by-element multiplication of $\mathbf{A}^N$ by **SLQ,** is

$$\mathbf{A}^{(SLQ)} = \begin{bmatrix} .1830 & .0668 & .0087 \\ .1377 & .3070 & .0707 \\ .1794 & .2074 & .2581 \end{bmatrix}.$$

Problem 10.6

This exercise illustrates the calculation of Cross-Industry Quotients (CIQ) using the national and regional data specified in problem 10.5. We calculate the cross industry quotients by $CIQ_{ij}^r = \left(\dfrac{x_i^r / x_i^n}{x_j^r / x_j^n} \right)$, but again set equal to 1 when the calculation of CIQ_{ij}^r exceeds 1, or, in matrix terms, the matrix of cross industry quotients in this case is $\mathbf{CIQ} = \begin{bmatrix} 1 & .8274 & 1 \\ 1 & 1 & 1 \\ .7508 & .6212 & 1 \end{bmatrix}$.

The corresponding estimate of the matrix of regional technical coefficients using SLQ, found by element-by-element multiplication of $\mathbf{A}^N$ by **CIQ** is $\mathbf{A}^{(CIQ)} = \begin{bmatrix} .1830 & .0553 & .0087 \\ .1377 & .3070 & .0707 \\ .1565 & .1497 & .2999 \end{bmatrix}$.

Problem 10.7

The exercise uses the RAS technique to generate a regional estimate using the national and regional data specified in problem 10.5 (and used in problem 10.6). The intermediate outputs vector for the regional economy is given by $\mathbf{u}^{(R)} = \mathbf{A}^{(R)}\mathbf{x}^{(R)} = \begin{bmatrix} 4{,}615.3 & 15{,}877.7 & 52{,}584.2 \end{bmatrix}'$ and the vector of intermediate inputs is given $\mathbf{v}^{(R)} = (\mathbf{i}'\mathbf{A}^{(R)}\hat{\mathbf{x}}^{(R)})' = \begin{bmatrix} 2{,}969.5 & 22{,}368.8 & 47{,}738.9 \end{bmatrix}'$.

Applying the RAS technique using $\mathbf{A}^N$, $\mathbf{u}^{(R)}$, $\mathbf{v}^{(R)}$, and $\mathbf{x}^{(R)}$, resulting estimate of the matrix of regional technical coefficients is $\mathbf{A}^{(RAS)} = \begin{bmatrix} .1241 & .0270 & .0059 \\ .0712 & .0945 & .0367 \\ .1640 & .1129 & .2370 \end{bmatrix}$

Problem 10.8

This exercise compares the estimates of regional technical coefficients from a matrix of national technical coefficients generated by simple location quotients (SLQ), cross industry quotients (CIQ), and RAS, respectively in problems 10.5, 10.6 and 10.7 in terms of mean absolute deviation from the actual regional technical coefficients. The mean absolute deviation (MAD) calculations for the three methods are: $MAD^{(SLQ)} = (\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left| a_{ij}^{(SLQ)} - a_{ij}^{(R)} \right| = .0606$;

$MAD^{(CIQ)} = (\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left| a_{ij}^{(CIQ)} - a_{ij}^{(R)} \right| = .0558$; and $MAD^{(RAS)} = (\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left| a_{ij}^{(RAS)} - a_{ij}^{(R)} \right| = .0073$. The

RAS technique produces the most accurate estimate in these examples since it shows the lowest value for *MAD* from the actual regional table.

Problem 10.9

This exercise compares the performance of estimates of a variety of nonsurvey estimation techniques in estimating the technical coefficients and associate Leontief inverse coefficients for a known region from a table of national coefficients. We use the three-sector, three-region Chinese MRIO data for 2000 specified in problem 3.6 to estimate regions 2 (South China) and 3 (Rest of China) from the national data.

If we adopt the same error metrics used in Table 10.2 and using *LQ*, *CIQ*, *FLQ*, *AFLQ*, *RPC*, and *RAS* techniques to estimate $\mathbf{A}^2$ (for region 2) and $\mathbf{A}^3$ (for region 3) from $\mathbf{A}^n$ (the national table), the results are the following.

Results for Region 2 (South China) using 2000 Chinese IRIO data.

	Intraregional Input Coefficients	Leontief Inverse
Survey	$\begin{bmatrix} 0.1279 & 0.1086 & 0.0340 \\ 0.1348 & 0.4299 & 0.2191 \\ 0.0394 & 0.0814 & 0.1255 \end{bmatrix}$	$\begin{bmatrix} 1.1889 & 0.2418 & 0.1069 \\ 0.3130 & 1.8828 & 0.4839 \\ 0.0827 & 0.1861 & 1.1933 \end{bmatrix}$
Using $\mathbf{A}^n$		
LQ	$\begin{bmatrix} 0.1252 & 0.1301 & 0.0336 \\ 0.1517 & 0.4605 & 0.2411 \\ 0.0411 & 0.0867 & 0.1235 \end{bmatrix}$	$\begin{bmatrix} 1.2033 & 0.3113 & 0.1317 \\ 0.3804 & 2.0378 & 0.5751 \\ 0.0940 & 0.2161 & 1.2039 \end{bmatrix}$
CIQ	$\begin{bmatrix} 0.1252 & 0.1263 & 0.0351 \\ 0.1517 & 0.4605 & 0.2411 \\ 0.0429 & 0.0842 & 0.1235 \end{bmatrix}$	$\begin{bmatrix} 1.2019 & 0.3018 & 0.1311 \\ 0.3806 & 2.0324 & 0.5743 \\ 0.0953 & 0.2099 & 1.2024 \end{bmatrix}$
FLQ	$\begin{bmatrix} 0.1076 & 0.1085 & 0.0301 \\ 0.1406 & 0.4076 & 0.2228 \\ 0.0369 & 0.0723 & 0.1061 \end{bmatrix}$	$\begin{bmatrix} 1.1598 & 0.2241 & 0.0950 \\ 0.3025 & 1.7994 & 0.4587 \\ 0.0724 & 0.1548 & 1.1598 \end{bmatrix}$
FLQA	$\begin{bmatrix} 0.1076 & 0.1109 & 0.0301 \\ 0.1406 & 0.4163 & 0.2228 \\ 0.0369 & 0.0739 & 0.1061 \end{bmatrix}$	$\begin{bmatrix} 1.1613 & 0.2328 & 0.0972 \\ 0.3077 & 1.8307 & 0.4667 \\ 0.0734 & 0.1609 & 1.1613 \end{bmatrix}$
RPC	$\begin{bmatrix} 0.1155 & 0.1199 & 0.0310 \\ 0.1350 & 0.4097 & 0.2145 \\ 0.0396 & 0.0837 & 0.1192 \end{bmatrix}$	$\begin{bmatrix} 1.1738 & 0.2531 & 0.1029 \\ 0.2978 & 1.8187 & 0.4533 \\ 0.0811 & 0.1841 & 1.1830 \end{bmatrix}$
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$		
LQ	$\begin{bmatrix} 0.1263 & 0.1324 & 0.0346 \\ 0.1530 & 0.4687 & 0.2483 \\ 0.0414 & 0.0882 & 0.1272 \end{bmatrix}$	$\begin{bmatrix} 1.2080 & 0.3242 & 0.1401 \\ 0.3933 & 2.0810 & 0.6077 \\ 0.0971 & 0.2257 & 1.2138 \end{bmatrix}$
CIQ	$\begin{bmatrix} 0.1263 & 0.1285 & 0.0361 \\ 0.1530 & 0.4687 & 0.2483 \\ 0.0433 & 0.0856 & 0.1272 \end{bmatrix}$	$\begin{bmatrix} 1.2066 & 0.3143 & 0.1394 \\ 0.3935 & 2.0751 & 0.6068 \\ 0.0984 & 0.2192 & 1.2122 \end{bmatrix}$

FLQ	$\begin{bmatrix} 0.1086 & 0.1105 & 0.0310 \\ 0.1418 & 0.4148 & 0.2295 \\ 0.0373 & 0.0736 & 0.1093 \end{bmatrix}$	$\begin{bmatrix} 1.1629 & 0.2321 & 0.1003 \\ 0.3110 & 1.8281 & 0.4819 \\ 0.0744 & 0.1608 & 1.1668 \end{bmatrix}$
FLQA	$\begin{bmatrix} 0.1086 & 0.1128 & 0.0310 \\ 0.1418 & 0.4237 & 0.2295 \\ 0.0373 & 0.0752 & 0.1093 \end{bmatrix}$	$\begin{bmatrix} 1.1645 & 0.2413 & 0.1028 \\ 0.3166 & 1.8611 & 0.4906 \\ 0.0754 & 0.1672 & 1.1685 \end{bmatrix}$
RPC	$\begin{bmatrix} 0.1165 & 0.1221 & 0.0319 \\ 0.1361 & 0.4169 & 0.2209 \\ 0.0400 & 0.0851 & 0.1228 \end{bmatrix}$	$\begin{bmatrix} 1.1772 & 0.2623 & 0.1089 \\ 0.3064 & 1.8488 & 0.4768 \\ 0.0834 & 0.1914 & 1.1912 \end{bmatrix}$

	Total Intraregional Intermediate Inputs	**Percentage Differences**[a]	**Average Percentage Difference**[b]
Survey	**0.3022 0.6199 0.3786**		
Using $\mathbf{A}^n$			
LQ	0.3180 0.6773 0.3982	5.24 9.25 5.17	6.55
CIQ	0.3198 0.6710 0.3997	5.84 8.23 5.56	6.55
FLQ	0.2852 0.5884 0.3591	-5.63 -5.08 -5.15	-5.29
FLQA	0.2852 0.6010 0.3591	-5.62 -3.05 -5.15	-4.61
RPC	0.2901 0.6133 0.3646	-4.00 -1.08 -3.69	-2.92
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$			
LQ	0.3208 0.6892 0.4102	6.15 11.18 8.34	8.56
CIQ	0.3226 0.6829 0.4117	6.76 10.15 8.74	8.55
FLQ	0.2876 0.5989 0.3699	-4.81 -3.40 -2.30	-3.50
FLQA	0.2876 0.6117 0.3699	-4.81 -1.34 -2.30	-2.82
RPC	0.2926 0.6241 0.3756	-3.17 0.67 -0.79	1.54[c]

	Intraregional Output Multipliers	**Percentage Differences**[d]	**Average Percentage Difference**
Survey	**1.5846 2.3108 1.7841**		
Using $\mathbf{A}^n$			
LQ	1.6778 2.5651 1.9108	5.88 11.01 7.10	8.00
CIQ	1.6779 2.5441 1.9078	5.89 10.10 6.94	7.64
FLQ	1.5347 2.1784 1.7135	-3.15 -5.73 -3.96	-4.28
FLQA	1.5425 2.2245 1.7252	-2.66 -3.73 -3.30	-3.23
RPC	1.5527 2.2559 1.7392	-2.01 -2.37 -2.51	-2.30
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$			
LQ	1.6584 2.6309 1.9617	7.18 13.85 9.96	10.33
CIQ	1.6985 2.6087 1.9584	7.18 12.89 9.77	9.95

FLQ	1.5482 2.2210 1.7419	-2.30 -3.88 -1.96	-2.71
FLQA	1.5565 2.2696 1.7619	-1.78 -1.78 -1.24	-1.60
RPC	1.5670 2.3025 1.7769	-1.11 -0.36 -0.40	-0.62

[a] This is $\{[(\mathbf{i}'\tilde{\mathbf{A}} - \mathbf{i}'\mathbf{A}) \oslash \mathbf{i}'\mathbf{A}] \times 100\}$, where "$\oslash$" indicates element-by-element division.

[b] This is a simple, unweighted average. Various kinds of weightings (e.g., using some measure of the size of each sector) are frequently used.

[c] This is the average of the absolute values of the differences, so that the negatives and positives do not cancel out.

[d] Calculated as $\{[(\mathbf{i}'\tilde{\mathbf{L}} - \mathbf{i}'\mathbf{L}) \oslash \mathbf{i}'\mathbf{L}] \times 100\}$.

Results for Region 3 (Rest of China) using 2003 Chinese IRIO data.

	Intraregional Input Coefficients	**Leontief Inverse**
Survey	$\begin{bmatrix} 0.1356 & 0.1494 & 0.0329 \\ 0.1050 & 0.3176 & 0.1945 \\ 0.0364 & 0.1016 & 0.1122 \end{bmatrix}$	$\begin{bmatrix} 1.1950 & 0.2773 & 0.1050 \\ 0.2046 & 1.5624 & 0.3498 \\ 0.0725 & 0.1902 & 1.1707 \end{bmatrix}$
Using $\mathbf{A}^n$		
LQ	$\begin{bmatrix} 0.1311 & 0.1362 & 0.0352 \\ 0.1293 & 0.3925 & 0.2055 \\ 0.0429 & 0.0905 & 0.1290 \end{bmatrix}$	$\begin{bmatrix} 1.1992 & 0.2861 & 0.1159 \\ 0.2853 & 1.7742 & 0.4301 \\ 0.0887 & 0.1984 & 1.1984 \end{bmatrix}$
CIQ	$\begin{bmatrix} 0.1311 & 0.1362 & 0.0352 \\ 0.1015 & 0.3925 & 0.1789 \\ 0.0387 & 0.0905 & 0.1290 \end{bmatrix}$	$\begin{bmatrix} 1.1886 & 0.2822 & 0.1059 \\ 0.2210 & 1.7506 & 0.3684 \\ 0.0757 & 0.1944 & 1.1910 \end{bmatrix}$
FLQ	$\begin{bmatrix} 0.1057 & 0.1288 & 0.0247 \\ 0.0643 & 0.2485 & 0.1133 \\ 0.0245 & 0.0772 & 0.0938 \end{bmatrix}$	$\begin{bmatrix} 1.1341 & 0.2001 & 0.0559 \\ 0.1030 & 1.3661 & 0.1735 \\ 0.0394 & 0.1218 & 1.1198 \end{bmatrix}$
FLQA	$\begin{bmatrix} 0.1252 & 0.1288 & 0.0272 \\ 0.0762 & 0.2485 & 0.1250 \\ 0.0290 & 0.0772 & 0.1035 \end{bmatrix}$	$\begin{bmatrix} 1.1632 & 0.2059 & 0.0640 \\ 0.1260 & 1.3723 & 0.1951 \\ 0.0485 & 0.1248 & 1.1343 \end{bmatrix}$
RPC	$\begin{bmatrix} 0.1223 & 0.1270 & 0.0328 \\ 0.1263 & 0.3835 & 0.2008 \\ 0.0397 & 0.0837 & 0.1193 \end{bmatrix}$	$\begin{bmatrix} 1.1810 & 0.2572 & 0.1026 \\ 0.2676 & 1.7322 & 0.4048 \\ 0.0786 & 0.1762 & 1.1786 \end{bmatrix}$
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$		
LQ	$\begin{bmatrix} 0.1251 & 0.1295 & 0.0335 \\ 0.1234 & 0.3732 & 0.1957 \\ 0.0409 & 0.0860 & 0.1228 \end{bmatrix}$	$\begin{bmatrix} 1.1844 & 0.2587 & 0.1029 \\ 0.2583 & 1.7022 & 0.3895 \\ 0.0806 & 0.1790 & 1.1830 \end{bmatrix}$
CIQ	$\begin{bmatrix} 0.1251 & 0.1295 & 0.0335 \\ 0.0969 & 0.3732 & 0.1703 \\ 0.0369 & 0.0860 & 0.1228 \end{bmatrix}$	$\begin{bmatrix} 1.1754 & 0.2557 & 0.0945 \\ 0.2005 & 1.6827 & 0.3344 \\ 0.0691 & 0.1758 & 1.1768 \end{bmatrix}$

FLQ	$\begin{bmatrix} 0.1009 & 0.1224 & 0.0235 \\ 0.0614 & 0.2363 & 0.1078 \\ 0.0234 & 0.0734 & 0.0893 \end{bmatrix}$	$\begin{bmatrix} 1.1262 & 0.1855 & 0.0510 \\ 0.0956 & 1.3402 & 0.1612 \\ 0.0366 & 0.1128 & 1.1123 \end{bmatrix}$
FLQA	$\begin{bmatrix} 0.1195 & 0.1224 & 0.0259 \\ 0.0727 & 0.2363 & 0.1190 \\ 0.0277 & 0.0734 & 0.0985 \end{bmatrix}$	$\begin{bmatrix} 1.1533 & 0.1905 & 0.0583 \\ 0.1168 & 1.3455 & 0.1809 \\ 0.0449 & 0.1154 & 1.1258 \end{bmatrix}$
RPC	$\begin{bmatrix} 0.1167 & 0.1207 & 0.0312 \\ 0.1206 & 0.3646 & 0.1912 \\ 0.0379 & 0.0796 & 0.1136 \end{bmatrix}$	$\begin{bmatrix} 1.1679 & 0.2334 & 0.0915 \\ 0.2432 & 1.6661 & 0.3679 \\ 0.0717 & 0.1596 & 1.1651 \end{bmatrix}$

	Total Intraregional Intermediate Inputs	**Percentage Differences**[a]	**Average Percentage Difference**[b]
Survey	**0.2771 0.5687 0.3396**		
Using $\mathbf{A}^n$			
LQ	0.3033 0.6192 0.3696	9.47 8.88 8.85	9.07
CIQ	0.2713 0.6192 0.3430	-2.07 8.88 1.02	3.99[c]
FLQ	0.1945 0.4545 0.2317	-29.81 -20.08 -31.76	-27.22
FLQA	0.2304 0.4545 0.2557	-16.84 -20.08 -24.70	-20.54
RPC	0.2883 0.5942 0.3529	4.05 4.49 3.92	4.15
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$			
LQ	0.2895 0.5887 0.3519	4.47 3.52 3.64	3.88
CIQ	0.2590 0.5887 0.3266	-6.54 3.52 -3.81	4.62
FLQ	0.1856 0.4321 0.2206	-33.02 -24.02 -35.03	-30.69
FLQA	0.2199 0.4321 0.2434	-20.63 -24.02 -28.31	-24.32
RPC	0.2751 0.5649 0.3360	-0.70 -0.66 -1.05	-0.81

	Intraregional Output Multipliers	**Percentage Differences**[d]	**Average Percentage Difference**
Survey	**1.4721 2.0299 1.6256**		
Using $\mathbf{A}^n$			
LQ	1.5732 2.2587 1.7444	6.87 11.27 7.31	8.48
CIQ	1.4853 2.2272 1.6654	0.90 9.72 2.45	4.36
FLQ	1.2765 1.6880 1.3492	-13.29 -16.84 -17.00	-15.71
FLQA	1.3377 1.7031 1.3934	-9.13 -16.10 -14.28	-13.17
RPC	1.5272 2.1656 1.6860	3.75 6.68 3.72	4.72
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$			
LQ	1.5233 2.1400 1.6754	3.48 5.42 3.07	3.99
CIQ	1.4450 2.1142 1.6057	-1.84 4.15 -1.22	2.41[c]
FLQ	1.2584 1.6384 1.3245	-14.51 -19.29 -18.52	-17.44
FLQA	1.3150 1.6514 1.3651	-10.67 -18.65 -16.02	-15.11

RPC	1.4828 2.0591 1.6244	0.73 1.44 -0.07	0.75[c]

[a] This is $\{[(\mathbf{i}'\tilde{\mathbf{A}} - \mathbf{i}'\mathbf{A}) \oslash \mathbf{i}'\mathbf{A}] \times 100\}$, where "$\oslash$" indicates element-by-element division.

[b] This is a simple, unweighted average. Various kinds of weightings (e.g., using some measure of the size of each sector) are frequently used.

[c] This is the average of the absolute values of the differences, so that the negatives and positives do not cancel out.

[d] Calculated as $\{[(\mathbf{i}'\tilde{\mathbf{L}} - \mathbf{i}'\mathbf{L}) \oslash \mathbf{i}'\mathbf{L}] \times 100\}$.

Problem 10.10

This exercise problem applies the RAS technique to generate a matrix of technical coefficients for the state of Washington using the U.S. matrix of technical coefficients as a starting point. The following are the 1997 matrix of technical coefficients and vector of total outputs for the State of Washington as well as the 2003 matrix of technical coefficients for the United States, where the sectors are defined as (1) agriculture, (2) mining, (3) construction, (4) manufacturing, (5) trade, transportation and utilities, (6) services, and (7) other:

$$\mathbf{A}^W = \begin{bmatrix} .1154 & .0012 & .0082 & .0353 & .0019 & .0033 & .0016 \\ .0008 & .0160 & .0057 & .0014 & .0022 & .0002 & .0001 \\ .0072 & .0084 & .0066 & .0043 & .0074 & .0196 & .0133 \\ .0868 & .0287 & .0958 & .0766 & .0289 & .0244 & .0205 \\ .0625 & .0278 & .0540 & .0525 & .0616 & .0317 & .0480 \\ .0964 & .1207 & .0704 & .0596 & .1637 & .1991 & .2224 \\ .0020 & .0031 & .0056 & .0019 & .0045 & .0051 & .0066 \end{bmatrix} \quad \mathbf{x}^W = \begin{bmatrix} 7{,}681.0 \\ 581.7 \\ 17{,}967.1 \\ 77{,}483.7 \\ 56{,}967.2 \\ 109{,}557.6 \\ 4{,}165.5 \end{bmatrix}$$

$$\mathbf{A}^{US} = \begin{bmatrix} .2225 & .0000 & .0012 & .0375 & .0001 & .0020 & .0010 \\ .0021 & .1360 & .0072 & .0453 & .0311 & .0003 & .0053 \\ .0034 & .0002 & .0012 & .0021 & .0035 & .0071 & .0214 \\ .1724 & .0945 & .2488 & .3204 & .0468 & .0572 & .1004 \\ .0853 & .0527 & .0912 & .0950 & .0643 & .0314 & .0526 \\ .0902 & .1676 & .1339 & .1261 & .1655 & .2725 & .1882 \\ .0101 & .0140 & .0103 & .0214 & .0206 & .0200 & .0247 \end{bmatrix}$$

To examine application of the RAS technique to estimate the Washington State table using the U.S. matrix of technical coefficients as a starting point, we first compute $\mathbf{Z}^W = \mathbf{A}^W \hat{\mathbf{x}}^W$ and then the vectors of total intermediate inputs and outputs for the real Washington State table:

$$\mathbf{u}(1) = \mathbf{Z}^W \mathbf{i} = \begin{bmatrix} 4{,}245.9 & 369.4 & 3{,}140.1 & 12{,}737.6 & 12{,}718 & 38{,}753.8 & 1{,}112.4 \end{bmatrix}'$$

$$\mathbf{v}(1) = (\mathbf{i}'\mathbf{Z}^W)' = \begin{bmatrix} 2{,}849.7 & 119.8 & 4{,}423 & 17{,}945.8 & 15{,}384.7 & 31{,}052.5 & 1{,}301.7 \end{bmatrix}'$$

Applying RAS using $\mathbf{A}^{US}$, $\mathbf{u}(1)$, $\mathbf{v}(1)$, and $\mathbf{x}^W$, the estimated matrix of technical coefficients for

Washington State is $^{US}\mathbf{A}^{W}=\begin{bmatrix} .2078 & .0000 & .0013 & .0299 & .0002 & .0027 & .0013 \\ .0001 & .0099 & .0005 & .0022 & .0031 & .0000 & .0005 \\ .0061 & .0004 & .0025 & .0032 & .0109 & .0177 & .0571 \\ .0526 & .0362 & .0883 & .0836 & .0246 & .0243 & .0456 \\ .0534 & .0415 & .0664 & .0508 & .0694 & .0273 & .0490 \\ .0493 & .1151 & .0852 & .0589 & .1561 & .2070 & .1531 \\ .0016 & .0028 & .0019 & .0029 & .0057 & .0044 & .0059 \end{bmatrix}$. The mean absolute deviation between the estimated and actual Washington State matrices of technical coefficients is $MAD=(\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left|{}^{US}a_{ij}^{W}-a_{ij}^{W}\right|=0.0098$.

