1

A Student's Guide to the Navier-Stokes Equations: A Supplement on the Bernoulli Equation

The Bernoulli equation is another important equation in fluid mechanics. It actually predates the Navier-Stokes equations by about a hundred years. It is very useful for engineering purposes but it does require some fairly significant assumptions. The most significant assumptions are that the flow is inviscid as well as something called irrotational.

There are a number of ways to derive Bernoulli's equation. It can be derived from energy conservation principles or from Newton's second law. The energy conservation approach is usually the easiest to grasp and understand. In particular, the mechanical energy equation, Equation 5.48 from the textbook, is the most useful starting point:

$$\rho\vec{V}\cdot\frac{D\vec{V}}{Dt} = \vec{V}\cdot\left(\vec{\nabla}\cdot\vec{\vec{T}}\right) + \rho\vec{V}\cdot\vec{g}$$

We can write the left hand side as (from Equation 5.46):

$$\underbrace{\frac{\rho}{2}\frac{D\left(\vec{V}\cdot\vec{V}\right)}{Dt}}_{=\rho\vec{V}\cdot\frac{D\vec{V}}{Dt}} = \vec{V}\cdot\left(\vec{\nabla}\cdot\vec{\vec{T}}\right) + \rho\vec{V}\cdot\vec{g}$$

We can write out the material derivative in the above equation in an Eulerian description to get:

$$\frac{\rho}{2}\left(\frac{\partial\left(\vec{V}\cdot\vec{V}\right)}{\partial t} + \vec{V}\cdot\vec{\nabla}\left(\vec{V}\cdot\vec{V}\right)\right) = \vec{V}\cdot\left(\vec{\nabla}\cdot\vec{\vec{T}}\right) + \rho\vec{V}\cdot\vec{g}$$

At steady state the time derivative goes away to give us:

$$\frac{\rho}{2}\vec{V} \cdot \vec{\nabla}\left(\vec{V} \cdot \vec{V}\right) = \vec{V} \cdot \left(\vec{\nabla} \cdot \vec{\vec{T}}\right) + \rho\vec{V} \cdot \vec{g}$$

We can move every term to the left hand side and factor out the $\vec{V}\cdot$ to get:

$$\vec{V} \cdot \left(\frac{\rho}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) - \vec{\nabla} \cdot \vec{\vec{T}} - \rho\vec{g}\right) = 0$$

The expression above holds under two circumstances: if $\vec{V} = 0$ (which is the trivial case) or if $\frac{\rho}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) - \vec{\nabla} \cdot \vec{\vec{T}} - \rho\vec{g} = 0$. The interesting case is when:

$$\frac{\rho}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) - \vec{\nabla} \cdot \vec{\vec{T}} - \rho\vec{g} = 0 \tag{1.1}$$

The stress tensor for an inviscid flow (where $\mu = 0$) just contains the pressure term, thus:

$$\vec{\vec{T}} = -p\vec{\vec{I}} \tag{1.2}$$

where $\vec{\vec{I}}$ is the identity matrix. Plugging Equation 1.2 into Equation 1.1 leads to:

$$\frac{\rho}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) - \underbrace{\vec{\nabla} \cdot \left(-p\vec{\vec{I}}\right)}_{=-\vec{\nabla}p} - \rho\vec{g} = 0$$

Recall from the textbook that the $\vec{\nabla}\cdot\left(-p\vec{\vec{I}}\right)$ term becomes a gradient of pressure. Now we have:

$$\frac{\rho}{2}\nabla\left(\vec{V} \cdot \vec{V}\right) + \vec{\nabla}p - \rho\vec{g} = 0$$

For a constant density flow (incompressible), we can sneak the $\frac{\rho}{2}$ in the first term into the gradient to get:

$$\nabla\left(\rho\frac{\vec{V} \cdot \vec{V}}{2}\right) + \vec{\nabla}p - \rho\vec{g} = 0 \tag{1.3}$$

