February 2, 2015

Problems for Chapter 18 of *Advanced Mathematics for Applications*

Matrices and Finite-Dimensional Linear Spaces

by Andrea Prosperetti

1. Show that, if $AX = XA$ for any matrix X, then the matrix A is a multiple of the identity.

2. Show by direct calculation for the 2×2 matrices A and B that $M_1 = AB$ and $M_2 = BA$ have the same eigenvalues. Does this result hold for A and B general $N \times N$ matrices?

3. Show that if A and B are real symmetric square matrices and if B is positive definite, then the generalized eigenvalues satisfying
$$\det(A - \lambda B) = 0$$
are real.

4. Using a scalar product defined by
$$(a, b) = \sum_{j=1}^{N} \overline{a}_j \, b_j$$
show that the eigenvalues of a Hermitian matrix (i.e. a matrix $A \equiv (a_{ij})$ such that $\overline{a}_{ji} = a_{ij}$) are real and that its eigenvectors corresponding to different eigenvalues are orthogonal. These results generalize the corresponding ones given in class for real symmetric matrices.

5. Show that the eigenvalues of a real skew-symmetric matrix (i.e. a matrix for which $a_{jj} = 0$, $a_{ij} = -a_{ji}\, i \neq j$) are either zero or purely imaginary.

6. Show that the eigenvalues of an orthogonal matrix have modulus equal to 1.

7. Find the eigenvalues and eigenvectors $\mathbf{x}_i$, $i = 1, 2, 3$, of the matrix
$$\mathsf{A} = \left| \begin{array}{ccc} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -1 & 1 \end{array} \right| .$$
Find then a vector $\mathbf{y}_1$ orthogonal to $\mathbf{x}_2$ and $\mathbf{x}_3$, a vector $\mathbf{y}_2$ orthogonal to $\mathbf{x}_1$ and $\mathbf{x}_3$ and a vector $\mathbf{y}_3$ orthogonal to $\mathbf{x}_1$ and $\mathbf{x}_2$. Show that these three vectors are eigenvectors of A^*.

8. Show that a real symmetric matrix A may be written in the form
$$A = \sum_{j=1}^{N} \lambda_j \, E_j, \qquad (*)$$
where the E_j's are non-negative definite matrices satisfying the conditions
$$E_i \, E_j = 0 \quad \text{if} \quad i \neq j, \qquad E_i^2 = E_i, \qquad (**)$$
and the λ_j's are the eigenvalues of A. [The notation used here means that, applying both the left-hand and the right-hand sides of the relations to any vector, one obtains a valid relation]. This representation is called the *spectral decomposition* of A and is described for general linear compact operators in section 21.3.2. [Hint: Since A is real and symmetric, it can be diagonalized. Once you have the diagonal form, it is obvious how to pick matrices such that (*) and (**) hold. At this point it is just a matter of going back ("un-diagonalize") and show that (*) and (**) are still satisfied].

9. Find the general expression for the n-th power of the matrix

$$M = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} ,$$

10. Given the matrices

$$M_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} , \quad M_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix},$$

calculate in closed form the matrices $A_i = \exp(\theta M_i)$, $i = 1, 2$, where θ is a constant.

11. Given the matrix

$$M = \begin{vmatrix} 2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{vmatrix} ,$$

calculate $M^{1/2}$ by diagonalizing etc.

12. Calculate in closed form $\exp A$ where the matrix A is given by

$$A = \begin{vmatrix} a & b \\ b & a \end{vmatrix},$$

with a, b given parameters.

13. Solve by matrix exponentiation the system

$$\dot{x} = y, \qquad \dot{y} = -x\,,$$

subject to the initial conditions $x(0) = x_0$, $y(0) = y_0$.

14. In $0 \le x \le \pi$ consider the system of equations

$$\frac{\partial \mathbf{u}}{\partial t} = \mathsf{B}\, \frac{\partial^2 \mathbf{u}}{\partial x^2}$$

where $\mathbf{u}$ is a vector with components $u_1(x,t), u_2(x,t), \dots u_N(x,t)$ and B is a constant $N \times N$ symmetric real matrix. The boundary conditions are $\mathbf{u}(0,t) = \mathbf{u}(\pi,t) = 0$ and the initial condition $\mathbf{u}(x,0) = \mathbf{f}(x)$, with $\mathbf{f}$ a given N-dimensional vector, $\mathbf{f} = (f_1, f_2, \dots, f_N)$. Solve this problem in general and then consider the special case in which $N = 2$, $f_1(x) = f_2(x)$ and

$$\mathsf{B} = c \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

with c a constant. [Hint: Start with a Fourier expansion in x.]

15. A Vandermonde matrix of order N is a square matrix with the structure

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ v_1 & v_2 & v_3 & \dots & v_N \\ v_1^2 & v_2^2 & v_3^2 & \dots & v_N^2 \\ \dots & \dots & \dots & \dots & \dots \\ v_1^{N-1} & v_2^{N-1} & v_3^{N-1} & \dots & v_N^{N-1} \end{vmatrix}$$

Show that the determinant of such a matrix is given by $\prod_{i>j}(v_i - v_j)$, namely by the product of all the differences $v_i - v_j$ with $i > j$.