Problem 10.11

This exercise extends the estimation considered in exercise problem 10.10, presuming that while we do not know all the technical coefficients for the Washington State economy, $\mathbf{A}^{W}$, we do know several, namely a_{11}^{W}, a_{62}^{W} and a_{65}^{W}. To use the RAS technique incorporating that we know these coefficients, we begin with defining a matrix of the exogenously specified technical coefficients: $\mathbf{K}=\begin{bmatrix} .1154 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .1207 & 0 & 0 & .1637 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The corresponding modified vectors of total intermediate inputs and outputs for the real Washington State table are found as:

$\mathbf{u}(1)=\mathbf{x}^{W}-\mathbf{Z}^{W}\mathbf{i}-\mathbf{K}\mathbf{x}^{W}=\begin{bmatrix}3{,}359.7 & 369.4 & 3{,}140.1 & 12{,}737.6 & 12{,}718 & 29{,}359.6 & 1{,}112.4\end{bmatrix}'$ and

$\mathbf{v}(1)=\mathbf{x}^{W}-\mathbf{i}'\mathbf{Z}^{W}-\mathbf{i}'\mathbf{K}\hat{\mathbf{x}}^{W}=\begin{bmatrix}1{,}963.5 & 49.6 & 4{,}423 & 17{,}945.8 & 6{,}060.7 & 31{,}052.5 & 1{,}301.7\end{bmatrix}'$.

Applying RAS using $\mathbf{A}^{(US)}$, $\mathbf{u}(1)$, $\mathbf{v}(1)$, and $\mathbf{x}^{W}$, and, this time, $\mathbf{K}$, the new estimated matrix of technical coefficients for Washington State is

$^{US}\mathbf{A}^{W}=\begin{bmatrix} .1154 & .0001 & .0017 & .0377 & .0002 & .0035 & .0017 \\ .0002 & .0096 & .0005 & .0023 & .0030 & .0000 & .0005 \\ .0096 & .0004 & .0025 & .0031 & .0100 & .0180 & .0576 \\ .0830 & .0337 & .0890 & .0809 & .0229 & .0248 & .0463 \\ .0850 & .0389 & .0675 & .0496 & .0649 & .0281 & .0502 \\ .0752 & .1207 & .0830 & .0551 & .1637 & .2044 & .1502 \\ .0026 & .0026 & .0020 & .0029 & .0053 & .0046 & .0060 \end{bmatrix}$. The mean absolute deviation between the estimated and actual Washington State matrices of technical coefficients is

$MAD = (\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left|{}^{US}a_{ij}^{W} - a_{ij}^{W}\right| = 0.0066$. In this case the constrained RAS procedure incorporating exogenous information improves the estimate considerably over the unconstrained case in problem 10.10.

Problem 10.12

This exercise further explores use of constrained RAS estimation developed in problem 10.11, this time assuming there is information from exogenous sources providing some alternative technical coefficients, namely a_{67}^{W}, a_{42}^{W} and a_{54}^{W} to those in problem 10.11. The procedure is the same, i.e., the new exogenously specified technical coefficients are given by

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .0287 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .0525 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .2224 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

. The revised vectors of total intermediate inputs and outputs for the real Washington State table are:

$$\mathbf{u}(1) = \mathbf{x}^{W} - \mathbf{Z}^{W}\mathbf{i} - \mathbf{K}\mathbf{x}^{W} = \begin{bmatrix}4{,}245.9 & 369.4 & 3{,}140.1 & 12{,}720.9 & 8{,}649.3 & 37{,}827.6 & 1{,}112.4\end{bmatrix}'$$

$$\mathbf{v}(1) = \mathbf{x}^{W} - \mathbf{i}'\mathbf{Z}^{W} - \mathbf{i}'\mathbf{K}\hat{\mathbf{x}}^{W} = \begin{bmatrix}2{,}849.7 & 103.1 & 4{,}423 & 13{,}877.1 & 15{,}384.7 & 31{,}052.5 & 375.5\end{bmatrix}'$$

Applying RAS using $\mathbf{A}^{US}$, $\mathbf{u}(1)$, $\mathbf{v}(1)$, $\mathbf{x}^{W}$, and $\mathbf{K}$ yields the new estimated matrix of technical coefficients for Washington State:

$${}^{US}\mathbf{A}^{W} = \begin{bmatrix} .2088 & .0000 & .0013 & .0298 & .0002 & .0027 & .0007 \\ .0001 & .0104 & .0005 & .0022 & .0031 & .0000 & .0003 \\ .0063 & .0005 & .0026 & .0033 & .0113 & .0184 & .0329 \\ .0530 & .0287 & .0895 & .0834 & .0251 & .0247 & .0258 \\ .0527 & .0433 & .0659 & .0525 & .0692 & .0272 & .0271 \\ .0485 & .1200 & .0844 & .0575 & .1554 & .2059 & .2224 \\ .0016 & .0030 & .0019 & .0029 & .0058 & .0045 & .0033 \end{bmatrix}$$

. The mean absolute deviation between the estimated and actual Washington State matrices of technical coefficients is $MAD = (\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left|{}^{US}a_{ij}^{W} - a_{ij}^{W}\right| = 0.0077$ —not as good an estimate as that obtained in problem 10.11, which resulted in $MAD = 0.0066$.

Finally, we can presume we can employ the exogenous information used in both problems 10.10 and 10.11 in a combined case. For this combined case, the new exogenously specified technical coefficients are given by $\mathbf{K} = \begin{bmatrix} .1154 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .0287 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .0525 & 0 & 0 & 0 \\ 0 & .1207 & 0 & 0 & .1637 & 0 & .2224 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

We once again compute the vectors of total intermediate inputs and outputs for the real Washington State table:

$$\mathbf{u}(1) = \mathbf{x}^W - \mathbf{Z}^W\mathbf{i} - \mathbf{K}\mathbf{x}^W = [3{,}359.7 \quad 369.4 \quad 3{,}140.1 \quad 12{,}720.9 \quad 8{,}649.3 \quad 28{,}433.4 \quad 1{,}112.4]'$$

$$\mathbf{v}(1) = \mathbf{x}^W - \mathbf{i}'\mathbf{Z}^W - \mathbf{i}'\mathbf{K}\hat{\mathbf{x}}^W = [1{,}963.5 \quad 32.9 \quad 4{,}423 \quad 13{,}877.1 \quad 6{,}060.7 \quad 31{,}052.5 \quad 375.5]'$$

Applying RAS using $\mathbf{A}^{US}$, $\mathbf{u}(1)$, $\mathbf{v}(1)$, $\mathbf{x}^W$, and $\mathbf{K}$, the new estimated matrix of technical coefficients for Washington State using both sets of exogenous information is

$${}^{US}\mathbf{A}^W = \begin{bmatrix} .1154 & .0001 & .0018 & .0376 & .0002 & .0036 & .0010 \\ .0002 & .0108 & .0005 & .0022 & .0031 & .0000 & .0003 \\ .0100 & .0005 & .0026 & .0031 & .0104 & .0187 & .0327 \\ .0847 & .0287 & .0908 & .0802 & .0234 & .0254 & .0259 \\ .0835 & .0423 & .0663 & .0525 & .0639 & .0277 & .0270 \\ .0746 & .1207 & .0822 & .0531 & .1637 & .2033 & .2224 \\ .0026 & .0030 & .0020 & .0028 & .0054 & .0047 & .0034 \end{bmatrix}$$

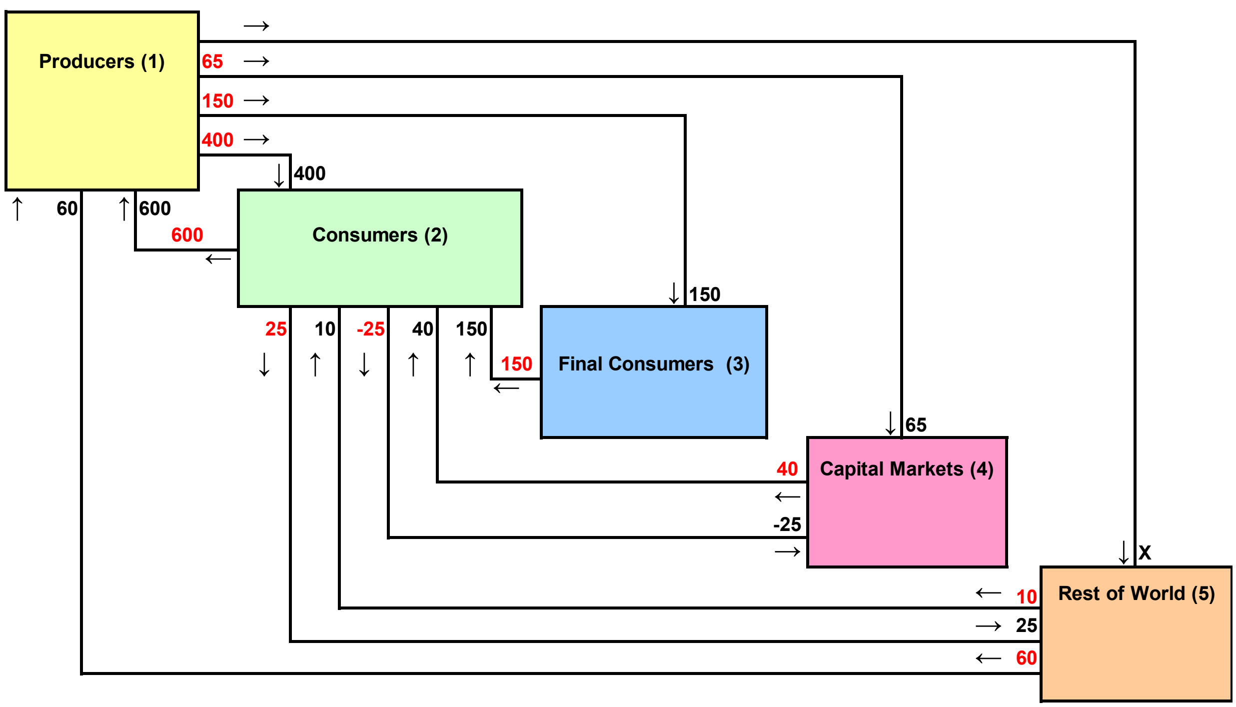
. The mean absolute deviation between the estimated and actual Washington State matrices of technical coefficients is $MAD = (\frac{1}{49})\sum_{i=1}^{7}\sum_{j=1}^{7}\left|{}^{US}a_{ij}^{W} - a_{ij}^{W}\right| = 0.0044$—perhaps not surprisingly better than either of the previous cases.

Chapter 11, Social Accounting Matrices

Chapter 11 expands the input–output framework to a broader class of economic analysis tools known as social accounting matrices (SAM) and other so-called "extended" input–output models to capture activities of income distribution in the economy in a more comprehensive and integrated way, including especially employment and social welfare features of an economy. The basic concepts of SAMs are explored and derived from the SNA introduced in Chapters 4 and 5, and the relationships between SAMs and input–output accounts are presented. The concept of SAM multipliers as well as the decomposition of SAM multipliers into components with specific economic interpretations are introduced and illustrated. Finally, techniques for balancing SAM accounts for internal accounting consistency are discussed and several illustrative applications of the use of SAMs are presented. The exercise problems for this chapter explore the construction of SAM accounts and models.

Problem 11.1

This exercise illustrates the relationships between a map of the circular flow of income and expenditure and a corresponding "macro-SAM." Consider a macro economy depicted in the figure below. Note the missing value, X, showing the exports from the Producers sector to the Rest of World sector. We can verify that this value is 45 from either the Producers balance equation, $X = (60+600)-(400+150+65) = 45$, or the Balance of Payment Account's Rest of World balance equation, $X = 25-(10+60) = -45$.



We can express the chart as a basic macro-SAM, where a sector defined as consumption combines both intermediate and final consumption as a single sector by the following:

	Prod	Cons	Cap	ROW	
Producers		550	65	45	660
Consumers	600	150	-25	25	750
Capital Markets		40			40
Rest of World	60	10			70
	660	750	40	70	

If we express consumption as in the figure, i.e., with consumption separated into intermediate consumers (2) and final consumers (3), the SAM becomes:

	Prod	Cons	Fin Con	Cap	ROW	
Producers		400	150	65	45	660
Consumers	600			-25	25	600
Final Consumers		150				150
Capital Markets		40				40
Rest of World	60	10				70
	660	600	150	40	70	

Problem 11.2

This exercise illustrates construction of a "fully articulated" SAM, i.e., including the interindustry detail provided by input-output accounts. For the economy depicted in problem 11.1, suppose the following input-output accounts are collected:

		Commodities		Industries		Final Demand	Totals	Grand Total
		Manuf.	Services	Manuf.	Services			
Commodities	Manuf.			94	96	110	300	660
	Services			94	117	148	360	
Industries	Manuf.	295	0				295	660
	Services	5	360				365	
Value Added				106	152	260		
Totals		300	360	295	365			
Grand Total		660		660				

To construct a "fully articulated" SAM, i.e., incorporating the interindustry detail provided by these input-output accounts, final demand must be allocated as part of consumer demand and commodity imports allocated to value added. There is no unique solution, but one such balanced fully articulated SAM is the following.

		Production		Consumption		Cap.	ROW	Total	
		Manuf.	Services	Manuf.	Services				
Production	Manuf.	0	0	158	96	28	18	300	660
	Services	0	0	94	203	37	25	360	
Consumption	Manuf.	284	0	0	0	-12	13	285	600
	Services	5	319	0	0	-13	4	315	
Capital		0	0	20	20	0	0	40	
ROW		10	41	3	5	0	0	60	
Total		300	360	276	324	40	60		
		660		600					

Problem 11.3

This exercise problem expands SAM accounts to include sectors defined for consumer demand and exports, using SAM developed in problem 11.2. Again, there is no unique solution, but the SAM must be balanced, i.e., row and column sums equal. One such SAM is the following:

		Commodities		Industries		Total Final Demand			Total	
		Manuf.	Services	Manuf.	Services	PCE	Cap.	Exports		
Commodities	Manuf.	0	0	94	96	64	28	18	300	660
	Services	0	0	94	117	86	37	25	360	
Industrieis	Manuf.	284	0	0	0	0	-12	13	285	600
	Services	5	319	0	0	0	-13	4	315	
Value Added	Consumer	0	0	73	77	0	0	0	150	250
	Capital	0	0	20	20	0	0	0	40	
	Imports	10	41	4	5	0	0	0	60	
Total		300	360	285	315	150	40	60		
		660		600		250				

Problem 11.4

This problem explores construction of a SAM matrix of total expenditure shares, $\overline{\mathbf{S}}$, and partitioning of $\overline{\mathbf{S}}$ to specify the SAM coefficient matrix, $\mathbf{S}$, and the "direct effect" multipliers using the table of SAM transactions developed in problem 11.3. First, we define the table of SAM transactions as $\overline{\mathbf{Z}}$ and the row or column totals of all transactions as $\overline{\mathbf{x}}$. Then we compute the matrix of total expenditure shares as

$$\overline{\mathbf{S}} = \overline{\mathbf{Z}}\hat{\overline{\mathbf{x}}}^{-1} = \left[\begin{array}{cccc:ccc} 0 & 0 & .331 & .305 & .427 & .7 & .3 \\ 0 & 0 & .331 & .371 & .573 & .925 & .417 \\ .95 & 0 & 0 & 0 & 0 & -.3 & .217 \\ .017 & .884 & 0 & 0 & 0 & -.325 & .067 \\ \hdashline 0 & 0 & .257 & .244 & 0 & 0 & 0 \\ 0 & 0 & .07 & .063 & 0 & 0 & 0 \\ .033 & .116 & .011 & .016 & 0 & 0 & 0 \end{array}\right].$$

Notice that $\overline{\mathbf{S}}$ is partitioned into interindustry sectors (commodities and industries) and sectors exogenous to interindustry activity (final demand and value added). If we assume final demand and value-added sectors are considered exogenous transactions to this economy, the SAM coefficient matrix, **S**, is the upper left partition of $\overline{\mathbf{S}}$, i.e., $\mathbf{S} = \begin{bmatrix} 0 & 0 & .331 & .305 \\ 0 & 0 & .331 & .371 \\ .95 & 0 & 0 & 0 \\ .017 & .884 & 0 & 0 \end{bmatrix}$.

The matrix of "direct effect" multipliers is then $\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-1} = \begin{bmatrix} 1.812 & .726 & .840 & .822 \\ .865 & 1.835 & .894 & .945 \\ 1.721 & .690 & 1.798 & .781 \\ .794 & 1.634 & .804 & 1.849 \end{bmatrix}$.

Problem 11.5

The exercise illustrates the construction of direct, indirect, cross, and total SAM multipliers in the additive form using a highly aggregated SAM for the developing nation of Sri Lanka:[2] The basic SAM accounts are reflected in the following:

Sri Lanka SAM 1970	Value Added	Insti-tutions	Indirect Taxes	Surplus/ Deficit	Pro-duction	Rest of World	Total
Value Added	-	-	-	11,473	-	-	11,473
Institutions	11,360	2,052	1,368	-	-	3	14,783
Indirect Taxes	-	389	-	885	-	94	1,368
Production	-	11,312	-	4,660	-	2,113	18,085
Surplus/Deficit	-	(425)	-	-	-	425	-
Rest of World	113	1,455	-	1,067	-	-	2,635
Total	11,473	14,783	1,368	18,085	-	2,635	

We define the full table of SAM transactions as $\overline{\mathbf{Z}}$ and the row and/or column totals as $\overline{\mathbf{x}}$. If we consider the sectors, Surplus/Deficit and Rest of World, as external to the SAM, we first reorder the table so that the Surplus/Deficit and Rest of World sectors become sectors 5 and 6 in the table by $\tilde{\mathbf{Z}} = \mathbf{R}\overline{\mathbf{Z}}\mathbf{R}'$ and $\tilde{\mathbf{x}} = \mathbf{R}\overline{\mathbf{x}}$ where

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

With the reordered sectors, we calculate $\overline{\mathbf{S}} = \tilde{\mathbf{Z}}\hat{\tilde{\mathbf{x}}}^{-1}$ and create partitions separating sectors 5 and 6 as the exogenous sectors. We can use the formulas developed in section 11.10.4 to compute the direct, indirect, cross, and total multipliers, respectively, as the following:

[2] Adapted from Pyatt and Round (1979), pp. 852-853.

$$\mathbf{N}_1 = \left[\begin{array}{cccc|cc} 5.7286 & 4.7756 & 4.7756 & 5.2105 & 0 & 0 \\ 7.2309 & 7.3209 & 7.3209 & 6.6610 & 0 & 0 \\ 0.5550 & 0.5605 & 1.5605 & 0.5772 & 0 & 0 \\ 7.4538 & 7.5279 & 7.5279 & 8.2133 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{N}_2 = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 26.9953 & 5.9390 \\ 0 & 0 & 0 & 0 & 34.7835 & 7.2176 \\ 0 & 0 & 0 & 0 & 3.2188 & 0.5400 \\ 0 & 0 & 0 & 0 & 42.5529 & 9.6888 \\ \hline -0.2506 & -0.2511 & -0.2511 & -0.2472 & 0 & 0 \\ 1.5382 & 1.5302 & 1.5302 & 1.6014 & 0 & 0 \end{array}\right]$$

$$\mathbf{N}_3 = \left[\begin{array}{cccc|cc} 1.5617 & 1.5182 & 1.5182 & 1.9068 & 0 & 0 \\ 1.4871 & 1.4301 & 1.4301 & 1.9388 & 0 & 0 \\ -0.0168 & -0.0224 & -0.0224 & 0.0271 & 0 & 0 \\ 2.8569 & 2.7889 & 2.7889 & 3.3954 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & -0.2075 \\ 0 & 0 & 0 & 0 & 6.2 & 1.3405 \end{array}\right]$$

$$\mathbf{N}_T = \left[\begin{array}{cccc|cc} 7.2903 & 6.2938 & 6.2938 & 7.1173 & 26.9953 & 5.9390 \\ 8.7180 & 8.7329 & 8.7329 & 8.5997 & 34.7835 & 7.2176 \\ 0.5382 & 0.5382 & 1.5382 & 0.6043 & 3.2188 & 0.5400 \\ 10.3106 & 10.3168 & 10.3168 & 11.6087 & 42.5529 & 9.6888 \\ \hline -0.2506 & -0.2511 & -0.2511 & -0.2472 & 0 & -0.2075 \\ 1.5382 & 1.5302 & 1.5302 & 1.6014 & 6.2 & 2.3405 \end{array}\right].$$

Problem 11.6

This problem illustrates use of the RAS technique to balance a SAM, i.e., to iteratively adjust the SAM transactions so that the row and column sums of the SAM are the same. Consider the unbalanced SAM given in the table below.

Suppose independent analysis indicates the total output of each sector; these are given in the additional column specified in the table.