The gravity term, i.e. the $\rho\vec{g}$ term, is usually considered just a constant value. In addition, the gravity is usually assumed to be in either the $z-$direction or the $y-$direction. As such, we can write the gravity term in the following manner (with z being the coordinate direction of gravity):

$$\rho\vec{g} = \vec{\nabla}\left(-\rho g_z z\right) \tag{1.4}$$

where g_z is the magnitude of the gravitational acceleration in the z−direction and z is the coordinate direction pointing in the opposite direction as gravity (i.e. pointing up instead of down, hence the negative sign). Writing the force as a gradient of some function is indicative of what is called a conservative force and is a requirement for the Bernoulli equation. To see how why this relationship works out, just perform the gradient operation from the right hand side of Equation 1.4:

$$\begin{aligned}
\vec{\nabla}\left(-\rho g_z z\right) &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(-\rho g_z z\right) \\
&= \left(\hat{i}\frac{\partial\left(-\rho g_z z\right)}{\partial x} + \hat{j}\frac{\partial\left(-\rho g_z z\right)}{\partial y} + \hat{k}\frac{\partial\left(-\rho g_z z\right)}{\partial z}\right) \\
&= 0\hat{i} + 0\hat{j} - \rho g_z\hat{k} = \rho\vec{g}
\end{aligned}$$

Note that $-\rho g_z$ is considered a constant and the $\vec{g}$ is defined as $\vec{g} = -g_z\hat{k}$.

Plugging Equation 1.4 into Equation 1.3 leads to:

$$\nabla\left(\rho\frac{\vec{V}\cdot\vec{V}}{2}\right) + \vec{\nabla}p + \vec{\nabla}\left(\rho g_z z\right) = 0 \tag{1.5}$$

Notice every term in Equation 1.5 is a gradient term. We can factor out the gradient to get:

$$\nabla\left(\rho\frac{\vec{V}\cdot\vec{V}}{2} + p + \rho g_z z\right) = 0$$

Since the gradient is equal to zero, the term inside the gradient must be a constant, thus we now have the Bernoulli equation:

$$\boxed{\rho\frac{\vec{V}\cdot\vec{V}}{2} + p + \rho g_z z = \text{constant}} \tag{1.6}$$

Equation 1.6 is the most common form of the Bernoulli equation (and the original version). It assumes an inviscid, incompressible flow with a conservative body force. In addition, although this is not obvious, it also assumes an irrotational flow. An irrotational flow is a flow that has zero vorticity, $\vec{\omega}$. Vorticity is defined as the curl of velocity, written as:

$$\vec{\omega} = \vec{\nabla} \times \vec{V} \tag{1.7}$$

To see how the Bernoulli equation also assumes an irrotational flow, consider Cauchy's first law:

$$\rho \frac{D\vec{V}}{Dt} = \vec{\nabla} \cdot \vec{\vec{T}} + \rho \vec{g}$$

We know from earlier that the right hand side can be written as the negative of the gradient of pressure and the gradient of $-\rho g_z z$, thus:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} p + \vec{\nabla}\left(-\rho g_z z\right)$$

Writing the above equation in non-conservative form by expanding out the material derivative and assuming a steady state, we get:

$$\rho \left(\cancelto{0}{\frac{\partial \vec{V}}{\partial t}} + \vec{V} \cdot \vec{\nabla} \vec{V} \right) = -\vec{\nabla} p + \vec{\nabla}\left(-\rho g_z z\right) \tag{1.8}$$

The advective term ($\vec{V} \cdot \vec{\nabla} \vec{V}$) can be written using the following vector identity:

$$\vec{V} \cdot \vec{\nabla} \vec{V} = \frac{1}{2} \vec{\nabla}\left(\vec{V} \cdot \vec{V}\right) - \overbrace{\vec{V} \times \underbrace{\left(\vec{\nabla} \times \vec{V}\right)}_{=0 \text{ if irrotational}}}^{\text{Lamb vector}} \tag{1.9}$$

The last term on the right is called the Lamb vector, named after Horace Lamb. The curl of velocity is contained in the last term, if the case of irrotational flow, $\vec{\nabla} \times \vec{V} = 0$. Plugging Equation 1.9 into Equation 1.8 gives us:

$$\frac{\rho}{2} \vec{\nabla}\left(\vec{V} \cdot \vec{V}\right) = -\vec{\nabla} p + \vec{\nabla}\left(-\rho g_z z\right)$$

This is essentially the same expression as we had earlier. We can move all of the terms to the left-hand side and factor out the gradient to get:

$$\vec{\nabla}\left(\rho \frac{\vec{V} \cdot \vec{V}}{2} + p + \rho g_z z\right) = 0$$

Thus leading to, again, the Bernoulli equation:

$$\boxed{\rho \frac{\vec{V} \cdot \vec{V}}{2} + p + \rho g_z z = \text{constant}} \tag{1.10}$$