	Prod.	Cons.	Capital	ROW	Totals	Estimated Totals
Producers	0	600	65	45	710	660
Consumers	700	0	-25	15	690	600
Capital	0	40	0	0	40	40
Rest of World	50	10	0	0	60	60
Totals	750	650	40	60	1,500	1,360

If we use the RAS technique to produce a balanced SAM with rows and columns both summing to the independent sector output estimates, the result is the following:

	Prod.	Cons.	Capital	ROW	Totals
Producers	-	560	40	60	660
Consumers	600	-	-	0	600
Capital	-	40	-	-	40
Rest of World	60	0	-	-	60
Totals	660	600	40	60	1,360

Problem 11.7

This problem explores the use of the RAS technique including additional exogenously specified information to balance a SAM using the unbalanced SAM given in problem 11.6. If, in addition to the estimated totals provided in the unbalanced table, we become aware that the elements $z_{23} = -25$, $z_{24} = 15$, and $z_{42} = 10$ in the balanced SAM are fixed, we can use the RAS procedure incorporating some fixed exogenous data for these elements (developed in chapter 10) to produce a balanced SAM:

	Prod.	Cons.	Capital	ROW	Totals
Producers	0	550	65	45	660
Consumer	610	0	-25	15	600
Capital	0	40	0	0	40
Rest of W	50	10	0	0	60
Totals	660	600	40	60	1,360

Problem 11.8

The problem explores development of direct, indirect, cross, and total SAM multipliers in their multiplicative form using a "macro-SAM" for the U.S. economy for 1988. The US SAM (as reported in Reinert and Roland-Holst, 1992, pp. 173-187) is the following:

US SAM 1988	Prod.	Comm.	Labor	Prop.	Enter-prises	House-holds	Govt.	Capital	ROW	Taxes	Errors	Total
Production		4831										4831
Commodities						3235	970	750	431			5386
Labor	2908											2908
Property	1556								117			1673
Enterprises				1589		95	93					1777
Households			2463		1045		556					4064
Government	377		445		138	587		96		18		1661
Capital					594	145			117		-10	846
Rest of World		537		84		2	42					665
Taxes		18										18
Errors & Omissions	-10											-10
Total	4831	5386	2908	1673	1777	4064	1661	846	665	18	-10	

If we consider the first five sectors as the endogenous sectors, the direct, indirect, cross, and total multipliers in their multiplicative form, respectively, are given by:

$$
\mathbf{M}_1 = \left[\begin{array}{ccccc|c}
1 & .897 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & \\
.602 & .54 & 1 & 0 & 0 & \mathbf{0} \\
.322 & .289 & 0 & 1 & 0 & \\
.306 & .274 & 0 & .95 & 1 & \\
\hline
 & & \mathbf{0} & & & \mathbf{I}
\end{array}\right],
$$

$$
\mathbf{M}_2 = \left[\begin{array}{cccccc|cccccc}
 & & & & & & .714 & .524 & .795 & .581 & 0 & 0 \\
 & & & & & & .796 & .584 & .887 & .648 & 0 & 0 \\
 & & & \mathbf{I} & & & .43 & .315 & .479 & .35 & 0 & 0 \\
 & & & & & & .23 & .169 & .256 & .363 & 0 & 0 \\
 & & & & & & .24 & .216 & .243 & .345 & 0 & 0 \\
\hline
0 & 0 & .847 & 0 & .588 & 0 & & & & & & \\
.078 & 0 & .153 & 0 & .078 & 0 & & & & & & \\
0 & 0 & 0 & 0 & .334 & 0 & & & & \mathbf{I} & & \\
0 & .10 & 0 & .05 & 0 & 0 & & & & & & \\
0 & .003 & 0 & 0 & 0 & 0 & & & & & & \\
-.002 & 0 & 0 & 0 & 0 & 0 & & & & & &
\end{array}\right],
$$

$$\mathbf{M}_3 = \left[\begin{array}{ccccc:cccccc} 1.206 & .331 & 3.296 & .167 & 3.455 & & & & & & \\ .23 & 1.369 & 3.675 & .186 & 3.851 & & & & & & \\ .124 & .199 & 2.984 & .1 & 2.079 & & & \mathbf{0} & & & \\ .071 & .132 & 1.136 & 1.066 & 1.191 & & & & & & \\ .078 & .135 & 1.2 & .068 & 2.246 & & & & & & \\ \hdashline & & & & & 3.469 & 1.931 & 2.669 & 2.488 & 0 & 0 \\ & & & & & .677 & 1.526 & .734 & .67 & 0 & 0 \\ & & & & & .413 & .333 & 1.441 & .452 & 0 & 0 \\ & & \mathbf{0} & & & .439 & .339 & .477 & 1.436 & 0 & 0 \\ & & & & & .013 & .01 & .014 & .012 & 1 & 0 \\ & & & & & -.007 & -.005 & -.008 & -.007 & 0 & 1 \end{array}\right] \text{ and}$$

$$\mathbf{M} = \left[\begin{array}{ccccc:cccccc} 4.301 & 4.189 & 3.296 & 3.448 & 3.455 & 3.415 & 2.640 & 3.713 & 3.321 & 0 & 0 \\ 3.68 & 4.67 & 3.675 & 3.844 & 3.851 & 3.807 & 2.943 & 4.14 & 3.703 & 0 & 0 \\ 2.589 & 2.521 & 2.984 & 2.075 & 2.079 & 2.056 & 1.589 & 2.235 & 1.999 & 0 & 0 \\ 1.463 & 1.444 & 1.136 & 2.197 & 1.191 & 1.177 & .91 & 1.28 & 1.322 & 0 & 0 \\ 1.509 & 1.488 & 1.2 & 2.201 & 2.246 & 1.237 & .995 & 1.319 & 1.352 & 0 & 0 \\ \hdashline 3.080 & 3.011 & 3.233 & 3.052 & 3.082 & 3.469 & 1.931 & 2.669 & 2.488 & 0 & 0 \\ .849 & .828 & .807 & .758 & .762 & .677 & 1.526 & .734 & .67 & 0 & 0 \\ .504 & .497 & .401 & .736 & .751 & .413 & .333 & 1.441 & .452 & 0 & 0 \\ .44 & .538 & .423 & .494 & .444 & .439 & .339 & .477 & 1.436 & 0 & 0 \\ .012 & .016 & .012 & .013 & .013 & .013 & .01 & .014 & .012 & 1 & 0 \\ -.009 & -.009 & -.007 & -.007 & -.007 & -0.007 & -0.005 & -0.008 & -0.007 & 0 & 1 \end{array}\right]$$

Problem 11.9
This exercise expands the development of SAM multipliers to the multiplicative form using once again the macro-SAM specified in problem 11.8. If we compute the direct multipliers in their additive form, we discover that they are the same as those in the multiplicative form, i.e., $\mathbf{M}_1 = \mathbf{N}_1$, which turns out to be always the case as discussed in section 11.10.5.

Problem 11.10
This problem explores development of multipliers for a SAM expanded with interindustry detail using the SAM for the U.S. (1988) introduced in problem 11.8, which is expanded with the interindustry detail shown in Table P11.10. If we consider the first nine sectors as the endogenous sectors, the resulting total multipliers are the following:

$\mathbf{i}'\mathbf{M} = [3.245\ \ 3.053\ \ 3.380\ \ 3.647\ \ 3.581\ \ 2.949\ \ 2.769\ \ 2.588\ \ 2.645\ \ 1.000\ \ 1.000\ \ 1.000\ \ 3.302\ \ 2.691\ \ 4.000\ \ 3.160\ \ 1.000\ \ 1.000]$

Table P11.10
SAM with Expanded Interindustry Detail for United States, 1988[3]

US SAM 1988 ($ billions)	1 Agric.	2 Mining	3 Const.	4 Nondur. Manuf	5 Durable Manuf.	6 Transp. & Util	7 Trade	8 Finance	9 Services	10 Labor	11 Propty	12 Enter-prises	13 House-holds	14 Govt.	15 Capital	16 Rest of World	17 Tariffs	18 Errors	Total
1 Agriculture	42	0	2	98	8	0	3	8	7	0	0	0	18	7	1	22	0	0	214
2 Mining	0	10	2	82	8	35	0	0	0	0	0	0	1	0	2	8	0	0	148
3 Construction	2	12	1	7	9	21	6	36	18	0	0	0	0	134	358	0	0	0	602
4 Nondurable Manuf.	30	1	35	370	83	37	24	14	149	0	0	0	453	38	4	93	0	0	1332
5 Durable Manuf.	4	3	175	55	480	19	7	4	81	0	0	0	236	97	296	187	0	0	1643
6 Transport & Utilities	5	1	17	66	65	78	46	31	84	0	0	0	310	34	13	26	0	0	774
7 Trade	8	1	72	57	73	11	14	7	50	0	0	0	529	11	56	43	0	0	932
8 Finance	10	3	10	18	25	14	52	20	79	0	0	0	771	16	22	25	0	0	1065
9 Services	5	1	53	68	74	31	124	93	214	0	0	0	917	632	0	27	0	0	2240
10 Labor	33	18	197	218	430	212	385	217	1198	0	0	0	0	0	0	0	0	0	2908
11 Property	60	56	32	142	69	207	147	511	332	0	0	0	0	0	0	117	0	0	1673
12 Enterprise	0	0	0	0	0	0	0	0	0	0	1589	0	96	92	0	0	0	0	1778
13 Households	0	0	0	0	0	0	0	0	0	2463	0	1046	0	556	0	0	0	0	4064
14 Government	8	12	7	28	18	35	127	113	30	445	0	138	587	0	96	0	16	0	1659
15 Capital	0	0	0	0	0	0	0	0	0	0	0	594	145	0	0	117	0	-10	846
16 Rest of World	8	31	0	115	295	75	0	12	2	0	83	0	2	42	0	0	0	0	665
17 Tariffs	0	0	0	8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	16
18 Errors & Omissions	0	0	-1	-1	-1	-1	-1	-2	-2	0	0	0	0	0	0	0	0	0	-10
Total	214	148	602	1332	1643	774	932	1065	2240	2908	1673	1778	4064	1659	846	665	16	-10	

[3] As reported in Reinert and Rolad-Holst (1992).

Chapter 12, Energy Input-Output Analysis

Chapter 12 explores the extension of the input–output framework to more detailed analysis of energy consumption associated with industrial production, including some of the complications that can arise when measuring input–output transactions in physical units of production rather than in monetary terms of the value of production.

The chapter reviews early efforts to develop energy input–output analysis and compares them with contemporary approaches and examines the strengths and limitations of the alternatives commonly used today. Special methodological considerations such as adjusting for energy conversion efficiencies are developed along with several illustrative applications, including estimation of the energy costs of goods and services, impacts of new energy technologies, and energy taxes.

Energy input-output analysis is increasingly being applied to global scale issues, such as the energy embodied in international trade of goods and services. Finally, the role of structural change of an input–output economy associated with changing patterns of energy use is illustrated, building on the more general approaches developed in Chapter 8.

The exercise problems for this chapter explore the use of input-output analysis to analyze the special case of energy production and use.

Problem12.1

This exercise problem develops two formulations of the energy input-output model from basic economic input-output accounts and supplemental information for tracking the flow of energy throughout an economy measured in physical units. Consider the following three-sector input-output economy; two sectors are energy sectors (oil is the primary energy sector and refined petroleum is the secondary energy sector):

Interindustry Transactions ($\$10^6$)	Oil	Refined Petroleum	Manuf.	Final Demand	Total Output
Crude Oil	0	20	0	0	20
Refined Petroleum	2	2	2	24	30
Manufacturing	0	0	0	20	20

The energy sector transactions are also measured in quadrillions of Btus in the following table:

Energy Sector Transactions (10^{15} Btus)	Oil	Refined Petroleum	Manuf.	Final Demand	Total Output
Crude Oil	0	20	0	0	20
Refined Petroleum	1	1	1	17	20

To formulate an energy input-output model from these data, we first define the customary Leontief economic transactions matrix, vector of final demands, and vector of total outputs, respectively, all measured in millions of dollars as: $\mathbf{Z}=\begin{bmatrix}0 & 20 & 0\\2 & 2 & 2\\0 & 0 & 0\end{bmatrix}$, $\mathbf{f}=\begin{bmatrix}0\\24\\20\end{bmatrix}$ $\mathbf{x}=\begin{bmatrix}20\\30\\20\end{bmatrix}$. We can now compute the economic matrix of technical coefficients as $\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}=\begin{bmatrix}0 & .667 & 0\\.10 & .067 & .1\\0 & 0 & 0\end{bmatrix}$ and the corresponding matrix of total requirements as $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}==\begin{bmatrix}1.077 & .769 & .077\\.115 & 1.154 & .115\\0 & 0 & 1\end{bmatrix}$.

The matrix of energy transactions in physical units (quadrillions of Btus) is $\mathbf{E}=\begin{bmatrix}0 & 20 & 0\\1 & 1 & 1\end{bmatrix}$; the vector of energy consumption in final demand, $\mathbf{q}=\begin{bmatrix}0\\17\end{bmatrix}$, and total energy consumption, $\mathbf{g}=\begin{bmatrix}20\\20\end{bmatrix}$, are also measured in quadrillions of Btus (often referred to as Quads). The matrix of implied energy prices, defined as the element-by-element division of **E** by the corresponding elements in the energy rows of **Z** where transactions are nonzero and zero otherwise, is $\mathbf{P}=\begin{bmatrix}0 & 1 & 0\\2 & 2 & 2\end{bmatrix}$.

The traditional energy input-output formulation specifies the direct energy requirements as $\boldsymbol{\varepsilon}=\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}+\tilde{\mathbf{Q}}$ where $\mathbf{D}=\mathbf{E}\hat{\mathbf{x}}^{-1}=\begin{bmatrix}0 & .667 & 0\\.05 & .033 & .05\end{bmatrix}$ and the elements of $\tilde{\mathbf{Q}}$ are defined as q_k / f_k for energy sectors and zero otherwise. In this case, $\tilde{\mathbf{Q}}=\begin{bmatrix}0 & 0 & 0\\0 & .708 & 0\end{bmatrix}$, so we find $\boldsymbol{\varepsilon}=\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}+\tilde{\mathbf{Q}}=\begin{bmatrix}.077 & .769 & .077\\.058 & .785 & .058\end{bmatrix}$. Note that this suggests a million dollars' worth of final demand for manufacturing in this economy would require production of 0.785 Quads of refined petroleum but only 0.769 Quads of crude oil, which is not sensible since the structure of this economy is that refined petroleum is a secondary energy sector receiving all its energy input from the primary energy sector, Crude Oil, so the primary and secondary energy consumption (aside from any energy conversion efficiencies) should be the same, often referred to as an energy conservation condition.

To formulate these data instead as a hybrid units energy input output model, we first define the matrix of transactions in hybrid units, the vector of final demands, and the vector of total outputs, respectively, as $\mathbf{Z}^*=\begin{bmatrix}0 & 20 & 0\\1 & 1 & 1\\0 & 0 & 0\end{bmatrix}$, $\mathbf{f}^*=\begin{bmatrix}0\\17\\20\end{bmatrix}$, and $\mathbf{x}^*=\begin{bmatrix}20\\20\\20\end{bmatrix}$ where energy rows are measured in Quads and nonenergy rows are measured in millions of dollars. We can now compute $\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 \\ .05 & .05 & .05 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{=1} == \begin{bmatrix} 1.056 & 1.111 & 0.056 \\ 0.056 & 1.111 & 0.056 \\ 0 & 0 & 1 \end{bmatrix}$ which is easy to see conforms to the energy conservation condition.

Problem 12.2

The problem illustrates the typical use of the traditional energy input-output model in public policy analysis. Consider the following input-output transactions table in value terms (millions of dollars) for two industries—*A* and *B*:

	A	*B*	Total Output
A	2	4	100
B	6	8	100

Suppose we have a direct energy requirements matrix for this economy that is given by:

$$\mathbf{D} = \begin{bmatrix} .2 & .3 \\ .1 & .4 \end{bmatrix} \begin{matrix} 10^{15} \text{ Btus of oil per million dollars of output} \\ 10^{15} \text{ Btus of coal per million dollars of output} \end{matrix}$$

If for simplicity we ignore energy consumption by final demand, we compute the total energy requirements matrix, $\boldsymbol{\varepsilon} = \mathbf{DL} = \begin{bmatrix} .225 & .336 \\ .129 & .440 \end{bmatrix}$ where the matrices of technical requirements and total requirements are, respectively, $\mathbf{A} = \begin{bmatrix} .02 & .04 \\ .06 & .08 \end{bmatrix}$ and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.023 & .044 \\ .067 & 1.09 \end{bmatrix}$.

Suppose further that the final demands for industries *A* and *B* are projected to be \$200 million and \$100 million respectively for the next year. The net increase in energy (both oil and gas) required to support this new final demand (again, neglecting energy consumed directly by final demand) is found by $\Delta\mathbf{g} = \mathbf{DL}\Delta\mathbf{f} = \begin{bmatrix} 28.5 \\ 19.8 \end{bmatrix}$. We can determine how much of the total energy produced to supply this net increase final demand is *direct* energy consumption $\Delta\mathbf{g}^{direct} = \mathbf{D}\Delta\mathbf{f} = \begin{bmatrix} 25.4 \\ 16.2 \end{bmatrix}$ where $\Delta\mathbf{f} = \mathbf{f}^{new} - \mathbf{f} = \begin{bmatrix} 200 \\ 100 \end{bmatrix} - \begin{bmatrix} 94 \\ 86 \end{bmatrix} = \begin{bmatrix} 106 \\ 14 \end{bmatrix}$. The amount of *indirect* energy consumption can be found as $\Delta\mathbf{g}^{indirect} = \Delta\mathbf{g} - \Delta\mathbf{g}^{direct} = \begin{bmatrix} 3.1 \\ 3.6 \end{bmatrix}$.

Finally, suppose an energy conservation measure in industry *B* causes the direct energy requirement of that industry for coal to be reduced from 0.4 to 0.3 (10^{15} Btus of coal per dollar of output of industry *B*). The resulting changes in direct and total energy requirements matrices are $\mathbf{D} = \begin{bmatrix} .2 & .3 \\ .1 & .3 \end{bmatrix}$ and $\mathbf{DL} = \begin{bmatrix} .225 & .336 \\ .122 & .331 \end{bmatrix}$, respectively. Hence the new change in total energy to support final demand, $\Delta\mathbf{f}$, is $\Delta\mathbf{g} = \mathbf{DL}\Delta\mathbf{f} = \begin{bmatrix} 28.5 \\ 17.6 \end{bmatrix}$. The direct portion is, once again,

$\Delta\mathbf{g}^{direct} = \mathbf{D}\Delta\mathbf{f} = \begin{bmatrix} 25.4 \\ 14.8 \end{bmatrix}$ so the indirect portion is $\Delta\mathbf{g}^{indirect} = \Delta\mathbf{g} - \Delta\mathbf{g}^{direct} = \begin{bmatrix} 3.1 \\ 2.8 \end{bmatrix}$.

Hence, the differences in total energy consumption before and after the energy conservation measure are given by $\begin{bmatrix} 28.5 \\ 19.8 \end{bmatrix} - \begin{bmatrix} 28.5 \\ 17.6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.2 \end{bmatrix}$; the differences in direct energy consumption are given by $\begin{bmatrix} 25.4 \\ 16.2 \end{bmatrix} - \begin{bmatrix} 25.4 \\ 14.8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.4 \end{bmatrix}$; and the differences in indirect energy consumption are given by $\begin{bmatrix} 3.1 \\ 3.6 \end{bmatrix} - \begin{bmatrix} 3.1 \\ 2.8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}$.

Problem 12.3

This problem uses the energy input-output formulation to illustrate computation of the total energy impacts of a change in nonenergy final demand. Consider the following input-output table ($\$10^6$):

	Transactions			Total Output
	Autos	Oil	Electricity	
Autos	2	6	1	10
Oil	0	0	20	20
Electricity	3	2	1	30

Assume that there is a matrix of implied inverse energy prices for this economy given by the following (inverse because the measure is millions of dollars per billion Btu rather than vice versa):

	Autos	Oil	Electricity	Final Demand
Oil	0	0	0.4082	0
Electricity	0.3333	0.2857	0.5	1.2912

We define the basic economic data of the matrix transactions and vectors of total outputs and final demands, respectively, as $\mathbf{Z} = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 0 & 20 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$, and

$$\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} - \begin{bmatrix} 9 \\ 20 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 24 \end{bmatrix}.$$

We can compute the energy transactions physical units (billions of Btus) by, first, defining the matrix of implied inverse energy prices from the table as $\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0.408 \\ 0.333 & 0.286 & 0.5 \end{bmatrix}$. If we multiply, element by element, **Q** and the energy rows of **Z**, the energy transactions matrix measured in physical units is $\mathbf{E} = \begin{bmatrix} 0 & 0 & 49 \\ 9 & 7 & 2 \end{bmatrix}$.

Using the energy sector elements of the computed economic final demands, $\begin{bmatrix} 0 \\ 24 \end{bmatrix}$, multiplied, element by element, by the inverse energy prices for final demand from the table, $\begin{bmatrix} 0 \\ 1.2912 \end{bmatrix}$, yields the vector of energy consumption in final demand measured in physical units, $\mathbf{q} = \begin{bmatrix} 0 \\ 31 \end{bmatrix}$, from which we can now compute the total energy consumption as $\mathbf{g} = \mathbf{Ei} + \mathbf{q} = \begin{bmatrix} 49 \\ 49 \end{bmatrix}$. We can express the energy flows as the energy rows in a hybrid units transactions matrix and corresponding vectors of final demands and total outputs:

$\mathbf{Z}^* = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 0 & 49 \\ 9 & 7 & 2 \end{bmatrix}$, $\mathbf{f}^* = \begin{bmatrix} 1 \\ 0 \\ 31 \end{bmatrix}$ and $\mathbf{x}^* = \begin{bmatrix} 10 \\ 49 \\ 49 \end{bmatrix}$ and $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} .2 & .122 & .02 \\ 0 & 0 & 1 \\ .9 & .143 & .041 \end{bmatrix}$. The direct energy requirements matrix is then defined as the energy rows of $\mathbf{A}^*$. For this economy $\mathbf{G} = \begin{bmatrix} 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$ so the direct energy coefficients can be found as

$$\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^* = \begin{bmatrix} 0 & 0 & 1 \\ .9 & .143 & .041 \end{bmatrix}$$

If a final demand vector of \$2 million worth of autos and 18 quadrillion Btus of electricity is presented to this economy, the total amount of energy (of each type) required to support this final demand is found by first retrieving the energy rows of $\mathbf{L}^*$ for

$\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.556 & 0.229 & 0.272 \\ 1.716 & 1.428 & 1.525 \\ 1.716 & 0.428 & 1.525 \end{bmatrix}$ which defines the total energy requirements matrix,

$\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^* = \begin{bmatrix} 1.716 & 1.428 & 1.526 \\ 1.716 & 0.428 & 1.526 \end{bmatrix}$. Then, for $\Delta\mathbf{f}^* = \begin{bmatrix} 2 \\ 0 \\ 18 \end{bmatrix}$ we compute the total energy consumption as $\Delta\mathbf{g} = \boldsymbol{\alpha}\Delta\mathbf{f}^* = \begin{bmatrix} 30.887 \\ 30.887 \end{bmatrix}$.

If we alternatively use the traditional energy input-output formulation, using the energy prices defined above for final demand we can first compute $\Delta\mathbf{f} = \begin{bmatrix} 2 \\ 0 \\ 23.2 \end{bmatrix}$ and then

$\Delta\mathbf{g} = \boldsymbol{\varepsilon}\Delta\mathbf{f} = \begin{bmatrix} 48.2 \\ 450.9 \end{bmatrix}$ for $\mathbf{A} = \begin{bmatrix} .2 & .3 & .033 \\ 0 & 0 & .667 \\ .3 & .1 & .033 \end{bmatrix}$, $\mathbf{L} = \begin{bmatrix} 1.385 & .451 & .359 \\ .308 & 1.174 & .821 \\ .462 & .262 & 1.231 \end{bmatrix}$ and

$\boldsymbol{\varepsilon} = \mathbf{DL} + \tilde{\mathbf{Q}} = \begin{bmatrix} .754 & .427 & 2.010 \\ 1.385 & .835 & 19.280 \end{bmatrix}$ where $\tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 18.587 \end{bmatrix}$ is the matrix of *implied inverse energy prices* for final demand. The elements of $\tilde{\mathbf{Q}} = [\tilde{q}_k]$ are defined by

$$\tilde{q}_k = \begin{cases} 1/p_{kf}, & \text{when energy sector } k \text{ and industry sector } j \text{ are the same sector} \\ 0, & \text{otherwise} \end{cases}$$

Problem 12.4

This problem explores the conditions for energy conservation in an input-output model. These conditions can be expressed as $\boldsymbol{\alpha}\hat{\mathbf{x}} = \boldsymbol{\alpha}\mathbf{Z} + \mathbf{G}$ where $\boldsymbol{\alpha}$ is the matrix of total energy coefficients, **Z** is the matrix of interindustry transactions, **x** is the vector of total outputs, and **G** is the matrix of primary energy outputs.

We can show that the hybrid-units formulation of the energy input-output model—that is, where **x** is replaced by $\mathbf{x}^*$ and **Z** is replaced by $\mathbf{Z}^*$—satisfies these conditions in general: $\boldsymbol{\alpha}\hat{\mathbf{x}}^* = \boldsymbol{\alpha}\mathbf{Z}^* + \mathbf{G}$ and $\mathbf{Z}^* = \mathbf{A}^*\hat{\mathbf{x}}^*$, so $\boldsymbol{\alpha}\hat{\mathbf{x}}^* = \boldsymbol{\alpha}\mathbf{A}^*\hat{\mathbf{x}}^* + \mathbf{G}$. Rearranging, this becomes $\boldsymbol{\alpha}(\mathbf{I} - \mathbf{A}^*)\hat{\mathbf{x}}^* = \mathbf{G}$ or $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^*)^{-1}$, which is the definition of the total energy requirements matrix in the hybrid units energy input-output formulation.

Given the following two tables of total energy coefficients, we adopt the convention that crude oil is a primary energy sector while refined petroleum and electricity are both secondary energy sectors.

Case 1	Crude Oil	Refined Petroleum	Electricity	Autos
Crude Oil	0	.6	.5	.3
Refined Petroleum	0	.4	.5	.2
Electricity	0	.2	0	.1

Case 2	Crude Oil	Refined Petroleum	Electricity	Autos
Crude Oil	0	.6	.5	.3
Refined Petroleum	0	.4	.2	.1
Electricity	0	.2	0	.1

Case 1 satisfies the energy conservation conditions, since $\boldsymbol{\alpha}_{ref.pet} + \boldsymbol{\alpha}_{elec.} = \boldsymbol{\alpha}_{crude} = [.6 \quad .5 \quad .3]$ i.e., the sum of all secondary energy consumed for energy type in the economy equals the total primary energy consumed by each energy sector. Case 2 fails to satisfy the energy conservation conditions, since $\boldsymbol{\alpha}_{ref.pet} + \boldsymbol{\alpha}_{elec.} = \begin{bmatrix} .6 & .2 & .2 \end{bmatrix} \neq \boldsymbol{\alpha}_{crude} = [.6 \quad .5 \quad .3]$.

Problem 12.5

This problem compares the total energy requirements matrices for the traditional and contemporary energy input-output formulations Consider an input-output economy defined (in $\$10^6$ units) by $\mathbf{Z} = \begin{bmatrix} 0 & 10 & 0 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 0 \\ 25 \\ 20 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 10 \\ 40 \\ 20 \end{bmatrix}$. The first two of the three industries are energy industries with patterns of output allocation expressed in energy terms (10^{15} Btus) for interindustry transactions, $\mathbf{E} = \begin{bmatrix} 0 & 40 & 0 \\ 5 & 5 & 15 \end{bmatrix}$, and for final demand, $\mathbf{g} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$. First, we compute the direct and total requirements as $\mathbf{A} = \begin{bmatrix} 0 & .25 & 0 \\ .5 & .13 & .25 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} 1.17 & .33 & .08 \\ .67 & 1.33 & .33 \\ 0 & 0 & 1 \end{bmatrix}$.

With the traditional energy input-output analysis formulation we have

$$\boldsymbol{\varepsilon} = \mathbf{DL} + \tilde{\mathbf{Q}} = \begin{bmatrix} .67 & 1.33 & .33 \\ .67 & .94 & .83 \end{bmatrix} \text{ where } \mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ .5 & .13 & .75 \end{bmatrix} \text{ and } \tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}.$$

With the hybrid unites formulation we have $\mathbf{Z}^* = \begin{bmatrix} 0 & 40 & 0 \\ 5 & 5 & 15 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{x}^* = \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix}$,

$$\mathbf{G} = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \end{bmatrix} \text{ and } \mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 \\ .13 & .13 & .75 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so } \boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.167 & 1.333 & 1.0 \\ .167 & 1.333 & 1.0 \end{bmatrix}.$$

Problem 12.6

This problem illustrates use of the hybrid units energy input-output model in impact analysis. Consider the following hybrid units transactions matrix and vector of total outputs, i.e., the first three rows of the energy sectors (oil, coal, and electricity) are measured in millions of Btu and the last row, manufacturing, is measured in millions of dollars: $\mathbf{Z}^* = \begin{bmatrix} 0 & 0 & 40 & 0 \\ 0 & 0 & 60 & 0 \\ 2 & 3 & 12 & 48 \\ 15 & 20 & 30 & 40 \end{bmatrix}$ and

$\mathbf{x}^* = \begin{bmatrix} 40 \\ 60 \\ 100 \\ 200 \end{bmatrix}$. Using $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1}$ we can compute $\mathbf{A}^* = \begin{bmatrix} 0 & 0 & .4 & 0 \\ 0 & 0 & .6 & 0 \\ .05 & .05 & .12 & .24 \\ .375 & .333 & .3 & .24 \end{bmatrix}$ and

$$\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.1024 & .0945 & .6299 & .1890 \\ .1535 & 1.1417 & .9449 & .2835 \\ .2559 & .2362 & 1.5748 & .4724 \\ .6767 & .6086 & 1.2795 & 1.6339 \end{bmatrix}.$$

If we project a final demand for manufactured goods will increase by \$200 billion, the change in final demand can be written as $(\Delta\mathbf{f}^*)' = \begin{bmatrix} 0 & 0 & 0 & 200 \end{bmatrix}$ so the corresponding change in total energy consumption can be expressed as $\Delta\mathbf{g} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^*)^{-1}\Delta\mathbf{f}^* = \begin{bmatrix} 18.2677 \\ 27.4016 \\ 0 \end{bmatrix}$ where

$\mathbf{G}(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The total primary energy intensity is $\mathbf{i}'\Delta\mathbf{g} = 94.4882$.

Problem 12.7

This problem illustrates the use of energy input-output analysis to evaluate the relative impact of alternative energy technologies on total energy consumption. For the economy specified in problem 12.6, two alternative technologies are proposed for generating electric power, which involve alternative new specifications for the matrix of technical coefficients depicting different "recipes" for electric power production in the economy, $\mathbf{A}^{*(I)}$ and $\mathbf{A}^{*(II)}$. For the original electric power generation column of the technical coefficients matrix is given by $\mathbf{A}^*$, suppose the two alternative changed columns of the technical coefficients matrix corresponding to the alternative technologies are given by $\mathbf{A}_{\bullet 3}^{*(I)} = \begin{bmatrix} .2 \\ .7 \\ .1 \\ .4 \end{bmatrix}$ and $\mathbf{A}_{\bullet 3}^{*(II)} = \begin{bmatrix} .5 \\ .4 \\ .12 \\ .4 \end{bmatrix}$ and a vector of new final demands of

$\Delta\mathbf{f}^* = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 30 \end{bmatrix}$ is presented to the economy.

To determine which economy [matrix incorporating the specifications $\mathbf{A}^*$, $\mathbf{A}^{*(I)}$ or $\mathbf{A}^{*(II)}$] reflects the most energy intensive manufacturing, i.e., which one of the two new technologies consumes the least primary energy per unit of final demand of manufacturing and how much less primary energy does that technology consume than the other to support final demand $\Delta\mathbf{f}^*$, first using $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1}$ for the alternative power generation technologies, *I* and *II*, we can specify the corresponding matrices of technical coefficients and total requirements as

$$\mathbf{A}^{*(I)} = \begin{bmatrix} 0 & 0 & .2 & 0 \\ 0 & 0 & .7 & 0 \\ .05 & .05 & .1 & .24 \\ .375 & .333 & .4 & .20 \end{bmatrix}, \ (\mathbf{I}-\mathbf{A}^{*(I)})^{-1} = \begin{bmatrix} 1.0506 & .0467 & .3113 & .0934 \\ .1770 & 1.1634 & 1.0895 & .3268 \\ .2529 & .2335 & 1.5564 & .4669 \\ .6927 & .6234 & 1.3781 & 1.6634 \end{bmatrix},$$

$$\mathbf{A}^{*(II)} = \begin{bmatrix} 0 & 0 & .5 & 0 \\ 0 & 0 & .4 & 0 \\ .05 & .05 & .12 & .24 \\ .375 & .333 & .4 & .2 \end{bmatrix} \text{ and } (\mathbf{I}-\mathbf{A}^{*(II)})^{-1} = \begin{bmatrix} 1.1313 & .1212 & .8081 & .2424 \\ .1051 & 1.0970 & .6465 & .1939 \\ .2626 & .2424 & 1.6162 & .4848 \\ .7054 & .6351 & 1.4562 & 1.6869 \end{bmatrix}.$$

If we designate the technical coefficients of original economy by $\mathbf{A}^{*(0)}$, the total energy consumption associate with the new final demand, $\Delta\mathbf{f}^*$, is $\Delta\mathbf{g}^{(0)} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I}-\mathbf{A}^{*(0)})^{-1}\Delta\mathbf{f}$ and for the technical coefficients modified by the two alternative technologies, $\Delta\mathbf{g}^{(I)} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I}-\mathbf{A}^{*(I)})^{-1}\Delta\mathbf{f}$ and $\Delta\mathbf{g}^{(II)} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I}-\mathbf{A}^{*(II)})^{-1}\Delta\mathbf{f}$, respectively where

$\mathbf{G}(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\Delta\mathbf{f}^* = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 30 \end{bmatrix}$, we can write

$\Delta\mathbf{g} = \left[\Delta\mathbf{g}^{(0)} \mid \Delta\mathbf{g}^{(I)} \mid \Delta\mathbf{g}^{(II)}\right] = \left[\begin{array}{c:c:c} 18.2677 & 9.0272 & 23.4343 \\ 27.4016 & 31.5953 & 18.7475 \\ 0 & 0 & 0 \end{array}\right]$ which provides the total energy of each fuel type to support $\Delta\mathbf{f}^*$.

The total primary energy intensity is given by $\mathbf{i}'(\Delta\mathbf{g}) = \begin{bmatrix} 45.6693 & 40.6226 & 42.1818 \end{bmatrix}$, so employment of technology *I* consumes 1.5592 less primary energy than employment of technology *II*. Both new technologies *I* and *II* are more efficient than the base technology.

Problem 12.8

This problem explores calculation of the total energy consumption in an economy associated with an energy saving manufacturing process technology, using the original energy-economy defined in problem 12.6. For the direct requirements matrix, $\mathbf{A}^*$, suppose an energy conserving manufacturing process is developed that can be depicted as a new column of the matrix of technical coefficients for manufacturing, given by $\mathbf{A}_{\bullet 4}^{*(new)} = \begin{bmatrix} 0 \\ 0 \\ .12 \\ .20 \end{bmatrix}$. The technical coefficient matrix incorporating the new manufacturing technology is $\mathbf{A}^{*(new)} = \begin{bmatrix} 0 & 0 & .2 & 0 \\ 0 & 0 & .7 & 0 \\ .05 & .05 & .1 & .12 \\ .375 & .333 & .4 & .2 \end{bmatrix}$ and

$$\mathbf{L}^{*(new)} = \begin{bmatrix} 1.0580 & .0546 & .5461 & .0819 \\ .0870 & 1.0819 & .8191 & .1229 \\ .1451 & .1365 & 1.3652 & .2048 \\ .5866 & .5276 & 1.1092 & 1.4164 \end{bmatrix}$$. So, for $\mathbf{\Delta g}^{(0)} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^{*(0)})^{-1}\mathbf{\Delta f}$ and $\mathbf{\Delta g}^{(new)} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^{*(new)})^{-1}\mathbf{\Delta f}$ we can write

$$\mathbf{i}'(\mathbf{\Delta g}) = \mathbf{i}'\left[\mathbf{\Delta g}^{(0} \;\vdots\; \mathbf{\Delta g}^{(new)}\right] = \left[\begin{array}{c:c} 18.2677 & 13.3788 \\ 27.4016 & 20.0683 \\ 0.0000 & 0.0000 \end{array}\right]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 45.6693 & 33.4471 \end{bmatrix}$$. Hence the primary energy saved by adopting the new technology is $45.6693 - 33.4471 = 12.2222$.

Problem 12.9

This problem explores the use of an energy input-output model in analyzing the implications of an oil supply reduction, again using the original energy economy introduced in problem 12.6 but with the added information that the energy prices to final demand are given by $\mathbf{p}_f = [p_{kf}] = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

From the original matrix of technical coefficients, $\mathbf{A}^{*(0)}$, we can compute

$$(\mathbf{I} - \mathbf{A}^{*(0)}) = \begin{bmatrix} 1 & 0 & -.4 & 0 \\ 0 & 1 & -.6 & 0 \\ -.05 & -.05 & .88 & -.24 \\ -.375 & -.333 & -.3 & .8 \end{bmatrix}$$. The GDP for the original economy can be found by

$GDP = \mathbf{i}'\tilde{\mathbf{Q}}\mathbf{f}^* = \mathbf{i}'\tilde{\mathbf{Q}}(\mathbf{I} - \mathbf{A}^{*(0)})\mathbf{x}^* = 105$ where $\tilde{\mathbf{Q}} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ and $(\mathbf{x}^*)' = \begin{bmatrix} 40 & 60 & 100 & 200 \end{bmatrix}$.

For a 10 percent reduction in availability of oil supply, the vector of total outputs becomes $(\mathbf{x}^*)' = \begin{bmatrix} 36 & 60 & 100 & 200 \end{bmatrix}$. Hence, we can compute the GDP as the sum of the corresponding final demand (measured in dollars) which we determine once again by $GDP = \mathbf{i}'\tilde{\mathbf{Q}}\mathbf{f}^* = \mathbf{i}'\tilde{\mathbf{Q}}(\mathbf{I} - \mathbf{A}^{*(0)})\mathbf{x}^* = 90.1$. The reduction in GDP due to the oil shortage is $105 - 90.2 = 14.9$. When the new technologies are incorporated into the technical coefficients matrix it becomes $\mathbf{A}^{*(new)} = \begin{bmatrix} 0 & 0 & .2 & 0 \\ 0 & 0 & .7 & 0 \\ .05 & .05 & .1 & .12 \\ .375 & .333 & .4 & .2 \end{bmatrix}$ and

$$(\mathbf{I}-\mathbf{A}^{*(new)}) = \begin{bmatrix} 1 & 0 & -.2 & 0 \\ 0 & 1 & -.7 & 0 \\ -.05 & -.05 & .9 & -.12 \\ -.375 & -.333 & -.4 & .8 \end{bmatrix}$$ and, as before, we compute GDP by

$GDP = \mathbf{i}'\tilde{\mathbf{Q}}\mathbf{f}^* = \mathbf{i}'\tilde{\mathbf{Q}}(\mathbf{I}-\mathbf{A}^{*(new)})\mathbf{x}^* = 205.6$. This turns out to be not a reduction at all but an increase in *GDP* of 100.6.

Problem 12.10

This problem uses energy input-output analysis to examine structural change using US input-output tables for two years. Below are 9-sector 1963 and 1980 input-output tables for the United States expressed in hybrid units (quadrillions of Btus for energy sectors and millions of dollars for non-energy sectors). The first five sectors are energy sectors: (1) coal, (2) oil, (3) refined petroleum products, (4) electricity, and (5) natural gas. The remaining four sectors are non-energy sectors: (6) natural resources, (7) manufacturing, (8) transportation, and (9) services.

1980	1	2	3	4	5	6	7	8	9	Total Output
1	0.0012	0.0000	0.0007	1.5464	0.0000	0.0000	0.0002	0.0000	0.0000	18,597
2	0.0001	0.0319	0.8960	0.0001	0.8707	0.0000	0.0001	0.0000	0.0000	36,842
3	0.0063	0.0024	0.0612	0.3344	0.0008	0.0005	0.0002	0.0023	0.0002	31,215
4	0.0026	0.0021	0.0035	0.0822	0.0020	0.0000	0.0001	0.0000	0.0001	7,827
5	0.0006	0.0461	0.0301	0.4856	0.0720	0.0001	0.0003	0.0000	0.0001	19,244
6	0.2092	1.4027	0.5040	7.8254	0.4350	0.0896	0.0628	0.0355	0.0289	6,194,571
7	2.6323	0.8480	2.4090	3.5155	0.1804	0.2672	0.3780	0.0493	0.0626	18,081,173
8	0.1773	0.0806	2.1831	4.8195	0.0794	0.0199	0.0251	0.1289	0.0141	2,240,904
9	1.8576	2.6159	2.7945	8.5173	1.2302	0.1831	0.1238	0.1224	0.2027	23,803,723

1963	1	2	3	4	5	6	7	8	9	Total Output
1	0.0019	0.0000	0.0008	1.7415	0.0010	0.0000	0.0004	0.0001	0.0000	12,476
2	0.0000	0.0423	0.7996	0.0007	0.9308	0.0000	0.0003	0.0000	0.0000	30,384
3	0.0015	0.0011	0.0600	0.1973	0.0031	0.0004	0.0003	0.0021	0.0002	19,878
4	0.0015	0.0007	0.0018	0.0963	0.0002	0.0000	0.0001	0.0000	0.0000	3,128
5	0.0001	0.0035	0.0330	0.7046	0.0919	0.0000	0.0003	0.0001	0.0001	13,194
6	0.0456	0.4582	0.5926	7.9623	0.6565	0.1111	0.0835	0.0415	0.0426	4,865,092
7	0.8684	0.4081	1.1700	1.0933	0.0937	0.2340	0.4035	0.0498	0.0496	11,333,710
8	0.1105	0.0655	1.1964	4.5632	0.3965	0.0231	0.0256	0.0863	0.0121	1,131,226
9	0.4794	2.2388	1.9461	8.0643	1.1016	0.1121	0.0881	0.1203	0.1721	10,588,385

To determine the amounts of the change in total energy use of each energy type between 1963 and 1980 and the components of that change that are attributable to change in production functions, to change in final demand, and to the interaction between the changes in production functions and final demand between the two years, we begin by calculating the total requirements matrices $\mathbf{L}^{*(80)} = (\mathbf{I} - \mathbf{A}^{*(80)})^{-1}$ and $\mathbf{L}^{*(63)} = (\mathbf{I} - \mathbf{A}^{*(63)})^{-1}$. From the available data we must calculate final demands as $\mathbf{f}^{*(80)} = \mathbf{x}^{*(80)} - \mathbf{A}^{*}\mathbf{x}^{*(80)}$ and $\mathbf{f}^{*(63)} = \mathbf{x}^{*(63)} - \mathbf{A}^{*}\mathbf{x}^{*(63)}$. The total requirements matrices and vectors of final demand are the following:

1980	1	2	3	4	5	6	7	8	9	Final Demand
1	1.0081	0.0059	0.016	1.718	0.0099	0.0003	0.0007	0.0002	0.0002	3,258
2	0.0164	1.0923	1.0933	1.0115	1.0301	0.0014	0.0015	0.0032	0.0006	-10,684
3	0.0115	0.0076	1.0851	0.4513	0.0106	0.001	0.0007	0.003	0.0004	10,461
4	0.0038	0.0035	0.0089	1.1062	0.006	0.0002	0.0003	0.0001	0.0001	3,155
5	0.0058	0.0578	0.098	0.6569	1.1341	0.0005	0.0008	0.0004	0.0003	4,066
6	0.7803	2.1173	3.4665	15.249	2.6707	1.154	0.1361	0.0722	0.0559	3,596,887
7	5.0854	2.9969	8.6363	27.246	3.696	0.5453	1.7127	0.1654	0.1614	7,804,130
8	0.479	0.3575	3.3772	9.3886	0.5332	0.0517	0.0613	1.1667	0.0288	925,557
9	3.5349	4.7573	10.33	30.909	6.5227	0.369	0.3197	0.2444	1.3022	15,022,410

1963	1	2	3	4	5	6	7	8	9	Final Demand
1	1.0058	0.0026	0.0094	1.9521	0.0049	0.0004	0.0011	0.0002	0.0002	2,199
2	0.0056	1.0532	0.9444	1.0968	1.0861	0.0012	0.0021	0.0026	0.0007	-2,359
3	0.0032	0.0033	1.0732	0.2727	0.0094	0.0008	0.0008	0.0026	0.0004	8,630
4	0.0019	0.0011	0.0037	1.1145	0.0016	0.0001	0.0002	0.0001	0.0001	1,037
5	0.0025	0.0061	0.0483	0.8888	1.1087	0.0004	0.0009	0.0004	0.0003	3,540
6	0.2843	0.8465	2.0483	14.209	1.88	1.1866	0.1853	0.0793	0.0749	2,820,771
7	1.6793	1.3446	4.2429	14.296	2.1657	0.4915	1.7788	0.1478	0.1362	4,989,750
8	0.2025	0.1894	1.7661	7.6651	0.7549	0.0485	0.0604	1.1074	0.0233	456,425
9	0.8733	3.1616	6.1679	21.334	5.0473	0.2267	0.2346	0.202	1.2401	6,933,979

If we denote the energy rows of $\mathbf{L}^{*(80)}$ as $\boldsymbol{\alpha}^{80}$, the vector of total energy output as $\mathbf{g}^{80}$, and final demand as $\mathbf{f}^{80}$ (now in all cases dropping the $*$ for simplicity) with the analogous designations for 1963, we can compute the changes in energy consumption as

$$\mathbf{g}^{80} - \mathbf{g}^{63} = \boldsymbol{\alpha}^{63}(\mathbf{f}^{80} - \mathbf{f}^{63}) + (\boldsymbol{\alpha}^{80} - \boldsymbol{\alpha}^{63})\mathbf{f}^{63} + (\boldsymbol{\alpha}^{80} - \boldsymbol{\alpha}^{63})(\mathbf{f}^{80} - \mathbf{f}^{63}) = \begin{bmatrix} 6{,}121.4 \\ 6{,}457.3 \\ 11{,}337.2 \\ 4{,}698.8 \\ 6{,}049.6 \end{bmatrix}$$

where the effect caused by changing final demand is $\boldsymbol{\alpha}^{63}(\mathbf{f}^{80} - \mathbf{f}^{63}) = \begin{bmatrix} 10{,}467.2 \\ 9{,}433.3 \\ 9{,}967.9 \\ 3{,}738.7 \\ 8{,}021.1 \end{bmatrix}$;

the effect caused by changes in production functions is $(\boldsymbol{\alpha}^{80} - \boldsymbol{\alpha}^{63})\mathbf{f}^{63} = \begin{bmatrix} -2460.4 \\ -1257.5 \\ 756.8 \\ 613.3 \\ -502.4 \end{bmatrix}$;

and the effect of interaction of final demand and production function changes is

$$(\boldsymbol{\alpha}^{80} - \boldsymbol{\alpha}^{63})(\mathbf{f}^{80} - \mathbf{f}^{63}) = \begin{bmatrix} -1885.4 \\ -1718.5 \\ 612.5 \\ 346.8 \\ -1469.1 \end{bmatrix}.$$

Chapter 13, Environmental Input–Output Analysis

Chapter 13 reviews the extension of the input–output framework to incorporate activities of environmental pollution generation and elimination associated with economic activities as well as the linkages of input–output to models of ecosystems. The chapter begins with the augmented Leontief model for incorporating pollution generation and elimination, from which many subsequent approaches have been developed.

The chapter then describes the now widespread application of input-output analysis to environmental life cycle assessment and establishing a "pollution footprint" for industrial activity. Environmental input-output is also now widely used to evaluate global environmental issues. The special case of a analyzing the relationship between global climate change and industrial activity with a carbon footprint is then explored along with using input-output to attribute pollution generation to the demands driving consumption compared with the more traditional attribution of pollution generation to the sectors of industrial production necessary to meet that demand.

The exercise problems for this chapter explore the features of environmentally extended input-output models and their applications.

Problem 13.1

This problem explores the basic features of a generalized input-output model configuration applied to assessing energy, pollution, and employment associated with industrial activity. Assume that we have the following direct coefficient matrices for energy, air pollution, and employment ($\mathbf{D}^e$, $\mathbf{D}^v$ and $\mathbf{D}^l$, respectively) for two industries, 1 and 2: $\mathbf{D}^e = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}$, $\mathbf{D}^v = \begin{bmatrix} 0.2 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}$ and $\mathbf{D}^l = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}$. Notice that industry 2 is both a high-polluting and high-employment industry.

Suppose that the local government has an opportunity to spend a total of \$10 million on a regional development project. Two projects are candidates: (1) Project 1 would spend appropriated dollars in the ratio of 60 percent to industry 1 and 40 percent to industry 2; the minimum size of this project is \$4 million; (2) Project 2 would spend appropriated dollars in the ratio of 30 percent to industry 1 and 70 percent to industry 2; the minimum size of this project is \$2 million. The government can adopt either project or a combination of the two projects (as long as the minimum size of each project is at least maintained and that the total budget is not overrun). In other words, we might describe the options available to the government as:

$$\begin{bmatrix} \beta_a \\ \beta_b \end{bmatrix} = \alpha_1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$$

where α_1 and α_2 are budgets allocated to projects 1 and 2, respectively. β_a and β_b are the total final demands presented to the regional economy by the combination of projects for industries *A* and *B*, respectively.

Suppose that four alternative compositions of these projects are being considered (1) $\begin{cases} \alpha_1 = 4 \\ \alpha_2 = 2 \end{cases}$, (2) $\begin{cases} \alpha_1 = 5 \\ \alpha_2 = 5 \end{cases}$, (3) $\begin{cases} \alpha_1 = 10 \\ \alpha_2 = 0 \end{cases}$ and (4) $\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 10 \end{cases}$. The following table of constraints describes the local regulation on energy consumption and environmental pollution in the region:

	Maximum Allowable Changes Collectively by All Industries
Oil Consumption (10^{15} Btus)	3.0
Coal Consumption (10^{15} Btus)	no limit
SO_2 Emissions (tons)	14.5
NO_x Emissions (tons)	10

Finally, suppose that the regional economy is currently described by the following input-output transactions table (in millions of dollars):

	A	*B*	Total Output
A	1	3	10
B	5	1	10

If we are interested in determining which of the proposed combinations of projects (1), (2), (3) or (4) permit the region to operate within the above constraints on energy consumption and air pollution emission and within the established budget constraint, we begin by retrieving the matrix of economic transactions, $\mathbf{Z} = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}$, and the vector of total outputs, $\mathbf{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, from the table to calculate the economic direct and total requirements matrices: $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .1 & .3 \\ .5 & .1 \end{bmatrix}$ and

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.364 & .455 \\ .758 & 1.364 \end{bmatrix}.$$

Then we define the direct impact matrix as a concatenation of the three individual impact matrices, $\mathbf{D}^e$, $\mathbf{D}^v$ and $\mathbf{D}^l$, as: $\mathbf{D} = \begin{bmatrix} \mathbf{D}^e \\ \mathbf{D}^v \\ \mathbf{D}^l \end{bmatrix} = \left[\begin{array}{cc} .1 & .2 \\ .2 & .3 \\ \hdashline .2 & .5 \\ .2 & .3 \\ \hdashline .2 & .5 \end{array}\right]$ from which we can compute the total impact coefficients matrix, $\mathbf{DL} = \left[\begin{array}{cc} .288 & .318 \\ .5 & .5 \\ \hdashline .652 & .773 \\ .500 & .500 \\ \hdashline .652 & .773 \end{array}\right]$. We can represent the four candidate projects by the following matrix, the columns of which are the final demand change vectors:

$$\Delta\mathbf{F} = \overbrace{\begin{bmatrix} 3 & 4.5 & 6 & 3 \\ 3 & 5.5 & 4 & 7 \end{bmatrix}}^{\textit{Projects}}.$$

Total allocated budgets can be represented by the column sums of $\Delta\mathbf{F}$, found as $\mathbf{i}'[\Delta\mathbf{F}]=\begin{bmatrix}6 & 10 & 10 & 10\end{bmatrix}$; that is, all four candidate projects satisfy the budget constraint of \$10 million. We can compute matrix of total impacts as

$$\Delta\mathbf{X}=\mathbf{DL}\Delta\mathbf{F}=\begin{bmatrix}\mathbf{x}^1 & \mathbf{x}^2 & \mathbf{x}^3 & \mathbf{x}^4\end{bmatrix}=\begin{bmatrix}1.8 & 3.0 & 3.0 & 3.1\\ 3.0 & 5.0 & 5.0 & 5.0\\ 4.3 & 7.2 & 7.0 & 7.4\\ 3.0 & 5.0 & 5.0 & 5.0\\ 4.3 & 7.2 & 7.0 & 7.4\end{bmatrix}$$

where the columns are the vectors of total impacts for each scenario 1, 2, 3, and 4, respectively. Note that Project 4, using 3.1×10^{15} Btus of oil, exceeds the established consumption limit of 3.0×10^{15} Btus.

If our goal is to maximize employment, Project 2 should be chosen since it produces the highest level of employment among the three feasible projects, i.e., from among the first three scenarios that comply with established energy or environmental constraints (from the bottom row of $\Delta\mathbf{X}$).

Problem 13.2

This problem illustrates construction of a generalized impact assessment model from available data. Consider a regional economy that has two primary industries, *A* and *B*. In producing these two products it was observed that in the previous year air pollution emissions associated with this industrial activity included 3 pounds of SO_2 and 1 pound of NO_x emitted per dollars' worth of output of industry *A*, and 5 pounds of SO_2 and 2 pounds of NO_x emitted per dollars' worth of output of industry *B*.

It was also observed that industries *A* and *B* consumed 1×10^6 tons and 6×10^6 tons of coal, respectively, during that year. Industry *A* also consumed 2×10^6 barrels of oil. Total employment in the region was 100,000 (40 percent of which were employed by industry *A* and the rest by industry *B*) and the regional planning agency constructed the following input-output table of interindustry activity and total output in the region (in \$$10^6$):

	A	*B*	Total Output
A	2	6	20
B	6	12	30

If the projected vector of final demands for the next year is $\mathbf{f}^{new}=\begin{bmatrix}15\\25\end{bmatrix}$, we can estimate for the next year the total consumption of each energy type (coal and oil), the total pollution emission (of each type), and the level of total employment by first by retrieving the matrix of economic transactions, $\mathbf{Z}=\begin{bmatrix}2 & 6\\ 6 & 12\end{bmatrix}$, and the vector of total outputs, $\mathbf{x}=\begin{bmatrix}20\\30\end{bmatrix}$, from the table to calculate the economic direct and total requirements matrices: $\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}=\begin{bmatrix}.1 & .2\\ .3 & .4\end{bmatrix}$ and $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}1.250 & .417\\ .625 & 1.875\end{bmatrix}$. From the available data we assemble the direct impact coefficient matrices for energy, emissions, and employment impact and concatenate them to yield an overall matrix of direct impact coefficients, $\mathbf{D} == \begin{bmatrix} 1 & 0 \\ .05 & .2 \\ \hline 3 & 5 \\ 1 & 2 \\ \hline .002 & .002 \end{bmatrix}$.

Now we can compute the total impacts as $\mathbf{x}^* = \mathbf{DLf}^{new} = \begin{bmatrix} \mathbf{x}^{e*} \\ \mathbf{x}^{v*} \\ \mathbf{x}^{l*} \end{bmatrix} = \begin{bmatrix} 29.167 \\ 12.708 \\ \hline 368.75 \\ 141.667 \\ \hline .171 \end{bmatrix}$. That is,

$\mathbf{x}^{e*} = \begin{bmatrix} 29.167 \\ 12.708 \end{bmatrix}$ shows 29,167,000 tons of coal and 12,708,000 barrels of oil will be consumed in production next year; $\mathbf{x}^{V*} = \begin{bmatrix} 368.75 \\ 141.667 \end{bmatrix}$ shows that 368,750,000 pounds of SO_2 and 141,667,000 pounds of NO_x will be emitted in the course of that industrial production; and $\mathbf{x}^{l*} = [0.171]$ shows that 171,000 workers will be employed. Total economic output is found as $\mathbf{x}^{new} = \mathbf{Lf}^{new} = \begin{bmatrix} 29.167 \\ 56.25 \end{bmatrix}$; that is, $x_1^{new} = \$29,167,000$ and $x_2^{new} = \$56,250,000$.

Problem 13.3

This problem explores typical regional planning consideration in application of a generalized input-output impact model. Suppose a regional planning agency initiates a regional development planning effort. Four projects are being considered that would represent government purchases of regionally produced products of the output of three industries, A, B, and C, which would appear as final demands presented to the regional economy, as depicted in the following table.

Regional Industry	Project Expenditure (millions of dollars)			
	Project 1	Project 2	Project 3	Project 4
A	2	4	2	2
B	2	0	0	2
C	2	2	4	3

Additional information is available, including the matrix of technical coefficients, $\mathbf{A} = \begin{bmatrix} 0.04 & 0.23 & 0.38 \\ 0.33 & 0.52 & 0.47 \\ 0 & 0 & 0.1 \end{bmatrix}$, and relationships between the following quantities and total output given by the following:

	Industry		
	A	B	C
Pollution emission (grams/$ output)	4.2	7	9.1
Energy Consumption (bbls oil/$ output)	7.6	2.6	0.5
Employment (workers/ $ output)	0.73	0.33	0.63

To determine which of the four projects contributes most to gross regional output, we begin by computing the total economic requirements matrix, $\mathbf{L} = \begin{bmatrix} 1.247 & .598 & .839 \\ .857 & 2.494 & 1.665 \\ 0 & 0 & 1.111 \end{bmatrix}$, and from the table we can assemble the direct impact matrix as $\mathbf{D} = \begin{bmatrix} 4.2 & 7 & 9.1 \\ 7.6 & 2.6 & .5 \\ .73 & .33 & .63 \end{bmatrix}$. The table of prospective project expenditures retrieved directly from the table is $\Delta\mathbf{F} = \begin{bmatrix} 2 & 4 & 2 & 2 \\ 2 & 0 & 0 & 2 \\ 2 & 2 & 4 & 3 \end{bmatrix}$ from which we can compute $\Delta\mathbf{X}^* = \mathbf{DL}\Delta\mathbf{F} = \begin{bmatrix} 112.986 & 95.527 & 123.618 & 138.27 \\ 67.98 & 69.341 & 68.44 & 79.236 \\ 8.628 & 8.496 & 9.832 & 10.49 \end{bmatrix}$, the corresponding total impacts where each column shows the total impacts of the corresponding column in $\Delta\mathbf{F}$ for each project. Since the sum of final demands equals the contribution to GRP, we can also note that Project 4 contributes most to gross regional product (GRP), i.e., that project shows the largest column sum of $\Delta\mathbf{F}$, $\mathbf{i}'[\Delta\mathbf{F}] = \begin{bmatrix} 6 & 6 & 6 & 7 \end{bmatrix}$. Project 4 also consumes the most energy (79.236×10^6 bbls of oil) and contributes the most to regional employment (10.490×10^6 workers).

Problem 13.4

This problem explores the potential tradeoffs between environmental and employment considerations using input-output analysis. Consider an input-output economy defined by interindustry transactions and total outputs, $\mathbf{Z} = \begin{bmatrix} 140 & 350 \\ 800 & 50 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1,000 \\ 1,000 \end{bmatrix}$.

Suppose this is an economy in deep economic trouble. The federal government has at its disposal policy tools that can be implemented to stimulate demand for goods from one sector or the other. Also suppose that the plants in sector 1 discharge 0.3 lbs. of airborne particulate substances for every dollar of output (0.3 lbs/$ output), while sector 2 pollutes at 0.5 lbs/$ output. Finally, let labor input coefficients be 0.005 and 0.07 for sectors 1 and 2, respectively.

To assess whether or not a conflict of interest would arise between unions and environmentalists in determining the sector toward which the government should direct its policy effort, first, from $\mathbf{Z}$ and $\mathbf{x}$ we compute $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .14 & .35 \\ .8 & .05 \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} 1.769 & .652 \\ 1.49 & 1.601 \end{bmatrix}$. Since from the data provided, $\mathbf{D} = \begin{bmatrix} .3 & .5 \\ .005 & .07 \end{bmatrix}$ we can compute $\mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.276 & .996 \\ .113 & .115 \end{bmatrix}$.

Therefore, for $\mathbf{f}^{new} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x}^* = \begin{bmatrix} 1.276 \\ \hdashline .113 \end{bmatrix}$, meaning that for each new dollar's worth of final demand for the output of sector 1, there will be 1.276 pounds of pollutant emitted and 0.113 new workers. Similarly, with $\mathbf{f}^{new} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we find $\mathbf{x}^* = \begin{bmatrix} .996 \\ \hdashline .115 \end{bmatrix}$, meaning that for each new dollar's worth of final demand for the output of sector 2, there will be 0.996 pounds of pollutant emitted and 0.115 new workers. Thus, there would not be a conflict between unions and environmentalists in this case; each dollar’s worth of new demand for sector 2 generates less pollution and also generates more employment (notice that this is true despite the fact that sector 2’s direct-pollution coefficient per dollar of output is larger than sector 1’s direct-pollution coefficient).

Problem 13.5

This problem explores the basic features of the pollution-activity augmented Leontief input-output formulation. Consider the following table of interindustry transactions and total industry outputs (the same transactions as in problem 13.4 but with different total outputs):

		Purchasing Sector		Total Output
		1	2	
Selling Sector	1	140	350	2,000
	2	800	50	1,850

An amount of pollution generated by sector 1 is 10 units and by sector 2 is 25 units. Pollution abatement reduced pollution by 5 units in sector 1 and 12 units in sector 2. Total pollution permitted by local regulation is 12 units. if final demands for both sectors increase by 100, we can use the pollution-activity-augmented Leontief formulation to determine is the level of output for each industry by first augmenting the basic economic transactions matrix, $\mathbf{Z} = \begin{bmatrix} 140 & 350 \\ 800 & 50 \end{bmatrix}$ with the pollution abatement and elimination data to yield

$$\bar{\mathbf{Z}} = \begin{bmatrix} 140 & 350 & 5 \\ 800 & 50 & 12 \\ 10 & 25 & 0 \end{bmatrix}.$$

Total pollution output is found by adding pollution generation in the interindustry matrix to the pollution tolerated (reflected as a negative value in final demand), i.e., we define $x_p = 10 + 25 - 12 = 23$, which we can augment to the total industry outputs vector to yield

$\overline{\mathbf{x}} = \begin{bmatrix} 2,000 & 1,850 & 23 \end{bmatrix}'$. The vector of final demands is $\overline{\mathbf{f}} = \overline{\mathbf{x}} - \overline{\mathbf{Z}}\mathbf{i} = \begin{bmatrix} 1,505 & 988 & -12 \end{bmatrix}'$ and the matrix of technical coefficients is $\overline{\mathbf{A}} = \overline{\mathbf{Z}}\hat{\overline{\mathbf{x}}}^{-1} = \begin{bmatrix} .07 & .189 & .217 \\ .4 & .027 & .522 \\ .005 & .014 & 0 \end{bmatrix}$, from which we can compute $\overline{\mathbf{L}} = (\mathbf{I} - \overline{\mathbf{A}})^{-1} = \begin{bmatrix} 1.178 & .234 & .378 \\ .491 & 1.133 & .698 \\ .013 & .016 & 1.011 \end{bmatrix}$.

For an increase in final demand of both sectors by 100, $\Delta\overline{\mathbf{f}} = \begin{bmatrix} 100 & 100 & 0 \end{bmatrix}'$, the changes in total outputs and pollution are found as $\Delta\overline{\mathbf{x}} = \begin{bmatrix} 141.2 & 162.4 & 2.9 \end{bmatrix}'$. Hence, the new levels of outputs and pollution are $\overline{\mathbf{x}}^{new} = \begin{bmatrix} 2,141.2 & 2,012.4 & 25.9 \end{bmatrix}'$.

Problem 13.6

This problem compares regional and national pollution, energy consumption, and employment impacts of a public works initiative. In problems 10.5 and 10.6 national and regional input-output tables are defined with three sectors (natural resources, manufacturing, and services) with the following matrices of technical coefficients and vectors of total outputs, respectively,

$\mathbf{A}^N = \begin{bmatrix} .1830 & .0668 & .0087 \\ .1377 & .3070 & .0707 \\ .1603 & .2409 & .2999 \end{bmatrix}$, $\mathbf{x}^N = \begin{bmatrix} 518,288.6 \\ 4,953,700.6 \\ 14,260,843.0 \end{bmatrix}$, $\mathbf{A}^R = \begin{bmatrix} .1092 & .0324 & .0036 \\ .0899 & .0849 & .0412 \\ .1603 & .1170 & .2349 \end{bmatrix}$ and $\mathbf{x}^R = \begin{bmatrix} 8,262.7 \\ 95,450.8 \\ 170,690.3 \end{bmatrix}$. We define the energy use, pollution, and employment coefficients that apply to both the regional and national economies in the following table:

Environmental, Energy, and Employment Impact Coefficients

	Industry		
	Nat. Res.	Manuf.	Services
Pollution emission (grams/$ output)	4.2	7	9.1
Energy Consumption (bbls oil/$ output)	7.6	2.6	0.5
Employment (workers/ $ output)	7.3	3.3	6.3

Suppose a major new public works initiative by the federal government is characterized by the following vector of increases in federal spending: $\Delta\mathbf{f}' = \begin{bmatrix} 250 & 3,000 & 7,000 \end{bmatrix}$, of which 20 percent will be spent in the region (assume the 20 percent applies linearly to all expenditures). We can determine the percentage changes in total impacts on pollution, energy use, employment, and total industrial output of each industry sector for the region compared with those of the nation as a whole by first defining, from the table, the common direct impact coefficients as

$$\mathbf{D} = \begin{bmatrix} 4.2 & 7 & 9.1 \\ 7.6 & 2.6 & .5 \\ 7.3 & 3.3 & 6.3 \end{bmatrix}.$$

The baseline environmental, energy, and employment impacts for the nation and the region, respectively, are found by $\mathbf{x}^{*N} = \mathbf{D}\mathbf{x}^N = [166{,}626{,}387.6 \quad 23{,}949{,}036.4 \quad 109{,}974{,}029.7\ ]'$ and $\mathbf{x}^{*R} = \mathbf{D}\mathbf{x}^R = [2{,}256{,}140.7 \quad 396{,}313.8 \quad 1{,}450{,}654.2\ ]'$. Then the total impacts, including the economic impacts of the new public works project for the nation and the region, respectively, are

$$\Delta\overline{\mathbf{x}}^N = \begin{bmatrix} \mathbf{D}^{*N} \\ \hline (\mathbf{I}-\mathbf{A}^N)^{-1} \end{bmatrix} \Delta\mathbf{f}^N = \begin{bmatrix} 10.7832 & 16.2768 & 14.7759 \\ 10.4543 & 5.2368 & 1.3729 \\ 12.5176 & 9.4836 & 10.1120 \\ \hline 1.2516 & 0.1306 & 0.0287 \\ 0.2881 & 1.5256 & 0.1576 \\ 0.3857 & 0.5549 & 1.4892 \end{bmatrix} \begin{bmatrix} 250 \\ 3{,}000 \\ 7{,}000 \end{bmatrix} = \begin{bmatrix} 154{,}957.3 \\ 27{,}934.5 \\ 102{,}364.1 \\ \hline 906.1 \\ 5{,}752.2 \\ 12{,}185.3 \end{bmatrix}$$

$$\Delta\overline{\mathbf{x}}^R = \begin{bmatrix} \mathbf{D}^{*R} \\ \hline (\mathbf{I}-\mathbf{A}^R)^{-1} \end{bmatrix} \Delta\mathbf{f}^R = \begin{bmatrix} 7.9150 & 9.5207 & 12.4438 \\ 9.0188 & 3.2720 & 0.8721 \\ 10.2453 & 5.0627 & 8.5550 \\ \hline 1.1281 & 0.0409 & 0.0075 \\ 0.1223 & 1.1048 & 0.0601 \\ 0.2550 & 0.1775 & 1.3178 \end{bmatrix} \begin{bmatrix} 50 \\ 600 \\ 1{,}400 \end{bmatrix} = \begin{bmatrix} 23{,}529.5 \\ 3{,}635.2 \\ 15{,}527.0 \\ \hline 91.5 \\ 753.1 \\ 1{,}964.2 \end{bmatrix}$$

where $\mathbf{\Delta f}^R = .2\mathbf{\Delta f}^N$. Hence, the comparative percentage changes from $\mathbf{x}^{*(N)}$ and $\mathbf{x}^{*(R)}$ are:

	Nation	Region
Nat. Res.	0.09	1.04
Manuf.	0.12	0.92
Services	0.09	1.07
Pollution	0.17	1.11
Energy	0.12	0.79
Employ.	0.09	1.15

Problem 13.7

This problem explores the implications of an energy shortage on economic performance using the regional economy specified in problem 13.6 prior to the projected final demand for that problem. Recall for that economy, the matrix of technical coefficients and vector of total outputs, respectively, were $\mathbf{A}^R = \begin{bmatrix} .1092 & .0324 & .0036 \\ .0899 & .0849 & .0412 \\ .1603 & .1170 & .2349 \end{bmatrix}$ and $\mathbf{x}^R = \begin{bmatrix} 8,262.7 \\ 95,450.8 \\ 170,690.3 \end{bmatrix}$. The matrix of direct impact coefficients was specified as $\mathbf{D} = \begin{bmatrix} 4.2 & 7 & 9.1 \\ 7.6 & 2.6 & .5 \\ 7.3 & 3.3 & 6.3 \end{bmatrix}$.

The levels of pollution, energy consumption, and employment accompanying the baseline levels of total industry output are found by

$\mathbf{x}^{*R} = \mathbf{D}\mathbf{x}^R = \begin{bmatrix} 4.2 & 7 & 9.1 \\ 7.6 & 2.6 & .5 \\ 7.3 & 3.3 & 6.3 \end{bmatrix} \begin{bmatrix} 8,262.7 \\ 95,450.8 \\ 170,690.3 \end{bmatrix} = \begin{bmatrix} 2,256,140.7 \\ 396,313.8 \\ 1,450,654.2 \end{bmatrix}$. We can show a ten percent reduction in energy availability defined by $\mathbf{x}^{*new} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .9 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2,256,140.7 \\ 396,313.8 \\ 1,450,654.2 \end{bmatrix} = \begin{bmatrix} 2,256,140.7 \\ 356,682.4 \\ 1,450,654.2 \end{bmatrix}$.

The corresponding limits on total industry output can be found conveniently as

$\mathbf{x}^{new} = \mathbf{D}^{-1}\mathbf{x}^{*new} = \begin{bmatrix} -0.0766 & 0.0732 & 0.1049 \\ 0.2301 & 0.2079 & -0.3488 \\ -0.0317 & -0.1937 & 0.2199 \end{bmatrix} \begin{bmatrix} 2,256,140.7 \\ 356,682.4 \\ 1,450,654.2 \end{bmatrix} = \begin{bmatrix} 5,362.0 \\ 87,210.5 \\ 178,367.8 \end{bmatrix}$, which in turn means

$\mathbf{f}^{new} = (\mathbf{I} - \mathbf{A})\mathbf{x}^{new} = \begin{bmatrix} 0.8908 & -0.0324 & -0.0036 \\ -0.0899 & 0.9151 & -0.0412 \\ -0.1603 & -0.1170 & 0.7651 \end{bmatrix} \begin{bmatrix} 5,362.0 \\ 87,210.5 \\ 178,367.8 \end{bmatrix} = \begin{bmatrix} 1,308.7 \\ 71,975.5 \\ 125,406.0 \end{bmatrix}$.

Finally, the change in GDP is $\mathbf{i}'\mathbf{f}^{new} - \mathbf{i}'\mathbf{f} = -2,637.7$ or a 1.31 percent reduction in GDP, where the original final demand is $\mathbf{f} = \mathbf{x} - \mathbf{A}\mathbf{x} = \begin{bmatrix} 3,653.3 \\ 79,571.8 \\ 118,102.9 \end{bmatrix}$.

Problem 13.8

This problem explores input-output analysis configured as a linear programming (LP) problem using alternative objective functions. Consider a traditional input-output economy is specified by the technical requirements matrix and vector of final demands, respectively, as $\mathbf{A} = \begin{bmatrix} .3 & .1 \\ .2 & .5 \end{bmatrix}$ and

$\mathbf{f} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, for which the vector of total outputs is found by $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \begin{bmatrix} 7.575 \\ 10.03 \end{bmatrix}$. The vector of value-added coefficients for this economy are found as $\mathbf{v} = \mathbf{x} - \mathbf{i}'\mathbf{A} = \begin{bmatrix} .5 \\ .4 \end{bmatrix}$.

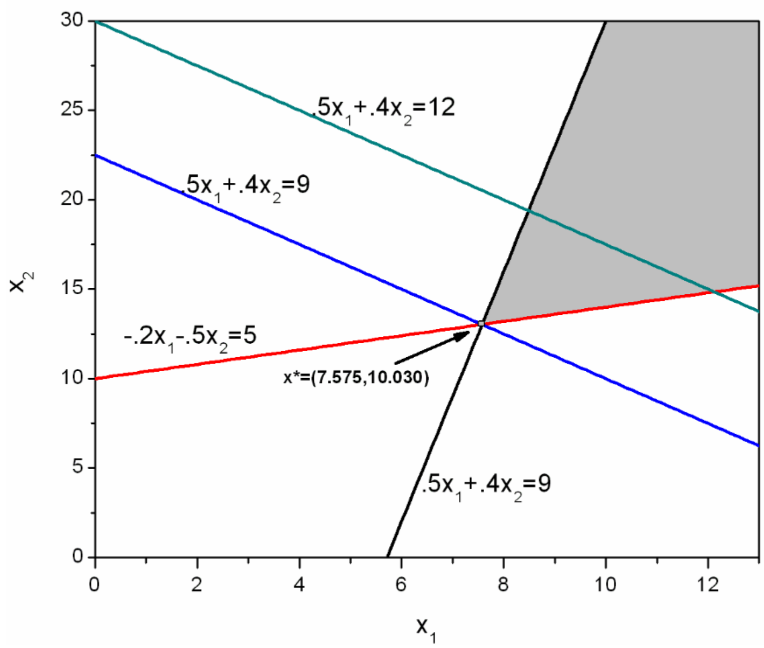
To specify this model as an equivalent LP formulation, we can write

$$\begin{gathered} Min\ .5x_1 + .4x_2 \\ .7x_1 - .1x_2 \geq 4 \\ -.2x_1 + .5x_2 \geq 5 \\ x_1, x_2 \geq 0 \end{gathered}$$

which we can interpret as minimizing the gross domestic product (the sum of all value added) subject to total industry production that at least satisfies all industry final demands. In matrix terms this is expressed as

$$\begin{gathered} Min\ \mathbf{v}'\mathbf{x} \\ (\mathbf{I} - \mathbf{A})\mathbf{x} \geq \mathbf{f} \\ \mathbf{x} \geq 0 \end{gathered}$$

The graphical solution is



Note that it turns out that the solution to this LP problem has a "dual" formulation (discussed in Chapter 13) of maximizing the value of total final demand (or maximizing gross domestic product) subject to the technical coefficients and supply availability of value-added factors, which may seem more intuitive. These dual LP problems have the same result such that maximized value of final demand equals the minimized cost of value-added factors and that value is the gross domestic product, or the familiar equality of national product to national income.

Now, suppose that for this economy pollution is generated at a rate of 2.5 units per dollar of output of industry 1 and 2 units per dollar of output of industry 2 for this economy. If we replace the objective function for the LP problem with minimizing pollution emissions instead of maximizing GDP, we can express this as

$$\begin{gathered} Min\ 2.5x_1 + 2x_2 \\ .7x_1 - .1x_2 \geq 4 \\ -.2x_1 + .5x_2 \geq 5 \\ x_1, x_2 \geq 0 \end{gathered}$$

Note that the solution to this LP problem is the same as with the original objective function, which should not be surprising, in this particular case, since the minimum production levels defined in the constraint equations are already determined uniquely by the input-output relationships $(\mathbf{I} - \mathbf{A}) \geq \mathbf{f}$.

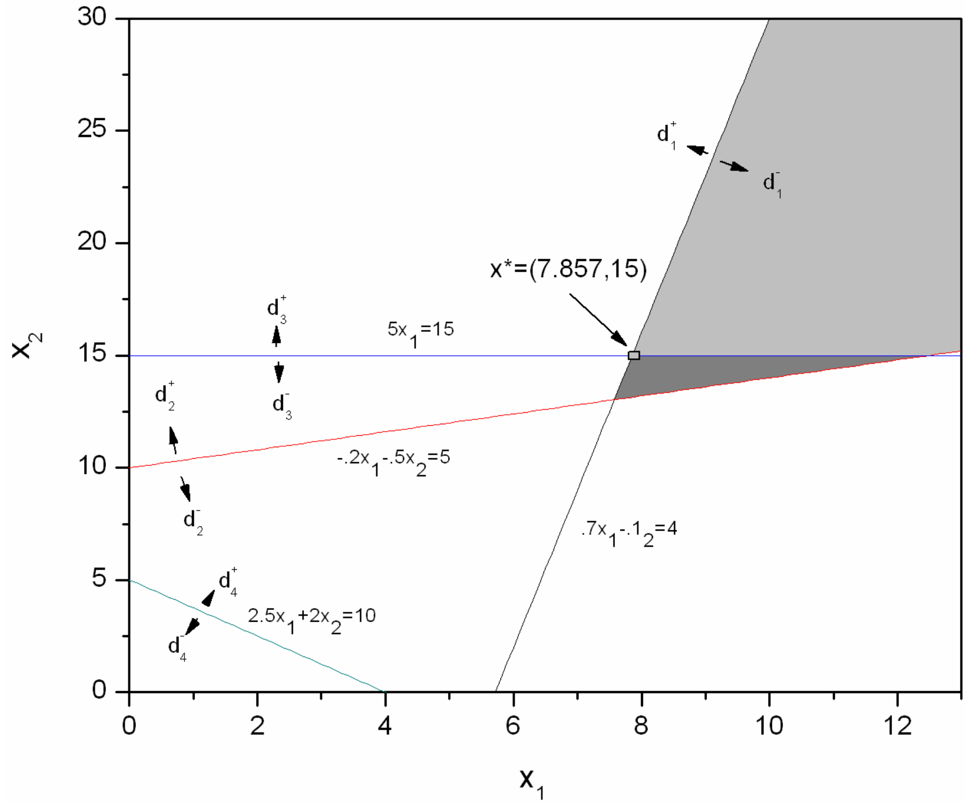
Problem 13.9

This problem expands the LP formulation for the economy specified in problem 13.8 to a goal programming (GP) formulation which can accommodate multiple objective functions. Suppose that, in addition to the environmental criteria specified in problem 13.8, we also know that employment is generated at a rate of 6 and 3 units per dollars' worth of output for industries 1 and 2, respectively, and that there is high priority employment target of 7.5 units for industry 2.

To formulate this situation as finding the vector of total outputs that meets the employment target for industry 2 as the highest priority, then meets final demand requirements for both industries as the next highest priority, and minimizes total pollution generation to the extent possible as the next priority, and, if possible, limiting pollution to a total level of 10 units between the two industries, we specify the following GP problem:

$$\begin{gathered} Min\ \ P_1(d_3^-) + P_2(d_1^- + d_2^-) + P_3(d_4^+) \\ .7x_1 - .1x_2 + d_1^+ + d_1^- = 4 \\ -.2x_1 - .5x_2 + d_2^+ + d_2^- = 5 \\ .5x_2 + d_3^+ + d_3^- = 7.5 \\ 2.5x_1 - 2x_2 + d_4^+ + d_4^- = 10 \end{gathered}$$

The graphical solution to this GP problem is shown below with the preferred vector of total outputs computed as $\mathbf{x}^* = (7.857, 15)$.



Problem 13.10

This problem illustrates the estimation of the change in U.S. carbon dioxide emissions between two reference years. We use highly aggregated (seven industry sectors) versions of the 1997 and 2007 U.S. input-output tables (industry-by-industry and assume industry-based technology, after redefinitions) provided in Appendix SD1.

First, we retrieve the supply and use matrices and, recalling that the supply matrix is the transpose of the use matrix, we can specify:

$$\mathbf{V}^{1997} = \begin{bmatrix} 258,234 & 0 & 0 & 45 & 0 & 796 & 0 \\ 0 & 162,842 & 0 & 9,119 & 0 & 0 & 0 \\ 0 & 0 & 762,267 & 0 & 0 & 0 & 0 \\ 0 & 1,007 & 0 & 3,752,428 & 0 & 28,347 & 3,789 \\ 126 & 381 & 0 & 0 & 2,262,980 & 259 & 742 \\ 0 & 240 & 0 & 0 & 11 & 6,577,434 & 1,866 \\ 1,002 & 0 & 0 & 0 & 76,744 & 165,709 & 1,326,951 \end{bmatrix}$$

$$\mathbf{V}^{2007} = \begin{bmatrix} 347,665 & 0 & 0 & 12 & 0 & 1,559 & 0 \\ 0 & 437065 & 0 & 24,850 & 0 & 0 & 0 \\ 0 & 0 & 1,436,071 & 0 & 0 & 0 & 0 \\ 0 & 828 & 0 & 5,176,967 & 0 & 33,134 & 5,050 \\ 439 & 15 & 0 & 0 & 3,784,910 & 2,194 & 1,382 \\ 0 & 149 & 0 & 0 & 10 & 12,218,068 & 1,377 \\ 1,413 & 0 & 0 & 0 & 120,309 & 296,914 & 2,348,118 \end{bmatrix}$$

$$\mathbf{U}^{1997} = \begin{bmatrix} 48,986 & 86 & 1,067 & 155,059 & 1,282 & 3,447 & 1,306 \\ 1,195 & 17,051 & 7,663 & 126,256 & 32,295 & 1,439 & 10,910 \\ 879 & 1,958 & 189 & 15,114 & 6,110 & 38,589 & 33,763 \\ 44,105 & 19,986 & 205,959 & 1,532,339 & 97,639 & 333,465 & 165,270 \\ 25,240 & 12,589 & 85,547 & 335,127 & 210,522 & 198,233 & 61,843 \\ 28,584 & 29,237 & 68,715 & 320,296 & 413,007 & 1,774,829 & 203,053 \\ 720 & 808 & 4,873 & 25,755 & 43,718 & 52,333 & 17,330 \end{bmatrix}$$

$$\mathbf{U}^{2007} = \begin{bmatrix} 64,432 & 127 & 1,841 & 206,823 & 1,524 & 5,602 & 3,832 \\ 1,995 & 41,923 & 13,422 & 428,689 & 83,277 & 4,020 & 24,343 \\ 1,764 & 3,806 & 188 & 16,102 & 12,568 & 129,186 & 58,171 \\ 62,374 & 40,138 & 343,216 & 1,954,459 & 210,331 & 542,427 & 327,316 \\ 37,563 & 18,993 & 150,821 & 489,276 & 393,027 & 363,846 & 112,297 \\ 34,714 & 54,852 & 118,577 & 386,380 & 745,794 & 3,643,107 & 433,804 \\ 823 & 1,401 & 1,611 & 38,698 & 59,939 & 90,726 & 26,504 \end{bmatrix}$$

We can compute the vectors of total industry outputs, $\mathbf{x} = \mathbf{Vi}$, total commodity outputs, $\mathbf{q} = \mathbf{i'V}$ for both years as:

$$\mathbf{x}^{1997} = \mathbf{V}^{1997}\mathbf{i} = \begin{bmatrix} 259,362 & 164,470 & 762,267 & 3,761,592 & 2,339,735 & 6,772,545 & 1,333,348 \end{bmatrix}$$

$$\mathbf{q}^{1997} = \mathbf{i'V}^{1997} \begin{bmatrix} 259,075 & 171,961 & 762,267 & 3,785,571 & 2,264,488 & 6,579,551 & 1,570,406 \end{bmatrix}'$$

$$\mathbf{x}^{2007} = \mathbf{V}^{2007}\mathbf{i} = \begin{bmatrix} 349,517 & 438,057 & 1,436,071 & 5,201,829 & 3,905,229 & 12,551,869 & 2,355,927 \end{bmatrix}'$$

$$\mathbf{q}^{2007} = \mathbf{i'V}^{2007} = \begin{bmatrix} 349,236 & 461,915 & 1,436,071 & 5,215,979 & 3,788,940 & 12,219,604 & 2,766,754 \end{bmatrix}$$

We can the compute the matrix of industry commodity requirements, **B**, and the matrix of commodity output proportions, **D**, for both years and under the assumption of an industry-by-industry model assuming industry-based technology we can specify the direct requirements matrix as $\mathbf{A} = \mathbf{DB}$ for each of the two years as:

$$\mathbf{A}^{1997} = \begin{bmatrix} 0.188273 & 0.000544 & 0.001409 & 0.041103 & 0.000568 & 0.000540 & 0.000996 \\ 0.004773 & 0.098467 & 0.010171 & 0.032766 & 0.013171 & 0.000320 & 0.008047 \\ 0.003389 & 0.011905 & 0.000248 & 0.004018 & 0.002611 & 0.005698 & 0.025322 \\ 0.169072 & 0.121839 & 0.268290 & 0.404378 & 0.042252 & 0.049956 & 0.123601 \\ 0.097359 & 0.076731 & 0.112182 & 0.089133 & 0.089964 & 0.029265 & 0.046382 \\ 0.110184 & 0.177859 & 0.090139 & 0.085177 & 0.176504 & 0.261988 & 0.152266 \\ 0.009150 & 0.011224 & 0.011481 & 0.011109 & 0.023286 & 0.014123 & 0.016394 \end{bmatrix}$$

$$\mathbf{A}^{2007}=\begin{bmatrix} 0.183530 & 0.000305 & 0.001287 & 0.039591 & 0.000413 & 0.000481 & 0.001643 \\ 0.006251 & 0.090990 & 0.009982 & 0.079768 & 0.020434 & 0.000509 & 0.010439 \\ 0.005047 & 0.008688 & 0.000131 & 0.003095 & 0.003218 & 0.010292 & 0.024691 \\ 0.177407 & 0.091459 & 0.237452 & 0.373278 & 0.054040 & 0.043692 & 0.138432 \\ 0.107608 & 0.043339 & 0.104929 & 0.094028 & 0.100577 & 0.029013 & 0.047656 \\ 0.099311 & 0.125233 & 0.082564 & 0.074299 & 0.190964 & 0.290211 & 0.184119 \\ 0.008570 & 0.007135 & 0.006298 & 0.011266 & 0.020864 & 0.014109 & 0.015542 \end{bmatrix}$$

The corresponding matrices of total requirements, $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}$, for the two years are:

$$\mathbf{L}^{1997}=(\mathbf{I}-\mathbf{A}^{1997})^{-1}=\begin{bmatrix} 1.252977 & 0.015617 & 0.027750 & 0.089978 & 0.007161 & 0.007798 & 0.014963 \\ 0.024627 & 1.122265 & 0.032976 & 0.068182 & 0.021330 & 0.006615 & 0.020653 \\ 0.008970 & 0.017526 & 1.005313 & 0.011175 & 0.006220 & 0.009337 & 0.029176 \\ 0.407744 & 0.285896 & 0.506676 & 1.768336 & 0.120680 & 0.133877 & 0.264423 \\ 0.187822 & 0.138849 & 0.188470 & 0.200829 & 1.125215 & 0.061665 & 0.094020 \\ 0.291291 & 0.346223 & 0.243804 & 0.289671 & 0.296740 & 1.393885 & 0.275576 \\ 0.025275 & 0.024644 & 0.026054 & 0.030630 & 0.032644 & 0.023243 & 1.026551 \end{bmatrix}$$

$$\mathbf{L}^{2007}=(\mathbf{I}-\mathbf{A}^{2007})^{-1}=\begin{bmatrix} 1.244908 & 0.010357 & 0.022731 & 0.082327 & 0.007667 & 0.006880 & 0.015992 \\ 0.049530 & 1.120132 & 0.053254 & 0.154067 & 0.038688 & 0.013456 & 0.039350 \\ 0.012215 & 0.013642 & 1.005408 & 0.011260 & 0.008746 & 0.016252 & 0.030428 \\ 0.407543 & 0.199923 & 0.430289 & 1.689458 & 0.140185 & 0.121968 & 0.280757 \\ 0.205969 & 0.086585 & 0.175881 & 0.205806 & 1.142868 & 0.064162 & 0.101938 \\ 0.288758 & 0.249039 & 0.226796 & 0.279912 & 0.339136 & 1.450302 & 0.335832 \\ 0.024442 & 0.015987 & 0.018918 & 0.029612 & 0.031088 & 0.023802 & 1.026593 \end{bmatrix}$$

Finally, we specify the vector of units of carbon dioxide emissions generated per dollar of total output in 1997 as $\mathbf{d}^{1997}=[2\quad 3\quad 4\quad 7\quad 10\quad 5\quad 4]'$. If we presume that the availability of new technology reduces the emissions per dollar of output in the year 2007 for the manufacturing sector by 10 percent and the construction sector by 15 percent, then we can specify the new emissions coefficients for 2007 as $\mathbf{d}^{2007}=[2\quad 3\quad 3.4\quad 6.3\quad 10\quad 5\quad 4]'$. Hence the total pollution impacts are $\Delta p=\mathbf{i}'[\mathbf{T}^{2007}\mathbf{f}^{2007}-\mathbf{T}^{19997}\mathbf{f}^{1997]}]=57{,}916{,}899$ where $\mathbf{T}^{2007}=[\mathbf{d}^{2007}]'\mathbf{L}^{2007}$ and $\mathbf{T}^{1997}=[\mathbf{d}^{1997}]'\mathbf{L}^{1997}$ or equivalently $[\mathbf{d}^{2007}]'\mathbf{x}^{2007}-[\mathbf{d}^{1997}]'\mathbf{x}^{1997}$. In this case the improvement in pollution coefficients (reduction pollution generation per dollar of total output) was offset by growth in output levels so the net result was an increase in pollution impacts.

Problem 13.11

This problem illustrates the attribution of pollution emissions, in this case CO_2 emissions, to either consumption or production using an interregional input-output (IRIO) model. Consider the 3 region 2 sector IRIO interindustry technical coefficients matrix defined by

$$\mathbf{A}=\left[\begin{array}{cc:cc:cc} .222 & .121 & .027 & .023 & .007 & .014 \\ .217 & .028 & .014 & .015 & .021 & .012 \\ \hdashline .02 & .025 & .126 & .088 & .019 & .019 \\ .002 & .025 & .06 & .141 & .002 & .019 \\ \hdashline .012 & .02 & .019 & .005 & .192 & .179 \\ .022 & .017 & .005 & .013 & .195 & .164 \end{array}\right]$$

. The corresponding Leontief inverse is then

$$\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}=\left[\begin{array}{cc:cc:cc} 1.335 & .17 & .048 & .045 & .025 & .033 \\ .3 & .1069 & .029 & .031 & .039 & .03 \\ \hdashline .043 & .04 & 1.155 & .125 & .038 & .039 \\ .018 & .037 & .085 & 1.217 & .015 & .034 \\ \hdashline .04 & .038 & .033 & .018 & 1.308 & .282 \\ .051 & .036 & .019 & .025 & .307 & 1.265 \end{array}\right]$$

. We define a vector of CO_2 emission coefficients as $\mathbf{g}'=\left[.9 \quad .4 \;\vdots\; .3 \quad 1.0 \;\vdots\; .2 \quad .7\right]$.

For a new vector of final demands presented to this IRIO economy, defined by

$$\mathbf{f}^{new}=\left[\begin{array}{c} 1500 \\ 2000 \\ \hdashline 55 \\ 40 \\ \hdashline 5 \\ 3 \end{array}\right]$$

, we can calculate the vector of the total CO_2 emissions associated with the total economic production for each sector in each region attributed to where the pollution is generated as

$$\mathbf{e}^{D}=\hat{\mathbf{g}}\mathbf{L}\mathbf{f}^{new}=\left[\begin{array}{c} 2112.0 \\ 1036.7 \\ \hdashline 63.9 \\ 153.9 \\ \hdashline 29.1 \\ 109.1 \end{array}\right]$$

. To attribute the emissions to consumption rather than production, i.e., where the consumption occurs that generates the demand for the production that generates the emissions, we specify the impacts as

$$\mathbf{e}^{C}=\mathbf{g}\mathbf{L}\hat{\mathbf{f}}^{new}=\left[\begin{array}{c} 2091.7 \\ 1320.2 \\ \hdashline 27.64 \\ 51.33 \\ \hdashline 2.7 \\ 3.1 \end{array}\right].$$

Note that both vectors sum to the same level of total CO_2 emissions, i.e., $\mathbf{i}'\mathbf{e}^{D}=\mathbf{i}'\mathbf{e}^{C}=34{,}967$, but $\mathbf{e}^{D}$ attributes the pollution generated to the sectors where the emissions were generated during production while $\mathbf{e}^{C}$ attributes the emissions to final consumers, i.e., final demand sectors that generated the demand for that industrial production and associate emissions. Since this economy is dominated by region 1, as is evident by total GDP in regions 1, 2, and 3 of 3500, 95, and 8, respectively (the sum of final demands in each region), the total emissions when attributable to final demand are 9 percent higher for region 1 and 63 and 96 percent lower in regions 2 and 3, respectively, when emissions are attributed to final consumption rather than the source of production.

Problem 13.12

This problem explores the same issues regarding attribution of CO_2 emissions attributed to consumption versus production as in Problem 13.11, but for a 3-region, 3-sector global IRIO model. Consider the Global IRIO transactions tables aggregated to 3 regions (the US, China, and Rest of World) and 3-Sector industry sectors (Agriculture and Mining, Manufacturing, and Services & Utilities) for the years 2005 and 2015 given in Appendix SD2.

First, we retrieve the matrices of IRIO transactions, **Z**, and total outputs, **x**, and specify for the two years:

$$\mathbf{Z}^{2005} = \left[\begin{array}{ccc|ccc|ccc} 355 & 247 & 155 & 2 & 1 & 0 & 18 & 11 & 7 \\ 114 & 1{,}173 & 1{,}199 & 1 & 17 & 3 & 19 & 195 & 80 \\ 263 & 810 & 4{,}985 & 1 & 5 & 4 & 21 & 87 & 159 \\ \hline 1 & 1 & 0 & 237 & 208 & 71 & 7 & 8 & 4 \\ 2 & 42 & 35 & 85 & 1{,}149 & 388 & 9 & 158 & 71 \\ 0 & 3 & 4 & 57 & 236 & 368 & 2 & 13 & 14 \\ \hline 42 & 197 & 32 & 10 & 59 & 4 & 1{,}800 & 1{,}352 & 702 \\ 18 & 264 & 190 & 10 & 213 & 42 & 573 & 6123 & 3{,}449 \\ 12 & 60 & 132 & 7 & 45 & 31 & 1{,}052 & 3{,}100 & 10{,}903 \end{array}\right] \quad \mathbf{x}^{2005} = \left[\begin{array}{c} 1{,}346 \\ 4{,}568 \\ 17{,}168 \\ \hline 846 \\ 2{,}675 \\ 1{,}867 \\ \hline 6{,}760 \\ 16{,}816 \\ 37{,}192 \end{array}\right]$$

$$\mathbf{Z}^{2015} = \left[\begin{array}{ccc|ccc|ccc} 503 & 269 & 224 & 10 & 5 & 2 & 35 & 20 & 13 \\ 108 & 1{,}180 & 1{,}228 & 3 & 57 & 14 & 27 & 250 & 122 \\ 374 & 974 & 7{,}023 & 6 & 22 & 19 & 39 & 133 & 304 \\ \hline 2 & 0 & 1 & 1{,}336 & 905 & 325 & 18 & 5 & 9 \\ 6 & 108 & 121 & 423 & 5{,}879 & 2{,}001 & 39 & 502 & 283 \\ 1 & 9 & 12 & 413 & 1{,}563 & 2{,}519 & 6 & 43 & 42 \\ \hline 39 & 148 & 33 & 52 & 254 & 16 & 3{,}415 & 1{,}968 & 1{,}096 \\ 19 & 280 & 225 & 25 & 533 & 119 & 868 & 7{,}857 & 4{,}714 \\ 15 & 63 & 196 & 18 & 111 & 69 & 1{,}635 & 3{,}981 & 15{,}278 \end{array}\right], \quad \mathbf{x}^{2015} = \left[\begin{array}{c} 1{,}838 \\ 5{,}284 \\ 23{,}699 \\ \hline 3{,}937 \\ 12{,}429 \\ 10{,}818 \\ \hline 11{,}465 \\ 22{,}489 \\ 52{,}297 \end{array}\right]$$

From $\mathbf{Z}^{2005}$, $\mathbf{x}^{2005}$, $\mathbf{Z}^{2015}$, and $\mathbf{x}^{2015}$, we can calculate the associated vectors of final demand:

$$\mathbf{f}^{2005} = \mathbf{x}^{2005} - \mathbf{Z}^{2005}\mathbf{i} = [551 \quad 1{,}768 \quad 10{,}835 \mid 310 \quad 736 \quad 1{,}171 \mid 2{,}563 \quad 5{,}934 \quad 21{,}851]$$

$$\mathbf{f}^{2015} = \mathbf{x}^{2015} - \mathbf{Z}^{2015}\mathbf{i} = [757 \quad 2{,}296 \quad 14{,}805 \mid 1{,}335 \quad 3{,}069 \quad 6{,}211 \mid 4{,}446 \quad 7{,}849 \quad 30{,}931].$$

Also, we can compute the matrices of technical coefficients, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, and total requirements, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ for each year as:

$$\mathbf{A}^{2005}=\left[\begin{array}{ccc:ccc:ccc}
0.2634 & 0.0541 & 0.0090 & 0.0025 & 0.0003 & 0.0002 & 0.0026 & 0.0006 & 0.0002\\
0.0849 & 0.2567 & 0.0698 & 0.0011 & 0.0062 & 0.0018 & 0.0027 & 0.0116 & 0.0021\\
0.1953 & 0.1772 & 0.2903 & 0.0014 & 0.0020 & 0.0019 & 0.0031 & 0.0051 & 0.0043\\ \hdashline
0.0006 & 0.0001 & 0.0000 & 0.2796 & 0.0778 & 0.0379 & 0.0010 & 0.0005 & 0.0001\\
0.0018 & 0.0093 & 0.0020 & 0.1006 & 0.4294 & 0.2078 & 0.0013 & 0.0094 & 0.0019\\
0.0002 & 0.0007 & 0.0002 & 0.0668 & 0.0881 & 0.1973 & 0.0003 & 0.0008 & 0.0004\\ \hdashline
0.0308 & 0.0432 & 0.0018 & 0.0116 & 0.0221 & 0.0022 & 0.2663 & 0.0804 & 0.0189\\
0.0135 & 0.0578 & 0.0111 & 0.0116 & 0.0796 & 0.0225 & 0.0847 & 0.3641 & 0.0927\\
0.0090 & 0.0131 & 0.0077 & 0.0078 & 0.0168 & 0.0168 & 0.1556 & 0.1844 & 0.2931
\end{array}\right]$$

$$\mathbf{A}^{2015}=\left[\begin{array}{ccc:ccc:ccc}
0.2736 & 0.0510 & 0.0094 & 0.0026 & 0.0004 & 0.0002 & 0.0030 & 0.0009 & 0.0002\\
0.0586 & 0.2232 & 0.0518 & 0.0008 & 0.0046 & 0.0013 & 0.0023 & 0.0111 & 0.0023\\
0.2034 & 0.1842 & 0.2964 & 0.0015 & 0.0017 & 0.0018 & 0.0034 & 0.0059 & 0.0058\\ \hdashline
0.0013 & 0.0000 & 0.0001 & 0.3394 & 0.0728 & 0.0301 & 0.0016 & 0.0002 & 0.0002\\
0.0034 & 0.0203 & 0.0051 & 0.1074 & 0.4730 & 0.1849 & 0.0034 & 0.0223 & 0.0054\\
0.0004 & 0.0017 & 0.0005 & 0.1048 & 0.1257 & 0.2329 & 0.0005 & 0.0019 & 0.0008\\ \hdashline
0.0210 & 0.0280 & 0.0014 & 0.0132 & 0.0204 & 0.0014 & 0.2978 & 0.0875 & 0.0209\\
0.0102 & 0.0529 & 0.0095 & 0.0063 & 0.0429 & 0.0110 & 0.0757 & 0.3494 & 0.0901\\
0.0082 & 0.0119 & 0.0083 & 0.0045 & 0.0089 & 0.0064 & 0.1426 & 0.1770 & 0.2921
\end{array}\right]$$

$$\mathbf{L}^{2005}=\left[\begin{array}{ccc:ccc:ccc}
1.3780 & 0.1078 & 0.0282 & 0.0059 & 0.0041 & 0.0022 & 0.0064 & 0.0050 & 0.0017\\
0.1996 & 1.3971 & 0.1407 & 0.0083 & 0.0239 & 0.0113 & 0.0123 & 0.0316 & 0.0097\\
0.4304 & 0.3810 & 1.4527 & 0.0090 & 0.0167 & 0.0103 & 0.0149 & 0.0253 & 0.0138\\ \hdashline
0.0027 & 0.0042 & 0.0013 & 1.4298 & 0.2151 & 0.1233 & 0.0035 & 0.0054 & 0.0017\\
0.0120 & 0.0305 & 0.0093 & 0.3099 & 1.8781 & 0.5020 & 0.0101 & 0.0335 & 0.0102\\
0.0023 & 0.0053 & 0.0017 & 0.1532 & 0.2245 & 1.3114 & 0.0023 & 0.0061 & 0.0022\\ \hdashline
0.0812 & 0.1098 & 0.0196 & 0.0441 & 0.0971 & 0.0383 & 1.4006 & 0.1995 & 0.0643\\
0.0767 & 0.1692 & 0.0478 & 0.0858 & 0.2842 & 0.1306 & 0.2440 & 1.6776 & 0.2283\\
0.0642 & 0.1005 & 0.0358 & 0.0591 & 0.1484 & 0.0873 & 0.3728 & 0.4834 & 1.4891
\end{array}\right]$$

$$\mathbf{L}^{2015}=\left[\begin{array}{ccc:ccc:ccc}
1.3923 & 0.0984 & 0.0261 & 0.0069 & 0.0041 & 0.0019 & 0.0075 & 0.0055 & 0.0020\\
0.1357 & 1.3229 & 0.0999 & 0.0070 & 0.0176 & 0.0076 & 0.0105 & 0.0282 & 0.0093\\
0.4393 & 0.3773 & 1.4556 & 0.0100 & 0.0151 & 0.0087 & 0.0169 & 0.0279 & 0.0175\\ \hdashline
0.0060 & 0.0090 & 0.0030 & 1.5737 & 0.2478 & 0.1216 & 0.0070 & 0.0116 & 0.0041\\
0.0252 & 0.0702 & 0.0226 & 0.4243 & 2.0900 & 0.5222 & 0.0270 & 0.0863 & 0.0289\\
0.0065 & 0.0164 & 0.0055 & 0.2847 & 0.3770 & 1.4060 & 0.0073 & 0.0205 & 0.0075\\ \hdashline
0.0571 & 0.0785 & 0.0140 & 0.0519 & 0.0912 & 0.0308 & 1.4639 & 0.2215 & 0.0727\\
0.0545 & 0.1398 & 0.0374 & 0.0610 & 0.1719 & 0.0699 & 0.2221 & 1.6349 & 0.2169\\
0.0491 & 0.0796 & 0.0315 & 0.0440 & 0.0933 & 0.0440 & 0.3513 & 0.4557 & 1.4824
\end{array}\right]$$

If we estimate CO_2 emission indices as $\mathbf{g}=[.2\quad .3\quad .1 \;\vdots\; .3\quad .5\quad .2 \;\vdots\; .1\quad .2\quad .1]$ per million US dollars and, for simplicity, we assume these indices do not change between 2005 and 2015, we have all we need to compute the vectors of generated emissions from producing sectors as

$\mathbf{e}^{p2005} = \hat{\mathbf{g}}\mathbf{L}^{2005}\mathbf{f}^{2005} = [269 \quad 1{,}370 \quad 1{,}717 \mid \quad 254 \quad 1{,}337 \quad 373 \mid \quad 676 \quad 3{,}363 \quad 3{,}719]$
$\mathbf{e}^{p2015} = \hat{\mathbf{g}}\mathbf{L}^{2015}\mathbf{f}^{2015} = [368 \quad 1{,}585 \quad 2{,}370 \mid 1{,}181 \quad 6{,}214 \quad 2{,}164 \mid 1{,}147 \quad 4{,}498 \quad 5{,}230]$.
The corresponding vectors of generated emissions attributed to consuming sectors are
$\mathbf{e}^{c2005} = \mathbf{g}\mathbf{L}^{2005}\hat{\mathbf{f}}^{2005} = [229 \quad 974 \quad 2{,}314 \mid \quad 200 \quad 838 \quad 695 \mid \quad 613 \quad 2{,}590 \quad 4{,}626]$
$\mathbf{e}^{c2015} = \mathbf{g}\mathbf{L}^{2015}\hat{\mathbf{f}}^{2015} = [303 \quad 1{,}238 \quad 3{,}050 \mid 1{,}025 \quad 3{,}852 \quad 3{,}750 \mid 1{,}108 \quad 3{,}593 \quad 6{,}836]$.

For convenience we compute the sums of emissions for all sectors in each region, defining the vectors of total regional emissions (from producing sectors) for the two years as $\mathbf{e}^{rp2005} = [3{,}356 \quad 1{,}965 \quad 7{,}758]$ and $\mathbf{e}^{rp2015} = [4{,}323 \quad 9{,}558 \quad 10{,}874]$.
The vectors of total regional emissions (attributed to consuming sectors) for the two years are $\mathbf{e}^{rc2005} = [3{,}518 \quad 1{,}733 \quad 7{,}828]$ and $\mathbf{e}^{rc2015} = [4{,}591 \quad 8{,}627 \quad 11{,}538]$. The percentage shifts for each region for attributing emissions to consumption rather than production are, for 2005 a 5 and 1 percent increase in the US and ROW, respectively, and a 12 percent decrease in China. For 2015 there is a 6 percent increase in both the US and ROW and a 10 percent decrease in China.

Chapter 14, Mixed and Dynamic Models

Chapter 14 describes so called mixed input-output models that are driven by a mix of output and final demand specifications rather than driven either solely by specification by final demand or total output. This chapter also introduces dynamic input–output models that more explicitly capture the role of capital investment and utilization in the production process. The exercise problems for this chapter illustrate key features of several mixed and dynamic model configurations.

Dynamic Models

Problem 14.1

This problem illustrates the basic structure of a dynamic input-output model. Consider an input-output economy with technical coefficients defined as $\mathbf{A} = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$ and capital coefficients defined as $\mathbf{B} = \begin{bmatrix} .01 & .003 \\ .005 & .020 \end{bmatrix}$. Current final demand is $\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ and the projections for the next three years for final demand are given by $\mathbf{f}^1 = \begin{bmatrix} 125 \\ 160 \end{bmatrix}$, $\mathbf{f}^2 = \begin{bmatrix} 150 \\ 175 \end{bmatrix}$ and $\mathbf{f}^3 = \begin{bmatrix} 185 \\ 200 \end{bmatrix}$.

For $\mathbf{A}$ and $\mathbf{B}$ as defined we specify the dynamic model as $\mathbf{B}\mathbf{x}^{t+1} = (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t - \mathbf{f}^t$ or $\mathbf{x}^t = (\mathbf{I} - \mathbf{A} + \mathbf{B})^{-1}(\mathbf{B}\mathbf{x}^{t+1} + \mathbf{f}^t)$, which we can write as $\mathbf{x}^t = \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^{t+1} + \mathbf{f}^t)$ where $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$. For this case we compute $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B}) = \begin{bmatrix} .69 & -.103 \\ -.205 & .48 \end{bmatrix}$ and $\mathbf{G}^{-1} = \begin{bmatrix} 1.548 & .332 \\ .661 & 2.225 \end{bmatrix}$. The "dynamic multipliers" are defined as $\mathbf{R} = \mathbf{G}^{-1}\mathbf{B} = \begin{bmatrix} .017 & .011 \\ .018 & .046 \end{bmatrix}$, $\mathbf{R}^2\mathbf{G}^{-1} = \begin{bmatrix} .001 & .002 \\ .003 & .006 \end{bmatrix}$ and $\mathbf{R}^3\mathbf{G}^{-1} = \begin{bmatrix} .00006 & .00009 \\ .00018 & .00029 \end{bmatrix}$.

Then we can construct the difference equations in matrix terms as

$$\mathbf{D} = \begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} & \mathbf{R}^3\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix} \text{ so that } \begin{bmatrix} \mathbf{x}^0 \\ \hdashline \mathbf{x}^1 \\ \hdashline \mathbf{x}^2 \\ \hdashline \mathbf{x}^3 \end{bmatrix} = \mathbf{D}\begin{bmatrix} \mathbf{f}^0 \\ \hdashline \mathbf{f}^1 \\ \hdashline \mathbf{f}^2 \\ \hdashline \mathbf{f}^3 \end{bmatrix}$$

or, for the base year and the three projected years, $\mathbf{x}^0 = \begin{bmatrix} 197.7 \\ 315 \end{bmatrix}$, $\mathbf{x}^1 = \begin{bmatrix} 257.7 \\ 468.3 \end{bmatrix}$, $\mathbf{x}^2 = \begin{bmatrix} 302.8 \\ 521.2 \end{bmatrix}$ and $\mathbf{x}^3 = \begin{bmatrix} 352.8 \\ 567.3 \end{bmatrix}$.

Problem 14.2

This problem illustrates the basic concepts of turnpike growth in a dynamic input-output model.

Consider the following closed dynamic input-output model, $\mathbf{Ax} + \mathbf{B}(\mathbf{x}^t - \mathbf{x}) = \mathbf{x}$ where $\mathbf{x}^t$ and $\mathbf{x}$ are the vectors of future total outputs and current total outputs, respectively $\mathbf{A} = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$ are the matrices of technical and capital coefficients, respectively.

Assume that $\mathbf{x}^t = \lambda\mathbf{x}$, where λ is some scalar (called the turnpike growth rate), the dynamic input-output model is expressed as $\mathbf{Ax} + \mathbf{B}(\lambda\mathbf{x} - \mathbf{x}) = \mathbf{x}$. Rearranging terms, this becomes $\mathbf{B}\lambda\mathbf{x} = (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}$ or $\mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x} = \lambda\mathbf{x}$, which we write more succinctly as $\mathbf{Qx}^t = \lambda\mathbf{x}^t$ where $\mathbf{Q} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B})$.

In this case we compute $\mathbf{B}^{-1} = \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$ and then $\mathbf{Q} = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$. To calculate the turnpike growth, we solve the characteristic equation $|\mathbf{Q} - \mathbf{I}| = 0$, and we find $\lambda_{\max} = 5$.

Problem 14.3

This problem illustrates the implications of changes in capital coefficients on the turnpike growth formulation of a dynamic input-output model. Consider the closed dynamic input-output model $\mathbf{Ax} + \mathbf{B}(\mathbf{x}^t - \mathbf{x}) = \mathbf{x}$, where $\mathbf{A} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$. Under the assumption of turnpike growth, we calculate $\mathbf{Q} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B}) = \begin{bmatrix} 10 & -2 \\ -3 & 7 \end{bmatrix}$ and solving the characteristic question $|\mathbf{Q} - \mathbf{I}| = 0$ find that the turnpike growth rate is $\lambda_{\max} = 11.37$.

If both the capital coefficients for the first industry (the first column of **B**) are changed to 0.1, then $\mathbf{B} = \begin{bmatrix} .1 & 0 \\ .1 & .1 \end{bmatrix}$. Hence, we find $\mathbf{Q} = \begin{bmatrix} 10 & -2 \\ -12 & 9 \end{bmatrix}$ and $\lambda_{\max} = 14.42$, which is an increase associated with the change in capital coefficients and indicates an improvement in the apparent overall "health" of the economy.

Problem 14.4

This problem illustrates the basic concepts of dynamic multipliers in dynamic input-output models. Consider an input-output economy with technical coefficients defined as $\mathbf{A} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.5 \end{bmatrix}$ and capital coefficients defined as $\mathbf{B} = \begin{bmatrix} .02 & .002 \\ .003 & .01 \end{bmatrix}$.

As in earlier problems, for **A** and **B** we specify the dynamic model as $\mathbf{Bx}^{t+1} = (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t - \mathbf{f}^t$ or $\mathbf{x}^t = (\mathbf{I} - \mathbf{A} + \mathbf{B})^{-1}(\mathbf{Bx}^{t+1} + \mathbf{f}^t)$, which we can write as $\mathbf{x}^t = \mathbf{G}^{-1}(\mathbf{Bx}^{t+1} + \mathbf{f}^t)$ where $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$. For this case, we compute

$\mathbf{G}=(\mathbf{I}-\mathbf{A}+\mathbf{B})=\begin{bmatrix} .78 & -.102 \\ -.303 & .49 \end{bmatrix}$ and $\mathbf{G}^{-1}=\begin{bmatrix} 1.395 & .29 \\ .863 & 2.220 \end{bmatrix}$, so the "dynamic multipliers" are

$\mathbf{R}=\mathbf{G}^{-1}\mathbf{B}=\begin{bmatrix} .029 & .006 \\ .024 & .024 \end{bmatrix}$, $\mathbf{R}^2\mathbf{G}^{-1}=\begin{bmatrix} .002 & .001 \\ .002 & .002 \end{bmatrix}$ and $\mathbf{R}^3\mathbf{G}^{-1}=\begin{bmatrix} .00006 & .00004 \\ .00010 & .00007 \end{bmatrix}$ such that

$\Delta\mathbf{x}^{-3}=\mathbf{R}^3\mathbf{G}^{-1}\Delta\mathbf{f}^0$, $\Delta\mathbf{x}^{-2}=\mathbf{R}^2\mathbf{G}^{-1}\Delta\mathbf{f}^0$, and $\Delta\mathbf{x}^{-1}=\mathbf{R}\mathbf{G}^{-1}\Delta\mathbf{f}^0$ or, in expanded matrix terms,

$$\begin{bmatrix} \mathbf{x}^{-3} \\ \mathbf{x}^{-2} \\ \mathbf{x}^{-1} \\ \mathbf{x}^{0} \end{bmatrix}=\begin{bmatrix} \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} & \mathbf{R}^3\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{f}^{-3} \\ \mathbf{f}^{-2} \\ \mathbf{f}^{-1} \\ \mathbf{f}^{0} \end{bmatrix}.$$

If we assume the current vector of final demands is $\mathbf{f}^0=\begin{bmatrix} 185 \\ 200 \end{bmatrix}$ and the vectors for final demand for the previous three years are given by $\mathbf{f}^{-1}=\begin{bmatrix} 150 \\ 175 \end{bmatrix}$, $\mathbf{f}^{-2}=\begin{bmatrix} 125 \\ 160 \end{bmatrix}$, and $\mathbf{f}^{-3}=\begin{bmatrix} 100 \\ 100 \end{bmatrix}$, we can specify $\Delta\mathbf{f}=\begin{bmatrix}100 & 100 & \vdots & 125 & 160 & \vdots & 150 & 175 & \vdots & 185 & 200\end{bmatrix}'$ and compute

$\Delta\mathbf{x}=\begin{bmatrix}177.9 & 325.4 & \vdots & 231.7 & 482.5 & \vdots & 272.6 & 539.9 & \vdots & 316.1 & 603.6\end{bmatrix}'$.

Mixed Models

Problem 14.5

This exercise problem illustrates the basic characteristics of a mixed input-output model. Consider an input-output economy specified by an interindustry transactions matrix, $\mathbf{Z}=\begin{bmatrix} 14 & 76 & 46 \\ 54 & 22 & 5 \\ 68 & 71 & 94 \end{bmatrix}$ and vector of final demands, $\mathbf{f}=\begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix}$ where the three industrial sectors are manufacturing, oil, and electricity.

Suppose the economic forecasts determine that total domestic output for oil and electricity will remain unchanged in the next year and final demand for manufactured goods will increase by 30 percent. That is, the projection is a mixture of total outputs and final demands rather than only final demands (or total outputs). In such a situation, we can construct a mixed input-output model by first determining the economy's total outputs as $\mathbf{x}=\mathbf{f}+\mathbf{Z}\mathbf{i}=\begin{bmatrix} 236 \\ 281 \\ 408 \end{bmatrix}$ so that we can compute the matrix of technical coefficients, $\mathbf{A}=\mathbf{Z}(\hat{\mathbf{x}})^{-1}=\begin{bmatrix} .059 & .270 & .113 \\ .229 & .078 & .012 \\ .288 & .253 & .230 \end{bmatrix}$. For sector 1 (manufactured goods), the level of final demand is exogenously specified and for sectors 2 and 3 (oil and electricity), levels of total output are specified for each sector, so we partition $\mathbf{A}$ as $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right] = \left[\begin{array}{c|cc} .059 & .27 & .113 \\ \hline .229 & .078 & .012 \\ .288 & .253 & .23 \end{array}\right]$, and with a vector of exogenously specified values $\begin{bmatrix} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{bmatrix}$ and the vector of endogenously determined values designated by $\begin{bmatrix} \mathbf{x} \\ \hline \mathbf{f} \end{bmatrix}$ we write $\mathbf{M}\begin{bmatrix} \mathbf{x} \\ \hline \mathbf{f} \end{bmatrix} = \mathbf{N}\begin{bmatrix} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{bmatrix}$ where $\mathbf{M} = \left[\begin{array}{c|c} (\mathbf{I}-\mathbf{A}_{11}) & \mathbf{0} \\ \hline -\mathbf{A}_{21} & -\mathbf{I} \end{array}\right] = \left[\begin{array}{c|cc} .941 & 0 & 0 \\ \hline -.229 & -1 & 0 \\ -.288 & 0 & -1 \end{array}\right]$ and $\mathbf{N} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}_{12} \\ \hline \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{array}\right] = \left[\begin{array}{c|cc} 1 & .270 & .113 \\ \hline 0 & -.922 & .012 \\ 0 & .253 & -.770 \end{array}\right]$.

It follows that $\mathbf{M}^{-1} = \left[\begin{array}{c|cc} 1.063 & 0 & 0 \\ \hline -.243 & -1 & 0 \\ -.306 & 0 & -1 \end{array}\right]$.

For the case where the economic forecasts determine that total domestic output for oil and electricity will remain unchanged in the next year and final demand for manufactured goods will increase by 30 percent, we specify $\begin{bmatrix} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 130 \\ \hline 281 \\ 408 \end{bmatrix}$ and find $\begin{bmatrix} \mathbf{x} \\ \hline \mathbf{f} \end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 267.9 \\ \hline 192.7 \\ 165.8 \end{bmatrix}$. That is, total output of manufactured goods will be 267.9, and final demands presented to the economy for oil and electricity are 192.7 and 165.8, respectively.

If instead the final demand for manufactured goods increased by 50 percent instead of 30 percent, we find the new projections of final demand for oil and electricity and the total output of manufacturing as $\begin{bmatrix} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 150 \\ \hline 281 \\ 408 \end{bmatrix}$ we find $\begin{bmatrix} \mathbf{x} \\ \hline \mathbf{f} \end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 289.2 \\ \hline 187.8 \\ 159.7 \end{bmatrix}$.

Problem 14.6

This problem explores modeling establishment of a new economic sector using input-output analysis revisiting the economy of problem 2.1. Consider the prospect of adding a new sector, finance and insurance (sector 3), to this economy. First, we can recall from problem 2.1 that, for this economy, the interindustry transactions matrix, $\mathbf{Z} = \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix}$, and the vector of total outputs, $\mathbf{x} = \begin{bmatrix} 1,000 \\ 800 \end{bmatrix}$, from which we can compute the matrix of technical coefficients, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} .500 & .438 \\ .320 & .450 \end{bmatrix}$, and the total requirements matrix, $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 4.074 & 3.241 \\ 2.370 & 3.704 \end{bmatrix}$.

Initially we know that the total output of this new sector will be $x_3 = \$900$ during the current year (its first year of operation), and that its needs for agricultural and manufactured goods are captured by technical coefficients $a_{13} = 0.001$ and $a_{23} = 0.07$. In the absence of any further information, we can estimate to be the impact of this new sector on the economy by first constructing a final demand vector by multiplying each of the new technical coefficients by the associated known total output to yield $\Delta\mathbf{f} = \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} (.001)(900) \\ (.07)(900) \end{bmatrix} = \begin{bmatrix} 0.9 \\ 63.0 \end{bmatrix}$. The impact of the new sector is found by $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f} = \begin{bmatrix} 207.8 \\ 235.47 \end{bmatrix}$.

Suppose we learn subsequently that: (1) that the agriculture and manufacturing sectors bought \$20 and \$40 in finance and insurance services last year from foreign firms (i.e., that they imported these inputs), and (2) that sector 3 will use \$15 of its own product for each \$100 worth of its output. We now have enough information to endogenize the sector into the interindustry transactions matrix, including specifying $a_{33} = 15/100 = 0.15$, $a_{31} = 20/x_1 = 20/1000 = 0.02$, and $a_{32} = 40/x_2 = 40/800 = 0.05$ so the new, expanded technical coefficient matrix becomes $\mathbf{A} = \begin{bmatrix} .500 & .438 & .001 \\ .320 & .450 & .070 \\ .020 & .050 & .150 \end{bmatrix}$ so the new total requirements matrix is $\mathbf{L} = \begin{bmatrix} 4.130 & 3.310 & .277 \\ 2.433 & 3.782 & .314 \\ .240 & .300 & 1.201 \end{bmatrix}$.

Hence, the new, expanded total outputs vector is $\mathbf{x} = \begin{bmatrix} 1{,}000 \\ 800 \\ 900 \end{bmatrix}$ so the new interindustry transactions matrix is found as $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}} = \begin{bmatrix} 500 & 350 & .9 \\ 320 & 360 & 63 \\ 20 & 40 & 135 \end{bmatrix}$. The third column of $\mathbf{Z}$ describes the interindustry purchases of the three sectors' outputs by the new finance and insurances services sector, which we can also describe as a new (at least in the first year) final demand to the expanded regional economy, $\Delta\mathbf{f} = \begin{bmatrix} 0.9 \\ 63.0 \\ 135.0 \end{bmatrix}$. We can now use the new expanded total requirements matrix, $\mathbf{L}$, to compute the total output in the economy to support introduction of the new economic sector, $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f} = \begin{bmatrix} 249.7 \\ 282.9 \\ 181.3 \end{bmatrix}$.

Problem 14.7

This problem illustrates use of a mixed input-output model applied to planning with availability of variable data, e.g., some estimated final demands for products of some sectors and some projected total outputs the balance of sectors in the economy. We revisit the Czaria economy from problem 7.1, for which $\mathbf{A} = \begin{bmatrix} 0.168 & 0.155 & 0.213 & 0.212 \\ 0.194 & 0.193 & 0.168 & 0.115 \\ 0.105 & 0.025 & 0.126 & 0.124 \\ 0.178 & 0.101 & 0.219 & 0.186 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 12{,}000 \\ 15{,}000 \\ 12{,}000 \\ 16{,}000 \end{bmatrix}$. Next year's projected total outputs in millions of dollars for agriculture, mining, and civilian manufacturing in Czaria are 4,558, 5,665 and 5,079, respectively, and final demand of military manufactured products is projected to be $2,050 million.

To compute the GDP and total gross production of the economy next year, we can fashion a mixed model by first reordering the industry sectors so that those with exogenously specified final demands are listed first (in this case only sector 3) and those with exogenously specified total outputs are listed second (in this case sectors 1, 2, and 4). With the reordered sectors, we can compute $\mathbf{M} = \left[\begin{array}{c|c} (\mathbf{I}-\mathbf{A}_{11}) & \mathbf{0} \\ \hline -\mathbf{A}_{21} & -\mathbf{I} \end{array}\right] = \left[\begin{array}{c|ccc} .832 & 0 & 0 & 0 \\ \hline -.194 & -1 & 0 & 0 \\ -.105 & 0 & -1 & 0 \\ -.178 & 0 & 0 & -1 \end{array}\right]$ and

$\mathbf{N} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}_{12} \\ \hline \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{array}\right] = \left[\begin{array}{c|ccc} 1 & .155 & .213 & .212 \\ \hline 0 & -.807 & .168 & .115 \\ 0 & .025 & -.874 & .124 \\ 0 & .101 & .219 & -.814 \end{array}\right]$, which satisfies the condition $\mathbf{M}\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{f} \end{array}\right] = \mathbf{N}\left[\begin{array}{c} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{array}\right]$ or $\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{f} \end{array}\right] = \mathbf{M}^{-1}\mathbf{N}\left[\begin{array}{c} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{array}\right]$ where $\left[\begin{array}{c} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{array}\right]$ is the vector of exogenously specified values and $\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{f} \end{array}\right]$ is the vector of endogenously determined values.

To compute the endogenously determined values we first compute $\mathbf{M}^{-1} = \left[\begin{array}{c|ccc} 1.202 & 0 & 0 & 0 \\ \hline -.233 & -1 & 0 & 0 \\ -.126 & 0 & -1 & 0 \\ -.214 & 0 & 0 & -1 \end{array}\right]$ and then $\mathbf{M}^{-1}\mathbf{N} = \left[\begin{array}{c|ccc} 1.202 & .186 & .256 & .255 \\ \hline -.233 & .771 & -.218 & -.164 \\ -.126 & -.045 & .847 & -.151 \\ -.214 & -.134 & -.265 & .769 \end{array}\right]$ so that

$\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{f} \end{array}\right] = \mathbf{M}^{-1}\mathbf{N}\left[\begin{array}{c} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{array}\right] = \left[\begin{array}{c} 6{,}058 \\ \hline 967 \\ 3{,}572 \\ 1{,}355 \end{array}\right]$ for $\left[\begin{array}{c} \bar{\mathbf{f}} \\ \hline \bar{\mathbf{x}} \end{array}\right] = \left[\begin{array}{c} 2{,}050 \\ \hline 4{,}558 \\ 5{,}665 \\ 5{,}079 \end{array}\right]$. Total output of sector 3 will be 6055, and amounts of sector 1, 2, and 4 production available for final demand are 969, 3573, and 1347, respectively. GDP is the sum of all final demands (7,944) and total gross production is the sum of all total outputs (21,360).

Problem 14.8

This problem illustrates use of a mixed input-output model applied to planning with availability of variable data, e.g., some estimated final demands for products of some sectors and some projected total outputs the balance of sectors in the economy. To illustrate the process, we use a highly aggregated industry by industry, industry technology-based input-output model for the

2005 U.S. economy specified a technical coefficients matrix, **A**, and make matrix, **V**, given for 7 industries: (1) agriculture, (2) mining, (3) construction, (4) manufacturing, (5) trade, transportation, and utility services, (6) services, and (7) other industries.

We first compute the baseline vector of total outputs as $\mathbf{x} = \mathbf{Vi} = \begin{bmatrix} 312,754 \\ 396,563 \\ 1,302,388 \\ 4,485,529 \\ 3,355,944 \\ 10,477,640 \\ 2,526,325 \end{bmatrix}$ and vector of total final demands as $\mathbf{f} = \mathbf{x} - \mathbf{Ax} = \begin{bmatrix} 47,244 \\ -118,692 \\ 1,150,094 \\ 1,574,473 \\ 2,026,508 \\ 5,697,200 \\ 2,079,011 \end{bmatrix}$. Note that the negative final demand for mining indicates net importation of products such as petroleum.

A	1	2	3	4	5	6	7
1	0.2258	0.0000	0.0015	0.0384	0.0001	0.0017	0.0007
2	0.0027	0.1432	0.0075	0.0675	0.0367	0.0004	0.0070
3	0.0051	0.0002	0.0010	0.0018	0.0037	0.0071	0.0215
4	0.1955	0.0877	0.2591	0.3222	0.0547	0.0566	0.1010
5	0.0819	0.0422	0.1011	0.0994	0.0704	0.0334	0.0487
6	0.0843	0.1276	0.1225	0.1172	0.1760	0.2783	0.2026
7	0.0099	0.0095	0.0093	0.0219	0.0215	0.0188	0.0240

V	1	2	3	4	5	6	7
1	310,868	0	0	65	0	1,821	0
2	0	373,811	0	22,752	0	0	0
3	0	0	1,302,388	0	0	0	0
4	0	0	0	4,454,957	0	26,106	4,467
5	0	808	0	0	3,354,043	47	1,046
6	0	556	0	0	152	10,473,161	3,771
7	4,657	1,410	0	4,111	115,428	339,582	2,061,136

Suppose our economic forecast projects, for 2010, a 10 percent growth in final demand for agriculture, mining, and construction, a 5 percent growth in final demand for manufactured goods, and a 6 percent growth in total output for the trade, transportation, utilities, services and other industries. So, the vector of exogenously specified data is $\begin{bmatrix} \bar{\mathbf{f}} \\ \hdashline \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 51{,}968 \\ -130{,}561 \\ 1{,}265{,}103 \\ 1{,}653{,}196 \\ \hdashline 3{,}557{,}300 \\ 11{,}106{,}299 \\ 2{,}677{,}904 \end{bmatrix}$. The sectors are already conveniently ordered such that the four sectors with exogenously specified final demands are listed first and the remaining three with exogenously specified total outputs follow, so we can compute

$$\mathbf{M} = \left[\begin{array}{c:c} (\mathbf{I}-\mathbf{A}_{11}) & \mathbf{0} \\ \hdashline -\mathbf{A}_{21} & -\mathbf{I} \end{array}\right] = \left[\begin{array}{rrrr:rrr} 0.7742 & 0.0000 & -0.0015 & -0.0384 & 0 & 0 & 0 \\ -0.0027 & 0.8568 & -0.0075 & -0.0675 & 0 & 0 & 0 \\ -0.0051 & -0.0002 & 0.9990 & -0.0018 & 0 & 0 & 0 \\ -0.1955 & -0.0877 & -0.2591 & 0.6778 & 0 & 0 & 0 \\ \hdashline -0.0819 & -0.0422 & -0.1011 & -0.0994 & -1 & 0 & 0 \\ -0.0843 & -0.1276 & -0.1225 & -0.1172 & 0 & -1 & 0 \\ -0.0099 & -0.0095 & -0.0093 & -0.0219 & 0 & 0 & -1 \end{array}\right] \text{ and}$$

$$\mathbf{N} = \left[\begin{array}{c:c} \mathbf{I} & \mathbf{A}_{12} \\ \hdashline \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{array}\right] = \left[\begin{array}{rrrr:rrr} 1 & 0 & 0 & 0 & 0.0001 & 0.0017 & 0.0007 \\ 0 & 1 & 0 & 0 & 0.0367 & 0.0004 & 0.0070 \\ 0 & 0 & 1 & 0 & 0.0037 & 0.0071 & 0.0215 \\ 0 & 0 & 0 & 1 & 0.0547 & 0.0566 & 0.1010 \\ \hdashline 0 & 0 & 0 & 0 & -0.9296 & 0.0334 & 0.0487 \\ 0 & 0 & 0 & 0 & 0.1760 & -0.7217 & 0.2026 \\ 0 & 0 & 0 & 0 & 0.0215 & 0.0188 & -0.9760 \end{array}\right]$$, which satisfies the condition $\mathbf{M}\begin{bmatrix} \mathbf{x} \\ \hdashline \mathbf{f} \end{bmatrix} = \mathbf{N}\begin{bmatrix} \bar{\mathbf{f}} \\ \hdashline \bar{\mathbf{x}} \end{bmatrix}$ or $\begin{bmatrix} \mathbf{x} \\ \hdashline \mathbf{f} \end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix} \bar{\mathbf{f}} \\ \hdashline \bar{\mathbf{x}} \end{bmatrix}$ where $\begin{bmatrix} \bar{\mathbf{f}} \\ \hdashline \bar{\mathbf{x}} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{x} \\ \hdashline \mathbf{f} \end{bmatrix}$ are the vectors of exogenously specified values and the vector of endogenously determined values, respectively. To compute the endogenously determined values we first compute

$$\mathbf{M}^{-1} = \left[\begin{array}{cccc:ccc} 1.3109 & 0.0077 & 0.0215 & 0.0752 & 0 & 0 & 0 \\ 0.0345 & 1.1794 & 0.0398 & 0.1195 & 0 & 0 & 0 \\ 0.0074 & 0.0005 & 1.0018 & 0.0031 & 0 & 0 & 0 \\ 0.3855 & 0.1551 & 0.3943 & 1.5138 & 0 & 0 & 0 \\ \hdashline -0.1479 & -0.0659 & -0.1439 & -0.1619 & -1 & 0 & 0 \\ -0.1610 & -0.1694 & -0.1759 & -0.1994 & 0 & -1 & 0 \\ -0.0218 & -0.0147 & -0.0185 & -0.0351 & 0 & 0 & -1 \end{array}\right] \text{ and then}$$

$$\mathbf{M}^{-1}\mathbf{N} = \left[\begin{array}{cccc:ccc} 1.3109 & 0.0077 & 0.0215 & 0.0752 & 0.0046 & 0.0067 & 0.0090 \\ 0.0345 & 1.1794 & 0.0398 & 0.1195 & 0.0500 & 0.0076 & 0.0212 \\ 0.0074 & 0.0005 & 1.0018 & 0.0031 & 0.0039 & 0.0073 & 0.0218 \\ 0.3855 & 0.1551 & 0.3943 & 1.5138 & 0.0900 & 0.0893 & 0.1628 \\ \hdashline -0.1479 & -0.0659 & -0.1439 & -0.1619 & 0.9178 & -0.0439 & -0.0687 \\ -0.1610 & -0.1694 & -0.1759 & -0.1994 & -0.1938 & 0.7088 & -0.2279 \\ -0.0218 & -0.0147 & -0.0185 & -0.0351 & -0.0241 & -0.0210 & 0.9719 \end{array}\right] \text{ so that}$$

$$\left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{f} \end{array}\right] = \mathbf{M}^{-1}\mathbf{N}\left[\begin{array}{c} \overline{\mathbf{f}} \\ \hdashline \overline{\mathbf{x}} \end{array}\right] = \left[\begin{array}{r} 333,767 \\ 414,773 \\ 1,426,583 \\ 4,748,959 \\ \hdashline 2,144,061 \\ 6,034,580 \\ 2,203,480 \end{array}\right] \text{ for } \left[\begin{array}{c} \overline{\mathbf{f}} \\ \hdashline \overline{\mathbf{x}} \end{array}\right] = \left[\begin{array}{r} 51,968 \\ -130,561 \\ 1,265,103 \\ 1,653,196 \\ \hdashline 3,557,300 \\ 11,106,299 \\ 2,677,904 \end{array}\right] \text{, specified above.}$$